Formalizing Commutative Algebra in Coq: Nakayama's Lemma*

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Abstract

We describe our formal proof of Nakayama's Lemma, a fundamental theorem in the mathematical field of commutative algebra. The statement and proof of this result involve several commutative-algebraic structures including commutative rings, ideals of these rings, and modules over them, and we also explain our process of formalizing these structures.

Keywords: Formalization of Mathematics, Formal Proof, Commutative Algebra, Commutative Ring, Local Ring, Ideal, Module over a Ring, Finitely Generated Module.

1 Introduction

The mathematical field of *commutative algebra* stems from the study of solutions to polynomial equations. Research in the field now centers around *commutative rings*—rings in which order does not affect multiplication, i.e., $x \cdot y = y \cdot x$ for any ring elements x and y—and fundamental algebraic objects associated to them: *ideals* of these rings, and *modules* over them. Commutative algebra has deep connections with other areas of theoretical mathematics, including number theory and algebraic geometry.

Commutative algebra also has broad applications to science and technology. For instance, it has been integral to advances in robotics [7], and has helped form our current understanding of the human genome [13]. The commutative-algebraic notion of a Gröbner basis, a special type of generating set for an ideal in a ring of polynomials, has become a fundamental computational tool in coding theory and cryptography (e.g., see [14]). A implementation of Buchberger's algorithm [4] for determining Gröbner bases of ideals in polynomial rings has been proved correct within the proof assistant Coq [5, 15], and an integrated formal development of the algorithm in Coq has also been carried out [12] (see also [6]).

Our goal is to newly formalize theoretical, rather than computational, commutative algebra in Coq. We formally prove *Nakayama's Lemma* [11, 3], an essential result in the field. In doing so, we formalize algebraic structures that are fundamental to higher-level algebra, such as *local rings* and *modules over commutative rings*, and *quotient rings and modules*. Rather than build upon some of the basic objects from abstract algebra, such as groups and rings, that have been formalized in Coq, e.g., in the *Mathematical Components*

Library [1], we start from scratch. The theory, including the formalization of all algebraic structures, makes up approximately 100 kB and 3300 lines of code.

To do: Verify whether MathComp only formalized finite algebraic structures. Drew is about as certain as he can be that this is the case.

The notion of a module over a ring is an extension of the linear-algebraic notion of a vector space over a field, ubiquitous in mathematics and its applications. Less frequently referred to as the *Krull-Azumaya theorem* [1] [10], Nakayama's Lemma describes one way that a finitely generated module over an arbitrary commutative ring acts like a vector space over a field. True to the convention that "lemma" often refers to a result serving as a stepping stone toward another goal, Nakayama's Lemma is applied widely throughout the field, and the result is typically introduced in a first graduate course in commutative algebra [2, 9, 8].

2 Mathematical Basis and Motivation

2.1 The Fundamental Algebraic Structures

Here, we give a brief description of the major mathematical structures from commutative algebra that are relevant to Nakayama's Lemma.

Commutative rings. In abstract algebra, the quintessential example of a commutative ring is the set of integers

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

using the natural definitions of addition and multiplication.

Adding two integers produces another, and the associative and commutative laws hold for addition. The integers form an *abelian group* under addition since $0 \in \mathbb{Z}$ is the *additive identity* in the sense that adding zero has no effect on any integer, and given any integer n, the integer $-n$ is its *additive inverse* in the sense that the sum of n and $-n$ is the additive identity 0.

The set of integers also forms a *ring* due to its properties of multiplication. It is closed under this binary operation, which satisfies associativity, and the distributive law governing the compatibility of addition and multiplication holds. We require rings to contain a *multiplicative identity*, and $1 \in \mathbb{Z}$ is such an element since $n \in \mathbb{Z}$ one has $n \cdot 1 = 1 \cdot n = n$. Even more, the integers form a *commutative ring* since $n \cdot m = m \cdot n$ for all integers n and m.

In general, a commutative ring is a set R with two binary operations, which we call *addition* and *mutiplication*, typically denoted $\cdot$ and $+$, respectively. As motivated by the properties of the ring of integers, addition, R must be an abelian group, multiplication must be associative, R must have a multiplicative identity, and the distributive law must hold, i.e., for all $r, s, t \in R$, $(r + s) \cdot t = r \cdot t + s \cdot t$ and $r \cdot (s + t) = r \cdot s + r \cdot t$.

Other familiar examples of commutative rings include the integers modulo a fixed integer $n > 0$, fields—commutative rings in which every nonzero element has a multiplicative inverse—such as the rings of rational numbers, real numbers, and complex numbers, and rings of polynomials in a variable x with integer coefficients, or with coefficients in a field.

Ideals of commutative rings. The concept of an ideal of a ring can be thought of as an extension of the notion of an integer n in the ring of integers $\mathbb{Z}$. An *ideal* of commutative ring R is a subset I of R that is itself an abelian group under addition, which also satisfies

*Source code for this work is available on the following site: `https://github.com/ku-sldg/algebra`

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[1] Hideyuki Matsumura explains in his text *Commutative Algebra* [9]: "This simple but important lemma is due to T. Nakayama, G. Azumaya, and W. Krull. Priority is obscure, and although it is usually called the Lemma of Nakayama, late Prof. Nakayama did not like the name."

the following "absorption" property: Given any element a of I, the product $x \cdot a$ is again in I for any ring element $x \in R$.

One can verify that given any integer n, the set $n\mathbb{Z}$ of its multiples forms an ideal of $\mathbb{Z}$. For instance, $2\mathbb{Z}$ consists of all even numbers, and is an abelian group under addition: the sum of two even numbers is even, the additive identity 0 is even, and the negative of an even number is even. Moreover, the absorption property holds since the product of any integer and an even number is again even. In fact, every ideal of the ring of integers has this form $n\mathbb{Z}$ for some integer n, though ideals in general commutative rings can have more complicated properties.

Since every integer n can be written as $1 \cdot n$, the ideal $1\mathbb{Z}$ is the entire ring $\mathbb{Z}$. One can see that given a commutative ring R itself satisfies the axioms required to be an ideal of R. We call an ideal I of R *proper* if it is strictly contained in R. The *zero ideal* consisting solely of its additive identity is a proper ideal of any commutative ring.

A *prime ideal* of a commutative ring is a proper ideal I with the following property: If the product $x \cdot y$ of ring elements x and y is in I, then $x \in I$ or $y \in I$. The naming convention is motivated by the ring of integers, where the prime ideals are precisely those of the form $p\mathbb{Z}$, where p is a prime number, along with the zero ideal.

A *maximal ideal* of a commutative ring is a proper ideal that is maximal with respect to inclusion, i.e., no other proper ideal strictly contains it. Returning to our example of the ring of integers, $6\mathbb{Z} \subsetneq 2\mathbb{Z}$ since every multiple of 6 is even, so $6\mathbb{Z}$ is not a maximal ideal of $\mathbb{Z}$. However, no proper ideal I contains $2\mathbb{Z}$: If $2\mathbb{Z} \subsetneq I \subsetneq \mathbb{Z}$, then I would necessarily contain an odd number n. Writing $n = 2k + 1$ for some integer k, we notice that since $-2k$ is in $2\mathbb{Z}$, it is also an element of the larger set I, and since I is an abelian group under addition, $(2k + 1) + (-2k) = 1$ is also in the ideal I. However, in this case, every integer $n = n \cdot 1$ is in I by absorption, so $I = \mathbb{Z}$ is not a proper ideal, a contradiction.

In fact, $3\mathbb{Z}$ is the only other maximal ideal of $\mathbb{Z}$ containing $6\mathbb{Z}$, and in general, the prime ideals in the ring of integers besides the zero ideal are $p\mathbb{Z}$, where p a prime number. It is not a coincidence that every maximal ideal of the ring of integers is also a prime ideal; the analogous statement can be proved in arbitrary commutative rings.

Local rings. A commutative ring is *local* if it has exactly one maximal ideal. Every field is local since the only proper ideal of a field is the zero ideal, though by our observations above, the ring of integers is not local. However, the set of all rational numbers that can be written with an odd denominator does form a subring of all rational numbers, and its unique maximal ideal consists of the elements with even numerator; in fact, this ring is the so-called *localization* of $\mathbb{Z}$ at the maximal ideal $2\mathbb{Z}$. The ring of integers modulo $n > 1$ is local if and only if n is a power of a prime number p, in which case the unique maximal ideal consists of all multiples of p.

The ring of polynomials over a field F in a variable x is not local; in fact, given any irreducible polynomial $f(x)$, the set of its multiples is a maximal ideal of the polynomial ring $F[x]$. On the other hand, the set of all formal power series in x over F is a local ring; its maximal ideal consists of the power series with no constant term.

Modules over commutative rings. Let R be a commutative ring. A *module over* R, or R*-module*, is an abelian group M under a binary operation $+$, and a scalar multiplication $R \times M \to M$ denoted $\cdot$, satisfying the following properties for all $r, s \in R$ and $u, v \in M$.

1. $r \cdot (u + v) = r \cdot u + r \cdot v$
2. $(r + s) \cdot u = r \cdot u + s \cdot u$
3. $(rs) \cdot u = r \cdot (s \cdot u)$
4. $1 \cdot u = u$

From this definition, one can see that a module over a field F is precisely an F-vector space, so the notion of a module over an arbitrary commutative ring extends that of a vector space over a field. Finitely generated vector spaces form the foundation for matrix algebra, and the extension of this notion to module theory is needed to state Nakayama's Lemma. We call an R-module M *finitely generated* if there exist a fixed finite number of elements $u_1, \ldots, u_n$ of M with the following property: Given any $w \in M$, there exist $r_1, \ldots, r_n \in R$ for which

$$w = r_1 u_1 + r_1 u_2 + \cdots + r_n u_n.$$

The set $\{u_1, \ldots, u_n\}$ is called a *generating set* for the M as an R-module.

When $R = F$ is a field and $M = V$ is a finite-dimensional vector space over F, one can choose $u_1, \ldots, u_n$ to be a basis for V, i.e., $n = \dim V$. In this case, the choice of scalar coefficients in the expression above for $w \in V$ is unique. When R is not a field, however, such an expression is typically not unique.

2.2 Nakayama's Lemma, Informal Statement

In order to state Nakayama's Lemma, we first explain some notation: If I is an ideal of a commutative ring R and M is an R-module, then IM is the set of elements of the form $a_1u_1 + a_2u_2 + \cdots + a_ku_k$, where, for some positive integer k, $a_1, \ldots, a_k \in I$ and $u_1, \ldots, u_k \in M$. Notice that due to the absorption property of ideals, IM is an R-module contained in M.

If an R-module M consists of only one element, this element must be its additive identity 0 as an abelian group under addition. The notation $M = 0$ means that we are in this situation.

Nakayama's Lemma. *Let R be a commutative local ring, and let $\mathfrak{m}$ denote its unique maximal ideal. If M is a finitely generated R-module and $M = \mathfrak{m}M$, then $M = 0$.*

When $R = F$ is a field, its unique maximal ideal is the zero ideal, and given any vector space $M = V$ over F, the only linear combination of vectors with coefficients in the zero ideal is the zero vector. Hence in this special case, the hypothesis that $M = \mathfrak{m}M$ is equivalent to the conclusion that $M = 0$. Hence Nakayama's Lemma describes one way that finitely generated modules over commutative local rings are similar to vector spaces.

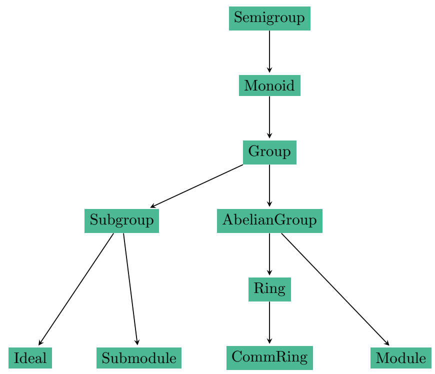
In general, the quotient $R/\mathfrak{m}$ of a local ring modulo its maximal ideal $\mathfrak{m}$ is a field, and the quotient of a module M modulo the submodule $\mathfrak{m}M$ is an $R/\mathfrak{m}$-module, i.e., $M/\mathfrak{m}M$ is a vector space over $R/\mathfrak{m}$. Nakayama's Lemma implies that if M is finitely generated, then bases for $M/\mathfrak{m}M$ corresponds, via lifting, to minimal sets of generators of M.

We point out that there are alternate statements of Nakayama's Lemma that do not require the hypothesis that R must be local. One can replace the unique maximal ideal with the Jacobson radical of the ring, which is the intersection of all maximal ideals. Alternatively, I is an arbitrary proper ideal of a commutative ring R and M is a finitely generated R-module for which $M = IM$, then this ensures the existence of a ring element r congruent to 1 modulo I such that $rM = 0$, i.e., $ru = 0$ for every $u \in M$.

3 Formalization

We start by describing our process of formalizing the required algebraic structures detailed in the previous section. Then, with that in hand, we move on to the formal proof of Nakayama's Lemma.

Figure 3.0.1: The hierarchy of our algebraic structures



3.1 Our Algebraic Hierarchy

Our foundation begins by defining a semigroup class, which declares a binary operation to be associative. From here, we build up through monoids, which introduce identities, to groups, which introduce inverses. Note the double equals "==" appearing in these definitions is notation for an arbitrary equivalence relation over the group's carrier set, which acts as equality.

```
Infix "==" := equiv (at level 60, no associativity).
Class Semigroup := {
  semigroup_assoc:
    forall (a b c: Carrier),
      a <o> b <o> c == a <o> (b <o> c);
}.
Class Monoid := {
  monoid_semigroup :> Semigroup equiv op;
  monoid_ident_l:
    forall (a: Carrier), ident <o> a == a;
  monoid_ident_r:
    forall (a: Carrier), a <o> ident == a;
}.
Class Group := {
  group_monoid :> Monoid equiv op ident;
  group_inv_l:
    forall (a: Carrier), inv a <o> a == ident;
  group_inv_r:
    forall (a: Carrier), a <o> inv a == ident;
}.
```

Lines such as "`monoid_semigroup :> Semigroup equiv op;`" simply coerce the monoid typeclass into a semigroup.

While in the end, our formal proof does not require calling upon quotients of algebraic structures, quotient rings and quotient modules are fundamental to commutative algebra, and one can use them to construct alternate proofs of Nakayama's Lemma. It is worth pointing out that we have formalized quotients of algebraic objects in Coq using typeclasses, which appear to work rather nicely.

An algebraic quotient is, roughly, the set of equivalence classes of an algebraic structure with respect to an equivalence relation on its elements, for which the set of equivalence classes inherit the same kind of algebraic structure. For example, consider the quotient of a group modulo a subgroup, i.e., a subset of elements of the group that it itself a group under the group operations. Under equivalence relation on the group, every element of the subgroup must be in the same equivalence class as the identity. With `P` the predicate for the subgroup, there are two ways to make an equivalence relation from this description.

```
Definition left_congru (a b: Carrier) :=
  P (inv a <o> b).
Definition right_congru (a b: Carrier) :=
  P (a <o> inv b).
```

When these two relations coincide, then we can prove that this common equivalence relation actually preserves the group structure. Subgroups which have this property are called *normal subgroups*.

```
Let normal_subgroup_congru_coincide :=
  forall (a b: Carrier),
    left_congru op inv P a b <->
    right_congru op inv P a b.

Theorem quotient_normal_subgroup_group:
  normal_subgroup_congru_coincide ->
  Group (left_congru op inv P) op ident inv.
```

The importance of quotients in commutative algebra motivates our use of equivalence relations to define the components of a group structure. If one were to instead use the regular Leibniz equality, it would be difficult to identify a quotient group and another group. However, by defining a group in terms of an arbitrary equivalence relation, in our theory a quotient group is simply defined as a group, but under an equivalence relation that is not the usual equality. Not much is lost due to Coq's rewrite tactics for setoids–types equipped with an equivalence relation— can still be called upon.

Moving onward, rings form the next step in our algebraic hierarchy; a ring has two binary operations: addition, which must be commutative, and multiplication, which need not be commutative in general. In our formulation, rings must have a multiplicative identity. Next, we formalized the definition of a commutative ring, further requiring commutativity of multiplication. At this point, we formalized the notion of an ideal of a commutative ring, a subgroup of the ring under addition that satisfies the absorption property under multiplication, i.e., ra is in the ideal for every element a of the ideal, and every element r of the commutative ring. We also used this to formalize the notion of a quotient ring R/I, where I is an ideal of a commutative ring R.

Next, we formalized the definition of a prime ideal, and then moved on to do the same for the notion of a maximal ideal, a proper ideal that is maximal with respect to inclusion. Below is the definition in Coq, which uses `P` as the predicate for the ideal.

```
Definition maximal_ideal :=
  exists (r: Carrier), (not (P r) /\
    forall (Q: Carrier -> Prop)
        (Q_proper: Proper (equiv ==> iff) Q)
        (Q_ideal: Ideal add zero minus mul Q),
      (forall (r: Carrier), P r -> Q r) ->
      (forall (r: Carrier), Q r) \/
        (forall (r: Carrier), Q r -> P r)).
```

We then were able to take advantage of the definition of a maximal ideal to formally define a local ring, i.e., a commutative ring with a single maximal ideal.

```
Definition local_ring :=
  exists (P: Carrier -> Prop)
      (P_proper: Proper (equiv ==> iff) P)
      (P_ideal: Ideal add zero minus mul P),
    maximal_ideal P /\
    (forall (Q: Carrier -> Prop)
        (Q_proper: Proper (equiv ==> iff) Q)
        (Q_ideal: Ideal add zero minus mul Q),
      maximal_ideal Q -> forall (r: Carrier), P r <-> Q r).
```

At this point we formalized the definition of a module over a commutative ring, the commutative-algebraic generalization of the notion of a vector space over a field. Nakayama's Lemma is a statement about finitely generated modules, and hence we must formalize the notion of a scalar combination of a finite collection of elements, $u_1, \ldots, u_n$, of an R-module M, i.e., expressions of the form $r_1 u_1 + r_1 u_2 + \cdots + r_n u_n$, where each $r_i \in R$.

In our formalization of scalar combinations, we use "list" to mean length-parameterized lists; since we don't use the simpler kind of lists, there are no name collisions. In our code, `M` is the type of module elements, `R` is the type of ring elements, act as coefficients, and `t A n` is a list whose elements are of type `A` and whose length is `n`.

```
Definition finitely_generated {n: nat}(generatingSet: t M n) :=
  forall (elt: M),
    exists (coeffs: t R n),
      elt =M= linear_combin coeffs generatingSet.
```

Next, given an ideal I of a commutative ring R and and an R-module M, we defined the submodule IM, the set consisting of all scalar combinations of elements of M whose coefficients are in I. We represented this in Coq as a predicate over M.

```
Context (P: R -> Prop).
Context {P_proper: Proper (Requiv ==> iff) P}.
Context {P_ideal: Ideal Radd Rzero Rminus Rmul P}.

Definition ideal_module (x: M): Prop :=
  exists (n: nat)(coeffs: t R n)(elts: t M n),
    Forall P coeffs /\
    x =M= linear_combin Madd Mzero action coeffs elts.
```

The use of "`Forall P coeffs`" ensures that every element of the coefficient list `coeffs` satisfies the predicate `P`.

3.2 Constructing the Formal Proof

Beyond formalizing the relevant structures from higher algebra, we also formally establish some basic theory. For instance, a *unit* of a commutative ring R is an element $x \in R$ with a multiplicative inverse, i.e., an element $x^{-1} \in R$ for which $x \cdot x^{-1}$ is the multiplicative identity $1 \in R$. In fact, it x is a unit, it has a unique inverse.

We formally proved that an ideal I of a commutative ring R that contains a unit x, then I must be the trivial ideal, i.e., the entire ring. The informal logic is as follows: By the absorption property, since $x \in I$, we have that $x \cdot x^{-1} = 1 \in I$. Hence for every element r of R, $r = r \cdot 1$ is also in I, i.e., $I = R$.

Every non-unit element of a commutative ring is contained in some maximal ideal. This fact relies on the Axiom of Choice. The following standard informal proof calls upon Zorn's lemma, which is equivalent to the Axiom of Choice, and says that given a partially ordered set S, if every chain in S has an upper bound, then S must have at least one maximal element.

> Set I_1 to be the principal ideal generated by an element x of a commutative ring R, i.e. the smallest ideal containing the element x, which consists of all R-multiples of x.
>
> If I_1 is not a maximal ideal, then there exists a strictly larger proper ideal I_2 of R, i.e., $x \in I_1 \subsetneq I_2 \subsetneq R$. Moreover, if I_2 is not maximal, then there exists a strictly larger proper ideal I_3 containing I_2.
>
> Continuing this process, if it terminates at some step, we have found a maximal ideal, and if not, one obtains an infinite chain of ideals containing x.
>
> $$x \in I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots \subsetneq R$$
>
> It is straightforward to verify that the increasing union of all I_k, $k \geq 1$, is by definition an ideal of R, and certainly contains x. Hence by Zorn's lemma, since every ascending chain of ideals containing x with respect to inclusion has its union as an upper bound, there exists a maximal ideal of R containing x.

This argument has potentially infinitely many steps, and chose to avoid this issue by including an axiom that in any non-unit x of a ring is contained in some maximal ideal.

```
Axiom comm_ring_nonunit_maximal_ideal:
  forall (x: Carrier),
    ~ is_unit equiv mul one x ->
    exists (P: Carrier -> Prop)(P_proper: Proper (equiv ==> iff) P)
           (P_ideal: Ideal add zero minus mul P),
      P x /\ maximal_ideal P.
```

We also used classical logic to prove that $1 - x$ is a unit whenever x is an element of a local ring that is not a unit. The proof is completed by way of contradiction, and uses the rule that $\neg\neg P \rightarrow P$.

To start describing our proof, we formalize the statement of Nakayama's Lemma in Coq.

Nakayama's Lemma. *Let R be a commutative local ring, and let $\mathfrak{m}$ denote its maximal ideal. Suppose that M is a finitely generated R-module. If $M = \mathfrak{m}M$, then $M = 0$, i.e., M must be the R-module containing only one element, its identity as an additive abelian group.*

```
Theorem nakayama:
  forall {n: nat}(basis: t M n),
    finitely_generated Mequiv Madd Mzero action basis ->
  (forall a: M, ideal_module_pred a) ->
  forall a: M, a =M= Mzero.
```

To use as a building block in the formal proof of Nakayama's Lemma, we first formally stated and proved a lemma that gives a concrete description of the elements of the submodule $\mathfrak{m}M$ appearing in the statement of the the theorem. This lemma applies more generally, to any finitely generated module M over a (not necessarily local) commutative ring, and any submodule of the form IM, for I an arbitrary ideal. By definition, IM consists of scalar combinations of elements of M with coefficients in I. The lemma states that the elements from M appearing in the expression can be chosen to be from any fixed finite generating set for M. In other words, given any generating set $u_1, \ldots, u_n \in M$ of a finitely generated R-module M, every element x of IM can be written as a scalar combination $a_1 \cdot u_1 + a_2 \cdot u_2 + \cdots + a_n \cdot u_n$, where each $a_i \in I$.

```
Lemma module_fin_gen_ideal_module:
  forall {n: nat}(generatingSet: t M n),
    finitely_generated Mequiv Madd Mzero action generatingSet ->
    forall {m: nat}(coeffs: t R m)(elts: t M m),
      Forall P coeffs ->
      exists (coeffs': t R n),
        linear_combin Madd Mzero action coeffs elts =M=
          linear_combin Madd Mzero action coeffs' generatingSet /\
        Forall P coeffs'.
```

The proof follows from a straightforward argument based on definitions, by induction on the number of elements in a fixed generating set for M, which we informally describe: By definition, $x \in IM$ can be written, for some positive integer k and elements $r_i \in R$ and $w_i \in M$, $1 \leq i \leq k$, as $x = r_1 \cdot w_1 + r_2 \cdot w_2 + \cdots + r_k \cdot w_k$. What's more, by definition of a finite generating set $u_1, \ldots, u_n$ for M, each w_i equals $c_{i1} \cdot u_1 + c_{i2} \cdot u_2 + \cdots + c_{in} \cdot u_n$ for appropriate choices of $c_{ij} \in R$. Hence, inductively applying associativity,

$$x = \sum_{i=1}^{k} r_i \cdot w_i = \sum_{i=1}^{k} r_i \cdot \left(\sum_{j=1}^{n} c_{ij} \cdot u_j \right) = \sum_{j=1}^{n} \sum_{i=1}^{k} r_i \cdot (c_{ij} \cdot u_j) = \sum_{j=1}^{n} \left(\sum_{i=1}^{k} c_{ij} \cdot r_i \right) \cdot u_j.$$

By absorption in ideals, each $c_{ij} \cdot r_i$ is in I, and since ideals are closed under addition, we inductively conclude that the coefficient $a_j := c_{1j} \cdot r_1 + c_{2j} \cdot r_2 + \cdots c_{kj} \cdot r_k$ of u_j is also in I.

Now we move forward to outline the formal proof of Nakayama's lemma. We proceed by induction on the number of elements in a generating set for the R-module M. The base case is the situation when M requires no generators, so M consists solely of the empty scalar combination, i.e., the empty sum, which, by convention, is the zero element. In other words, $M = 0$ by assumption, and this is also precisely the conclusion of Nakayama's Lemma. Hence the statement trivially, without using the hypothesis that $M = \mathfrak{m}M$.

We turn to the inductive step. Fixing an arbitrary nonnegative integer n, we assume that Nakayama's Lemma holds in the case that M has a generating set consisting of n elements.

Now, fix an R-module M generated by $u_1, \ldots, u_{n+1} \in M$. By assumption, $M = \mathfrak{m}M$, and in particular, the generator u_1 is an element of the submodule $\mathfrak{m}M$. The lemma described above guarantees the existence of ring elements $a_1, \ldots, a_n \in \mathfrak{m}$ for which

$$u_1 = a_1 \cdot u_1 + a_2 \cdot u_2 + \cdots + a_{n+1} \cdot u_{n+1}.$$

Collecting the u_1 terms on the left-hand side of the equation, applying the existence of additive inverses in R, and distributivity of scalar multiplication for modules, we see that

$$(1 - a_1) \cdot u_1 = a_2 \cdot u_2 + \cdots + a_{n+1} \cdot u_{n+1}. \tag{3.1}$$

Since a proper ideal contains no units and a_1 is taken to be in $\mathfrak{m}$, it is not a unit. Hence, by the formalized statement mentioned earlier, $1 - a_1$ is a unit of R. Let $b_1 \in R$ denote its multiplicative inverse, so that $b_1 \cdot (1 - a_1) = 1$. Multiplying this element on the either side of (3.1), we deduce the following.

$$\begin{aligned}
b_1 \cdot ((1 - a_1) \cdot u_1) &= b_1 \cdot (a_2 \cdot u_2 + \cdots + a_{n+1} \cdot u_{n+1}) \\
(b_1 \cdot (1 - a_1)) \cdot u_1 &= b_1 \cdot (a_2 \cdot u_2) + \cdots + b_1 \cdot (a_{n+1} \cdot u_{n+1}) \\
u_1 = 1 \cdot u_1 &= (b_1 \cdot a_2) \cdot u_2 + \cdots + (b_1 \cdot a_{n+1}) \cdot u_{n+1}
\end{aligned}$$

In particular, the generator u_1 can be written as a scalar combination of the n generators $u_2, \ldots, u_{n+1}$, and is superfluous. Hence M can be generated by n elements, and $M = 0$ by the inductive hypothesis.

Emily: Reflecting, I think it would be very valuable to include the formal proof. Pieces would work, though it is short, and it could be "broken up" among the description of the informal argument. As is, our description of the proof focuses on the informal argument, which is not new/interesting.

Showing that the inductive hypothesis holds in Coq is more work than in this informal proof, but this extra work is just a lot of bookkeeping.

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