CROSS PRODUCT (Vector Product)

Dot product (Scalar Product)

Learning Objectives

1). To use *cross products* to:

i) calculate the angle (θ) between two vectors and,

~~ii) determine a vector normal to a plane.~~

2). To do an *engineering estimate* of these quantities.

Purpose

Four primary uses of the cross product are to:

1) calculate the angle (θ) between two vectors,
2) ~~determine a vector normal to a plane,~~
3) calculate the moment of a force about a point, ~~and~~
4) ~~calculate the moment of a force about a line.~~

Definition

Magnitude of cross product

$$\boxed{|\bar{A}\times\bar{B}| = |\bar{B}\times\bar{A}| = |\bar{A}|\ |\bar{B}|\ \sin\theta}$$

In words, $|\bar{A}\times\bar{B}|$ = the component of $\bar{B}$ perpendicular to $\bar{A}$ multiplied by the magnitude of $\bar{A}$; or vice versa.

Angle (θ) Between Two Vectors

$$\boxed{\theta = \sin^{-1}\left[\frac{|\bar{A}\times\bar{B}|}{|\bar{A}|\ |\bar{B}|}\right]}$$

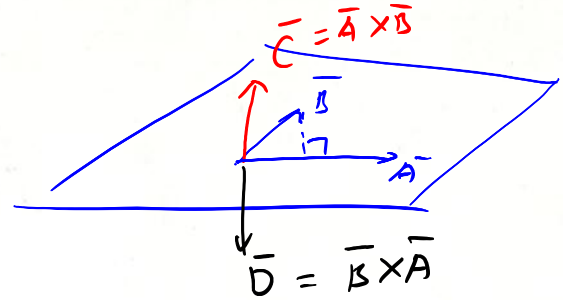
Recall: $\theta = \cos^{-1}\left(\frac{\bar{A}\cdot\bar{B}}{|\bar{A}|\ |\bar{B}|}\right)$

Unit Normal Vector

$$\boxed{\text{Unit Normal} = \frac{\bar{A}\times\bar{B}}{|\bar{A}\times\bar{B}|}}$$

Direction of cross product

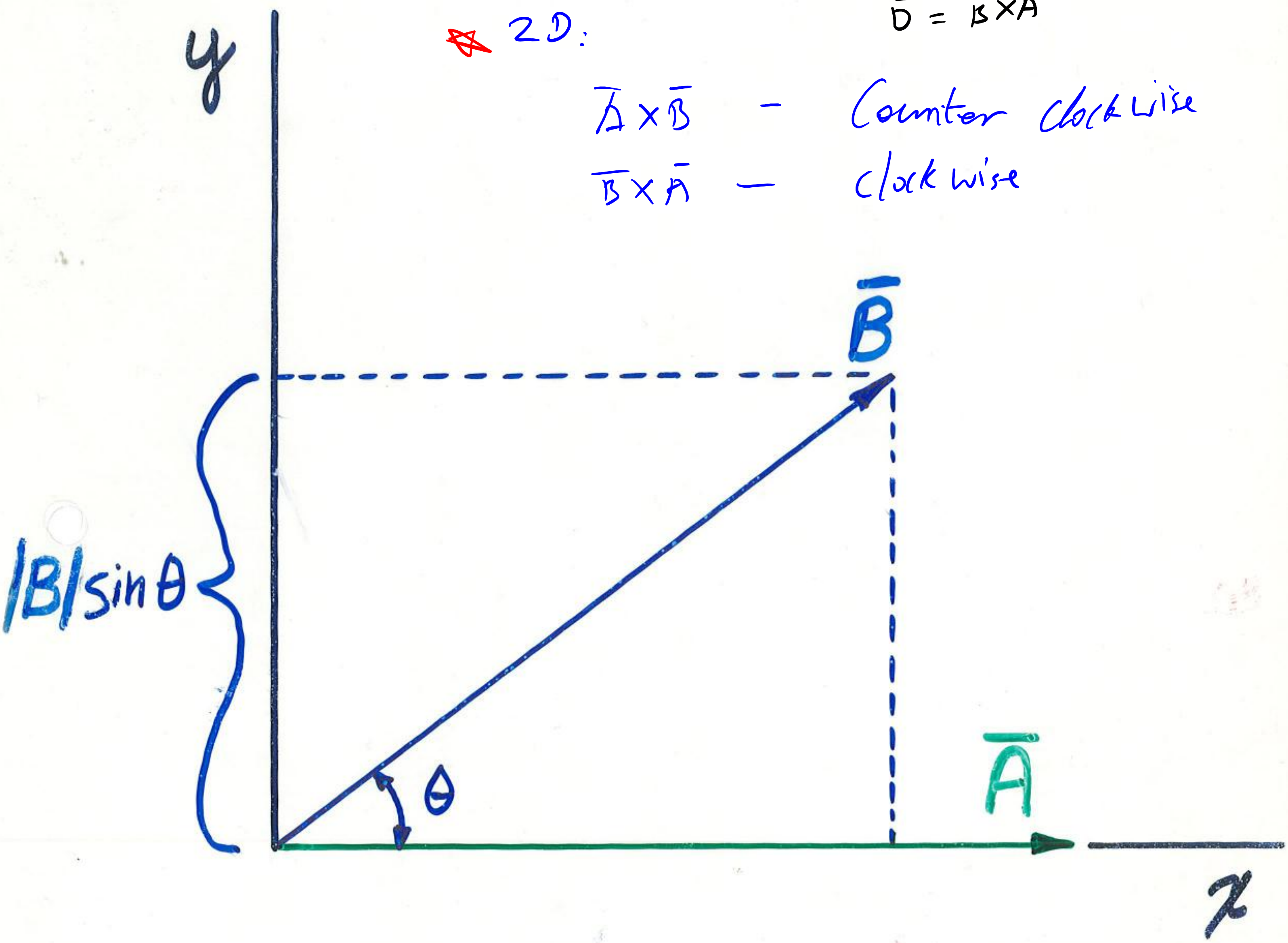
3D:



2D:

$\bar{A} \times \bar{B}$ — Counter clockwise

$\bar{B} \times \bar{A}$ — clockwise



REMEMBER: $(\bar{A} \times \bar{B}) = -(\bar{B} \times \bar{A})$

Use **RHR** to determine direction

Mechanics (assuming a right-handed coordinate system)

$$\bar{A} \times \bar{B} = (A_x\bar{i} + A_y\bar{i} + A_z\bar{k}) \times (B_x\bar{i} + B_y\bar{j} + B_z\bar{k})$$

$$\boxed{\bar{A} \times \bar{B} = (A_yB_z - B_yA_z)\,\bar{i} - (A_xB_z - B_xA_z)\,\bar{j} + (A_xB_y - B_xA_y)\,\bar{k}}$$

Recall,

$$\bar{i} \times \bar{i} = \bar{j} \times \bar{j} = \bar{k} \times \bar{k} = 0 \qquad \bar{i} \times \bar{k} = -\bar{j} \qquad \bar{k} \times \bar{i} = \bar{j}$$

$$\bar{i} \times \bar{j} = k \qquad \bar{j} \times \bar{i} = -k \qquad \bar{j} \times \bar{k} = \bar{i} \qquad \bar{k} \times \bar{j} = -\bar{i}$$

Note: Given $\bar{C} = \bar{A} \times \bar{B}$, $\bar{C}$ is perpendicular to the plane containing vectors $\bar{A}$ and $\bar{B}$.

Basic Properties

1). $\bar{A} \times \bar{B} = -\bar{B} \times \bar{A}$, Use right hand rule to determine direction of resultant vector.

2). $p(\bar{A} \times \bar{B}) = (p\bar{A}) \times \bar{B} = \bar{A} \times (p\bar{B})$

$$\bar{A} \times (\bar{B} + \bar{C}) = (\bar{A} \times \bar{B}) + (\bar{A} \times \bar{C})$$

$$(\bar{A} + \bar{B}) \times \bar{C} = (\bar{A} \times \bar{C}) + (\bar{B} \times \bar{C})$$

3). If $|\bar{A} \times \bar{B}| = 0$, then $\theta = 0^\circ$

If $|\bar{A} \times \bar{B}| = |A||B|$, then $\theta = 90^\circ$

4). $\bar{A} \times \bar{A} = \bar{0}$

MOMENT ABOUT A POINT

Learning Objectives

1). To use *cross products* to calculate the moment of a force about a point.

3). To do an *engineering estimate* of this quantity.

Definition $\overline{M}_o = \overline{r}_{op} \times \overline{F}$

Moment About a Point: a measure of the tendency of a force to turn a body to which the force is applied.

Magnitude

$$|\overline{M}_o| = r_\perp |\overline{F}| = \left[|\overline{r}_{op}| \sin\theta\right] |\overline{F}| = |\overline{r}_{op}||\overline{F}| \sin\theta$$

Direction: RHR ⟶ CW or CCW

or

$$|\overline{M}_o| = |\overline{r}_{op}| F_\perp = |\overline{r}_{op}| \left[|\overline{F}| \sin\theta\right] = |\overline{r}_{op}| \; |\overline{F}| \sin\theta$$

or

$$|\overline{M}_o| = |\overline{r}_{op} \times \overline{F}| = |\overline{r}_{op}| \; |\overline{F}| \sin\theta$$

$$= |(r_x \overline{i} + r_y \overline{j}) \times (F_x \overline{i} + F_y \overline{j})|$$

$$= |(r_x F_y - r_y F_x)\overline{k}|$$

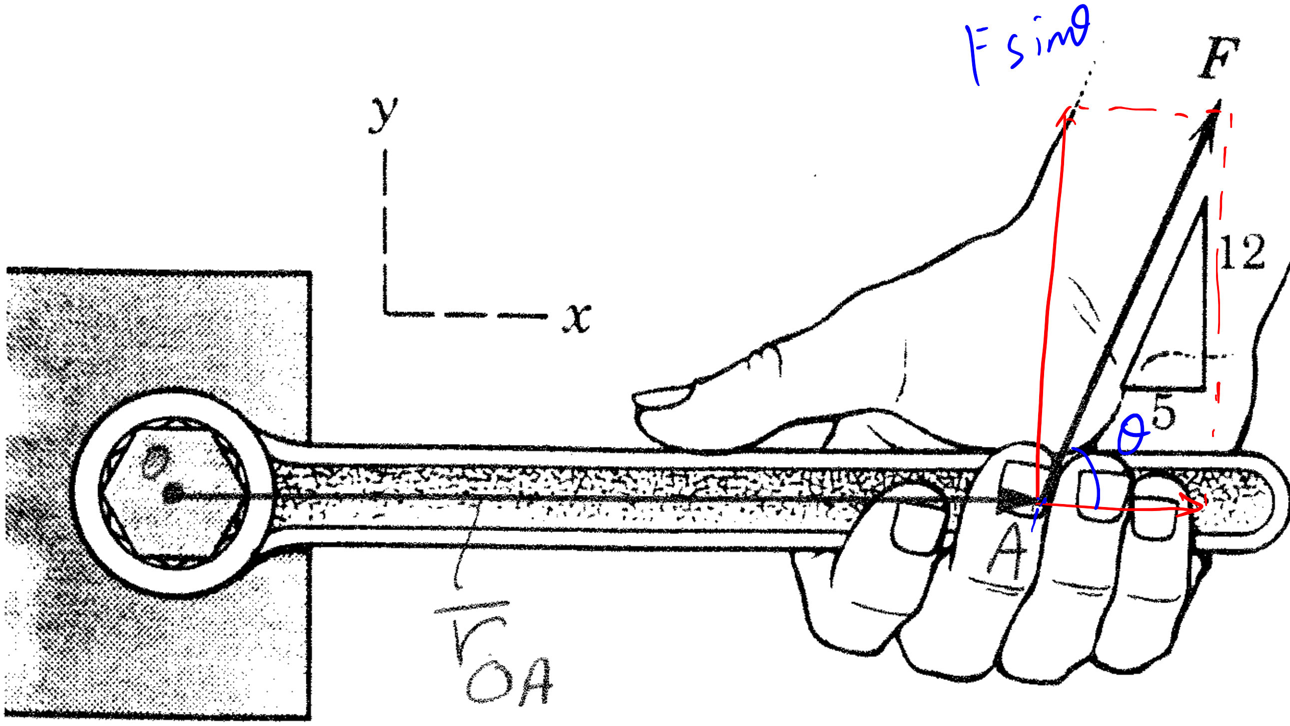
Comments

1). Direction of the moment can be determined by the *right hand rule*.

2). Point P can be any point along the *line of action* of the force without altering the resultant moment $(\overline{M_O})$.

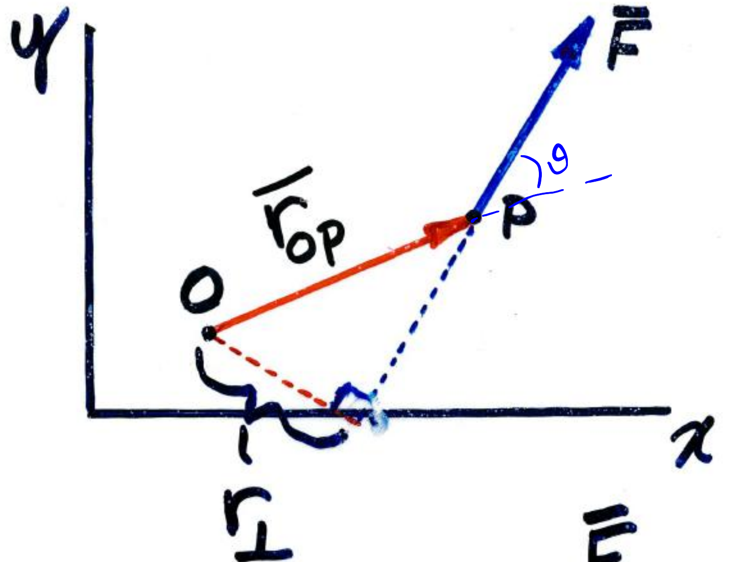
3). Use *trigonometric relationships* for calculating the moment about a point for *2-D problems*.

4). Use *vectors* and *cross products* when calculating the moment about a point for *3-D problems*.



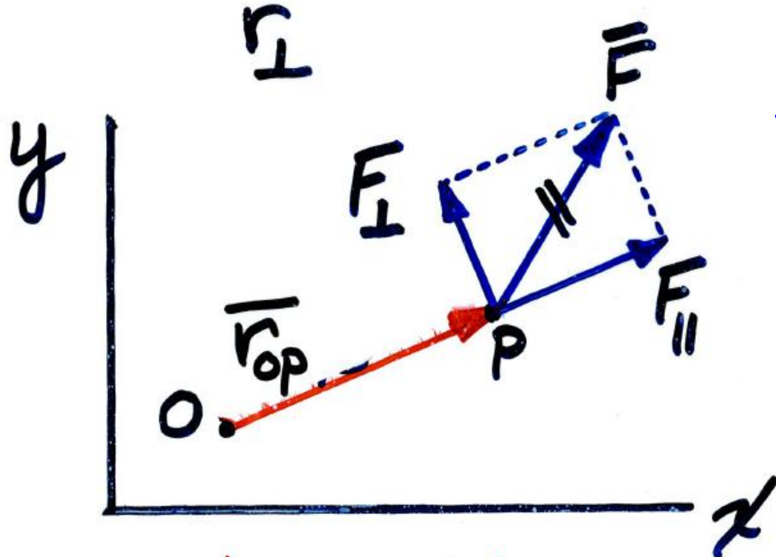
F sinθ
F
y
x
12
5
θ
O
A
$\bar{r}_{OA}$

$$|\bar{m}_o| = |\bar{r}_{op}|\,|\bar{F}| \sin\theta$$

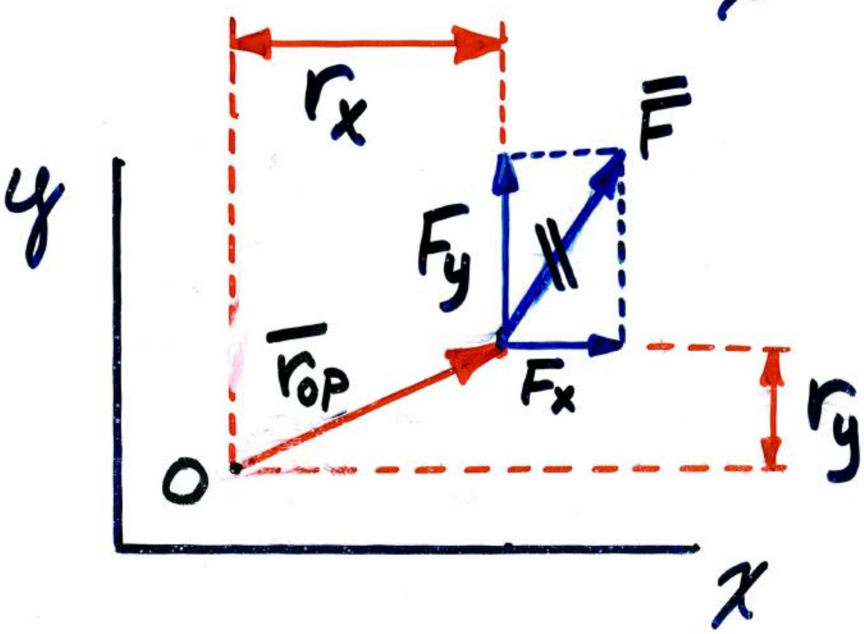


$$r_{\perp} = |\bar{r}_{op}| \sin\theta$$

– moment arm



$$|F_{\perp}| = |\bar{F}| \sin\theta$$



Moment about a Point

Example 1

Given: Angled bar OA is loaded with a 90N force as shown.

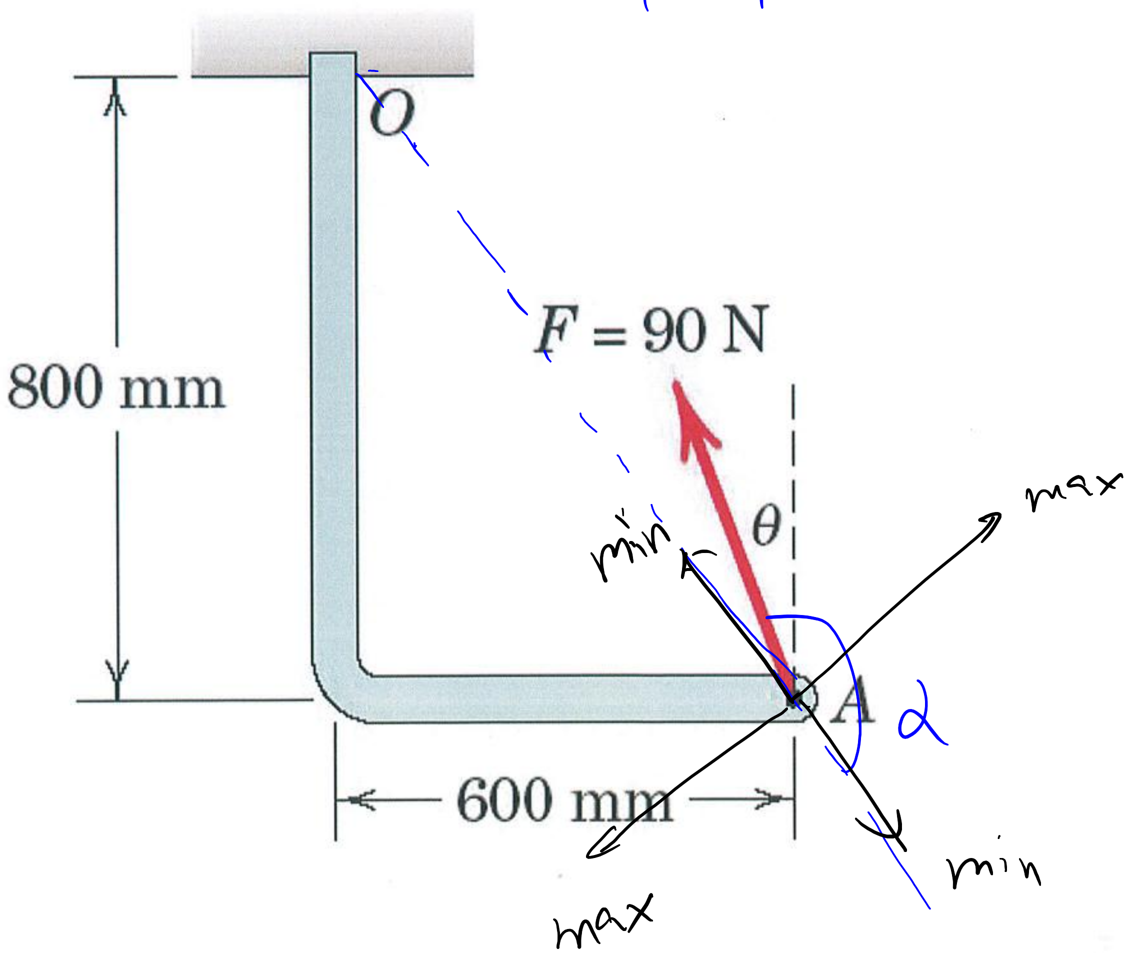
Find:

a) Determine the angle θ which maximizes the magnitude of the moment about point O.

b) Determine the angle θ which minimizes the magnitude of the moment about point O.

c) For the angle θ shown, estimate the magnitude of the moment about point O.

$$|\bar{m}| = |OA|\,|\bar{F}|\,\sin\alpha$$



Moment about a Point

Example 2

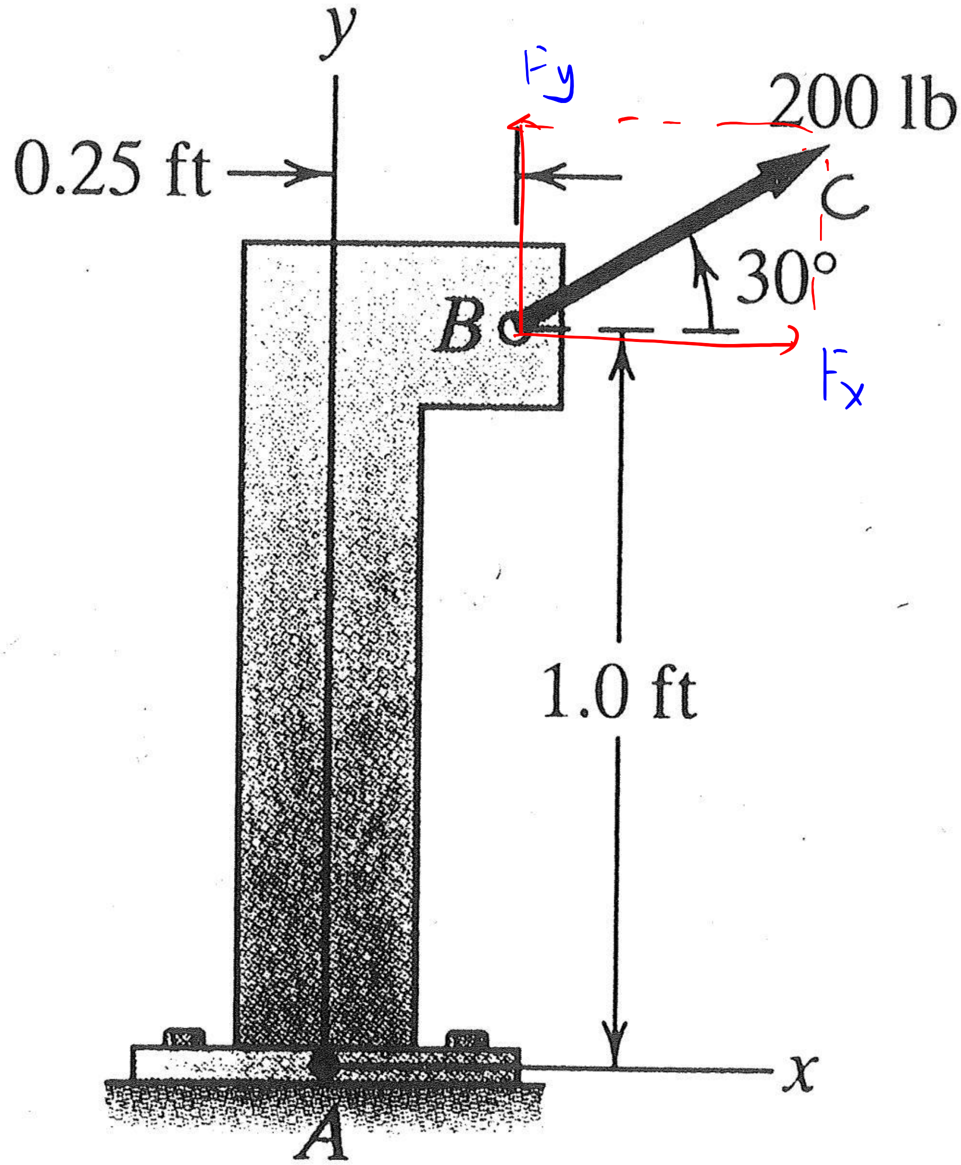
Given: Angled bar AB has a 200 lb load applied at B.

Find:

~~a) Estimate the magnitude of the moment about fixed support A.~~

b) Calculate the moment about fixed support A.

$$\overline{M}_A = \overline{r}_{AB} \times \overline{F} = \overline{r}_{AB} \times F_x \bar{i} + \overline{r}_{AB} \times F_y \bar{j}$$



b) $$\overline{M}_A = \overline{r}_{AB} \times F_x \bar{i} + \overline{r}_{AB} \times F_y \bar{j}$$

$$= \underbrace{-(1.0)(200 \cos 30^\circ)\,\bar{k}}_{CW}$$

$$+ \underbrace{(0.25)(200 \sin 30^\circ)\,\bar{k}}_{CCW}$$

$$= \underbrace{-148.2\,\bar{k} \quad \text{lb}\cdot\text{ft}}_{CW}$$

Moment about a Point

Example 3

Given: Angled bar AO is loaded by cable AB. The tension in cable AB is T_{AB} = 1.2 kN.

Find:

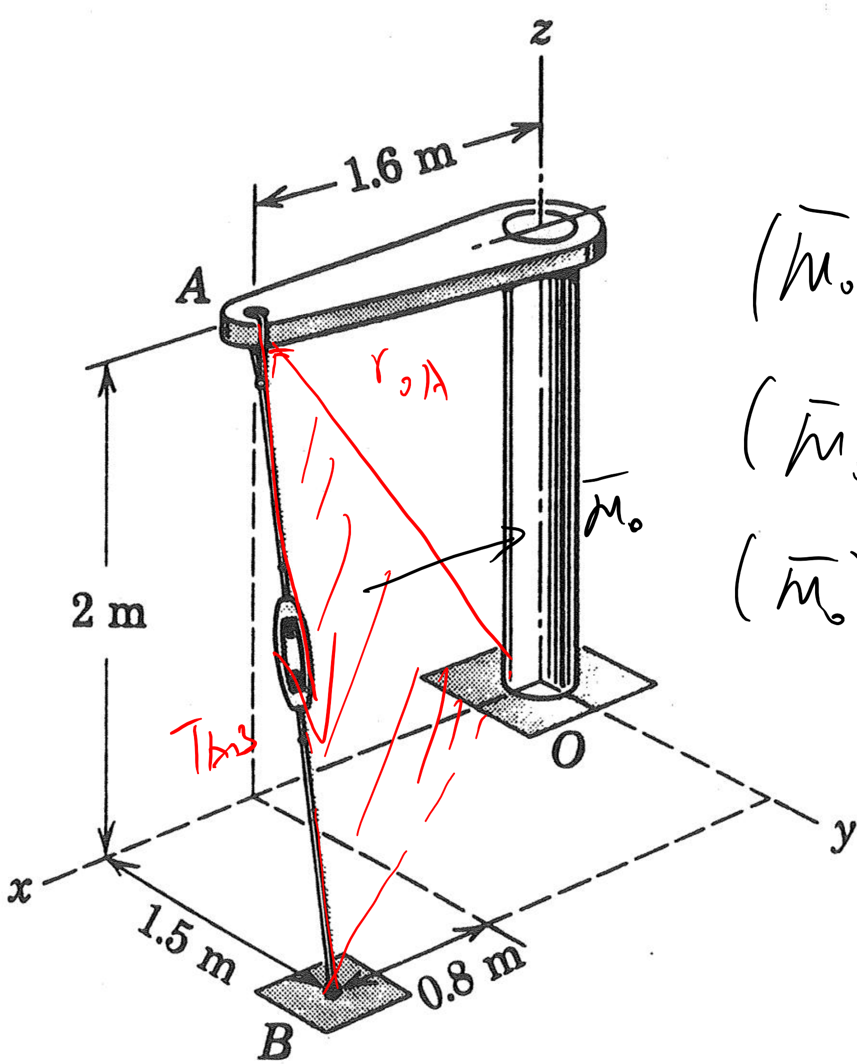
Signs of each Component

a) Estimate the ~~magnitude and component origins~~ of moment vector $\bar{M}_O$.

b) Calculate the moment vector about point O ($\bar{M}_O$) using $\bar{r}_{OA}$.

c) Calculate the moment vector about point O ($\bar{M}_O$) using $\bar{r}_{OB}$.

d) How do the solutions for parts (b) and (c) compare?



$(\bar{M}_o)_x$ −

$(\bar{M}_o)_y$ +

$(\bar{M}_o)_z$ +

b) $\overline{M}_o = \overline{r}_{OA} \times \overline{T}_{AB}$

$\overline{r}_{OA} = 1.6\overline{i} + 2\overline{k}$ m

$\overline{T}_{AB} = T_{AB}\,\overline{u}_{AB} = (1.2)\left(\dfrac{0.8\overline{i} + 1.5\overline{j} - 2\overline{k}}{\sqrt{(0.8)^2 + (1.5)^2 + (-2)^2}}\right)$

$= 0.366\overline{i} + 0.686\overline{j} - 0.914\overline{k}$ kN

$\overline{M}_o = \overline{r}_{OA} \times \overline{T}_{AB}$

$= -1.37\overline{i} + 2.19\overline{j} + 1.10\overline{k}$ kN·m

c) $\overline{r}_{OB} = 2.4\overline{i} + 1.5\overline{j}$ m

$\overline{M}_o = \overline{r}_{OB} \times \overline{T}_{AB}$

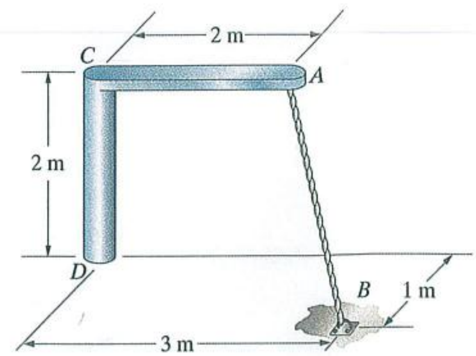
$= -1.37\overline{i} + 2.19\overline{j} + 1.10\overline{k}$ kN·m

$|\overline{M}_o| = 2.81$ kN·m

d) They produce the same results.

Moment about a Point

Example 4



Given: Cable AB exerts a 2 kN tension on bar DCA at point A.

Find:

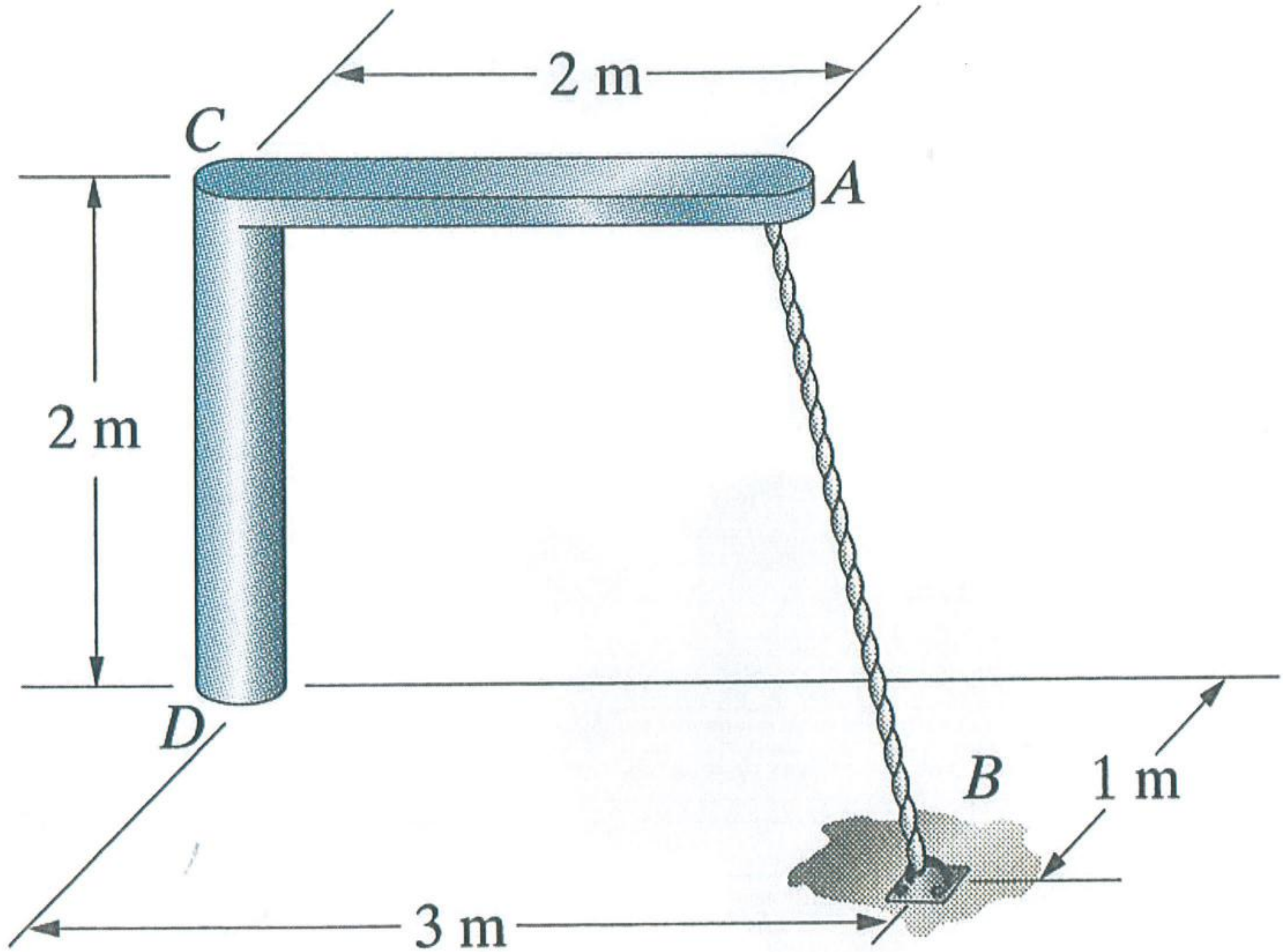
a) Write a vector expression for the tension of cable AB $\left(\overline{T}_{AB}\right)$.

b) Predict the signs of the components of the moment about point D due to cable AB.

c) What is the largest possible magnitude for M_D?

d) Calculate the moment about base D due to cable T_{AB}. Did you correctly predict the signs of each component?

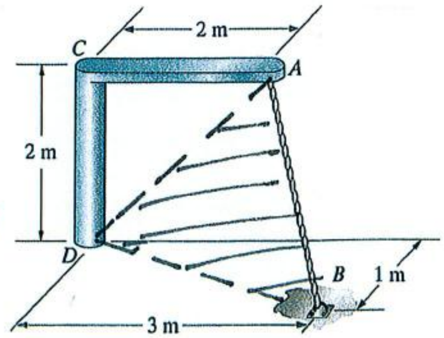
e) Repeat part (d) for M_C.


2 m
C
A
2 m
D
B
1 m
3 m

a) $\bar{T}_{AB} = T_{AB}\,\bar{u}_{AB}$

$$= 2\,\frac{(1\bar{i} + 3\bar{j} - 2\bar{k})}{\left[1^2 + 3^2 + (-2)^2\right]^{1/2}}$$

$$\boxed{\bar{T}_{AB} = 0.534\bar{i} + 1.604\bar{j} - 1.068\bar{k}\ \text{kN}}$$



b) $\bar{i} = (-)$ $\quad \bar{j} = (+)$ $\quad \bar{k} = (-)$

c) $|\bar{M}_D|_{max} = (|\bar{r}_{DA}|\,|\bar{T}_{AB}|) = 2\,(2.83) = \boxed{5.66\ \text{kN-m}}$

$\bar{r}_{DA} = 2\bar{j} + 2\bar{k} \qquad |\bar{r}_{DA}| = \sqrt{8} = 2.83\ m$

d) $\bar{M}_D = \bar{r}_{DA} \times \bar{T}_{AB} = (2\bar{j} + 2\bar{k}) \times (0.534\bar{i} + 1.604\bar{j} - 1.068\bar{k})$

$$\boxed{\bar{M}_D = -5.344\bar{i} + 1.068\bar{j} - 1.068\bar{k}\ \text{kN-m}}$$

$|\bar{M}_D| = \left[(-5.344)^2 + (1.068)^2 + (-1.068)^2\right]^{1/2} = 5.55\ \text{kN-m}$

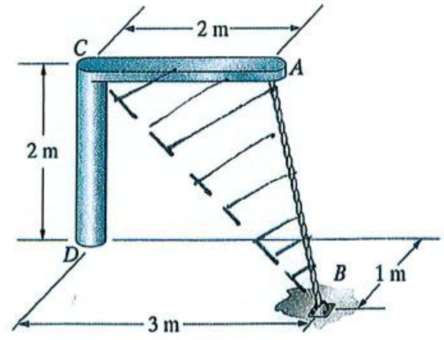
Note:

Since $|\bar{M}_D|$ is close to $|\bar{M}_D|_{max} \Rightarrow \theta \approx 90^\circ$

e)

Estimate

$\bar{i} = (-) \qquad \bar{j} = 0 \qquad \bar{k} = (-)$



$$\bar{M}_c = \bar{r}_{CA} \times \bar{T}_{AB}$$

$$= 2\bar{j} \times (0.534\bar{i} + 1.604\bar{j} - 1.068\bar{k})$$

$$\boxed{\bar{M}_c = -2.136\,\bar{i} - 1.068\bar{k} \text{ kN-m}}$$

<u>Note:</u>

$$|\bar{M}_c|_{max} = |\bar{r}_{CA}|\,|\bar{T}_{AB}| = 2(2) = 4 \text{ kN-m}$$

$$|\bar{M}_c|_{Actual} = [2.136^2 + (-1.068)^2]^{1/2} = 2.28 \text{ kN-m}$$

$\therefore$ Angle θ <u>not</u> near 90°

Moment about a Point
Group Quiz 1

Group #: ___________ Group Members: 1) ____________________________
(Present Only)

Date: _____________ Period: ________ 2) ____________________________

3) ____________________________

4) ____________________________

Given: A force of 800N acts on a bracket as shown.

Find:

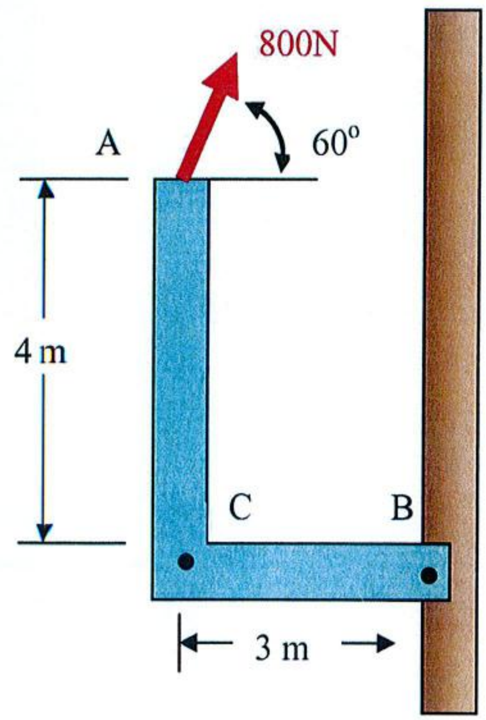
a) Estimate M_C.

b) Calculate M_C.

c) Estimate M_B.

d) Calculate M_B.

e) What is M_A?



Solution:

ME 270 – Basic Mechanics I – Group Quiz

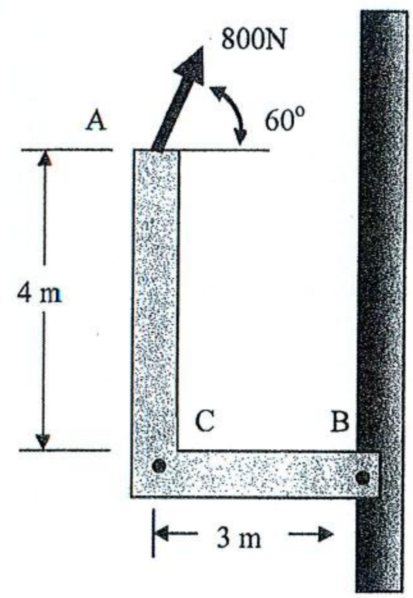
Your Name: SOLUTION Group Members: 1) ______________

Date: ______________ Period: __________ 2) ______________

3) ______________

4) ______________

Given: A force of 800N acts on a bracket as shown.



Find:

(a) Estimate M_C

(b) Calculate M_C

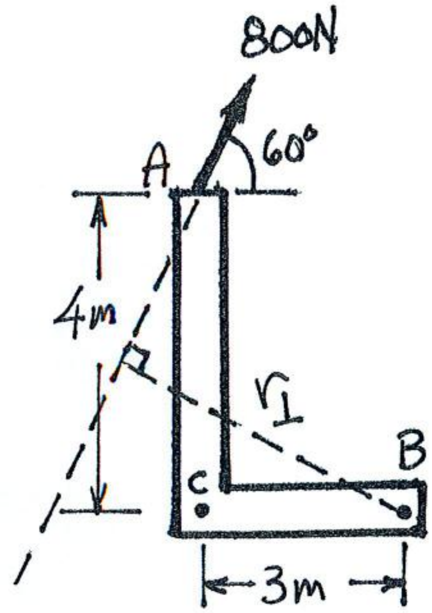
(c) Estimate M_B

(d) Calculate M_B

(e) What is M_A?

Solution:

c)



a) $\boxed{M_C \approx -1600 \text{ N-m}}$

b) $M_C = -(4)(800 \cos 60°) = \boxed{-1600 \text{ N-m}}$

c) $M_B \approx -(5)(800) = \boxed{-4000 \text{ N-m}}$

d) $M_B = -(4)(800 \cos 60°)$
$\quad -(3)(800 \sin 60°)$

$= -1600 - 2078$

$\boxed{M_B = -3678 \text{ or } -3680 \text{ N-m}}$

e) $M_A = 0$ N-m (The moment arm is zero)