Formalizing Commutative Algebra in Coq: Nakayama's Lemma

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Abstract

We describe our formal proof of Nakayama's Lemma, a fundamental theorem in the mathematical field of commutative algebra. The statement and proof of this result involve several commutative-algebraic structures including commutative rings, ideals of these rings, and modules over them, and we also explain our process of formalizing these structures. Source code for this work is available on the following site: https://github.com/ku-sldg/algebra.

Keywords: formalization of mathematics, formal proof, commutative algebra, commutative ring, local ring, ideal, module over a ring, finitely generated module

1 Introduction

The mathematical field of *commutative algebra* stems from the study of solutions to polynomial equations. Research in the field now centers around *commutative rings*—rings in which order does not affect multiplication, i.e., $x \cdot y = y \cdot x$ for any ring elements x and y—and fundamental algebraic objects

associated to them: *ideals* of these rings, and *modules* over them. Commutative algebra has deep connections with other areas of theoretical mathematics, including number theory and algebraic geometry.

Commutative algebra also has broad applications to science and technology. For instance, it has been integral to advances in robotics [7], and has helped form our current understanding of the human genome [1]. The commutative-algebraic notion of a Gröbner basis, a special type of generating set for an ideal in a ring of polynomials, has become a fundamental computational tool in coding theory and cryptography (e.g., see [1]). A implementation of Buchberger's algorithm [4] for determining Gröbner bases of ideals in polynomial rings has been proved correct within the proof assistant Coq [1, 5], and an integrated formal development of the algorithm in Coq has also been carried out [1] (see also [6]).

Our goal is to newly formalize theoretical, rather than computational, commutative algebra in Coq. We formally prove *Nakayama's Lemma* [1, 3], an essential result in the field. In doing so, we formalize algebraic structures that are fundamental to higher-level algebra, such as *local rings* and *modules over commutative rings*, and *quotient rings and modules*. Rather than build upon some of the basic objects from abstract algebra, such as groups and rings, that have been formalized in Coq, e.g., in the *Mathematical Components Library* [1], we start from scratch. The theory, including the formalization of all algebraic structures, makes up approximately 100 kB, and 3300 lines of code.

The notion of a module over a ring is an extension of the linear-algebraic notion of a vector space over a field, ubiquitous in mathematics and its applications. Less frequently referred to as the *Krull-Azumaya theorem* [1] [1], Nakayama's Lemma describes one way that a finitely generated module over

[1] Hideyuki Matsumura explains in his text *Commutative Algebra* [9]: "This simple but important lemma is due to T. Nakayama, G. Azumaya, and W. Krull. Priority is obscure, and although it is usually called the Lemma of Nakayama, late Prof. Nakayama did not like the name."

an arbitrary commutative ring acts like a vector space over a field. True to the convention that "lemma" often refers to a result serving as a stepping stone toward another goal, Nakayama's Lemma is applied widely throughout the field, and the result is typically introduced in a first graduate course in commutative algebra [2, 8, 9].

2 Mathematical Basis and Motivation

2.1 The Fundamental Algebraic Structures

Here, we give a brief description of the major mathematical structures from commutative algebra that are relevant to Nakayama's Lemma.

Commutative rings

In abstract algebra, the quintessential example of a commutative ring is the set of integers

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

using the natural definitions of addition and multiplication.

Adding two integers produces another, and the associative and commutative laws hold for addition. The integers form an *abelian group* under addition since $0 \in \mathbb{Z}$ is the *additive identity* in the sense that adding zero has no effect on any integer, and given any integer n, the integer $-n$ is its *additive inverse* in the sense that the sum of n and $-n$ is the additive identity 0.

The set of integers also forms a *ring* due to its properties of multiplication. It is closed under this binary operation, which satisfies associativity, and the distributive law governing the compatibility of addition and multiplication holds. We require rings to contain a *multiplicative identity*, and $1 \in \mathbb{Z}$ is such an element since $n \in \mathbb{Z}$ one has $n \cdot 1 = 1 \cdot n = n$. Even more, the integers form a *commutative ring* since $n \cdot m = m \cdot n$ for all integers n and m.

In general, a commutative ring is a set R with two binary operations, which we call *addition* and *multiplication*, typically denoted $\cdot$ and $+$, respectively. As motivated by the properties of the ring of integers, addition, R must be an abelian group, multiplication must be associative, R must have a multiplicative identity, and the distributive law must hold, i.e., for all $r, s, t \in R$, $(r + s) \cdot t = r \cdot t + s \cdot t$ and $r \cdot (s + t) = r \cdot s + r \cdot t$.

Other familiar examples of commutative rings include the integers modulo a fixed integer $n > 0$, fields—commutative rings in which every nonzero element has a multiplicative inverse—such as the rings of rational numbers $\mathbb{Q}$, real numbers $\mathbb{R}$, and complex numbers $\mathbb{C}$, and the rings of polynomials in a variable x with integer coefficients, or with coefficients in a field.

A *subgroup* of a group G is a subset H of G that is itself a group when the binary operation of G is restricted to H. Similarly, a *subring* of a ring R is a subset S of R that forms a group when the operations on R are restricted to S, using the same multiplicative identity. One has the sequence of subrings $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

Ideals of commutative rings

The concept of an ideal of a ring can be thought of as an extension of the notion of an integer n in the ring of integers $\mathbb{Z}$. An *ideal* of commutative ring R is a subset I of R that is itself an abelian group under addition, which also satisfies the following "absorption" property: Given any element a of I, the product $x \cdot a$ is again in I for any ring element $x \in R$.

One can verify that given any integer n, the set $n\mathbb{Z}$ of its multiples forms an ideal of $\mathbb{Z}$. For instance, $2\mathbb{Z}$ consists of all even numbers, and is an abelian group under addition: the sum of two even numbers is even, the additive identity 0 is even, and the negative of an even number is even. Moreover, the absorption property holds since the product of any integer and an even

number is again even. In fact, every ideal of the ring of integers has this form $n\mathbb{Z}$ for some integer n, though ideals in general commutative rings can have more complicated properties.

Since every integer n can be written as $1 \cdot n$, the ideal $1\mathbb{Z}$ is the entire ring $\mathbb{Z}$. One can see that given a commutative ring R itself satisfies the axioms required to be an ideal of R. We call an ideal I of R *proper* if it is strictly contained in R. The *zero ideal* consisting solely of its additive identity is a proper ideal of any commutative ring.

A *prime ideal* of a commutative ring is a proper ideal I with the following property: If the product $x \cdot y$ of ring elements x and y is in I, then $x \in I$ or $y \in I$. The naming convention is motivated by the ring of integers, where the prime ideals are precisely those of the form $p\mathbb{Z}$, where p is a prime number, along with the zero ideal.

A *maximal ideal* of a commutative ring is a proper ideal that is maximal with respect to inclusion, i.e., no other proper ideal strictly contains it. Returning to our example of the ring of integers, $6\mathbb{Z} \subsetneq 2\mathbb{Z}$ since every multiple of 6 is even, so $6\mathbb{Z}$ is not a maximal ideal of $\mathbb{Z}$. However, no proper ideal I contains $2\mathbb{Z}$: If $2\mathbb{Z} \subsetneq I \subsetneq \mathbb{Z}$, then I would necessarily contain an odd number n. Writing $n = 2k + 1$ for some integer k, we notice that since $-2k$ is in $2\mathbb{Z}$, it is also an element of the larger set I, and since I is an abelian group under addition, $(2k + 1) + (-2k) = 1$ is also in the ideal I. However, in this case, every integer $n = n \cdot 1$ is in I by absorption, so $I = \mathbb{Z}$ is not a proper ideal, a contradiction.

In fact, $3\mathbb{Z}$ is the only other maximal ideal of $\mathbb{Z}$ containing $6\mathbb{Z}$, and in general, the prime ideals in the ring of integers besides the zero ideal are $p\mathbb{Z}$, where p a prime number. It is not a coincidence that every maximal ideal of the ring of integers is also a prime ideal; the analogous statement can be proved in arbitrary commutative rings.

Local rings

A commutative ring is *local* if it has exactly one maximal ideal. Every field is local since the only proper ideal of a field is the zero ideal, though by our observations above, the ring of integers is not local. However, the set of all rational numbers that can be written with an odd denominator does form a subring of all rational numbers, and its unique maximal ideal consists of the elements with even numerator; in fact, this ring is the so-called *localization* of $\mathbb{Z}$ at the maximal ideal $2\mathbb{Z}$. The ring of integers modulo $n > 1$ is local if and only if n is a power of a prime number p, in which case the unique maximal ideal consists of all multiples of p.

The ring of polynomials over a field F in a variable x is not local; in fact, given any irreducible polynomial $f(x)$, the set of its multiples is a maximal ideal of the polynomial ring $F[x]$. On the other hand, the set of all formal power series in x over F is a local ring; its maximal ideal consists of the power series with no constant term.

Modules over commutative rings

Let R be a commutative ring. A *module over* R, or R*-module*, is an abelian group M under a binary operation $+$, and a scalar multiplication $R \times M \to M$ denoted $\cdot$, satisfying the following properties for all $r, s \in R$ and $u, v \in M$.

1. $r \cdot (u + v) = r \cdot u + r \cdot v$
2. $(r + s) \cdot u = r \cdot u + s \cdot u$
3. $(rs) \cdot u = r \cdot (s \cdot u)$
4. $1 \cdot u = u$

From this definition, one can see that a module over a field F is precisely an F-vector space, so the notion of a module over an arbitrary commutative ring extends that of a vector space over a field. An R*-submodule* N of a module

M is simply a subgroup of M that inherits the scalar multiplication of N, i.e., $r \cdot w$ is an element of N whenever $w \in N$. Therefore, returning to the case that F is a field, the F-submodules of a vector space over F are precisely its vector subspaces.

Finitely generated vector spaces form the foundation for matrix algebra, and the extension of this notion to module theory is needed to state Nakayama's Lemma. We call an R-module M *finitely generated* if there exist a fixed finite list of elements $u_1, \ldots, u_n$ of M such that every element of M is a scalar combination of the u_i. In other words, given any $w \in M$, there exist $r_1, \ldots, r_n \in R$ for which

$$w = r_1 u_1 + r_1 u_2 + \cdots + r_n u_n.$$

The set $\{u_1, \ldots, u_n\}$ is called a *generating set* for the M as an R-module.

It is straightforward to see that a finitely generated module over a field is simply a finite dimensional vector space. Moreover, if V is a finite dimensional vector space over a field F, one can choose the generators $u_1, \ldots, u_n$ to be a basis for V, so that $n = \dim V$. In this case, the choice of scalar coefficients in the expression above for $w \in V$ is unique. When R is not a field, however, such an expression is typically not unique.

2.2 Nakayama's Lemma, Informal Statement

In order to informally state Nakayama's Lemma, we first explain some of notational conventions. In this discussion, suppose that I is an ideal of a commutative ring R, and that M is an R-module. Then IM denotes the set of all scalar combinations of elements in M with coefficients in I. That is, IM is the

set of elements of the form, for some positive integer k,

$$a_1u_1 + a_2u_2 + \cdots + a_ku_k$$

where $a_1, \ldots, a_k \in I$ and $u_1, \ldots, u_k \in M$. Observe the absorption property of ideals implies that IM is an R-submodule of M.

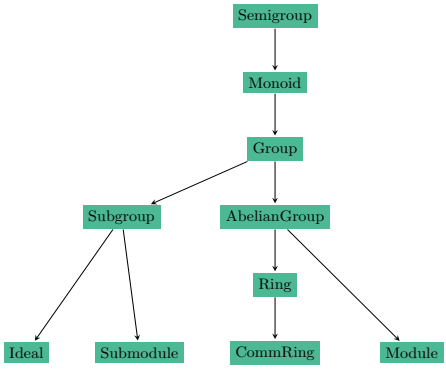
If an R-module M consists of just one element, this element must be its additive identity 0, by virtue of the fact that M is an abelian group under addition. The notation $M = 0$ is used to indicate that we are in this situation.

Nakayama's Lemma. *Let R be a commutative local ring, and let $\mathfrak{m}$ denote its unique maximal ideal. If M is a finitely generated R-module and $M = \mathfrak{m}M$, then $M = 0$.*

When $R = F$ is a field, its unique maximal ideal is the zero ideal. Given any vector space $M = V$ over F, the only linear combination of vectors with coefficients in the zero ideal is the zero vector. Hence in this case, regardless of the choice of vector space, the hypothesis that $M = \mathfrak{m}M$ is equivalent to the conclusion that $M = 0$. In particular, the hypothesis that V is finitely generated, which is equivalent to the assumption that it is a finite-dimensional F-vector space, is not required in this setting. Therefore, Nakayama's Lemma describes one way that finitely generated modules over commutative local rings are similar to vector spaces.

The quotient $R/\mathfrak{m}$ of a commutative ring R modulo any maximal ideal $\mathfrak{m}$ is a field; in fact, this property characterizes maximal ideals. Moreover, given an arbitrary R-module M, its quotient modulo the submodule $\mathfrak{m}M$ is an $R/\mathfrak{m}$-module, i.e., $M/\mathfrak{m}M$ is a vector space over $R/\mathfrak{m}$. Nakayama's Lemma implies

Fig. 3.0.1 The hierarchy of our algebraic structures in Coq



that if M is finitely generated, then bases for $M/\mathfrak{m}M$ corresponds, via lifting, to minimal sets of generators of M.

We point out that there are alternate statements of Nakayama's Lemma that do not require the hypothesis that R must be local. One can replace the unique maximal ideal with the Jacobson radical of the ring, which is the intersection of all maximal ideals. Alternatively, I is an arbitrary proper ideal of a commutative ring R and M is a finitely generated R-module for which $M = IM$, then this ensures the existence of a ring element r congruent to 1 modulo I such that $rM = 0$, i.e., $ru = 0$ for every $u \in M$.

3 Our Algebraic Hierarchy

In this section we describe our process of formalizing the necessary algebraic structures, detailed in the previous section, in Coq. Then, with these formal definitions in hand, we move on in the next section to detail our formal proof of Nakayama's Lemma.

3.1 Semigroups

Appearing at the top of Figure 3.0.1, our foundation begins by defining a semigroup class, which declares a binary operation to be associative. From

here, we build up through monoids, which introduce identities, to groups, which introduce inverses. Note the double equals, "==", appearing in these definitions is notation for an arbitrary equivalence relation over the group's carrier set, which acts as equality.

```
Infix "==" := equiv (at level 60, no associativity).
Class Semigroup := {
  semigroup_assoc:
    forall (a b c: Carrier),
      a <o> b <o> c == a <o> (b <o> c);
}.
Class Monoid := {
  monoid_semigroup :> Semigroup equiv op;
  monoid_ident_l:
    forall (a: Carrier), ident <o> a == a;
  monoid_ident_r:
    forall (a: Carrier), a <o> ident == a;
}.
Class Group := {
  group_monoid :> Monoid equiv op ident;
  group_inv_l:
    forall (a: Carrier), inv a <o> a == ident;
  group_inv_r:
    forall (a: Carrier), a <o> inv a == ident;
}.
```

The line "`monoid_semigroup :> Semigroup equiv op;`" simply coerces the monoid typeclass into a semigroup, and similar lines perform analogous functions.

3.2 Algebraic Quotients

While in the end, our formal proof does not call upon quotients of algebraic structures, alternative proofs of Nakamaya's Lemma take advantage of the structures quotient rings and quotient modules. As algebraic quotient structures are also fundamental to commutative algebra, it is worth pointing out that we have formalized quotients of groups, rings, and modules using typeclasses, which appear to work rather smoothly.

An algebraic quotient is, roughly, the set of equivalence classes of an algebraic structure with respect to an equivalence relation on its elements, for which the set of equivalence classes inherit the same kind of algebraic structure. As an example, consider the quotient of a group modulo a subgroup, i.e., a subset of elements of the group that it itself a group using the same operations. Under this equivalence relation on the group, every element of the subgroup must be in the same equivalence class as the identity. For instance, after taking the quotient of the ring of integers $\mathbb{Z}$ by the subgroup $n\mathbb{Z}$ consisting of all multiples of an integer n, one obtains the group of integers modulo n, often denoted $\mathbb{Z}/n\mathbb{Z}$. In general, with `P` the predicate for the subgroup, there are two ways to make an equivalence relation from this description.

```
Definition left_congru (a b: Carrier) :=
  P (inv a <o> b).
Definition right_congru (a b: Carrier) :=
  P (a <o> inv b).
```

When these two equivalence relations coincide, then we can prove that this common relation actually preserves the group structure. Subgroups of a group that satisfy this property are called *normal subgroups*.

```
Let normal_subgroup_congru_coincide :=
  forall (a b: Carrier),
```

```
    left_congru op inv P a b <->
    right_congru op inv P a b.

Theorem quotient_normal_subgroup_group:
  normal_subgroup_congru_coincide ->
  Group (left_congru op inv P) op ident inv.
```

The utility–and, as a consequence, the ubiquity–of quotients in algebra motivates our choice to use equivalence relations to define the components of a group structure. If one were to instead use the traditional Leibniz equality, it would be difficult to identify a quotient group with another group. However, by defining a group in terms of an arbitrary equivalence relation, in our theory a quotient group is simply defined as a group, but under an equivalence relation that is not the usual equality. Not much is lost in adopting this convention, thanks to Coq's rewrite tactics for setoids, which are types equipped with an equivalence relation.

3.3 Rings and their Ideals

Moving onward, rings form the next step in our algebraic hierarchy; a ring has two binary operations: addition, which must be commutative, and multiplication, which need not be commutative in general. In our formulation, rings must have a multiplicative identity. Next, we formalize the definition of a commutative ring, further requiring commutativity of multiplication.

At this point, we formalize several algebraic structures defined in terms of commutative rings. First is the definition of an ideal of a commutative ring, a subgroup of the ring under addition that satisfies the absorption property under multiplication, and with this, also the notion of a quotient ring R/I, where I is an ideal of a commutative ring R. We also formalize the definition

of a prime ideal, and that of a maximal ideal–a proper ideal that is maximal with respect to inclusion. Below is the definition of latter in Coq, which uses `P` as the predicate for the ideal.

```
Definition maximal_ideal :=
  exists (r: Carrier), (not (P r) /\
    forall (Q: Carrier -> Prop)
        (Q_proper: Proper (equiv ==> iff) Q)
        (Q_ideal: Ideal add zero minus mul Q),
      (forall (r: Carrier), P r -> Q r) ->
      (forall (r: Carrier), Q r) \/
        (forall (r: Carrier), Q r -> P r)).
```

Next, we employ the above definition to formally define a local ring, i.e., a commutative ring with a single maximal ideal.

```
Definition local_ring :=
  exists (P: Carrier -> Prop)
      (P_proper: Proper (equiv ==> iff) P)
      (P_ideal: Ideal add zero minus mul P),
    maximal_ideal P /\
    (forall (Q: Carrier -> Prop)
        (Q_proper: Proper (equiv ==> iff) Q)
        (Q_ideal: Ideal add zero minus mul Q),
      maximal_ideal Q -> forall (r: Carrier), P r <-> Q r).
```

3.4 Modules over Rings

With the fundamental definitions for commutative rings and their ideals formalized, we move on to build our formal module theory. We start by formalizing the definition of a module over a commutative ring, the

commutative-algebraic generalization of the notion of a vector space over a field. Nakayama's Lemma is a statement about finitely generated modules, and hence we formalized the notion of a scalar combination of a finite collection of elements, $u_1, \ldots, u_n$ a module M over a commutative ring R, i.e., expressions of the form $r_1u_1 + r_1u_2 + \cdots + r_nu_n$, where each r_i is an element of R.

In our formalization of scalar combinations, we use "list" to mean length-parameterized lists; since we don't use the simpler kind of lists, there are no name collisions. In the code excerpted below, `M` is the type of module elements, `R` is the type of ring elements, acting as coefficients, and `t A n` is a list whose elements are of type `A` and whose length is `n`.

```
Definition finitely_generated {n: nat}(genSet: t M n) :=
  forall (elt: M),
    exists (coeffs: t R n),
      elt =M= linear_combin coeffs genSet.
```

Finally, we define the submodule IM of a module M over a commutative ring R, where I is an arbitrary ideal of R. Recall that when R is local and I is its unique maximal ideal, this submodule appears in the statement of Nakayama's Lemma. In general, IM is the set consisting of all scalar combinations of elements of M whose coefficients are in I. We represent this in Coq as a predicate over M.

```
Context (P: R -> Prop).
Context {P_proper: Proper (Requiv ==> iff) P}.
Context {P_ideal: Ideal Radd Rzero Rminus Rmul P}.

Definition ideal_module (x: M): Prop :=
  exists (n: nat)(coeffs: t R n)(elts: t M n),
```

```
Forall P coeffs /\
x =M= linear_combin Madd Mzero action coeffs elts.
```

The use of "`Forall P coeffs`" ensures that every element of the coefficient list `coeffs` satisfies the predicate `P`.

4 Constructing the Formal Proof

Not only to aid in the formal proof of Nakayama's lemma, but also to add to the body of formalized theory in commutative algebra, along with our formal definitions of algebraic structures, we also establish some basic theory of these objects in Coq. For instance, consider the notion of a *unit* of a commutative ring R, which is an element $a \in R$ with a multiplicative inverse, i.e., there exists an element $a^{-1} \in R$ for which $x^{-1} \cdot x$ is the multiplicative identity $1 \in R$. In fact, it x is a unit, its inverse is unique.

Suppose that I is an ideal of a commutative ring R. Then if I is the trivial ideal, i.e., $I = R$, then it, of course, contains a unit, namely 1. We formally establish the converse of this statement: If I contains a unit a, then $I = R$. The informal proof logic is as follows: Due to the absorption property of I, $a^{-1} \cdot a = 1 \in I$. Hence for every element r of R, $r = r \cdot 1$ is also in I, i.e., $I = R$. On the other hand, if $I = R$, then the multiplicative identity 1 is a unit in I.

We also use classical logic to prove that $1 - x$ is a unit whenever x is an element of a local ring that is not a unit. The proof is completed by way of contradiction, and uses the rule that $\neg\neg P \to P$.

4.1 Dealing with the Axiom of Choice

Every non-unit element of a commutative ring is contained in some maximal ideal, a fact that relies on the Axiom of Choice. This fact can be derived from from the weaker statement that every commutative ring contains a maximal

ideal: Indeed, assume this, and fix a non-unit x of a commutative ring R. Then the quotient $R/\langle x\rangle$ of R modulo the principal ideal generated by x is a commutative ring, and so contains a maximal ideal, which corresponds precisely to a maximal ideal of R containing x by the lattice isomorphism theorem for rings.

The following standard informal proof of the fact that every commutative rings contains a maximal ideal calls upon Zorn's lemma, which is equivalent to the Axiom of Choice (assuming Zermelo-Fraenkel Set Theory), and which says that if every chain of elements from a partially ordered set has an upper bound in the set, then the set must contain at least one maximal element.

Fix a commutative ring R. Since $1 \neq 0$ in R, the zero ideal is proper. If the zero ideal is maximal, then we have identified a maximal ideal of R. If not, then there must exist a proper ideal I_1 of R containing more than just the element 0.

If I_1 is maximal, then we have found a maximal ideal, but if not, then there exists a proper ideal I_2 of R that strictly contains I_1. Likewise, if I_2 is maximal, we are done, and if not, there exists a proper ideal I_3 such that $I_3 \supsetneq I_2$. Continuing this pattern, we either identify a maximal ideal of R, or we construct an infinite chain of proper ideals of R:

$$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots \subsetneq I_n \subsetneq \cdots . \tag{4.1}$$

More precisely, assume by induction that there exist proper, non-maximal ideals I_k of R, $1 < k \leq n$, such that each I_{k-1} is strictly contained in I_k. Since I_n is not maximal, there exists a proper ideal I_{n+1} of R such that $I_n \subsetneq I_{n+1}$. Then either I_{n+1} is maximal, or we have extended the chain by one additional link.

It is straightforward to verify, by definition, that an increasing union of ideals is again an ideal. Hence if, in this process, no maximal ideal is identified, i.e., we have ensured the existence of a chain of the form (4.1), then the increasing union of the I_k is again an ideal.

> Let X denote the set of all proper ideals of R, partially ordered by inclusion. We have shown that every chain in X has an upper bound in X, so Zorn's lemma ensures the existence of a maximal element of X, i.e., a maximal ideal of R.

This argument potentially requires infinitely many steps, posing an implementation problem. Indeed, suppose one were to write a Coq proof tactic called `generate_larger_ideal` that allows one to move from a non-maximal, proper ideal I that contains a fixed non-unit x to a strictly larger ideal J, which is again proper, and also contains x.

After fixing a proper ideal I_1, provided that it is not maximal, our first invocation of `generate_larger_ideal` will produce another ideal, I_2. It is possible that I_2 is also non-maximal, requiring that we invoke `generate_larger_ideal` again, producing I_3, which again could be non-maximal, and so on, which could lead to an infinite number of calls to the tactic `generate_larger_ideal` Of course, this would occupy any machine's processor(s) forever, and further, the boundlessly increasing number of ideals in the proof state would fill any machine's memory.

Even if it were possible not to overwhelm a system with these infinitely many calls to a single proof tactic, one would still have to invoke more tactics to finish the proof with an application of Zorn's lemma, which finally guarantees us a maximal ideal containing x. To summarize, we are forced to run the following proof script which will never get past calling `generate_larger_ideal` infinitely many times.

```
(* must be called infinitely often *)
repeat generate_larger_ideal.
(* called after the above non-terminating call finishes *)
zorns_lemma.
```

We choose to avoid this infinite proof script issue by including an axiom that any non-unit x of a ring must be contained in some maximal ideal.

```
Axiom comm_ring_nonunit_maximal_ideal:
  forall (x: Carrier),
    ~ is_unit equiv mul one x ->
    exists (P: Carrier -> Prop)
      (P_proper: Proper (equiv ==> iff) P)
            (P_ideal: Ideal add zero minus mul P),
      P x /\ maximal_ideal P.
```

4.2 The Formal Statement

We now present our formal statement of Nakayama's Lemma in Coq, and describe a lemma called upon in our proof.

Nakayama's Lemma. *Let R be a commutative local ring, and let $\mathfrak{m}$ denote its maximal ideal. Suppose that M is a finitely generated R-module. If $M = \mathfrak{m}M$, then $M = 0$, i.e., M must be the R-module containing only one element, its identity as an additive abelian group.*

```
Context {P_ideal: Ideal Radd Rzero Rminus Rmul P}.
Context {P_maxideal: maximal_ideal Requiv Radd ... Rmul P}.
Context {R_local: local_ring Requiv Radd Rzero Rminus Rmul}.

Let ideal_module_pred := ideal_module Mequiv Madd Mzero action P.

Theorem nakayama:
  forall {n: nat}(basis: t M n),
    finitely_generated Mequiv Madd Mzero action basis ->
```

```
  (forall a: M, ideal_module_pred a) ->
  forall a: M, a =M= Mzero.
```

To use as a building block in the formal proof of Nakayama's Lemma, we formally state and prove a lemma that gives a concrete description of the elements of the submodule $\mathfrak{m}M$ appearing in its statement. This lemma applies more generally, to an arbitrary finitely generated module M over a (not necessarily local) commutative ring, and any submodule of the form IM, for I an arbitrary ideal.

By definition, IM consists of scalar combinations of elements of M with coefficients in I. The lemma states that the elements from M appearing in such an expression can be chosen to be from any fixed finite generating set for M. In other words, given any generating set $u_1, \ldots, u_n \in M$ for a finitely generated R-module M, every element x of IM can be written as a scalar combination $a_1 \cdot u_1 + a_2 \cdot u_2 + \cdots + a_n \cdot u_n$, where each $a_i \in I$.

```
Lemma module_fin_gen_ideal_module:
  forall {n: nat}(genSet: t M n),
    finitely_generated Mequiv Madd Mzero action genSet ->
    forall {m: nat}(coeffs: t R m)(elts: t M m),
      Forall P coeffs ->
      exists (coeffs': t R n),
        linear_combin Madd Mzero action coeffs elts =M=
          linear_combin Madd Mzero action coeffs' genSet /\
        Forall P coeffs'.
```

This lemma's proof follows from a straightforward argument based on definitions, by induction on the number of elements in a fixed generating set for M, which we informally describe: By definition, $x \in IM$ can be written, for some positive integer k and elements $r_i \in R$ and $w_i \in M$, $1 \leq i \leq k$, as

$x = r_1 \cdot w_1 + r_2 \cdot w_2 + \cdots + r_k \cdot w_k$. What's more, by definition of a finite generating set $u_1, \ldots, u_n$ for M, each w_i equals $c_{i1} \cdot u_1 + c_{i2} \cdot u_2 + \cdots + c_{in} \cdot u_n$ for appropriate choices of $c_{ij} \in R$. Hence, inductively applying associativity,

$$\begin{aligned} x &= \sum_{i=1}^{k} r_i \cdot w_i = \sum_{i=1}^{k} r_i \cdot \left(\sum_{j=1}^{n} c_{ij} \cdot u_j \right) \\ &= \sum_{j=1}^{n} \sum_{i=1}^{k} r_i \cdot (c_{ij} \cdot u_j) = \sum_{j=1}^{n} \left(\sum_{i=1}^{k} c_{ij} \cdot r_i \right) \cdot u_j. \end{aligned}$$

By the absorption property of ideals, each $c_{ij} \cdot r_i$ is in I, and since ideals are closed under addition, we inductively conclude that the coefficient $a_j := c_{1j} \cdot r_1 + c_{2j} \cdot r_2 + \cdots c_{kj} \cdot r_k$ of u_j is also in I.

4.3 The Formal Proof

We now outline our formal proof of Nakayama's lemma, and proceed by way of induction on the number of elements in a fixed generating set for the finitely generated R-module M. The base case in this situation is when M requires no generators, so M consists solely of the empty scalar combination, i.e., the empty sum, which, by convention, is the zero element. In other words, $M = 0$ by assumption, and this is also precisely the conclusion of Nakayama's Lemma. Hence the statement holds trivially, without using the hypothesis that $M = \mathfrak{m}M$.

We turn to the inductive step. Fixing an arbitrary nonnegative integer n, we assume that Nakayama's Lemma holds in the case that M has a generating set consisting of n elements.

Now, fix an R-module M generated by $u_1, \ldots, u_{n+1} \in M$. By assumption, $M = \mathfrak{m}M$, and in particular, the generator u_1 is an element of the submodule $\mathfrak{m}M$. The lemma described in Section 4.2 guarantees the existence of ring

elements $a_1, \ldots, a_n \in \mathfrak{m}$ for which

$$u_1 = a_1 \cdot u_1 + a_2 \cdot u_2 + \cdots + a_{n+1} \cdot u_{n+1}.$$

Collecting the u_1 terms on the left-hand side of the equation, calling on the existence of additive inverses in R, and distributivity of scalar multiplication for modules, we see that

$$(1 - a_1) \cdot u_1 = a_2 \cdot u_2 + \cdots + a_{n+1} \cdot u_{n+1}. \tag{4.2}$$

In Coq, we work backward to establish (4.2). The comment following each tactic appearing in the code below offers a brief indication of its effects on the goal.

```
(* (1 - a1) u1 = ... *)
assert ((Rone [+] Rminus a1) <.> u1 =M=
  linear_combin Madd Mzero action coeffs' generatingSet').
{ (* 1 u1 + (-a1) u1 = ... *)
  setoid_rewrite (module_distrib_Radd Radd Rmul Rone
    Mequiv Madd Mzero Mminus action).
  (* u1 + (-a1) u1 = ... *)
  setoid_rewrite (module_Rone Radd Rmul Rone
    Mequiv Madd Mzero Mminus action).
  (* u1 - a1 u1 = ... *)
  setoid_rewrite (module_minus_l Requiv Radd ... Rone
    Mequiv Madd Mzero Mminus action).
  symmetry. (* ... = u1 - a1 u1 *)
  (* ... + a1 u1 = u1 *)
```

```
apply (group_move_r Mequiv Madd Mzero Mminus).
(* u1 = ... + a1 u1 *)
symmetry.
assumption. }
```

Since a proper ideal contains no units and a_1 is taken to be in $\mathfrak{m}$, it is not a unit. Hence, by the formalized statement mentioned early in Section 4, $1-a_1$ is a unit of R. Let $b_1 \in R$ denote its multiplicative inverse, so that $b_1 \cdot (1-a_1) = 1$. Multiplying this element on the either side of (4.2), we deduce the following.

$$b_1 \cdot ((1 - a_1) \cdot u_1) = b_1 \cdot (a_2 \cdot u_2 + \cdots + a_{n+1} \cdot u_{n+1}) \tag{4.3}$$

$$(b_1 \cdot (1 - a_1)) \cdot u_1 = b_1 \cdot (a_2 \cdot u_2) + \cdots + b_1 \cdot (a_{n+1} \cdot u_{n+1}) \tag{4.4}$$

$$u_1 = 1 \cdot u_1 = (b_1 \cdot a_2) \cdot u_2 + \cdots + (b_1 \cdot a_{n+1}) \cdot u_{n+1} \tag{4.5}$$

```
pose proof
  (local_comm_ring_sub_1_nonunit Requiv Radd ... Rone
    R_local x1 Hx1_nonunit) as H1ma1_unit.
(* Getting to u1 = (b1 coeffs') . genSet *)
(* 1 - a1 is a unit *)
inversion_clear H1ma1_unit as [b1 Hb1].
(* b1 (1 - a1) = 1 *)
setoid_rewrite (commutative Requiv Rmul) in Hb1.
```

This block of code has the effect of naming b_1 as the inverse of the unit $1 - a_1$, and storing in the hypothesis `Hb1` the equation $b_1 \cdot (1 - a_1) = 1$. The remainder of the code goes on to multiply on the left by b_1 in the equation `H` in order to establish (4.3). Going on and applying the axioms of a module over a commutative ring, along with hypothesis `Hb1`, we can simplify the left hand side of `H` to be just u_1, as in (4.5).

```
apply (module_op_l Requiv Mequiv action) with (r:=b1) in H.
(* H: (b1 (1 - a1)) u1 = ... *)
setoid_rewrite <- (module_distrib_Rmul Radd Rmul Rone
  Mequiv Madd Mzero Mminus action) in H.
(* H: (1) u1 = ... *)
setoid_rewrite Hb1 in H.
(* H: u1 = ... *)
setoid_rewrite (module_Rone Radd Rmul Rone
  Mequiv Madd Mzero Mminus action) in H.
```

In particular, the generator u_1 can be written as a scalar combination of the n generators $u_2, \ldots, u_{n+1}$, and is superfluous. Hence M can be generated by n elements, and $M = 0$ by the inductive hypothesis. Though showing that the inductive hypothesis holds is more technical in Coq than in this informal proof, the extra steps are essentially bookkeeping.

5 Conclusion

Our work here is evidence that Coq is amendable to the formalization of abstract algebraic notions from commutative algebra, which are often non-discrete, and their theory. There are several natural directions in which to move from here, a first step being to further develop the formal theory of quotient rings, and to formalize the isomorphism theorems for rings.

Our current body of work can be extended to include further essential concepts from commutative algebra, e.g., module homomorphisms; direct sums and free modules; tensor products of modules and exact sequences; projective, injective, and flat modules; direct and inverse limits of modules; and/or graded rings and modules. In this process, building formal proofs of other fundamental theorems involving these notions will likely be valuable in effectively designing

their formal definitions, as we noticed in building our algebraic heirarchy for Nakayama's Lemma.

Finally, integrating within the formalized commutative algebra theory some of the computational algebra that has been formally developed, such as that regarding Gröbner bases [1, 4], could also be a fruitful direction to pursue.

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