Finite frames for K4.3 × S5 are decidable

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Abstract

If a modal logic L is finitely axiomatisable, then it is of course decidable whether a finite frame is a frame for L: one just has to check the finitely many axioms in it. If L is not finitely axiomatisable, then this might not be the case. For example, it is shown in [7] that the finite frame problem is undecidable for every L between the product logics $\mathbf{K} \times \mathbf{K} \times \mathbf{K}$ and $\mathbf{S5} \times \mathbf{S5} \times \mathbf{S5}$. Here we show that the finite frame problem for the modal product logic $\mathbf{K4.3} \times \mathbf{S5}$ is decidable. $\mathbf{K4.3} \times \mathbf{S5}$ is outside the scope of both the finite axiomatisation results of [4], and the non-finite axiomatisability results of [11]. So it is not known whether $\mathbf{K4.3} \times \mathbf{S5}$ is finitely axiomatisable. Here we also discuss whether our results bring us any closer to either proving non-finite axiomatisability of $\mathbf{K4.3} \times \mathbf{S5}$, or finding an explicit, possibly infinite, axiomatisation of it.

Keywords: products of modal logics, finite frame problem, axiomatisation

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1 Introduction and results

The product construction as a combination method for modal logics was introduced in [13,14,4], and has been extensively studied ever since. Modal products are connected to several other multi-dimensional logical formalisms, see [3,9] for surveys and references. Here we consider only two-dimensional products, but the definitions can be generalised to higher dimensions. In what follows we assume that the reader is familiar with basic notions of propositional multi-modal logic and its possible world (or relational) semantics, and we use these

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without explicit references. For concepts and statements not defined or proved here, consult, for example, [1,2].

Given two Kripke frames $\mathfrak{F}_0 = \langle W_0, R_0 \rangle$ and $\mathfrak{F}_1 = \langle W_1, R_1 \rangle$, their *product* is defined to be the 2-frame

$$\mathfrak{F}_0 \times \mathfrak{F}_1 = \langle W_0 \times W_1, \bar{R}_0, \bar{R}_1 \rangle,$$

where $W_0 \times W_1$ is the Cartesian product of W_0 and W_1 and, for all $x, x' \in W_0$, $y, y' \in W_1$,

$$\begin{array}{lll} \langle x, y \rangle \bar{R}_0 \langle x', y' \rangle & \text{iff} & x R_0 x' \text{ and } y = y', \\ \langle x, y \rangle \bar{R}_1 \langle x', y' \rangle & \text{iff} & y R_1 y' \text{ and } x = x'. \end{array}$$

Frames of this form will be called *product frames* throughout. Now let L_0 and L_1 be Kripke complete modal logics in the languages with $\Box_0$ and $\Box_1$, respectively. Their product $L_0 \times L_1$ is then the set of all bimodal formulas, in the language having both $\Box_0$ and $\Box_1$, that are valid in all product frames $\mathfrak{F}_0 \times \mathfrak{F}_1$, where $\mathfrak{F}_0$ is a frame for L_0, and $\mathfrak{F}_1$ is a frame for L_1. (Here we assume that $\Box_0$ is interpreted by $\bar{R}_0$, while $\Box_1$ is interpreted by $\bar{R}_1$.) Note that $L_0 \times L_1$ always contains the *fusion* $L_0 \oplus L_1$ of L_0 and L_1: the smallest normal bimodal logic that contains L_0 for $\Box_0$ and L_1 for $\Box_1$. Therefore, *any* product frame $\mathfrak{F}_0 \times \mathfrak{F}_1$ for $L_0 \times L_1$ is such that $\mathfrak{F}_i$ is a frame for L_i, for $i = 0, 1$.

A modal product logic $L_0 \times L_1$ is Kripke complete by definition: it is defined as a set of formulas that are valid in some class $\mathcal{C}$ of frames. However, there are frames for $L_0 \times L_1$ that are not in $\mathcal{C}$. So even if it is decidable whether a finite 2-modal frame is in $\mathcal{C}$ or not, the *finite frame problem* for $L_0 \times L_1$ is not necessarily decidable. If $L_0 \times L_1$ is finitely axiomatisable, then it is of course decidable whether a finite frame is a frame for $L_0 \times L_1$: one just has to check the finitely many axioms in it. But if $L_0 \times L_1$ is not finitely axiomatisable, then this might not be the case, even if the component logics L_0 and L_1 are both finitely axiomatisable, and so the class of product frames for $L_0 \times L_1$ is decidable. We do not know two-dimensional examples of this kind, but there are non-finitely axiomatisable higher dimensional product logics with undecidable finite frame problems (such as $\mathbf{K} \times \mathbf{K} \times \mathbf{K}$ and $\mathbf{S5} \times \mathbf{S5} \times \mathbf{S5}$), see [7].

Below we summarise the known results on the axiomatisation problem for two-dimensional product logics:

(1) If both unimodal logics L_0 and L_1 are such that their classes of Kripke frames are definable by recursive sets of first-order sentences, then their product $L_0 \times L_1$ is a recursively enumerable bimodal logic [4].

(2) If both L_0 and L_1 are finitely axiomatisable by modal formulas having universal Horn first-order correspondents, then $L_0 \times L_1$ is finitely axiomatisable [4]. For example, if each L_i is either $\mathbf{K}$ (the logic of all frames), or $\mathbf{K4}$ (the logic of all transitive frames), or $\mathbf{S4}$ (the logic of all reflexive and transitive frames), or $\mathbf{S5}$ (the logic of all equivalence frames), then $L_0 \times L_1$ is finitely axiomatisable.

(3) The result in (2) cannot be generalised to products of logics axiomatised by formulas having universal (but not necessarily Horn) first-order components.

A counterexample is the finitely axiomatisable modal logic **K4.3**, determined by the frames $\langle W, R\rangle$, where R is transitive and *weakly connected*:

$$\forall x, y, z \in W \left(xRy \wedge xRz \to (y = z \vee yRz \vee zRy) \right).$$

(A rooted transitive and weakly connected relation is a *linearly ordered sequence of clusters*.) As shown in [11], there are product logics with a 'linear' first component that are not axiomatisable finitely: For example, if L_0 is any of the logics **K4.3**, **S4.3**, $\mathsf{Logic\_of}\{\langle\omega, \leq\rangle\}$, and L_1 is any of the logics **K**, **K4**, **S4**, **GL**, **Grz**, then $L_0 \times L_1$ is not axiomatisable using finitely many propositional variables.

However, there are recursively enumerable product logics that are outside the scope of both (2) and (3) above, so it is not known whether they are finitely axiomatisable or not. A notable example is $\mathbf{K4.3} \times \mathbf{S5}$. In this paper we show the following:

Theorem 1.1 *It is decidable whether a finite 2-frame is a frame for* $\mathbf{K4.3} \times \mathbf{S5}$.

It is clearly enough to decide the frame problem for finite *rooted* 2-frames. As both being transitive and weakly connected, and being an equivalence relation are first-order definable, the respective classes of all frames for **K4.3** and **S5** are closed under ultraproducts. As **K4.3** and **S5** are modal logics, their classes of frames are also closed under point-generated subframes. So, by [10, Thm.2.10], we obtain that, for every finite rooted 2-frame $\mathfrak{F}$, $\mathfrak{F}$ is a frame for $\mathbf{K4.3} \times \mathbf{S5}$ iff $\mathfrak{F}$ is a p-morphic image of a product frame for $\mathbf{K4.3} \times \mathbf{S5}$. So it is enough to show the following:

Theorem 1.2 *It is decidable whether a finite rooted 2-frame is a p-morphic image of a product frame for* $\mathbf{K4.3} \times \mathbf{S5}$.

Note that if every finite frame for $\mathbf{K4.3} \times \mathbf{S5}$ were the p-morphic image of a *finite* product frame for $\mathbf{K4.3} \times \mathbf{S5}$, then we could enumerate finite frames for $\mathbf{K4.3} \times \mathbf{S5}$. As $\mathbf{K4.3} \times \mathbf{S5}$ is recursively enumerable, we can always enumerate those finite frames that are not frames for $\mathbf{K4.3} \times \mathbf{S5}$. So this would provide us with a decision algorithm for the finite frame problem. However, take, say, the 2-frame $\mathfrak{F} = \langle W, \leq, W \times W\rangle$, where $W = \{x, y\}$ and $x \leq x \leq y \leq y$. Then it is easy to see that $\mathfrak{F}$ is a p-morphic image of $\langle\omega, \leq\rangle \times \langle\omega, \omega \times \omega\rangle$, but there is no finite product frame $\mathfrak{G}$ for $\mathbf{K4.3} \times \mathbf{S5}$ such that $\mathfrak{F}$ is a p-morphic image of $\mathfrak{G}$.

To explain our decision algorithm, now we have a closer look at some properties of 2-frames for $\mathbf{K4.3} \oplus \mathbf{S5}$, that is, where the first relation is transitive and weakly connected, and the second relation is an equivalence. To emphasise these facts, the transitive and weakly connected relations in our 2-frames will always be denoted by $\leq$, and the equivalence relations by $\sim$. This will not necessarily mean that $\leq$ is reflexive: there might be 'reflexive' points in our frames with $x \leq x$, and some other 'irreflexive' ones with $y \not\leq y$. (This is a slight abuse of notation, as we will also denote by $\leq$ the usual — reflexive and antisymmetric — linear order on the natural numbers.) So from now on, let

$\mathfrak{F} = \langle W, \leq, \sim \rangle$ be a 2-frame for $\mathbf{K4.3} \oplus \mathbf{S5}$. We will use the following notation:

$$
\begin{array}{lll}
C_x = \{x' : x \leq x' \text{ and } x' \leq x\}, & & \\
x < y \quad \text{iff} \quad x \leq y \text{ and } y \not\leq x, & & \\
x \ll y \quad \text{iff} \quad x < y \text{ and } \forall x'\,(x \leq x' < y \to x' \in C_x), & & \\
[x, y] = \{u : x \leq u \leq y\}, & & [x, y) = \{u : x \leq u < y\}, \\
(x, y] = \{u : x < u \leq y\}, & & (x, y) = \{u : x < u < y\}.
\end{array}
$$

Observe that if x is irreflexive, then C_x is not the '$\leq$-cluster' of x in the usual sense, but $C_x = \emptyset$. Also, the above 'intervals' are not the usual ones either, as $x \notin [x, y]$ or $x \notin [x, y)$ for irreflexive x. For any $X \subseteq W$, we let

$$
\begin{aligned}
&\mathit{min}\, X = \{x \in X : \text{there is no } x' \in X \text{ with } x' < x\}, \text{ and} \\
&\mathit{max}\, X = \{x \in X : \text{there is no } x' \in X \text{ with } x < x'\}.
\end{aligned}
$$

Note that $\mathit{min}\, X$ and $\mathit{max}\, X$ are nonempty, whenever X is finite and nonempty. For any $n > 0$ and $X, Y \subseteq W$, we let

$$
\begin{aligned}
X \overset{n}{\rightsquigarrow} Y \quad \text{iff} \quad & \forall x_1, \dots, x_n \in X \, \big(x_1 \leq \dots \leq x_n \to \\
& \exists y_1, \dots, y_n \in Y \, (y_1 \leq \dots \leq y_n \land \bigwedge_{1 \leq i \leq n} x_i \sim y_i)\big).
\end{aligned}
$$

For $n = 1$, we omit the superscript and write $X \rightsquigarrow Y$:

$$
X \rightsquigarrow Y \qquad \text{iff} \qquad \forall x \in X \; \exists y \in Y \; x \sim y.
$$

If $X = \{x\}$ then we write $x \rightsquigarrow Y$ instead of $\{x\} \rightsquigarrow Y$. Clearly, as $\sim$ is transitive, $\rightsquigarrow$ is a transitive relation on the subsets of W: if $X \rightsquigarrow Y$ and $Y \rightsquigarrow Z$, then $X \rightsquigarrow Z$. Note that if $x \not\leq x$ then $C_x = \emptyset$, and so $C_x \rightsquigarrow Y$ always holds. Observe that $X \rightsquigarrow Y$ does not always follow from $X \overset{2}{\rightsquigarrow} Y$, as there might exist some $x \in X$ with neither $x \leq x'$ nor $x' \leq x$, for any $x' \in X$.

Next, we introduce some important properties of our 2-frames, expressed in the first-order frame-correspondence language having binary predicate symbols $\leq$ and $\sim$. First of all, let

$$
\mathsf{sq}(x, y, z, w) \qquad \text{iff} \qquad x \sim y \leq z \land x \leq w \sim z.
$$

When $\mathsf{sq}(x, y, z, w)$ holds, we visualise this fact with the picture

$$
\begin{tikzcd}
y \arrow[r, "\leq"] \arrow[d, dash, "\sim"'] & z \arrow[d, dash, "\sim"] \\
x \arrow[r, "\leq"'] & w
\end{tikzcd}
$$

The locations of x, y, z, w in this picture motivate the notation for the remaining first-order properties of our frames (l = left, r = right, u = up, d = down):

$$
\begin{aligned}
\psi_u(x, y, z, w) &: \quad \mathsf{sq}(x, y, z, w) \land [y, z) \rightsquigarrow [x, w] \\
\psi_d(x, y, z, w) &: \quad \mathsf{sq}(x, y, z, w) \land [x, w) \rightsquigarrow [y, z] \\
\psi_b(x, y, z, w) &: \quad \psi_u(x, y, z, w) \land \psi_d(x, y, z, w)
\end{aligned}
$$

$$
\begin{aligned}
\psi_{u^2}(x,y,z,w): &\quad \mathsf{sq}(x,y,z,w) \wedge [y,z] \overset{2}{\rightsquigarrow} [x,w] \\
\psi_{d^2}(x,y,z,w): &\quad \mathsf{sq}(x,y,z,w) \wedge [x,w] \overset{2}{\rightsquigarrow} [y,z] \\
\psi_{(u,d^2)}(x,y,z,w): &\quad \mathsf{sq}(x,y,z,w) \wedge \\
&\qquad\qquad \forall a \left(a \in [y,z) \to \exists b \left(b \in [x,w] \wedge \psi_{d^2}(b,a,z,w)\right)\right) \\
\Phi_l: &\quad \forall x,y,z \left(x \sim y \le z \ \to\ \exists w\, \psi_b(x,y,z,w)\right) \\
\Phi_r^+: &\quad \forall x,w,z \left(x \le w \sim z \ \to\ \exists y \left(\psi_{u^2}(x,y,z,w) \wedge \right.\right. \\
&\qquad\qquad\qquad \left.\left.\psi_{d^2}(x,y,z,w) \wedge \psi_{(u,d^2)}(x,y,z,w)\right)\right) \\
\Phi: &\quad \Phi_l \wedge \Phi_r^+
\end{aligned}
$$

Observe that $\psi_u(x,y,z,w)$ follows from $\psi_{(u,d^2)}(x,y,z,w)$.

Now we are in a position to formulate our main result:

Theorem 1.3 *For every finite rooted 2-frame $\mathfrak{F} = \langle W, \le, \sim \rangle$ for* $\mathbf{K4.3} \oplus \mathbf{S5}$, *$\mathfrak{F}$ is a p-morphic image of a product frame for* $\mathbf{K4.3} \times \mathbf{S5}$ *iff Φ holds in $\mathfrak{F}$.*

The formula Φ is quite complex (Π_3). Figure 1 shows that we cannot hope for a much simpler one: $\mathfrak{F}$ is a frame for $\mathbf{S4.3} \oplus \mathbf{S5}$, where Φ_r^+ fails (see the indicated x, w, z), but Φ_l,

$$
\begin{aligned}
&\forall x,w,z \left(x \le w \sim z \ \to\ \exists y \left(\psi_{u^2}(x,y,z,w) \wedge \psi_{d^2}(x,y,z,w)\right)\right), \text{ and} \\
&\forall x,w,z \left(x \le w \sim z \ \to\ \exists y \left(\psi_{u^2}(x,y,z,w) \wedge \psi_{(u,d^2)}(x,y,z,w)\right)\right)
\end{aligned}
$$

all hold (the arrows and ellipses represent the reflexive, transitive and weakly connected $\le$, and the triangles and circles the $\sim$-equivalence classes).

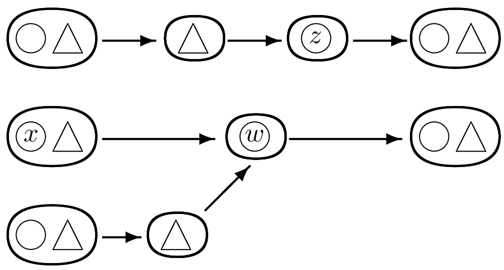


Fig. 1. A frame $\mathfrak{F}$ showing that something like Φ is needed.

The paper is organised as follows. The main steps of the proof of Theorem 1.3 are discussed in Section 2. The more technical claims and lemmas are proved in Section 3. Finally, in Section 4 we discuss some related open problems, possible extensions of our results, and also whether they bring us any closer to either proving non-finite axiomatisability of $\mathbf{K4.3} \times \mathbf{S5}$, or finding an explicit, possibly infinite, axiomatisation of it.

2 P-morphic images of product frames for K4.3 $\times$ S5

We begin with a general observation about p-morphic images of transitive and weakly connected frames.

Claim 2.1 *Let f be a p-morphism from some transitive and weakly connected frame $\mathfrak{F}_0 = \langle W_0, \leq_0 \rangle$ onto a frame $\mathfrak{F}_1 = \langle W_1, \leq_1 \rangle$. For all $a, b \in W_0$, $x_1, \dots, x_n \in W_1$, if $a \leq_0 b$ and $f(a) \leq_1 x_1 \leq_1 \cdots \leq_1 x_n <_1 f(b)$, then there exist $c_1, \dots, c_n \in W_0$ such that $a \leq_0 c_1 \leq_0 \cdots \leq_0 c_n <_0 b$ and $f(c_i) = x_i$, for $i = 1, \dots, n$.*

Proof. Take some $a, b \in W_0$, $x_1, \dots, x_n \in W_1$ such that $a \leq_0 b$ and $f(a) \leq_1 x_1 \leq_1 \cdots \leq_1 x_n <_1 f(b)$. By the backward condition on f, there exists $c_1, \dots, c_n \in W_0$ such that $a \leq_0 c_1 \leq_0 \cdots \leq_0 c_n$ and $f(c_i) = x_i$, for $i = 1, \dots, n$. As $\leq_0$ is transitive, we have $a \leq_0 c_n$. As $\leq_0$ is weakly connected, we have either $c_n = b$, or $b \leq_0 c_n$, or $c_n \leq_0 b$. But $f(c_n) = x_n <_1 f(b)$, so the first two cases cannot hold. Therefore, $c_n <_0 b$ follows. ☐

It is straightforward to check that Φ holds in every product frame for **K4.3** $\times$ **S5**. And, using Claim 2.1, it is not hard to check either that Φ is preserved under taking p-morphic images of frames for **K4.3** $\oplus$ **S5**. So we have:

Proposition 2.2 *If $\mathfrak{F}$ is a p-morphic image of a product frame for* **K4.3** $\times$ **S5**, *then Φ holds in $\mathfrak{F}$.*

We have to work a bit more to prove the other direction of Theorem 1.3. Given a rooted 2-frame $\mathfrak{F} = \langle W, \leq, \sim \rangle$ for **K4.3** $\oplus$ **S5**, we will define a 'p-morphism game' between two players $\forall$ (male) and $\exists$ (female) over $\mathfrak{F}$. In this game, $\exists$ constructs step-by-step, (special) homomorphisms from larger and larger **K4.3** $\times$ **S5**-product frames to $\mathfrak{F}$, and $\forall$ tries to challenge her by pointing out possible 'defects': reasons why her current homomorphism is not an onto p-morphism yet. Versions of such games are used for building complete representations in algebraic logic [5,6], and in connection with axiomatisation problems of multi-dimensional modal logics [3,8].

We will then show that if Φ holds in a finite rooted frame $\mathfrak{F}$ for **K4.3** $\oplus$ **S5**, then $\exists$ has a winning strategy in the ω-step game over $\mathfrak{F}$. Before defining the rules of the game, let us introduce some notions we will use throughout. Given a rooted 2-frame $\mathfrak{F} = \langle W, \leq, \sim \rangle$ for **K4.3** $\oplus$ **S5** and $0 < m, n < \omega$, we call an $n \times m$ matrix

$$\langle x_j^i \in W : i < m,\, j < n \rangle$$

a *perfect grid*, if either $m = 1$ and $x_i^0 \sim x_j^0$ for all $i, j < n$, or $m > 1$ and the following hold:

(pg1) $x_j^i \sim x_k^i$, for all $i < m$, $j, k < n$,

(pg2) either $x_j^i \ll x_j^{i+1}$ or $x_j^i \in C_{x_j^{i+1}}$, for all $i < m - 1$, $j < n$,

(pg3) for all $i < m - 1$, $j < n$, if $x_j^i \ll x_j^{i+1}$ then for all $k < n$, either $C_{x_j^i} \rightsquigarrow C_{x_k^i}$ or $C_{x_j^i} \rightsquigarrow C_{x_k^{i+1}}$.

(See Figure 2 for an example, where the arrows and ellipses represent $\leq$, and the triangles and circles the $\sim$-equivalence classes.)

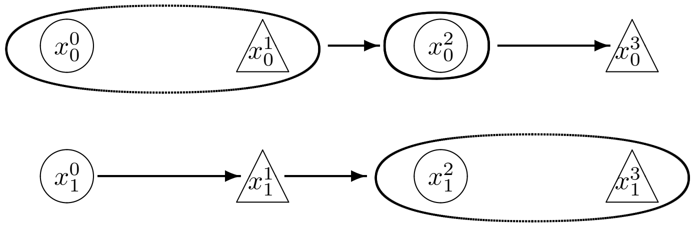


Fig. 2. A perfect grid $\langle x_j^i : i < 4,\, j < 2\rangle$.

Observe that if $\langle x_j^i : i < m,\, j < n\rangle$ is a perfect grid, then for all $k < \ell \leq m$, $\langle x_j^i : k \leq i \leq \ell,\, j < n\rangle$ is a perfect grid as well. If $m = 2$ then we call the $2n$-tuple $\langle x_0^0, \ldots, x_{n-1}^0, x_0^1, \ldots, x_{n-1}^1\rangle$ a *perfect atomic grid*. Clearly, if $m > 1$ and $\langle x_j^i : i < m,\, j < n\rangle$ is a perfect grid, then $\langle x_0^i, \ldots, x_{n-1}^i, x_0^{i+1}, \ldots, x_{n-1}^{i+1}\rangle$ is a perfect atomic grid, for each $i < m - 1$.

Given an $n \times m$ matrix $\bar{x} = \langle x_j^i : i < m,\, j < n\rangle$ and an $n \times k$ matrix $\bar{y} = \langle y_j^i : i < k,\, j < n\rangle$ such that $x_j^{m-1} = y_j^0$, for all $j < n$, their *union* $\bar{x} \sqcup \bar{y}$ is the $n \times (m + k - 1)$ matrix $\langle z_j^i : i < m + k - 1,\, j < n\rangle$, defined by taking, for all $j < n$,

$$z_j^i = \begin{cases} x_j^i, & \text{if } i < m, \\ y_j^{i-m+1}, & \text{if } m - 1 \leq i < m + k - 1. \end{cases}$$

It is easy to see the following claim:

Claim 2.3 *If $\bar{x} = \langle x_j^i : i < m,\, j < n\rangle$ and $\bar{y} = \langle y_j^i : i < k,\, j < n\rangle$ are perfect grids such that $x_j^{m-1} = y_j^0$, for all $j < n$, then $\bar{x} \sqcup \bar{y}$ is a perfect grid as well.*

Given a rooted 2-frame $\mathfrak{F} = \langle W, \leq, \sim\rangle$ for $\mathbf{K4.3}{\oplus}\mathbf{S5}$, we define an $\mathfrak{F}$-*network* to be a tuple $N = \langle U^N, <^N, V^N, f^N\rangle$ such that the following hold:

- $U^N = \{u_0, \ldots, u_m\}$ for some $m < \omega$,
- $<^N$ is an irreflexive linear order on U^N with $u_0 <^N \cdots <^N u_m$,
- $V^N = \{v_0, \ldots, v_n\}$ for some $n < \omega$,
- f^N is a function from $U^N \times V^N$ to W such that $\langle f^N(u_i, v_j) : i \leq m,\, j \leq n\rangle$ is a perfect grid.

It is not hard to see, using (pg1) and (pg2), that if N is an $\mathfrak{F}$-network, then f^N is a homomorphism from the product frame $\langle U^N, <^N\rangle \times \langle V^N, V^N \times V^N\rangle$ to $\mathfrak{F}$.

Now we define a *game* $\mathcal{G}_\omega(\mathfrak{F})$ between $\forall$ and $\exists$. They build a countable sequence of $\mathfrak{F}$-networks $N_0 \subseteq N_1 \subseteq \cdots \subseteq N_k \subseteq \ldots$. (Here $N_k \subseteq N_{k+1}$ means that $U^{N_k} \subseteq U^{N_{k+1}}$, $<^{N_k} \subseteq <^{N_{k+1}}$, $V^{N_k} \subseteq V^{N_{k+1}}$, and $f^{N_k} \subseteq f^{N_{k+1}}$.) In round 0, $\forall$ picks a root r of $\mathfrak{F}$, and $\exists$ responds with $U^{N_0} = \{u_0\}$, $<^{N_0} = \emptyset$, $V^{N_0} = \{v_0\}$, and $f^{N_0}(u_0, v_0) = r$.

In round k $(0 < k < \omega)$, some sequence $N_0 \subseteq \cdots \subseteq N_{k-1}$ of $\mathfrak{F}$-networks has already been built. $\forall$ picks

- a pair $\langle u, v\rangle \in U^{N_{k-1}} \times V^{N_{k-1}}$, and
- a point $w \in W$ such that either (a) $f^{N_{k-1}}(u, v) \leq w$, or (b) $f^{N_{k-1}}(u, v) \sim w$.

In case (a), $\exists$ can respond in two ways. If there is some $u' \in U^{N_{k-1}}$ with $u <^{N_{k-1}} u'$ and $f^{N_{k-1}}(u', v) = w$, then she responds with $N_k = N_{k-1}$. Otherwise, she responds (if she can) with some $\mathfrak{F}$-network $N_k \supseteq N_{k-1}$ such that

- $U^{N_{k-1}} \cup \{u^+\} \subseteq U^{N_k}$ and $f^{N_k}(u^+, v) = w$, for some fresh point u^+, and
- $V^{N_k} = V^{N_{k-1}}$.

In case (b), again $\exists$ can respond in two ways. If there is some $v' \in V^{N_{k-1}}$ with $f^{N_{k-1}}(u, v') = w$, then she responds with $N_k = N_{k-1}$. Otherwise, she responds (if she can) with some $\mathfrak{F}$-network $N_k \supseteq N_{k-1}$ such that

- $V^{N_k} = V^{N_{k-1}} \cup \{v^+\}$ and $f^{N_k}(u, v^+) = w$, for some fresh point v^+.

If $\exists$ can respond in each round k for $k < \omega$ then *she wins the play*. We say that $\exists$ *has a winning strategy in* $\mathcal{G}_\omega(\mathfrak{F})$ if she can win all plays, whatever moves $\forall$ takes in the rounds.

Proposition 2.4 *Let $\mathfrak{F}$ be a countable rooted 2-frame for* **K4.3** $\oplus$ **S5**. *If $\exists$ has a winning strategy in $\mathcal{G}_\omega(\mathfrak{F})$, then $\mathfrak{F}$ is a p-morphic image of a product frame for* **K4.3** $\times$ **S5**.

Proof. Consider a play of the game $\mathcal{G}_\omega(\mathfrak{F})$ when $\forall$ eventually picks all possible pairs and corresponding $\leq$- or $\sim$-connected points in $\mathfrak{F}$ (since $\mathfrak{F}$ is countable, he can do this). If $\exists$ uses her strategy, then she succeeds to construct a countable ascending chain of $\mathfrak{F}$-networks whose union gives a p-morphism from some **K4.3** $\times$ **S5**-product frame onto $\mathfrak{F}$. $\square$

Proposition 2.5 *Let $\mathfrak{F}$ be a finite rooted 2-frame for* **K4.3** $\oplus$ **S5** *such that Φ holds in $\mathfrak{F}$. Then $\exists$ has a winning strategy in $\mathcal{G}_\omega(\mathfrak{F})$.*

Proof. We prove that, for all $k < \omega$, $\exists$ can survive round k in every play, no matter what moves $\forall$ takes in the rounds. We prove this by induction on k. For $k = 0$ this is obvious. So assume inductively that some sequence $N_0 \subseteq \cdots \subseteq N_{k-1}$ of $\mathfrak{F}$-networks has already been built, for some $0 < k < \omega$. Suppose that $U^{N_{k-1}} = \{u_0, \ldots, u_m\}$ such that $u_0 <^{N_{k-1}} \cdots <^{N_{k-1}} u_m$, and $V^{N_{k-1}} = \{v_0, \ldots, v_n\}$. Next, $\forall$ picks some $\langle u, v\rangle \in U^{N_{k-1}} \times V^{N_{k-1}}$ and $w \in W$. There are several cases, depending on how $f^{N_{k-1}}(u, v)$ and w are related. In each case we show how $\exists$ can respond with an N_k satisfying the requirements. We omit those cases where $\exists$'s response is fully determined by the rules of the game.

Case (a).1. $f^{N_{k-1}}(u, v) \leq w$, for all $u' \in U^{N_{k-1}}$, if $u <^{N_{k-1}} u'$ then $f^{N_{k-1}}(u', v) \neq w$, but there exists $u^* \in U^{N_{k-1}}$ such that $u <^{N_{k-1}} u^*$ and $f^{N_{k-1}}(u^*, v) \not\leq w$.

By the IH, $f^{N_{k-1}}$ is a homomorphism, and so $f^{N_{k-1}}(u, v) \leq f^{N_{k-1}}(u^*, v)$ follows. Thus, by weak connectedness of $\leq$, we have $w < f^{N_{k-1}}(u^*, v)$. Therefore, as $U^{N_{k-1}}$ is finite, there are $<^{N_{k-1}}$-successor points $u', u'' \in U^{N_{k-1}}$ such that

$$f^{N_{k-1}}(u', v) \leq w < f^{N_{k-1}}(u'', v). \tag{1}$$

To simplify notation, we let $x_i = f^{N_{k-1}}(u', v_i)$, $y_i = f^{N_{k-1}}(u'', v_i)$, for all $i \leq n$. By the IH, we have that

$$\langle x_0, \ldots, x_n, y_0, \ldots, y_n \rangle \text{ is a perfect atomic grid.} \tag{2}$$

We may assume that $v = v_0$, and so we have $x_0 \ll y_0$ by (1) and (2). Therefore, by (pg3), for each $i \leq n$, we have either $C_{x_0} \rightsquigarrow C_{x_i}$ or $C_{x_0} \rightsquigarrow C_{y_i}$. We now define w_i, for each $i \leq n$ (see Figure 3). Let $w_0 = w$, so by (1) and (2), we have $w_0 \in C_{x_0}$. For every $0 < i \leq n$,

- if $C_{x_0} \rightsquigarrow C_{x_i}$, then we choose some $w_i \in C_{x_i}$ with $w_0 \sim w_i$, and
- if $C_{x_0} \not\rightsquigarrow C_{x_i}$, then $C_{x_0} \rightsquigarrow C_{y_i}$ and we choose some $w_i \in C_{y_i}$ with $w_0 \sim w_i$.

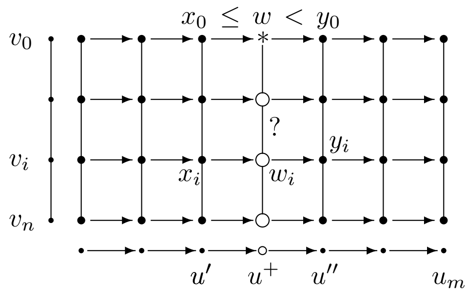


Fig. 3. Case (a).1 of the p-morphism game.

Claim **2.5.1**

(i) $\langle x_0, \ldots, x_n, w_0, \ldots, w_n \rangle$ *is a perfect atomic grid.*

(ii) $\langle w_0, \ldots, w_n, y_0, \ldots, y_n \rangle$ *is a perfect atomic grid.*

Proof. Let us prove (pg3) first. (i): Let $i \leq n$ be such that $x_i \ll w_i$. Then $w_i \notin C_{x_i}$, so by the definition of w_i, we have

$$C_{x_0} \not\rightsquigarrow C_{x_i}, \tag{3}$$

$w_i \in C_{y_i}$, and so

$$x_i \ll y_i. \tag{4}$$

Take some $j < n$. There are two cases:

- $w_j \in C_{y_j}$. Then, by (4) and (2), either $C_{x_i} \rightsquigarrow C_{x_j}$ or $C_{x_i} \rightsquigarrow C_{y_j} = C_{w_j}$.
- $w_j \notin C_{y_j}$. Then $C_{x_0} \rightsquigarrow C_{x_j}$ by the definition of w_j. Therefore, $C_{x_j} \not\rightsquigarrow C_{x_i}$ follows by (3), and so $C_{x_i} \rightsquigarrow C_{x_j}$ by Claim 3.1.

(ii): Let $i \le n$ be such that $w_i \ll y_i$. Then $w_i \notin C_{y_i}$, so

$$C_{x_0} \rightsquigarrow C_{x_i}, \tag{5}$$
$$w_i \in C_{x_i}, \tag{6}$$

and so (4) holds. Take some $j < n$. There are two cases:

- $w_j \in C_{x_j}$. Then by (6), (4) and (2), either $C_{w_i} = C_{x_i} \rightsquigarrow C_{x_j} = C_{w_j}$ or $C_{w_i} = C_{x_i} \rightsquigarrow C_{y_j}$.
- $w_j \notin C_{x_j}$. Then $C_{x_0} \not\rightsquigarrow C_{x_j}$ by the definition of w_j. Therefore, $C_{x_i} \not\rightsquigarrow C_{x_j}$ follows by (5), and so we have $C_{w_i} = C_{x_i} \rightsquigarrow C_{y_j}$ by (6), (4) and (2).

As (pg1) and (pg2) clearly hold in both cases, the proof of Claim 2.5.1 is completed. $\square$

Now take a fresh point u^+. Let $U^{N_k} = U^{N_{k-1}} \cup \{u^+\}$, let $<^{N_k} \supseteq <^{N_{k-1}}$ be such that $u' <^{N_k} u^+ <^{N_k} u''$, and let $f^{N_k}(u^+, v_i) = w_i$, for $i < n$. By Claim 2.5.1, the obtained N_k is an $\mathfrak{F}$-network extending N_{k-1} as required.

Case (a).2. $f^{N_{k-1}}(u, v) \le w$, and for all $u' \in U^{N_{k-1}}$, if $u <^{N_{k-1}} u'$ then $f^{N_{k-1}}(u', v) \le w$ and $f^{N_{k-1}}(u', v) \ne w$.

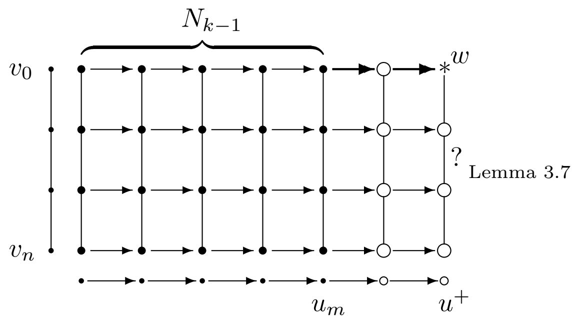


Fig. 4. Case (a).2 of the p-morphism game.

Then $f^{N_{k-1}}(u_m, v) \le w$. We may assume that $v = v_0$ (see Figure 4). By the IH, we have $f^{N_{k-1}}(u_m, v_i) \sim f^{N_{k-1}}(u_m, v_j)$, for all $i, j \le n$. So, by Lemma 3.7, there exists $t > 0$ and a perfect grid $\bar{z} = \langle z_j^\ell : \ell \le t,\, j \le n \rangle$ such that $z_j^0 = f^{N_{k-1}}(u_m, v_j)$, for $j \le n$, and $z_0^t = w$. By the IH, $\bar{f} = \langle f^{N_{k-1}}(u_i, v_j) : i \le m,\, j \le n \rangle$ is a perfect grid, and so by Claim 2.3, $\bar{f} \sqcup \bar{z}$ is a perfect grid as well. Therefore, if we define

- $U^{N_k} = U^{N_{k-1}} \cup \{u_\ell^+ : 0 < \ell \le t\}$, $u^+ = u_t^+$,
- $f^{N_k}(u_\ell^+, v_j) = z_j^\ell$, for $0 < \ell \le t$, $j \le n$,

then we obtain an $\mathfrak{F}$-network N_k extending N_{k-1} as required.

Case (b). $f^{N_{k-1}}(u, v) \sim w$, and $w \ne f^{N_{k-1}}(u, v')$ for all $v' \in V^{N_{k-1}}$.

Suppose $u = u_p$ for some $p \le m$ (see Figure 5). By the IH, $\langle f^{N_{k-1}}(u_i, v_j) : i \le p,\, j \le n \rangle$ is a perfect grid, and $w \sim f^{N_{k-1}}(u_p, v) \sim f^{N_{k-1}}(u_p, v_n)$. So by

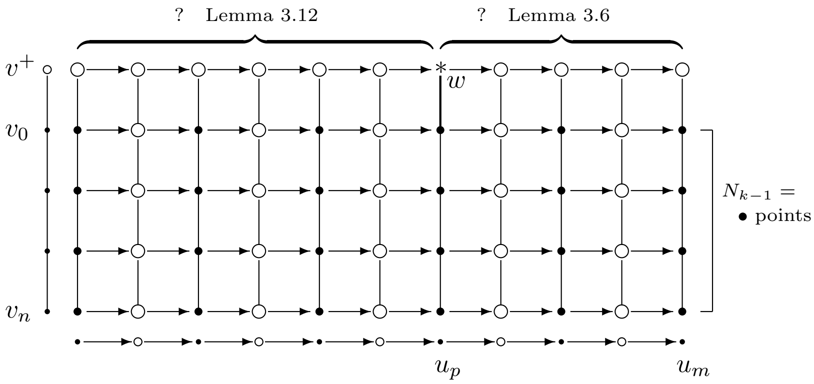


Fig. 5. Case (b) of the p-morphism game.

Lemma 3.12, there exist $s_i < \omega$ ($i \leq p$) and a perfect grid $\bar{z} = \langle z_j^\ell : \ell \leq s_p,\ j \leq n+1 \rangle$ such that $0 = s_0 < s_1 < \cdots < s_p$, $z_{n+1}^{s_p} = w$, and $z_j^{s_i} = f^{N_{k-1}}(u_i, v_j)$, for $i \leq p$, $j \leq n$.

By the IH, $\langle f^{N_{k-1}}(u_{p+i}, v_j) : i \leq m-p,\ j \leq n \rangle$ is a perfect grid as well. As we have $w \sim f^{N_{k-1}}(u_p, v) \sim f^{N_{k-1}}(u_p, v_n)$, by Lemma 3.6 there exist $t_i < \omega$ ($i \leq m-p$) and a perfect grid $\bar{y} = \langle y_j^t : t \leq t_{m-p},\ j \leq n+1 \rangle$ such that $0 = t_0 < t_1 < \cdots < t_{m-p}$, $y_{n+1}^0 = w$, and $y_j^{t_i} = f^{N_{k-1}}(u_{p+i}, v_j)$, for $i \leq m-p$, $j \leq n$.

By Claim 2.3, $\bar{z} \sqcup \bar{y} = \langle x_j^\ell : \ell \leq s_p + t_{m-p} - 1,\ j \leq n+1 \rangle$ is a perfect grid, and therefore by defining

- $U^{N_k} = U^{N_{k-1}} \cup \{u_\ell^+ : \ell < s_p + t_{m-p} - 1,\ \ell \neq s_i, s_p + t_j \text{ for } i \leq p,\ j \leq m-p\}$,
- $V^{N_k} = V^{N_{k-1}} \cup \{v^+\}$,
- $f^{N_k}(u_\ell^+, v_j) = x_j^\ell$, for $u_\ell^+ \in U^{N_k}$, $j \leq n$, and
- $f^{N_k}(u_p, v^+) = w$, $f^{N_k}(u_\ell^+, v^+) = x_{n+1}^\ell$, for $u_\ell^+ \in U^{N_k}$,

we obtain an $\mathfrak{F}$-network N_k extending N_{k-1} as required. This completes the proof of Proposition 2.5. $\square$

3 How Φ helps $\exists$ to have a winning strategy in $\mathcal{G}_\omega(\mathfrak{F})$

In this section we state and prove the claims and lemmas that are used in the proof of Proposition 2.5. The material is divided into two subsections. In Section 3.1 we discuss those statements that describe plays of the game played 'on the left', that is, when $\exists$ makes use of the the fact that the finite frame $\mathfrak{F}$ validates Φ_l. Then in Section 3.2 we describe those plays of the game that are played 'on the right', that is, when $\exists$ also needs to use the conjunct Φ_r^+ of Φ.

Throughout, $\mathfrak{F} = \langle W, \leq, \sim \rangle$ is a finite rooted 2-frame for **K4.3** $\oplus$ **S5**. We begin with two claims that are very important throughout:

Claim 3.1 *Suppose that Φ_l holds in $\mathfrak{F}$, and let $x, y \in W$ be such that $x \sim y$. Then, either $C_x \rightsquigarrow C_y$ or $C_y \rightsquigarrow C_x$.*

Proof. Suppose that $C_x \not\rightsquigarrow C_y$, that is, there is some $a \in C_x$ with $a \not\rightsquigarrow C_y$. Then $y \sim x \leq a$, and so by Φ_l, there is some b such that $\psi_d(y, x, a, b)$ holds. Therefore, $y \leq b$ and $a \sim b$, so $b \notin C_y$, and so $y < b$. Thus, $C_y \subseteq [y, b)$, and so $C_y \rightsquigarrow [x, a] = C_x$ follows by $\psi_d(y, x, a, b)$. $\square$

As $\rightsquigarrow$ is a transitive relation on the subsets of W, we obtain the following:

Claim 3.2 *Suppose that Φ_l holds in $\mathfrak{F}$, let $\emptyset \neq X \subseteq W$ be finite such that $x \sim y$ for all $x, y \in X$, and let $\mathcal{C} = \{C_x : x \in X\}$. Then $\langle \mathcal{C}, \rightsquigarrow \rangle$ is a finite linearly ordered chain of '$\rightsquigarrow$-clusters'. In particular,*

(i) *there is $x_i \in X$ such that C_{x_i} is $\rightsquigarrow$-initial in $\mathcal{C}$: $C_{x_i} \rightsquigarrow C$ for all $C \in \mathcal{C}$;*

(ii) *there is $x_f \in X$ such that C_{x_f} is $\rightsquigarrow$-final in $\mathcal{C}$: $C \rightsquigarrow C_{x_f}$ for all $C \in \mathcal{C}$.*

3.1 Playing on the left

We start with formulating and proving a general structural property of finite frames validating Φ_l (Lemma 3.3). Then in Lemma 3.4 we show that this structural property can be generalised to extensions of perfect atomic grids. This property is then used in Lemma 3.5 to help $\exists$ maintaining a perfect grid, whenever $\forall$ challenges to extend a perfect atomic grid with a '$\leq$-move' (see Case (a).2 in the proof of Prop. 2.5). Then Lemma 3.5 is used as the base case in the inductive proof of Lemma 3.6. Finally, Lemma 3.6 is used in the inductive proof of Lemma 3.7. This last lemma states that any perfect grid can be extended by $\exists$, whenever $\forall$ plays a '$\leq$-move' of the above kind.

Given $x, y, z, w, a \in W$, we write $\mathsf{left}(x, y, z, w, a)$ if the following hold:

(le1) $\mathsf{sq}(x, y, z, w)$ and $x \leq a \leq w$,

(le2) $C_y \rightsquigarrow C_a$,

(le3) $[x, a) \rightsquigarrow C_y$,

(le4) either $a \in C_w$, or $C_a \rightsquigarrow C_y$, or $C_a \rightsquigarrow C_z$,

(le5) $(a, w) \rightsquigarrow C_z$.

Lemma 3.3 *Suppose that Φ_l holds in $\mathfrak{F}$. For all $x, y, z \in W$, if $x \sim y \leq y \ll z$ then there exist w^*, a^* such that $\mathsf{left}(x, y, z, w^*, a^*)$ holds.*

Proof. By Φ_l, there exists w with $\psi_b(x, y, z, w)$. If $w \in C_x$ then let $w^* = a^* = w$, and we clearly have $\mathsf{left}(x, y, z, w^*, a^*)$ as required.

So suppose that

$$\text{there is no } w \in C_x \text{ with } \psi_b(x, y, z, w), \tag{7}$$

and let

$$w^+ \in max\{w : x < w \text{ and } \psi_b(x, y, z, w)\} \tag{8}$$

(as $\mathfrak{F}$ is finite, there is such w^+ by Φ_l and (7)). Now there are two cases: either $[x, w^+) \rightsquigarrow C_y$, or $[x, w^+) \not\rightsquigarrow C_y$.

<u>*Case 1.*</u> $[x, w^+) \rightsquigarrow C_y$.

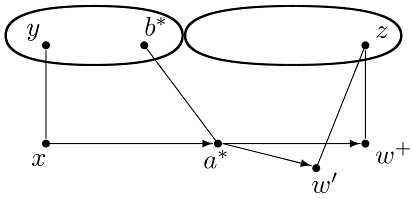
As $\psi_b(x, y, z, w^+)$ and $y \ll z$, we have $C_y \rightsquigarrow [x, w^+]$. As $y \leq y$, there exists $a \in [x, w^+]$ with $a \rightsquigarrow C_y$. Let

$$a^* \in max\{a \in [x, w^+] : a \rightsquigarrow C_y\} \tag{9}$$

(there is such a^* as $\mathfrak{F}$ is finite). We claim that

$$\mathsf{left}(x, y, z, w^+, a^*), \tag{10}$$

and so $w^* = w^+$ will do. Indeed, we clearly have $x \leq a^* \leq w^+$, so we have (le1) by (8). (le2): Let $b^* \in C_y$ be such that $a^* \sim b^*$. By Φ_l, there exists w' with $\psi_b(a^*, b^*, z, w')$.



We claim that

$$\psi_b(x, y, z, w'). \tag{11}$$

Indeed, on the one hand, if $b \in [y, z)$ then $b \in [b^*, z)$, and so $b \rightsquigarrow [a^*, w']$ by $\psi_b(a^*, b^*, z, w')$. As $x \leq a^*$, this implies $b \rightsquigarrow [x, w']$. On the other hand, if $a \in [x, w')$ then there are two cases:

- $a \in [x, a^*)$. Then $a \in [x, w^+)$, and so $a \rightsquigarrow [y, z]$ by (8).
- $a = a^*$ or $a \in [a^*, w')$. Then $a \rightsquigarrow [b^*, z] = [y, z]$ by $\psi_b(a^*, b^*, z, w')$.

So in both cases we have $a \rightsquigarrow [y, z]$, and so (11) is proved.

Now (7) and (11) imply that $x < w'$. Therefore, by (11) and (8), we have $w^+ \not\leq w'$. As $x \leq w^+$ and $x \leq w'$, by the weak connectedness of $\leq$ we have

$$\text{either } w' = w^+ \text{ or } w' \leq w^+. \tag{12}$$

Now we can show (le2), that is, $C_y \rightsquigarrow C_{a^*}$. Take some $b \in C_y$. Then $b \in [b^*, z)$, and so by $\psi_b(a^*, b^*, z, w')$, we have $b \rightsquigarrow [a^*, w']$. By (12), this implies $b \rightsquigarrow [a^*, w^+]$, that is, $b \sim a$ for some $a \in [a^*, w^+]$. Thus, $a \in [x, w^+]$ and $a \rightsquigarrow C_y$, and so by (9), we have $a^* \not\leq a$. As we also have $a^* \leq a$, this implies $a \in C_{a^*}$, as required in (le2).

(le3): As we are in the case when $[x, w^+) \rightsquigarrow C_y$, we also have $[x, a^*) \rightsquigarrow C_y$ by $a^* \leq w^+$, and so (le3) holds.

(le4) and (le5): If $a^* \in C_{w^+}$ then (le4) holds. If $a^* < w^+$, then take any $a \in [a^*, w^+)$. As $a \in [x, w^+)$ and we are in the case when $[x, w^+) \rightsquigarrow C_y$, we have $a \rightsquigarrow C_y$, proving $C_{a^*} \rightsquigarrow C_y$, and so (le4). Moreover, by (9), we also have $a^* \not\leq a$, and so $a \in C_{a^*}$ follows. Therefore, $a^* \ll w^+$, and so $\emptyset = (a^*, w^+) \rightsquigarrow C_z$, as required in (le5), completing the proof of (10).

<u>*Case 2.*</u> $[x, w^+) \not\rightsquigarrow C_y$.
Then there is some $r \in [x, w^+)$ with $r \not\rightsquigarrow C_y$. Let

$$r^* \in \mathit{min}\{r \in [x, w^+) : r \not\rightsquigarrow C_y\} \tag{13}$$

(there is such r^* as $\mathfrak{F}$ is finite). As $\psi_b(x, y, z, w^+)$ by (8), we have

$$r^* \rightsquigarrow C_z. \tag{14}$$

Now let $s^* \in C_z$ be such that $r^* \sim s^*$. By Φ_l, there is w^* with $\psi_b(r^*, s^*, z, w^*)$. Thus, we have

$$[r^*, w^*) \rightsquigarrow C_z. \tag{15}$$

We also need to define a^*. To this end, we claim that

$$\{a \in [x, r^*] : a \rightsquigarrow C_y\} \text{ is not empty.} \tag{16}$$

Indeed, by Φ_l and $y \le y$, there is a such that $\psi_b(x, y, y, a)$ holds. Thus, $a \sim y$ and $[x, a) \rightsquigarrow C_y$, and so $a \neq r^*$ and $r^* \not\le a$ follow from (13). As $x \le r^*$ and $x \le a$, the weak connectedness of $\le$ implies that $a \le r^*$, proving (16). Now let

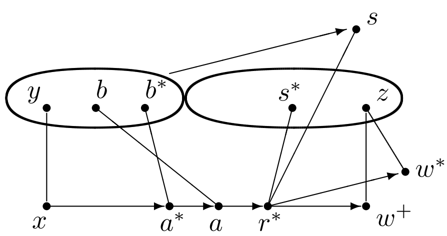
$$a^* \in max\{a \in [x, r^*] : a \rightsquigarrow C_y\} \tag{17}$$

(there is such a^* by (16) and the finiteness of $\mathfrak{F}$). We claim that

$$\mathsf{left}(x, y, z, w^*, a^*). \tag{18}$$

Indeed, we have $x \le a^* \le r^* \le w^*$, so (le1) holds.

(le2): As $a^* \rightsquigarrow C_y$ by (17), there is $b^* \in C_y$ be such that $a^* \sim b^*$. By Φ_l, there is s with $\psi_b(b^*, a^*, r^*, s)$, and so $b^* \le s$. As $r^* \sim s$ and $r^* \not\rightsquigarrow C_y$ by (13), we have $s \notin C_y = C_{b^*}$, and so $b^* < s$ follows. Now take any $b \in C_y$. Then $b \in [b^*, s)$, and so $\psi_b(b^*, a^*, r^*, s)$ implies that there is some $a \in [a^*, r^*]$ with $a \sim b$. Therefore, $a \in [x, r^*]$ and $a \rightsquigarrow C_y$, so $a^* \not< a$ by (17). But we also have $a^* \le a$, and so $a \in C_{a^*}$ follows, as required in (le2).



(le3): As $a^* \le r^* < w^+$ by (17), we have $[x, a^*) \rightsquigarrow C_y$ by (13).

For (le4) and (le5), first we claim that

$$\text{either } C_{a^*} = C_{r^*} \text{ or } a^* \ll r^*. \tag{19}$$

Indeed, we have $a^* \le r^*$ by (17). Suppose that $C_{a^*} \neq C_{r^*}$, and let $a \in [a^*, r^*)$. Then $a \in [x, w^+)$ and $a < r^*$, so $a \rightsquigarrow C_y$ follows by (13). As $a \in [x, r^*]$, we have $a^* \not< a$ by (17). Therefore, $a \in C_{a^*}$ follows from $a^* \le a$, as required in (19).

(le5): $(a^*, w^*) \rightsquigarrow C_z$ follows from (14), (15) and (19).

(le4): If $a^* \in C_{w^*}$, then (le4) holds. If $a^* < w^*$, then by (19) there are two cases:

- $C_{a^*} = C_{r^*}$. Then $r^* < w^*$ and $C_{a^*} \subseteq [r^*, w^*)$. So $C_{a^*} \rightsquigarrow C_z$ follows by (15).
- $a^* \ll r^*$. Then $C_{a^*} \rightsquigarrow C_y$ follows by (13).

So (le4) holds in both cases, completing the proof of (18). $\square$

Lemma 3.4 *Suppose that Φ_l holds in $\mathfrak{F}$, and let $\langle x_0, \ldots, x_{n-1}, y_0, \ldots, y_{n-1}\rangle$ be a perfect atomic grid for some $n > 0$. For all $x \in W$, if $x \sim x_0$ then there exists y such that $y \sim y_0$ and one of the following* (I) *or* (II) *holds:*

(I) *Either $y \in C_x$ and for all $j < n$, if $x_j \ll y_j$ then $C_{x_j} \rightsquigarrow C_x = C_y$.*

(II) *Or $x < y$ and:*

(a) *For all $j < n$, if $x_j \in C_{y_j}$ or $x_j \not\leq x_j$, then $[x, y) \rightsquigarrow C_{y_j}$.*

(b) *For all $j < n$, if $x_j \leq x_j \ll y_j$, then there is a_j with $\mathsf{left}(x, x_j, y_j, y, a_j)$, that is,*

$$
\begin{aligned}
&\mathsf{sq}(x, x_j, y_j, y) \textit{ and } x \leq a_j \leq y, && (20)\\
&C_{x_j} \rightsquigarrow C_{a_j}, && (21)\\
&[x, a_j) \rightsquigarrow C_{x_j}, && (22)\\
&\textit{either } a_j \in C_y, \ \textit{ or } C_{a_j} \rightsquigarrow C_{x_j}, \ \textit{ or } C_{a_j} \rightsquigarrow C_{y_j}, && (23)\\
&(a_j, y) \rightsquigarrow C_{y_j}. && (24)
\end{aligned}
$$

Proof. There are two cases:

<u>*Case 1.*</u> For all $j < n$, either $x_j \in C_{y_j}$ or $x_j \not\leq x_j$.
By (pg1) and Claim 3.2, there is $i < n$ such that

$$C_{y_i} \text{ is } \rightsquigarrow\text{-initial in } \{C_{y_j} : j < n\}. \tag{25}$$

By Φ_l, there is some y such that

$$\psi_b(x, x_i, y_i, y). \tag{26}$$

There are two cases, either $y \in C_x$, or $x < y$:

- $y \in C_x$. As for all $j < n$ with $x_j \ll y_j$, we have $x_j \not\leq x_j$, it follows that $\emptyset = C_{x_j} \rightsquigarrow C_x = C_y$, as required in (I).
- $x < y$. Then $[x, y) \rightsquigarrow [x_i, y_i]$ by (26). As either $x_i \in C_{y_i}$ or $x_i \not\leq x_i$, we have $[x_i, y_i] = C_{y_i}$ by (pg2). Therefore, by (25) and the transitivity of $\rightsquigarrow$, it follows that $[x, y) \rightsquigarrow C_{y_j}$, for all $j < n$, as required in (II).

<u>*Case 2.*</u> There is some $j < n$ such that $x_j \leq x_j \ll y_j$.
By (pg1) and Claim 3.2, there exists some $f < n$ such that C_{x_f} is $\rightsquigarrow$-final in $\{C_{x_j} : j < n,\, x_j \leq x_j \ll y_j\}$. Also, there is $i < n$ such that C_{y_i} is $\rightsquigarrow$-initial in $\{C_{y_j} : j < n,\, x_j \leq x_j \ll y_j, \text{ and } C_{x_f} \rightsquigarrow C_{x_j}\}$. Observe that then

$$
\begin{aligned}
&C_{y_i} \text{ is } \rightsquigarrow\text{-initial in } \{C_{y_j} : j < n,\, x_j \leq x_j \ll y_j, \text{ and } C_{x_i} \rightsquigarrow C_{x_j}\}, \text{ and} && (27)\\
&C_{x_i} \text{ is } \rightsquigarrow\text{-final in } \{C_{x_j} : j < n,\, x_j \leq x_j \ll y_j\}. && (28)
\end{aligned}
$$

Now, by Lemma 3.3, there exist y^*, a^* such that

$$\mathsf{left}(x, x_i, y_i, y^*, a^*). \tag{29}$$

There are two cases, either $y^* \in C_x$, or $x < y^*$. If $y^* \in C_x$, then we let $y = y^*$, and claim that (I) holds. Indeed, by (29) we have $a^* \in C_x$, and so $C_{x_i} \rightsquigarrow C_{a^*} = C_x = C_y$, again by (29). Thus by (28), $C_{x_j} \rightsquigarrow C_x = C_y$ for all $j < n$ with $x_j \leq x_j \ll y_j$. Also, if $j < n$ is such that $x_j \not\leq x_j$, then $C_{x_j} = \emptyset$, and so $C_{x_j} \rightsquigarrow C_x = C_y$, as required in (I).

So suppose that $x < y^*$. We will define some y, and show that

$$\mathsf{sq}(x, x_i, y_i, y) \text{ and } x \leq a^* \leq y, \text{ and} \tag{30}$$

$$(a^*, y) \rightsquigarrow C_{y_j}, \text{ for all } j < n. \tag{31}$$

Then

$$\mathsf{left}(x, x_i, y_i, y, a^*) \tag{32}$$

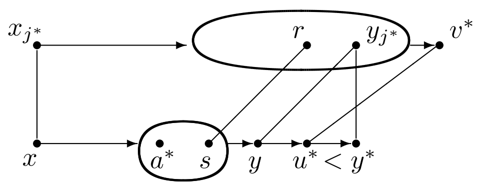
will follow from (29), as the other conjuncts in $\mathsf{left}(x, x_i, y_i, y, a^*)$ do not depend on y, but only on a^*. (Observe that (31) is more than what is required in $\mathsf{left}(x, x_i, y_i, y, a^*)$: it is for all $j < n$, not just for i.)

To this end, we consider three cases:

- $y_i \rightsquigarrow C_{a^*}$. Then we choose some $y \in C_{a^*}$ such that $y_i \sim y$, and so (30)–(31) clearly hold.
- $y_i \not\rightsquigarrow C_{a^*}$ and $(a^*, y^*) \rightsquigarrow C_{y_j}$, for all $j < n$. Then we let $y = y^*$, and (30)–(31) clearly hold.
- $y_i \not\rightsquigarrow C_{a^*}$ and $(a^*, y^*) \not\rightsquigarrow C_{y_j}$, for some $j < n$. Then let

  $$u^* \in \mathit{min}\{u \in (a^*, y^*) : u \not\rightsquigarrow C_{y_j} \text{ for some } j < n\} \tag{33}$$

  (there is such u^* as $\mathfrak{F}$ is finite), and let $j^* < n$ be such that $u^* \not\rightsquigarrow C_{y_{j^*}}$. As $(a^*, y^*) \rightsquigarrow C_{y_i}$ follows from (29), we then have $C_{y_i} \not\rightsquigarrow C_{y_{j^*}}$. Therefore, by (27), we have $C_{x_i} \not\rightsquigarrow C_{x_{j^*}}$, and so $C_{x_i} \rightsquigarrow C_{y_{j^*}}$ follows by $x_i \ll y_i$ and (pg3). We also have $C_{x_i} \rightsquigarrow C_{a^*}$ by (29). Therefore, there are $r \in C_{y_{j^*}}$ and $s \in C_{a^*}$ such that $r \sim s$. By Φ_l, there is v^* such that $\psi_b(r, s, u^*, v^*)$ holds. As $u^* \not\rightsquigarrow C_{y_{j^*}}$ by (33), we have $y_{j^*} < v^*$. So by $\psi_b(r, s, u^*, v^*)$, there is some $y \in [s, u^*]$ such that $y \sim y_{j^*}$. Now, as $s \in C_{a^*}$, we have $x \leq a^* \leq s \leq y$, and so (30) follows from (pg1). Also, as $y \leq u^* < y^*$, we have (31) by (33).



So we proved that y satisfies (30)–(32) in all three cases. Note that y is defined such that

$$\text{if } y_i \rightsquigarrow C_{a^*} \text{ then } y \in C_{a^*}. \tag{34}$$

Next, we show that (30)–(32) imply that (II) holds for y. The following claim will be used several times:

Claim **3.4.1** *If $a^* < y$ and $j < n$ is such that $C_{x_i} \rightsquigarrow C_{y_j}$, then $C_{a^*} \rightsquigarrow C_{y_j}$.*

Proof. By (32), we have $C_{x_i} \rightsquigarrow C_{a^*}$. If $C_{x_i} \rightsquigarrow C_{y_j}$, there exist $u \in C_{a^*}$, $v \in C_{y_j}$ with $u \sim v$. So by Claim 3.1, we have either $C_{a^*} \rightsquigarrow C_{y_j}$ or $C_{y_j} \rightsquigarrow C_{a^*}$. If $C_{y_j} \rightsquigarrow C_{a^*}$ were the case, then we would have $y_j \rightsquigarrow C_{a^*}$, and so $y_i \rightsquigarrow C_{a^*}$ would follow by (pg1). By (34), we would have $y \in C_{a^*}$, contradicting $a^* < y$. Therefore, we have $C_{a^*} \rightsquigarrow C_{y_j}$. □

Proof of (II)(a): Let $j < n$ be such that $x_j \in C_{y_j}$ or $x_j \not\leq x_j$.
By $x_i \ll y_i$ and (pg3), we have

$$C_{x_i} \rightsquigarrow C_{y_j}. \tag{35}$$

Now there are two cases: either $a^* \in C_y$, or $a^* < y$. In each case, we claim to have $[x, y) \rightsquigarrow C_{y_j}$, as required in (II)(a). Indeed,

- $a^* \in C_y$. Then $[x, y) = [x, a^*)$, and we have $[x, a^*) \rightsquigarrow C_{x_i}$ by (32). So $[x, y) \rightsquigarrow C_{y_j}$ follows by (35).
- $a^* < y$. Then we have:
  · $[x, a^*) \rightsquigarrow C_{x_i}$ by (32), and so $[x, a^*) \rightsquigarrow C_{y_j}$ by (35);
  · $C_{a^*} \rightsquigarrow C_{y_j}$ by (35) and Claim 3.4.1;
  · $(a^*, y) \rightsquigarrow C_{y_j}$ by (31).

Proof of (II)(b): Let $j < n$ be such that $x_j \leq x_j \ll y_j$.
There are two cases, either $[x, a^*) \rightsquigarrow C_{x_j}$, or $[x, a^*) \not\rightsquigarrow C_{x_j}$. In both cases, first we define a_j and then show that (20)–(24) (that is, $\mathsf{left}(x, x_j, y_j, y, a_j)$) hold.

- $[x, a^*) \rightsquigarrow C_{x_j}$. Then we let $a_j = a^*$, and we clearly have (20) and (22). By (28), we have $C_{x_j} \rightsquigarrow C_{x_i}$, and by (32), we have $C_{x_i} \rightsquigarrow C_{a_j}$. So $C_{x_j} \rightsquigarrow C_{a_j}$ follows, proving (21). We have (24) by (31). Finally, let us prove (23), that is, either $a_j \in C_y$, or $C_{a_j} \rightsquigarrow C_{x_j}$ or $C_{a_j} \rightsquigarrow C_{y_j}$: Suppose that $a_j = a^* < y$. By (32), there are two cases: either $C_{a^*} \rightsquigarrow C_{x_i}$ or $C_{a^*} \rightsquigarrow C_{y_i}$.
  · $C_{a^*} \rightsquigarrow C_{x_i}$. Then, by $x_i \ll y_i$ and (pg3), we have either $C_{x_i} \rightsquigarrow C_{x_j}$ or $C_{x_i} \rightsquigarrow C_{y_j}$, so (23) follows.
  · $C_{a^*} \rightsquigarrow C_{y_i}$. If $C_{x_i} \rightsquigarrow C_{x_j}$, then $C_{y_i} \rightsquigarrow C_{y_j}$ follows by (27), and so we have $C_{a^*} \rightsquigarrow C_{y_j}$. If $C_{x_i} \not\rightsquigarrow C_{x_j}$, then by $x_i \ll y_i$ and (pg3), we have $C_{x_i} \rightsquigarrow C_{y_j}$. So by Claim 3.4.1, we have $C_{a^*} \rightsquigarrow C_{y_j}$, as required in (23).
- $[x, a^*) \not\rightsquigarrow C_{x_j}$. By Lemma 3.3, there are a_j, y_j^* such that

$$\mathsf{left}(x, x_j, y_j, y_j^*, a_j). \tag{36}$$

  We claim that $\mathsf{left}(x, x_j, y_j, a_j)$ as well, that is, (20)–(24) hold. Indeed, by (36), we have $x \leq a_j$ and $[x, a_j) \rightsquigarrow C_{x_j}$. As $x \leq a^*$ and $[x, a^*) \not\rightsquigarrow C_{x_j}$, by the weak connectivity of $\leq$ it follows that

$$x \leq a_j < a^* \leq y, \tag{37}$$

  as required in (20). As (21) and (22) do not depend on y, they hold because of (36). Next, by (32), we have $[x, a^*) \rightsquigarrow C_{x_i}$, and so $C_{x_i} \not\rightsquigarrow C_{x_j}$ follows from

$[x, a^*) \not\rightsquigarrow C_{x_j}$. So by $x_i \ll y_i$ and (pg3), we have

$$C_{x_i} \rightsquigarrow C_{y_j}, \tag{38}$$

and so

$$[x, a^*) \rightsquigarrow C_{y_j}. \tag{39}$$

For (23): We have $C_{a_j} \rightsquigarrow C_{y_j}$ by (37) and (39). For (24): (37) and (39) imply $(a_j, a^*) \rightsquigarrow C_{y_j}$. So if $a^* \in C_y$, then $(a_j, y) \rightsquigarrow C_{y_j}$ follows. If $a^* < y$, then $C_{a^*} \rightsquigarrow C_{y_j}$ follows by (38) and Claim 3.4.1. Also, we have $(a^*, y) \rightsquigarrow C_{y_j}$ by (31). Therefore, $(a_j, y) \rightsquigarrow C_{y_j}$ holds, as required.

So we proved (II)(b), and the proof of Lemma 3.4 is completed. $\square$

Lemma 3.5 *Suppose that Φ_l holds in $\mathfrak{F}$, and let $\langle x_0, \dots, x_{n-1}, y_0, \dots, y_{n-1}\rangle$ be a perfect atomic grid for some $n > 0$. For all $x \in W$, if $x \sim x_0$ then there exist $k > 0$ and a perfect grid $\langle z_j^\ell : \ell \le k,\ j \le n\rangle$ such that $z_j^0 = x_j$, $z_j^k = y_j$, for $j < n$, and $z_n^0 = x$.*

Proof. By Lemma 3.4, there is y such that either (I) or (II) of the lemma holds. If (I) holds, that is, $y \in C_x$, then let $k = 1$, $z_n^0 = x$, and $z_n^1 = y$. Of course, we let $z_j^0 = x_j$ and $z_j^1 = y_j$, for $j < n$. It is straightforward to show that $\langle z_0^0, \dots, z_n^0, z_0^1, \dots, z_n^1\rangle$ is a perfect atomic grid.

Suppose that (II) holds, that is $x < y$, and for all $j < n$ with $x_j \le x_j \ll y_j$, we have some a_j as in (II)(b). Then let $k > 0$, and $z_n^0, \dots z_n^k$ be such that $x = z_n^0 \ll \cdots \ll z_n^k = y$ (that is, we take a point from each $\le$-cluster between x and y). Of course, we let $z_j^0 = x_j$, $z_j^k = y_j$, for all $j < n$. Next, we define a number $\ell_j < k$, for every $j < n$ as follows:

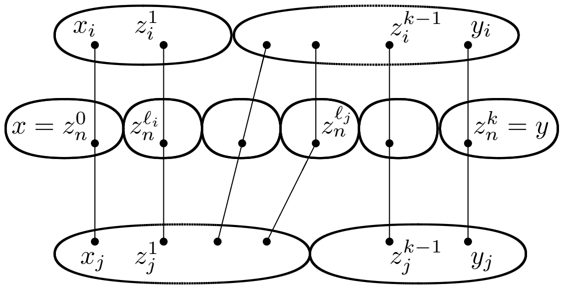
- If $x_j \in C_{y_j}$ or $x_j \not\le x_j$, then let $\ell_j = 0$.
- If $x_j \le x_j \ll y_j$, then there are several cases, depending on the location of a_j in $[x, y]$:
  - · If $a_j \in C_y$, then let $\ell_j = k - 1$.
  - · If $a_j < y$ and $C_{a_j} \rightsquigarrow C_{x_j}$, then let ℓ_j be such that $z_n^{\ell_j} \in C_{a_j}$.
  - · If $a_j < y$, $C_{a_j} \not\rightsquigarrow C_{x_j}$, and $a_j \in C_x$, then let $\ell_j = 0$.
  - · If $a_j < y$, $C_{a_j} \not\rightsquigarrow C_{x_j}$, and $x < a_j$, then let ℓ_j be such that $z_n^{\ell_j+1} \in C_{a_j}$.

The following claim is a straightforward consequence of (II)(a) and (22)–(24) in (II)(b):

Claim **3.5.1**
(ii) *Either $C_{z_n^0} \rightsquigarrow C_{x_i}$, or ($\ell_j = 0$ and $C_{z_n^0} \rightsquigarrow C_{y_j}$).*
(ii) *$z_n^\ell \rightsquigarrow C_{x_i}$ and $C_{z_n^\ell} \rightsquigarrow C_{x_i}$, for all ℓ with $0 < \ell \le \ell_j$.*
(iii) *$z_n^\ell \rightsquigarrow C_{y_i}$ and $C_{z_n^\ell} \rightsquigarrow C_{y_i}$, for all ℓ with $\ell_j < \ell < k$.*

We use Claim 3.5.1(ii) and (iii) to define z_j^ℓ, for each $0 < \ell < k$ and $j < n$:

- If $0 < \ell \le \ell_j$, then choose $z_j^\ell \in C_{x_j}$ such that $z_n^\ell \sim z_j^\ell$.
- If $\ell_j < \ell < k$, then choose $z_j^\ell \in C_{y_j}$ such that $z_n^\ell \sim z_j^\ell$.



As a consequence of Claim 3.5.1, and (21), we obtain the following:

CLAIM **3.5.2** *For all $j < n$,*
(i) *either $C_{z_n^0} \rightsquigarrow C_{z_j^0}$ or $C_{z_n^0} \rightsquigarrow C_{z_j^1}$;*
(ii) $C_{z_n^\ell} \rightsquigarrow C_{z_j^\ell}$, *whenever $0 < \ell < k$;*
(iii) *if $x_j \ll y_j$ then either $C_{z_j^{\ell_j}} \rightsquigarrow C_{z_n^{\ell_j}}$ or $C_{z_j^{\ell_j}} \rightsquigarrow C_{z_n^{\ell_j+1}}$.*

Now we claim that $\langle z_j^\ell : \ell \le k,\ j \le n\rangle$ is a perfect grid as required. Indeed (pg1) and (pg2) clearly hold. Let us prove that (pg3) holds as well, that is, for all $\ell < k$, $i, j \le n$,

$$\text{if } z_i^\ell \ll z_i^{\ell+1} \text{ then either } C_{z_i^\ell} \rightsquigarrow C_{z_j^\ell} \text{ or } C_{z_i^\ell} \rightsquigarrow C_{z_j^{\ell+1}}. \tag{40}$$

If $i = j$, this clearly holds. Otherwise, there are three cases:

- $i = n$, $j < n$. Then (40) holds by Claim 3.5.2(i) and (ii).
- $i < n$, $j = n$. If $z_i^\ell \ll z_i^{\ell+1}$ then $\ell = \ell_i$ and (40) holds by Claim 3.5.2(iii).
- $i, j < n$. Again, if $z_i^\ell \ll z_i^{\ell+1}$ then $\ell = \ell_i$, and so either $C_{z_i^{\ell_i}} \rightsquigarrow C_{z_n^{\ell_i}}$ or $C_{z_i^{\ell_i}} \rightsquigarrow C_{z_n^{\ell_i+1}}$, by Claim 3.5.2(iii). Now either $C_{z_i^{\ell_i}} \rightsquigarrow C_{z_j^{\ell_i}}$ or $C_{z_i^{\ell_i}} \rightsquigarrow C_{z_j^{\ell_i+1}}$ follow by Claim 3.5.2(i) and (ii),

completing the proof of Lemma 3.5. □

Lemma 3.6 *Suppose that Φ_l holds in $\mathfrak{F}$, and let $\langle x_j^i : i \le m,\ j < n\rangle$ be a perfect grid, for some $m, n < \omega$, $n > 0$. For all $x \in W$, if $x \sim x_0^0$ then there exist $t_i < \omega$ $(i \le m)$ and a perfect grid $\langle z_j^\ell : \ell \le t_m,\ j \le n\rangle$ such that $0 = t_0 < t_1 < \cdots < t_m$, $z_j^{t_i} = x_j^i$, for $i \le m$, $j < n$, and $z_n^0 = x$.*

Proof. It is by induction on m. For $m = 0$ the statement is obvious. Suppose the statement holds for some $m < \omega$. Let $\langle x_j^i : i \le m+1,\ j < n\rangle$ be a perfect grid, and let $x \in W$ be such that $x \sim x_0^0$. Then $\langle x_j^i : i \le m,\ j < n\rangle$ is a perfect grid, and so by the IH, there exist $t_i < \omega$, for $i \le m$, and a perfect grid $\bar{z} = \langle z_j^\ell : \ell \le t_m,\ j \le n\rangle$ such that $0 = t_0 < t_1 < \cdots < t_m$, $z_j^{t_i} = x_j^i$, for $i \le m$, $j < n$, and $z_n^0 = x$. We also have that $\langle x_0^m, \ldots, x_{n-1}^m, x_0^{m+1}, \ldots, x_{n-1}^{m+1}\rangle$ is a perfect atomic grid, and $z_n^{t_m} \sim z_0^{t_m} = x_0^m$. So by Lemma 3.5, there exist $k > 0$ and a perfect grid $\bar{y} = \langle y_j^\ell : \ell \le k,\ j \le n\rangle$ such that $y_j^0 = x_j^m$, for $j < n$, $y_n^0 = z_n^{k_m}$ and $y_j^k = x_j^{m+1}$, for $j < n$. By Claim 2.3, $\bar{z} \sqcup \bar{y}$ is a perfect grid as required. □

Lemma 3.7 *Suppose that Φ_l holds in $\mathfrak{F}$, and let $\langle y_j : j \leq n\rangle$ be such that $y_i \sim y_j$ for $i, j \leq n$. For all $y \in W$, if $y_0 \leq y$ then there exist $t > 0$ and a perfect grid $\langle z_j^\ell : \ell \leq t,\ j \leq n\rangle$ such that $z_0^t = y$ and $z_j^0 = y_j$, for $j \leq n$.*

Proof. It is by induction on n. If $n = 0$, then take $t > 0$ and $z_0^0, \dots, z_0^t$ such that $y_0 = z_0^0$, $y = z_0^t$, either $z_0^0 \in C_{z_0^1}$ or $z_0^0 \ll z_0^1$, and $z_0^\ell \ll z_0^{\ell+1}$, for all $1 \leq \ell < t$. Then $\langle z_0^0, \dots, z_0^t\rangle$ is clearly a perfect grid.

Now suppose that the statement holds for some $n < \omega$. Let $\langle y_j : j \leq n+1\rangle$ be such that $y_i \sim y_j$ for $i, j \leq n+1$, and take some $y \in W$ with $y_0 \leq y$. By the IH, there exist $m > 0$ and a perfect grid $\langle x_j^i : i \leq m,\ j \leq n\rangle$ such that $x_0^m = y$ and $x_j^0 = y_j$, for $j \leq n$. As $y_{n+1} \sim y_0 = x_0^0$, by Lemma 3.6 there exist $t_i < \omega$ ($i \leq m$) and a perfect grid $\bar{z} = \langle z_j^\ell : \ell \leq t_m,\ j \leq n+1\rangle$ such that $0 = t_0 < t_1 < \cdots < t_m$, $z_j^{t_i} = x_j^i$, for $i \leq m$, $j \leq n$, and $z_{n+1}^0 = y_{n+1}$. Therefore, $z_0^{t_m} = x_0^m = y$, $z_j^0 = z_j^{t_0} = x_j^0 = y_j$, for $j \leq n$, and $z_{n+1}^0 = y_{n+1}$, showing that $\bar{z}$ is a perfect grid as required. $\Box$

3.2 Playing on the right

Similarly to Section 3.1, here we start with formulating and proving a general structural property of finite frames validating Φ (Lemma 3.8). Observe that the 'right' conjunct Φ_r^+ of Φ is kind of 'stronger' than its 'left' conjunct Φ_l. Perhaps this is why the 'right' property below is considerably simpler than the corresponding 'left' property (see Lemma 3.3 above). Then in Lemma 3.10 we show that this structural property can be generalised to extensions of perfect atomic grids. This property is then used in Lemma 3.11 to help $\exists$ maintaining a perfect grid, whenever $\forall$ challenges to extend a perfect atomic grid with a '$\sim$-move' (see Case (b) in the proof of Prop. 2.5). Finally, Lemma 3.11 is used as the base case in the inductive proof of Lemma 3.12 that, together with Lemma 3.6, show that any perfect grid can be extended by $\exists$, whenever $\forall$ plays a '$\sim$-move'.

Given $x, y, z, w \in W$, we write $\mathsf{right}(x, y, z, w)$ if the following hold:

(r1) $\mathsf{sq}(x, y, z, w)$,

(r2) either $x \in C_w$ or $C_x \rightsquigarrow C_y$,

(r3) either $y \in C_z$, or $C_y \rightsquigarrow C_x$, or $C_y \rightsquigarrow C_w$,

(r4) $(y, z) \rightsquigarrow C_w$.

Lemma 3.8 *Suppose that Φ holds in $\mathfrak{F}$. For all $x, w, z \in W$, if $x \leq x \ll w \sim z$ then there exists y^* such that $\mathsf{right}(x, y^*, z, w)$ holds.*

Proof. If $C_x \rightsquigarrow C_z$, then there is $y^* \in C_z$ with $x \sim y^*$. It is straightforward to see that $\mathsf{right}(x, y^*, z, w)$ holds. So suppose that

$$C_x \not\rightsquigarrow C_z, \tag{41}$$

and let

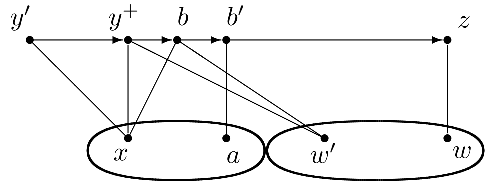
$$y^+ \in min\{y : \psi_{all}(x, y, z, w)\}, \tag{42}$$

where $\psi_{all}(x, y, z, w)$ is a shorthand for

$$\psi_{u^2}(x, y, z, w) \wedge \psi_{d^2}(x, y, z, w) \wedge \psi_{(u,d^2)}(x, y, z, w).$$

(As $\mathfrak{F}$ is finite, there is such y^+ by Φ_r^+.) Now there are two cases: either $[y^+, z) \rightsquigarrow C_w$, or $[y^+, z) \not\rightsquigarrow C_w$.

<u>*Case 1.*</u> $[y^+, z) \rightsquigarrow C_w$.
We claim that $\mathsf{right}(x, y^+, z, w)$ holds, and so $y^* = y^+$ will do. Indeed, we clearly have (r1). (r3) and (r4) hold by $[y^+, z) \rightsquigarrow C_w$. For (r2): By (41), there is some $a \in C_x$ with $a \not\rightsquigarrow C_z$. We have $\psi_{d^2}(x, y^+, z, w)$ by (42), and so $x \leq x \leq a < w$ implies that there are b, b' such that $y^+ \leq b \leq b' \leq z$, $x \sim b$, and $a \sim b'$. Thus $b' \notin C_z$, and so $b \leq b' < z$ follows. Now $[y^+, z) \rightsquigarrow C_w$ implies that $b \rightsquigarrow C_w$, and so $y^+ \rightsquigarrow C_w$ follows from $y^+ \sim x \sim b$. Therefore, there is some $w' \in C_w$ with $y^+ \sim w'$. By Φ_r^+, there is y' such that $\psi_{all}(x, y', y^+, w')$.



It is straightforward to check that $\psi_{all}(x, y', z, w)$ also holds. So by (42), we have $y' \in C_{y^+}$, and so $C_x \rightsquigarrow C_{y^+}$ follows by $x \leq x < w$ and $\psi_{d^2}(x, y', y^+, w')$, completing the proof of (r2).

<u>*Case 2.*</u> $[y^+, z) \not\rightsquigarrow C_w$.
Then let

$$b^+ \in max\{b \in [y^+, z) : b \not\rightsquigarrow C_w\}. \tag{43}$$

(there is such b^+ as $\mathfrak{F}$ is finite). We have $\psi_{(u,d^2)}(x, y^+, z, w)$ by (42), so there is $a^+ \in [x, w]$ such that $a^+ \sim b^+$ and

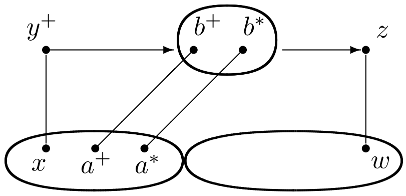
$$[a^+, w) \overset{2}{\rightsquigarrow} [b^+, z]. \tag{44}$$

By (43), we have $b^+ \not\rightsquigarrow C_w$, and so $a^+ \in C_x$.

We claim that there exists b^* such that

$$b^* \in C_{b^+} \cup \{b^+\},\ b^* \not\rightsquigarrow C_w \text{ and } b^* \not\rightsquigarrow C_z. \tag{45}$$

Indeed, if $b^+ \not\rightsquigarrow C_z$ then (45) holds for $b^* = b^+$. So suppose that $b^+ \rightsquigarrow C_z$. As by (43) we also have $b^+ \not\rightsquigarrow C_w$, it follows that $C_z \not\rightsquigarrow C_w$. So by Claim 3.1, we have $C_w \rightsquigarrow C_z$, and so $C_x \not\rightsquigarrow C_w$ follows by (41). Also by (41), there is some $a^* \in C_x$ such that $a^* \not\rightsquigarrow C_z$. By $C_w \rightsquigarrow C_z$, we also have $a^* \not\rightsquigarrow C_w$. As $a^+ \leq a^* \leq a^* < w$, by (44) there exists $b^* \in [b^+, z]$ with $a^* \sim b^*$. As $a^* \not\rightsquigarrow C_z$, we have $b^* \not\rightsquigarrow C_z$ and $b^* \notin C_z$. Thus $b^* \in [b^+, z) \subseteq [y^+, z)$ follows. As $a^* \not\rightsquigarrow C_w$, we also have $b^* \not\rightsquigarrow C_w$. Therefore, by (43), we obtain that $b^* \in C_{b^+}$, as required in (45).



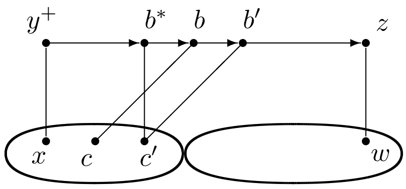
So take some b^* satisfying (45). By (43), we have

$$b^* \in max\{b \in [y^+, z) : b \not\rightsquigarrow C_w\}. \tag{46}$$

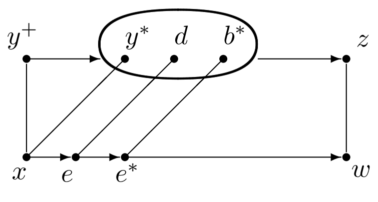
We claim that

$$C_x \rightsquigarrow C_{b^*}. \tag{47}$$

Indeed, as we have $\psi_{(u,d^2)}(x, y^+, z, w)$ by (42), there is $c' \in [x, w]$ such that $[c', w) \stackrel{2}{\rightsquigarrow} [b^*, z]$ and $c' \sim b^*$. As $b^* \not\rightsquigarrow C_w$, it follows that $c' \in C_x$. Now take any $c \in C_x$. Then $c' \leq c \leq c' < w$, and so there exist b, b' such that $b^* \leq b \leq b' \leq z$, $c \sim b$ and $c' \sim b'$. Thus $b' \sim b^*$ and by (45) we have $b' \notin C_z$ and $b' \not\rightsquigarrow C_w$. Therefore, $y^+ \leq b^* \leq b \leq b' < z$ follows, and by (46) we have that $b' \in C_{b^*}$. Therefore, $b \in C_{b^*}$ as well, as required in (47).



Now by (47), there is $y^* \in C_{b^*}$ such that $x \sim y^*$. We claim that $\mathsf{right}(x, y^*, z, w)$ holds. Indeed, (r1) is clear, (r2) is (47), and (r4) holds by (46). For (r3): We show that $C_{y^*} \rightsquigarrow C_x$. Take some $d \in C_{y^*} = C_{b^*}$. Then $y^+ \leq d \leq b^* < z$. As by (42) we have $\psi_{u^2}(x, y^+, z, w)$, this implies that there exist e, e^* such that $x \leq e \leq e^* \leq w$, $e \sim d$ and $e^* \sim b^*$.



As $b^* \not\rightsquigarrow C_w$ by (46), we have $e^* \in C_x$, and so $e \in C_x$ follows, as required. $\square$

The following claim will be useful in subsequent proofs:

Claim 3.9 *Suppose that Φ_r^+ holds in $\mathfrak{F}$. If $y^+ \in min\{y : \psi_u(x, y, z, w)\}$, then $C_x \rightsquigarrow C_{y^+}$.*

Proof. If $C_x = \emptyset$, then this holds. So take some $a \in C_x$. As $a \leq x \sim y^+$, by Φ_r^+ there exists b such that $\psi_{(u,d^2)}(a, b, y^+, x)$, and so $\psi_u(a, b, y^+, x)$. As $x \leq a \sim b$, by Φ_r^+ again, there exists y' such that $\psi_u(x, y', b, a)$. So we have $y' \le b \le y^+$, and $[y', y^+) \cup \{y^+\} \rightsquigarrow C_x$. So it is straightforward to check that $\psi_u(x, y', z, w)$ holds. Therefore, by $y^+ \in min\{y : \psi_u(x, y, z, w)\}$, we have $y' \not< y^+$, and so $y' \in C_{y^+}$. Therefore, $b \in C_{y^+}$ follows, proving $C_x \rightsquigarrow C_{y^+}$. □

Lemma 3.10 *Suppose that Φ holds in $\mathfrak{F}$, and let $\langle x_0, \dots, x_{n-1}, y_0, \dots, y_{n-1}\rangle$ be a perfect atomic grid for some $n > 0$. For all $y \in W$, if $y \sim y_0$ then there exists x such that, for every $j < n$, $\mathsf{right}(x_j, x, y, y_j)$ holds, that is,*

$$\mathsf{sq}(x_j, x, y, y_j), \tag{48}$$

$$\textit{either } x_j \in C_{y_j} \textit{ or } C_{x_j} \rightsquigarrow C_x, \tag{49}$$

$$\textit{either } x \in C_y, \textit{ or } C_x \rightsquigarrow C_{x_j}, \textit{ or } C_x \rightsquigarrow C_{y_j}, \tag{50}$$

$$(x, y) \rightsquigarrow C_{y_j}. \tag{51}$$

Proof. By (pg1), Φ_l and Claim 3.2, there is $i < n$ such that

$$C_{y_i} \text{ is } \rightsquigarrow\text{-initial in } \{C_{y_j} : j < n\}. \tag{52}$$

We claim that there exists x^* such that

$$\mathsf{sq}(x_i, x^*, y, y_i), \tag{53}$$

$$C_{x_i} \rightsquigarrow C_{x^*}, \tag{54}$$

$$\text{either } x^* \in C_y, \text{ or } C_{x^*} \rightsquigarrow C_{x_i}, \text{ or } C_{x^*} \rightsquigarrow C_{y_i}, \tag{55}$$

$$(x^*, y) \rightsquigarrow C_{y_i}. \tag{56}$$

Indeed, if $x_i \le x_i \ll y_i$ then such an x^* exists by Lemma 3.8. If $x_i \in C_{y_i}$ or $x_i \not\le x_i$, then let $x^* \in min\{x' : \psi_u(x_i, x', y, y_i)\}$ (there exists such x^* by Φ_r^+ and the finiteness of $\mathfrak{F}$). Then (53), (55), and (56) follow from $\psi_u(x_i, x^*, y, y_i)$ and $[x_i, y_i] = C_{y_i}$, and (54) follows from Claim 3.9.

Now we consider two cases:

Case 1. For all $j < n$, if $x_j \le x_j \ll y_j$ then $C_{x_j} \rightsquigarrow C_{x_i}$.
Then we let $x = x^*$, and claim that (48)–(51) hold, for all $j < n$. Indeed, take some $j < n$. Then (48) is clear. For (49): If $x_j \in C_{y_j}$ or $x_j \not\le x_j$, then (49) clearly holds. If $x_j \le x_j \ll y_j$ then $C_{x_j} \rightsquigarrow C_{x_i}$, so (49) follows from (54). For (50): By (55), there are three cases:

- $x \in C_y$. Then (50) holds.
- $C_x \rightsquigarrow C_{y_i}$. Then $C_x \rightsquigarrow C_{y_j}$ by (52).
- $C_x \rightsquigarrow C_{x_i}$ and $C_x \not\rightsquigarrow C_{y_i}$. Then $x_i \le x_i \ll y_i$, and by (pg3) we have either $C_{x_i} \rightsquigarrow C_{x_j}$ or $C_{x_i} \rightsquigarrow C_{y_j}$. So (50) follows by the transitivity of $\rightsquigarrow$.

Finally, (51) follows from (56) and (52).

Case 2. There is some $j < n$ with $x_j \le x_j \ll y_j$ and $C_{x_j} \not\rightsquigarrow C_{x_i}$.
By (pg1), Φ_l and Claim 3.2, there is $f < n$ such that

$$C_{x_f} \text{ is } \rightsquigarrow\text{-final in } \{C_{x_j} : j < n,\ x_j \le x_j \ll y_j\}. \tag{57}$$

We claim that

$$C_{x_f} \rightsquigarrow C_{y_i}. \tag{58}$$

Indeed, if $x_i \in C_{y_i}$ or $x_i \not\leq x_i$, then this holds by $x_f \ll y_f$ and (pg3). If $x_i \leq x_i \ll y_i$, then $C_{x_f} \not\rightsquigarrow C_{x_i}$ by our assumption on Case 2 and (57), and so $C_{x_f} \rightsquigarrow C_{y_i}$ follows again by $x_f \ll y_f$ and (pg3).

As $x_f \leq x_f \sim x^*$, by Φ_r^+ and the finiteness of $\mathfrak{F}$, there is some x such that

$$x \in min\{x' : \psi_u(x_f, x', x^*, x_f)\}. \tag{59}$$

We claim that, for all $j < n$, we have $\mathsf{right}(x_j, x, y, y_j)$, that is, (48)–(51) hold. Indeed, take some $j < n$. Then (48) is clear. For (49): By (59) and Claim 3.9, we have that $C_{x_f} \rightsquigarrow C_x$. If $x_j \notin C_{y_j}$, then $C_{x_j} \rightsquigarrow C_x$ follows by (57).

In order to show (50) and (51), we claim that

$$\text{either } x \in C_y, \text{ or } [x, y) \rightsquigarrow C_{y_i}. \tag{60}$$

Indeed, suppose that $x \notin C_y$ and take some $a \in [x, y)$. There are three cases:

- $a \in [x, x^*) \cup \{x^*\}$. Then $a \rightsquigarrow C_{x_f}$ by (59), and so $a \rightsquigarrow C_{y_i}$ follows by (58).
- $x^* \notin C_y$ and $a \in C_{x^*}$. Then by (55), either $a \rightsquigarrow C_{y_i}$, or $a \rightsquigarrow C_{x_i}$. In the latter case, either $C_{x_i} = C_{y_i}$, or $C_{x_i} \rightsquigarrow C_{x_f}$ by (57), and so $a \rightsquigarrow C_{y_i}$ follows by (58).
- $a \in (x^*, y)$. Then $a \rightsquigarrow C_{y_i}$ by (56).

Now let us show (50): If $x \notin C_y$, then we have $C_x \rightsquigarrow C_{y_i}$ by (60), and so $C_x \rightsquigarrow C_{y_j}$ follows by (52). And for (51): We have $(x, y) \rightsquigarrow C_{y_i}$ by (60), and so $(x, y) \rightsquigarrow C_{y_j}$ follows by (52). □

Lemma 3.11 *Suppose that Φ holds in $\mathfrak{F}$, and let $\langle x_0, \ldots, x_{n-1}, y_0, \ldots, y_{n-1}\rangle$ be a perfect atomic grid for some $n > 0$. For all $y \in W$, if $y \sim y_0$ then there exist $k > 0$ and a perfect grid $\langle z_j^\ell : \ell \leq k, j \leq n\rangle$ such that $z_j^0 = x_j$, $z_j^k = y_j$, for $j < n$, and $z_n^k = y$.*

Proof. By Lemma 3.10, there is x such that $\mathsf{right}(x_j, x, y, y_j)$ holds, for every $j < n$. If $x \in C_y$ then let $k = 1$, $z_n^0 = x$, $z_n^1 = y$, and $z_j^0 = x_j$, $z_j^1 = y_j$, for all $j < n$. It is straightforward to show that $\langle z_0^0, \ldots, z_n^0, z_0^1, \ldots, z_n^1\rangle$ is a perfect atomic grid.

If $x < y$, then let $k > 0$ and $z_n^0, \ldots z_n^k$ be such that $x = z_n^0 \ll \cdots \ll z_n^k = y$ (that is, we take a point from each $\leq$-cluster between x and y). Of course, we let $z_j^0 = x_j$, and $z_j^k = y_j$, for all $j < n$. Next, for each $j < n$, we have $(x, y) \rightsquigarrow C_{y_j}$ by (51). Therefore, for each $0 < \ell < k$, there exists $z_j^\ell \in C_{y_j}$ such that $z_n^\ell \sim z_j^\ell$. We claim that $\langle z_j^\ell : \ell \leq k, j \leq n\rangle$ is a perfect grid as required. Indeed (pg1) and (pg2) clearly hold. Let us prove that (pg3) holds as well, that is, for all $\ell < k$, $i, j \leq n$, if $z_i^\ell \ll z_i^{\ell+1}$ then either $C_{z_i^\ell} \rightsquigarrow C_{z_j^\ell}$ or $C_{z_i^\ell} \rightsquigarrow C_{z_j^{\ell+1}}$. Indeed, if $i = j$, this clearly holds. Otherwise, there are three cases:

- $i = n$, $j < n$. Then $C_{z_n^0} = C_x$, and we have either $C_x \rightsquigarrow C_{x_j} = C_{z_j^0}$ or $C_x \rightsquigarrow C_{y_j} = C_{z_j^1}$ by (50). Also, if $0 < \ell < k$ then $C_{z_n^\ell} \subseteq (x, y) \rightsquigarrow C_{y_j} = C_{z_j^\ell}$ by (51).

- $i < n$, $j = n$. If $z_i^\ell \ll z_i^{\ell+1}$, then $\ell = 0$ and $x_i \ll y_i$, and so $C_{z_i^0} = C_{x_i} \rightsquigarrow C_x = C_{z_n^0}$ by (49).
- $i, j < n$. Again, if $z_i^\ell \ll z_i^{\ell+1}$ then $\ell = 0$ and $x_i \ll y_i$. So by (pg3), either $C_{z_i^0} = C_{x_i} \rightsquigarrow C_{x_j} = C_{z_j^0}$ or $C_{z_i^0} = C_{x_i} \rightsquigarrow C_{y_j} = C_{z_j^1}$,

completing the proof of Lemma 3.11. □

Lemma 3.12 *Suppose that* Φ *holds in* $\mathfrak{F}$*, and let* $\langle x_j^i : i \le m,\ j < n\rangle$ *be a perfect grid, for some* $m, n < \omega$*,* $n > 0$*. For all* $x \in W$*, if* $x \sim x_0^m$ *then there exist* $s_i < \omega$ *(*$i \le m$*) and a perfect grid* $\langle z_j^\ell : \ell \le s_m,\ j \le n\rangle$ *such that* $0 = s_0 < s_1 < \cdots < s_m$*,* $z_j^{s_i} = x_j^i$*, for* $j < n$*,* $i \le m$*, and* $z_n^{s_m} = x$*,*

Proof. It is by induction on m. For $m = 0$ the statement is obvious. Suppose the statement holds for some $m < \omega$. Let $\langle x_j^i : i \le m+1,\ j < n\rangle$ be a perfect grid, and let $x \in W$ be such that $x \sim x_0^m$. Then $\langle x_j^i : 1 \le i \le m+1,\ j < n\rangle$ is a perfect grid, and so by the IH, there exist $s_i < \omega$, for $1 \le i \le m+1$, and a perfect grid $\bar{z} = \langle z_j^\ell : 1 \le \ell \le s_{m+1},\ j \le n\rangle$ such that $1 = t_1 < t_2 < \cdots < t_{m+1}$, $z_j^{t_i} = x_j^i$, for $1 \le i \le m+1$, $j < n$, and $z_n^{t_{m+1}} = x$. We also have that $\langle x_0^0, \ldots, x_{n-1}^0, x_0^1, \ldots, x_{n-1}^1\rangle$ is a perfect atomic grid, and $z_n^{t_1} \sim z_0^{t_1} = x_0^1$. So by Lemma 3.11, there exist $k > 0$ and a perfect grid $\bar{y} = \langle y_j^\ell : \ell \le k,\ j \le n\rangle$ such that $y_j^0 = x_j^0$, for $j < n$, $y_j^k = x_j^1$, for $j < n$, and $y_n^k = z_n^1$. By Claim 2.3, $\bar{y} \sqcup \bar{z}$ is a perfect grid as required. □

4 Discussion

Our results can be extended to **S4.3** $\times$ **S5**, even with some simplifications to the formula Φ. Theorem 1.3 also holds for $\mathsf{Logic\_of}\{\langle\omega, <\rangle\} \times$ **S5**. However, as the class of all frames for $\mathsf{Logic\_of}\{\langle\omega, <\rangle\}$ is not closed under ultraproducts, it is not known whether $\mathsf{Logic\_of}\{\langle\omega, <\rangle\} \times$ **S5** has other finite frames as well, frames that are not p-morphic images of product frames. It would also be interesting to know whether any of the logics (such as the decidable **K4.3** $\times$ **K**, or the undecidable but recursively enumerable **K4.3** $\times$ **K4**) that are within the scope of the non-finite axiomatisability results of [11] has a decidable finite frame problem.

Are we any closer to either proving non-finite axiomatisability of **K4.3** $\times$ **S5**, or finding an explicit, possibly infinite, axiomatisation of it? On the one hand, a way of proving that a product logic L is not finitely axiomatisable is constructing a sequence $\langle\mathfrak{F}_n : n < \omega\rangle$ of *finite* frames such that no $\mathfrak{F}_n$ is a frame for L, but some countable elementary substructure $\mathfrak{G}$ of a non-trivial ultraproduct of the $\mathfrak{F}_n$ is a p-morphic image of a product frame for L. Since the formula Φ we use to decide the finite frame problem for **K4.3** $\times$ **S5** is a first-order formula in the frame-correspondence language, if it fails in every $\mathfrak{F}_n$ then, by Łos' theorem, it fails in any ultraproduct as well, and so it fails in $\mathfrak{G}$. But Φ holds in every product frame and preserved under p-morphic images. So our result implies that we cannot hope for an argument of this kind to work, and have to do something else, possibly constructing *infinite* $\mathfrak{F}_n$.

On the other hand, it can be shown that our first-order formula Φ is not reflected under ultrafilter extensions, and so not modally definable. However, there is a bimodal formula φ such that

- for every 2-frame $\mathfrak{F}$ for $\mathbf{K4.3} \oplus \mathbf{S5}$, if Φ holds in $\mathfrak{F}$, then φ is valid in $\mathfrak{F}$;
- for every finite 2-frame $\mathfrak{F}$ for $\mathbf{K4.3} \oplus \mathbf{S5}$, if φ is valid in $\mathfrak{F}$, then Φ holds in $\mathfrak{F}$.

So if L_φ is the smallest normal bimodal logic containing $\mathbf{K4.3} \oplus \mathbf{S5}$ and φ, then we have $L_\varphi \subseteq \mathbf{K4.3} \times \mathbf{S5}$. However, in order to show the converse inclusion, one would need to show that L_φ has the *finite model property*. And we have no idea about that. Note that it is not known either whether $\mathbf{K4.3} \times \mathbf{S5}$ has the finite model property w.r.t. arbitrary (not necessarily product) frames. $\mathbf{K4.3}^t \times \mathbf{S5}$ lacks the finite model property [12], where $\mathbf{K4.3}^t$ is the temporal extension of **K4.3** with a 'past box'. Note that $\mathbf{K4.3}^t \times \mathbf{S5}$ (and so $\mathbf{K4.3} \times \mathbf{S5}$) is decidable [12].

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