$\mathbf{S5}^3$ lacks the fmp (another proof)

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Abstract

We show, using the idea of Németi [1], that all 3-modal logics between $[\mathbf{S5}, \mathbf{S5}, \mathbf{S5}]$ and $\mathbf{S5}^3$ lack the fmp.

Theorem 1. *All 3-modal logics between* $[\mathbf{S5}, \mathbf{S5}, \mathbf{S5}]$ *and* $\mathbf{S5}^3$ *lack the fmp.*

Proof. Let $\Phi(p, d_{01}, d_{02}, d_{12})$ be the conjunction of the following formulas:

(a) $\Box(p \wedge \Diamond_0(p \wedge d_{02}) \to d_{02})$ ("p^{-1} is a function")

(b) $\Box(p \leftrightarrow \Diamond_2 p)$ ("p is binary")

(c) $\Box\Diamond_1 p$ ("$Dom\ p = \top$")

(d) $\neg\Diamond_1(\Diamond_0 p \wedge d_{01})$ ("$Rng\ p \neq \top$")

(1) $\Box\,(\Diamond_0 d_{01} \wedge \Diamond_2 d_{02})$

(2) $\Box\,[(\Diamond_0 d_{12} \leftrightarrow d_{12}) \wedge (\Diamond_1 d_{02} \leftrightarrow d_{02}) \wedge (\Diamond_2 d_{01} \leftrightarrow d_{01})]$

(3) $\Box\,[(d_{01} \wedge d_{02} \to d_{12}) \wedge (d_{12} \wedge d_{02} \to d_{01})]$

(4) $\Box(d_{12} \wedge \Diamond_1(d_{12} \wedge p) \to p)$

Lemma 1. Φ *is* $\mathbf{S5}^3$*-satisfiable.*

Proof. Let $\mathfrak{F}$ be the universal product frame on $\omega \times \omega \times \omega$. Let

$$\begin{aligned} \upsilon(p) &= \{(x, x+1, z) : x, z \in \omega\} \\ \upsilon(d_{ij}) &= \{(x_0, x_1, x_2) : x_0, x_1, x_2 \in \omega,\ x_i = x_j\} \quad (i < j < 3). \end{aligned}$$

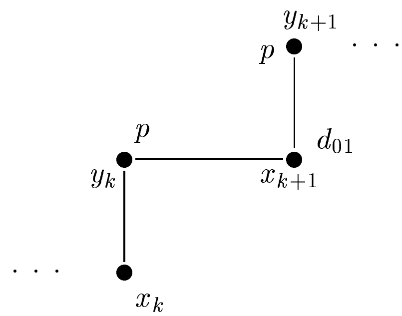
Then (say) $(\mathfrak{F}, \upsilon), (0,0,0) \models \Phi$. $\Box$

Lemma 2. Φ *is not satisfiable in finite frames for* $[\mathbf{S5}, \mathbf{S5}, \mathbf{S5}]$.

Proof. Assume $\mathfrak{F} = (W, R_0, R_1, R_2)$ is a frame for $[\mathbf{S5}, \mathbf{S5}, \mathbf{S5}]$, that is, the R_i $(i < 3)$ are commuting equivalence relations on W. Suppose $\mathfrak{M}$ is a model on $\mathfrak{F}$ and $\mathfrak{M}, x \models \Phi$. We show that $\mathfrak{F}$ must be infinite. For each $n \in \omega$ we define a formula φ_n and worlds x_n, y_n of $\mathfrak{F}$ as follows.

$$\begin{aligned} \varphi_0 &= \neg\Diamond_1(\Diamond_0 p \wedge d_{01}) \\ \varphi_{n+1} &= \Diamond_1(\Diamond_0(\varphi_n \wedge p) \wedge d_{01}). \end{aligned}$$

Let $x_0 = x$. Assume x_k is already defined. By (c) and (1), there are y_k, x_{k+1} such that $x_k R_1 y_k R_0 x_{k+1}$, $y_k \models p$ and $x_{k+1} \models d_{01}$.

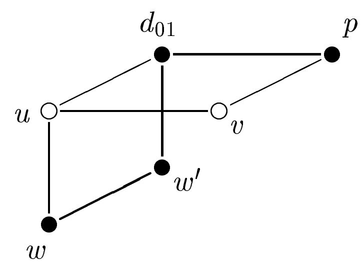


CLAIM 2.1. $(\forall n \in \omega)$ $y_n \models \varphi_n$.

Proof. By induction on n. □

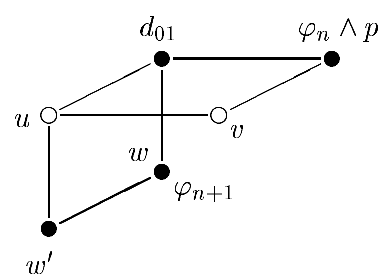
CLAIM 2.2. $(\forall n \in \omega)$ *if* $w \models \varphi_n$ *and* wR_2w' *then* $w' \models \varphi_n$ *as well.*

Proof. By induction on n: Assume $w' \not\models \varphi_0$. Then, by commutativity, there are u, v with:



Then $u \models d_{01}$ by (2), $v \models p$ by (b), thus $w \not\models \varphi_0$.

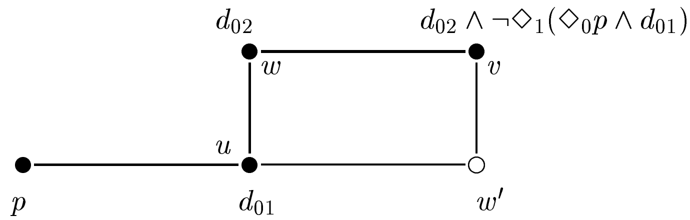
Now assume $w \models \varphi_{n+1}$. Then, by commutativity, there are u, v with:



Then $u \models d_{01}$ by (2), $v \models \varphi_n \wedge p$ by (b) and the induction hypothesis, thus $w' \models \varphi_{n+1}$. □

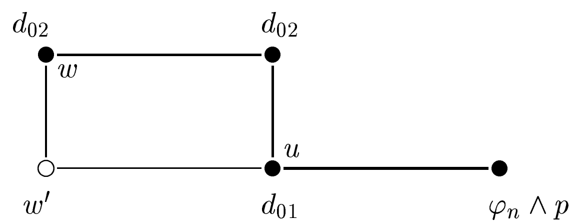
CLAIM 2.3. $(\forall n \in \omega)$ $\mathfrak{M} \models \Box(d_{02} \wedge \Diamond_0(d_{02} \wedge \varphi_n) \to \varphi_n)$.

Proof. For $n = 0$: let w such that $w \models d_{02} \wedge \Diamond_0(d_{02} \wedge \neg\Diamond_1(\Diamond_0 p \wedge d_{01}))$, and assume that $w \models \Diamond_1(\Diamond_0 p \wedge d_{01})$ also holds. Then there are u, v and, by commutativity, w' such that:



Then $u \models d_{02}$ and $w' \models d_{02}$, by (2). Thus $u \models d_{12}$, by (3); and $w' \models d_{12}$, by (2). Therefore $w' \models d_{01}$, again by (3), contradicting $v \models \neg\Diamond_1(\Diamond_0 p \wedge d_{01})$.

For $n+1$: assume $w \models d_{02} \wedge \Diamond_0[d_{02} \wedge \Diamond_1(\Diamond_0(\varphi_n \wedge p) \wedge d_{01})]$. Then there is some u and, by commutativity, there is a w' with:

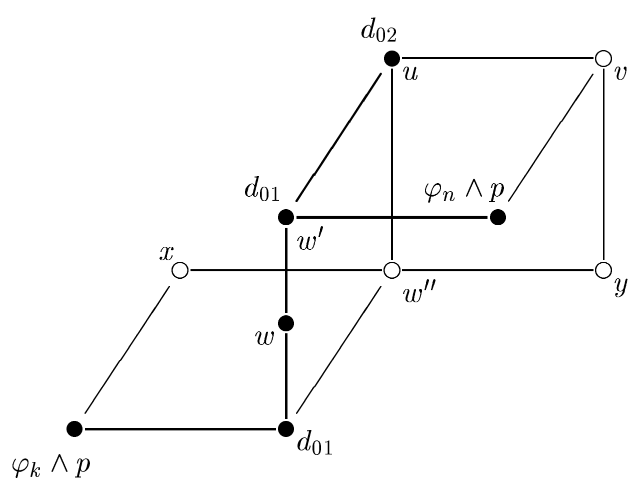


Then $u \models d_{02}$ and $w' \models d_{02}$, by (2). Therefore $u \models d_{12}$, by (3). Thus $w' \models d_{12}$, by (2); and $w' \models d_{01}$, again by (3). Thus $w' \models \Diamond_0(\varphi_0 \wedge p) \wedge d_{01}$, implying $w \models \Diamond_1(\Diamond_0(\varphi_n \wedge p) \wedge d_{01})$. $\square$

Claim 2.4. $(\forall k, n \in \omega,\ k < n)(\forall w)\quad w \not\models \varphi_k \wedge \varphi_n$.

Proof. Induction on k. For $n > 0, k = 0$: if $w \models \varphi_n$ then w R_1-sees a d_{01}-world which R_0-sees a p-world; if $w \models \varphi_0$ then w does not R_1-see a d_{01}-world which R_0-sees a p-world.

Assume $w \models \varphi_{k+1} \wedge \varphi_{n+1}$. Then there is w' with $w' \models d_{01} \wedge \Diamond_0(\varphi_n \wedge p)$. By (1), there is u with $w' R_2 u$ and $u \models d_{02}$. Thus, by commutativity, there are v, x, w'', y such that:



Then $u \models d_{01}$, by (2); thus $u \models d_{12}$, by (3); and $v \models d_{12}$, again by (2). Further, $v \models \varphi_n$, by Claim 2.2; $y \models \varphi_n$, by definition of φ_n. On the other hand, $w'' \models d_{01} \wedge d_{02}$, by (2); thus $w'' \models d_{12}$, by (3). Therefore, $y \models d_{12}$, by (2). Finally, $y \models p$, by (4). Since $x \models \varphi_k \wedge p$, by Claim 2.2 and (b), we obtain that x and y are such that

$$xR_0y, \quad x \models \varphi_k \wedge p \quad \text{and} \quad y \models \varphi_n \wedge p.$$

By (1), there is some s with yR_2s and $s \models d_{02}$. By commutativity, there is some w''' with:

By (b) and Claim 2.2, $w''' \models \varphi_k \wedge p$ and $s \models \varphi_n \wedge p$. By (a), $w''' \models d_{02}$ follows. Then, by Claim 2.3, $w''' \models \varphi_n$. Thus $w''' \models \varphi_k \wedge \varphi_n$, contradicting the induction hypothesis. $\square$

Now Lemma 2 clearly follows from Claims 2.1 and 2.4. $\square$

Finally, Theorem 1 follows from Lemmas 1 and 2. $\square$

References

[1] I. Németi, *Neither CA_α nor Gs_α is generated by its finite members if $\alpha \geq 3$*, Preprint, Math. Inst. Hung. Acad. Sci., 1984.