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PRODUCTS OF 'TRANSITIVE' MODAL LOGICS

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Abstract. We solve a major open problem concerning algorithmic properties of products of 'transitive' modal logics by showing that products and commutators of such standard logics as **K4**, **S4**, **S4.1**, **K4.3**, **GL**, or **Grz** are undecidable and do not have the finite model property. More generally, we prove that no Kripke complete extension of the commutator [**K4**, **K4**] with product frames of arbitrary finite or infinite depth (with respect to both accessibility relations) can be decidable. In particular, if $\mathcal{C}_1$ and $\mathcal{C}_2$ are classes of transitive frames such that their depth cannot be bounded by any fixed $n < \omega$, then the logic of the class $\{\mathfrak{F}_1 \times \mathfrak{F}_2 \mid \mathfrak{F}_1 \in \mathcal{C}_1, \quad \mathfrak{F}_2 \in \mathcal{C}_2\}$ is undecidable. (On the contrary, the product of, say, **K4** and the logic of all transitive Kripke frames of depth $\leq n$, for some fixed $n < \omega$, is decidable.) The complexity of these undecidable logics ranges from r.e. to co-r.e. and Π^1_1-complete. As a consequence, we give the first known examples of Kripke incomplete commutators of Kripke complete logics.

§1. Introduction. Products of modal (in particular, temporal, spatial, epistemic, description, etc.) logics—or, more generally, multi-modal languages interpreted in various product-like structures—are very natural and clear formalisms arising in both pure logic and numerous applications; see, e.g., [29, 8, 3, 30, 12, 1, 6, 38]. For example, dynamic topological logics of [2, 24, 25, 7] or spatio-temporal logics of [38, 15] are interpreted in structures of the form $(T, <) \times (W, R)$ where $(T, <)$ models the flow of time (say, $(\omega, <)$) and (W, R) is a quasi-order (a frame for **S4**) representing the topological space, with the **S4**-box being understood as the interior operator over this space. By interpreting W as a domain of objects whose properties may change over time, one can also use such product frames as models for (fragments of) first-order temporal and modal logics, temporal data or knowledge bases.

Introduced in the 1970s [32, 33], products of modal logics have been intensively studied over the last decade; for a comprehensive exposition and further references see [11]. The landscape of the obtained results that are relevant to the decision problem for these logics can be briefly outlined as follows:

1. The product of finitely many logics, whose Kripke frames are definable by recursive sets of first-order sentences, is recursively enumerable [12].
2. Products of two standard logics, where at least one component logic is determined by a class of frames of finite bounded depth (like **S5**), are usually decidable. This condition can be considerably weakened: product logics are often decidable when, in order to check satisfiability of a formula

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φ, it is enough to consider only those product frames where the depth of one of the components is bounded by some finite number which can be effectively computed from φ. This result covers multi-modal **K** and **S5** as well as products with tense extensions of multi-modal **K** or temporal logics of metric spaces [12, 13, 11, 30, 22].

3. Products of two 'linear transitive' logics are undecidable whenever the depth of frames for both component logics cannot be bounded by any fixed $n < \omega$; examples are products of **K4.3**, **S4.3**, **GL.3** or $\mathsf{Log}\,(\omega, <)$ (the logic of the frame $(\omega, <)$) [28, 31, 35].
4. Products of more than two modal logics are usually undecidable. In fact, no logic between $\mathbf{K} \times \mathbf{K} \times \mathbf{K}$ and $\mathbf{S5} \times \mathbf{S5} \times \mathbf{S5}$ is decidable [20].

Thus, the main gap in our knowledge about the decision problem for product logics is the computational behaviour of products of two 'transitive' logics whose 'depth' is not bounded by any fixed $n < \omega$ and at least one component logic has *branching* frames. Many natural and useful logics, such as $\mathbf{S4} \times \mathbf{S4}$ and $\mathbf{S4.3} \times \mathbf{S4}$, belong to this group. Apart from item 3 above, the only known result in this direction concerns products with $\mathsf{Log}\,(\omega, <)$. Namely, [11, Theorem 7.24] showed that the product logics $\mathsf{Log}\,(\omega, <) \times \mathbf{K4}$ and $\mathsf{Log}\,(\omega, <) \times \mathbf{S4}$ are not decidable. However, that proof was rather tailor-made for this special case. On the one hand, it heavily used the linearity and discreteness of $(\omega, <)$. On the other hand, the proof reduced the undecidable but recursively enumerable Post's correspondence problem to the satisfiability problem for the logics in question. Since products like $\mathbf{K4} \times \mathbf{K4}$ or $\mathbf{S4.3} \times \mathbf{S4}$ are recursively enumerable by item 1 above, there was no hope to 'simply extend the proof' to these cases.

In this paper, we introduce a novel technique for dealing with products of logics with transitive branching frames. Our main new result is that all products—and quite often even the commutators—of two Kripke complete modal logics with transitive frames of arbitrary finite or infinite depth are undecidable, in many cases these products are not axiomatisable and do not enjoy the (abstract) finite model property, and sometimes they are even Π^1_1-hard. Precise formulations are given in Section 3. These results solve a number of open problems from [12, 27, 6, 11].

To a certain extent, the obtained results are optimal. For example, the product of, say, **K4** and the logic of all transitive Kripke frames of depth $\leq n$, for some fixed $n < \omega$, is decidable. This can be proved using the method of quasi-models similarly to [11, Theorem 6.10].

Modal logic is usually praised for being reasonably expressive and yet computationally manageable. Although the series of 'negative' results from the 1970–1980s produced a zoo of 'monstrous' modal logics for any taste (see, e.g., [5]), basically all of those 'monsters' were *artificial*. The *standard, natural* modal logics are reasonably simple. The results of this paper show that *simple and natural combinations* of standard modal logics can be extremely complex. For example, the *undecidable* product logic $\mathbf{K4} \times \mathbf{K4}$ is defined syntactically by the axioms of classical propositional logic, the modal axioms

$$\boxminus(p \to q) \to (\boxminus p \to \boxminus q) \qquad \boxbar(p \to q) \to (\boxbar p \to \boxbar q)$$

$$\boxminus p \to \boxminus\boxminus p \qquad \boxbar p \to \boxbar\boxbar p$$

$$\diamondminus\boxbar p \to \boxbar\diamondminus p \qquad \diamondbar\diamondminus p \leftrightarrow \diamondminus\diamondbar p$$

and the inference rules modus ponens, substitution and necessitation $\varphi/\boxminus\varphi$ and $\varphi/\boxbar\varphi$. Its semantical definition is equally natural and transparent (see below for details).

As a ‘by-product,’ we also obtain natural *Kripke incomplete* logics, such as the logic $[\mathbf{K4}, \mathbf{GL.3}]$ which can be obtained by adding to $\mathbf{K4} \times \mathbf{K4}$ the well-known axioms

$$\boxbar(\boxbar p \to p) \to \boxbar p \qquad \boxbar(p \wedge \boxbar p \to q) \vee \boxbar(q \wedge \boxbar q \to p).$$

The structure of the paper is as follows. Section 2 provides all the relevant definitions. Section 3 lists the obtained results. The proofs are given in Sections 4 and 5. Roughly, the scheme is as follows. First, in Section 4, we present a formula φ_∞ which ‘forces’ the existence of ‘$n \times m$-rectangles,’ for all $n, m < \omega$, in any frame for $\mathbf{K4} \times \mathbf{K4}$. Then, in Section 5.1, we use these rectangles to encode points of the $\omega \times \omega$-grid, a kind of universal structure where one can represent one’s favourite undecidable master problem, be it the (non)halting problem for Turing or register machines, a tiling (or domino) problem, or Post’s correspondence problem. In this paper we obtain our undecidability results using Turing machines (encoded in Section 5.2) and tilings (encoded in Section 5.3). Finally, in Section 6 we discuss the obtained results and future directions of research.

§2. Products and commutators. Given unimodal Kripke frames $\mathfrak{F}_1 = (W_1, R_1)$ and $\mathfrak{F}_2 = (W_2, R_2)$, their *product* is defined to be the bimodal frame

$$\mathfrak{F}_1 \times \mathfrak{F}_2 \quad = \quad (W_1 \times W_2, R_h, R_v),$$

where $W_1 \times W_2$ is the Cartesian product of W_1 and W_2 and, for all $u, u' \in W_1$, $v, v' \in W_2$,

$$(u,v)R_h(u',v') \quad \text{iff} \quad uR_1u' \text{ and } v = v',$$
$$(u,v)R_v(u',v') \quad \text{iff} \quad vR_2v' \text{ and } u = u'.$$

Bimodal frames of this form will be called *product frames* throughout. Let L_1 be a normal (uni)modal logic in the language with the box $\boxminus$ and the diamond $\diamondminus$. Let L_2 be a normal (uni)modal logic in the language with the box $\boxbar$ and the diamond $\diamondbar$. Assume also that both L_1 and L_2 are Kripke complete. Then the *product* of the logics L_1 and L_2 is the (Kripke complete) bimodal logic $L_1 \times L_2$ in the language $\mathcal{ML}_2$ with the boxes $\boxminus$, $\boxbar$ and the diamonds $\diamondminus$, $\diamondbar$ which is characterised by the class of product frames $\mathfrak{F}_1 \times \mathfrak{F}_2$, where $\mathfrak{F}_i$ is a frame for L_i, $i = 1, 2$. (Here we assume that $\boxminus$ and $\diamondminus$ are interpreted by R_h, while $\boxbar$ and $\diamondbar$ are interpreted by R_v.)

A good starting point in understanding the behaviour of product logics is to find basic principles that hold for every product frame $(W_1 \times W_2, R_h, R_v)$:

- *left commutativity*: $\forall x \forall y \forall z \left(xR_v y \wedge yR_h z \to \exists u \, (xR_h u \wedge uR_v z)\right)$,
- *right commutativity*: $\forall x \forall y \forall z \left(xR_h y \wedge yR_v z \to \exists u \, (xR_v u \wedge uR_h z)\right)$,
- *Church–Rosser property*: $\forall x \forall y \forall z \left(xR_v y \wedge xR_h z \to \exists u \, (yR_h u \wedge zR_v u)\right)$.

These properties can also be expressed by the $\mathcal{ML}_2$-formulas

$$\Diamond_v \Diamond_h p \to \Diamond_h \Diamond_v p, \qquad \Diamond_h \Diamond_v p \to \Diamond_v \Diamond_h p, \qquad \Diamond_h \Box_v p \to \Box_v \Diamond_h p. \tag{1}$$

Given Kripke complete unimodal logics L_1 and L_2, their *commutator* $[L_1, L_2]$ is the smallest normal modal logic in the language $\mathcal{ML}_2$ which contains L_1, L_2 and the axioms (1).

Clearly, we always have $[L_1, L_2] \subseteq L_1 \times L_2$. However, sometimes more information can be drawn. First, since the axioms in (1) are Sahlqvist formulas, the commutator of two canonical logics is always canonical [12], and so Kripke complete (like, e.g., $[\mathbf{K4}, \mathbf{K4}]$ and $[\mathbf{K4.3}, \mathbf{S4}]$). As we will see later on in this paper, not all commutators are Kripke complete; examples are $[\mathbf{K4}, \mathbf{GL.3}]$ and $[\mathbf{GL}, \mathbf{Grz.3}]$ (see Corollary 4.2 below). Second, using the Kripke completeness of the commutators, it is shown in [12, 11] that for certain pairs of logics, their commutators and products actually coincide: for example,

$$[\mathbf{K4}, \mathbf{K4}] \; = \; \mathbf{K4} \times \mathbf{K4} \qquad \text{and} \qquad [\mathbf{S4}, \mathbf{S4}] \; = \; \mathbf{S4} \times \mathbf{S4}.$$

On the other hand, the Kripke complete $[\mathbf{K4.3}, \mathbf{K4}]$ does not coincide with $\mathbf{K4.3} \times \mathbf{K4}$; see [11, Theorem 5.15].

Although product logics $L_1 \times L_2$ are Kripke complete by definition, there can be (and, in general, there are) other, *non-product*, frames for $L_1 \times L_2$. This gives rise to two different types of the finite model property. As usual, a bimodal logic L (in particular, a product logic $L_1 \times L_2$) is said to have the (*abstract*) *finite model property* (*fmp*, for short) if, for every $\mathcal{ML}_2$-formula $\varphi \notin L$, there is a finite frame $\mathfrak{F}$ for L such that $\mathfrak{F} \not\models \varphi$. (By a standard argument, this is equivalent to saying that $\mathfrak{M} \not\models \varphi$ for some *finite model* $\mathfrak{M}$ for L; see, e.g., [5].) And we say that $L_1 \times L_2$ has the *product finite model property* (*product fmp*, for short) if, for every $\mathcal{ML}_2$-formula $\varphi \notin L_1 \times L_2$, there is a finite *product* frame $\mathfrak{F}$ for $L_1 \times L_2$ such that $\mathfrak{F} \not\models \varphi$.

Clearly, the product fmp implies the fmp. Examples of product logics having the product fmp (and so the fmp) are $\mathbf{K} \times \mathbf{K}$, $\mathbf{K} \times \mathbf{S5}$, and $\mathbf{S5} \times \mathbf{S5}$ (see [11] and references therein). On the other hand, there are product logics, such as $\mathbf{K4} \times \mathbf{S5}$ and $\mathbf{S4} \times \mathbf{K}$, that do enjoy the (abstract) fmp [12, 34], but lack the product fmp [11]. In general, it is well known that many product logics with at least one 'transitive' (but not 'symmetric') component do not have the product fmp (see, e.g., [11, Theorems 5.32, 5.33, and 7.10]). A simple $\mathcal{ML}_2$-formula that can be used to show that many such logics do not have the product fmp is as follows:

$$\Box_h^+ \Diamond_v p \wedge \Box_h^+ \Box_v (p \to \Diamond_h \Box_h^+ \neg p),$$

where $\Box_h^+ \psi$ abbreviates $\psi \wedge \Box_h \psi$. Note that this formula (as well as the others known so far) is satisfiable in appropriate finite (in fact, very small) non-product frames for the logics in question.

§3. Main results. From now on we only consider products and commutators of '*transitive*' (uni)modal logics, that is, normal extensions of $\mathbf{K4}$. In other words, we deal with extensions of the bimodal logic $[\mathbf{K4}, \mathbf{K4}] = \mathbf{K4} \times \mathbf{K4}$. In this section we list the main results of the paper and illustrate them by drawing some consequences. The proofs are provided in Sections 4 and 5.

Given a transitive frame $\mathfrak{F} = (W, R)$, a point $x \in W$ is said to be of *depth* $n < \omega$ in $\mathfrak{F}$ if there is a path $x = x_0 R x_1 R \dots R x_n$ of points from distinct clusters[1] in $\mathfrak{F}$ (that is, $x_{i+1} R x_i$ does not hold for any $i < n$) and there is no such path of greater length. If for every $n < \omega$ there is a path of n points from distinct clusters starting from x, then we say that x is of *infinite depth*, or x is of *depth* ∞. The *depth of* $\mathfrak{F}$ is defined to be the supremum of the depths of its points (with $n < \infty$ for all $n < \omega$). For instance, $\mathfrak{F}$ is of infinite depth if it contains points of arbitrary finite depth. By the *depth* of a bimodal frame (W, R_1, R_2) with transitive R_1, R_2 we understand the minimal depth of (W, R_1) and (W, R_2).

Given classes $\mathcal{C}_1$ and $\mathcal{C}_2$ of frames, we let

$$\mathcal{C}_1 \times \mathcal{C}_2 \quad = \quad \{\mathfrak{F}_1 \times \mathfrak{F}_2 \mid \mathfrak{F}_1 \in \mathcal{C}_1,\ \mathfrak{F}_2 \in \mathcal{C}_2\}.$$

Denote by $\mathsf{Log}\,(\mathcal{C})$ the normal modal logic of a class $\mathcal{C}$ of frames. If $\mathcal{C}$ consists of a single frame $\mathfrak{F}$ then we write $\mathsf{Log}\,\mathfrak{F}$ instead of $\mathsf{Log}\,(\{\mathfrak{F}\})$. Recall that a logic L is *Kripke complete* if $L = \mathsf{Log}\,(\mathcal{C})$ for some class $\mathcal{C}$ of frames.

The main result of this paper is the following:

THEOREM 1. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be classes of transitive frames both containing frames of arbitrarily large finite or infinite depth. Then* $\mathsf{Log}\,(\mathcal{C}_1 \times \mathcal{C}_2)$ *is undecidable.*

More generally, if L is any Kripke complete bimodal logic containing $[\mathbf{K4}, \mathbf{K4}]$ *and having product frames of arbitrarily large finite or infinite depth, then L is undecidable.*

We obtain this theorem as a consequence of more general Theorems 2 and 3 below. To formulate them, we require some terminology. We remind the reader that a bimodal frame (W, R_1, R_2) is called *rooted* if there exists $r \in W$ such that $W = \{u \in W \mid r(R_1 \cup R_2)^* u\}$, where R^* denotes the reflexive and transitive closure of R. Fix some propositional variables h and v. Given a Kripke model $\mathfrak{M}$ based on $\mathfrak{F} = (W, R_1, R_2)$, define new relations $\bar{R}_1^{\mathfrak{M}}$ and $\bar{R}_2^{\mathfrak{M}}$ by taking, for all $x, y \in W$,

$$x \bar{R}_1^{\mathfrak{M}} y \quad \text{iff} \quad \exists z \in W\ \big[x R_1 z \text{ and } \big((\mathfrak{M}, x) \models h \iff (\mathfrak{M}, z) \models \neg h\big) \tag{2}$$
$$\text{and (either } z = y \text{ or } z R_1 y)\big],$$

$$x \bar{R}_2^{\mathfrak{M}} y \quad \text{iff} \quad \exists z \in W\ \big[x R_2 z \text{ and } \big((\mathfrak{M}, x) \models v \iff (\mathfrak{M}, z) \models \neg v\big) \tag{3}$$
$$\text{and (either } z = y \text{ or } z R_2 y)\big].$$

In other words, $x \bar{R}_1^{\mathfrak{M}} y$ iff $x R_1 y$ and either x, y are of different 'horizontal colours' in the sense that h is true in precisely one of them, or x, y are of the same h-colour (i.e., $x \models h$ iff $y \models h$), but there is a point z of different h-colour such that $x R_1 z R_1 y$. Clearly, we always have $\bar{R}_i \subseteq R_i$ $(i = 1, 2)$.

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[1] A set $X \subseteq W$ is called a *cluster* in $\mathfrak{F}$ if $X = \{x\} \cup \{y \in W \mid xRy \text{ and } yRx\}$ for some $x \in W$.

For every point $x \in W$, define its *horizontal* and *vertical ranks* $hr^{\mathfrak{M}}(x)$ and $vr^{\mathfrak{M}}(x)$ in $\mathfrak{M}$ as follows:

$$hr^{\mathfrak{M}}(x) = \begin{cases} n, & \text{if the length of the longest } \bar{R}_1^{\mathfrak{M}}\text{-path} \\ & \quad \text{starting from } x \text{ is } n < \omega, \\ \infty, & \text{otherwise,} \end{cases} \tag{4}$$

$$vr^{\mathfrak{M}}(x) = \begin{cases} n, & \text{if the length of the longest } \bar{R}_2^{\mathfrak{M}}\text{-path} \\ & \quad \text{starting from } x \text{ is } n < \omega, \\ \infty, & \text{otherwise.} \end{cases} \tag{5}$$

Note that, say, $hr^{\mathfrak{M}}(x)$ is not the same as the depth of x in the frame $(W, \bar{R}_1^{\mathfrak{M}})$. For example, if xR_1y, yR_1x and x, y are of different h-colours then $x\bar{R}_1^{\mathfrak{M}}x$ and $hr^{\mathfrak{M}}(x) = \infty$.

For our constructions in Sections 4 and 5, points of *finite* horizontal and vertical ranks will be of particular importance. For $k < \omega$, we call a rooted bimodal frame $\mathfrak{F} = (W, R_1, R_2)$ for $[\mathbf{K4}, \mathbf{K4}]$ a *k-chessboard* if there is a model $\mathfrak{M}$ based on $\mathfrak{F}$ and such that the following conditions are satisfied:

- **(cb1)** for all $x, y \in W$ with xR_1y, $(\mathfrak{M}, x) \models v$ iff $(\mathfrak{M}, y) \models v$;
- **(cb2)** for all $x, y \in W$ with xR_2y, $(\mathfrak{M}, x) \models h$ iff $(\mathfrak{M}, y) \models h$; and
- **(cb3)** there is $x \in W$ such that $hr^{\mathfrak{M}}(x) = vr^{\mathfrak{M}}(x) = k$.

Clearly, if $\mathfrak{F}$ is a k-chessboard then it is an n-chessboard for any $n < k$. Observe that the product of any two rooted transitive frames of depths at least k is always a k-chessboard. Further, it is not hard to see that for any model $\mathfrak{M}$ based on a rooted frame for $[\mathbf{K4}, \mathbf{K4}]$ that satisfies **(cb1)** and **(cb2)**, $(W, \bar{R}_1^{\mathfrak{M}}, \bar{R}_2^{\mathfrak{M}})$ is a (not necessarily rooted) frame for $[\mathbf{K4}, \mathbf{K4}]$, that is,

$$\text{both } \bar{R}_1^{\mathfrak{M}} \text{ and } \bar{R}_2^{\mathfrak{M}} \text{ are transitive,} \tag{tran}$$

$$\bar{R}_1^{\mathfrak{M}} \text{ and } \bar{R}_2^{\mathfrak{M}} \text{ commute, and} \tag{com}$$

$$\bar{R}_1^{\mathfrak{M}} \text{ and } \bar{R}_2^{\mathfrak{M}} \text{ are Church–Rosser.} \tag{chro}$$

A rooted frame $\mathfrak{F}$ for $[\mathbf{K4}, \mathbf{K4}]$ is called an *∞-chessboard* if there is an $\mathfrak{M}$ based on $\mathfrak{F}$ which satisfies **(cb1)**, **(cb2)** and contains points x_k with $hr^{\mathfrak{M}}(x_k) = vr^{\mathfrak{M}}(x_k) = k$ for every $k < \omega$. Clearly, an ∞-chessboard is a k-chessboard, for every $k < \omega$, and

$$\text{an } \infty\text{-chessboard is always infinite.} \tag{6}$$

Typical examples of ∞-chessboards are products of transitive frames where each component is

- either a frame containing an infinite descending chain with a root, say, $(\{\infty\} \cup \omega, >)$ or $(\{\infty\} \cup \mathbb{Z}, >)$;
- or a frame containing the infinite n-ary tree for some $n \geq 2$ as a subframe;
- or an infinite 'xmas tree' with arbitrarily long finite branches (that is, an ω-type ascending chain where a branch of length n starts at point n, for every $n < \omega$).

(For more details see the proof of Corollary 4.1 in Section 5.) Note, however, that a product of transitive frames of infinite depth is not necessarily an

∞-chessboard. For instance, it is not hard to see that if one of the components is

- an infinite frame of finite width (that is, without antichains of more than n points, for some fixed $n < \omega$) containing no infinite descending chain (in particular, the infinite ascending chain $(\omega, <)$),

then the product is *not* an ∞-chessboard. As we will see in Section 4, there is a formula that is satisfiable in precisely those frames for $[\mathbf{K4}, \mathbf{K4}]$ that are ∞-chessboards.

THEOREM 2. *Let L be any bimodal logic containing $[\mathbf{K4}, \mathbf{K4}]$ and having an ∞-chessboard among its frames. Then L*

(i) *does not have the (abstract) fmp, and*
(ii) *is undecidable.*

Observe that Theorem 2 does not require L to be Kripke complete.

THEOREM 3. *Let $\mathcal{C}$ be a class of frames for $[\mathbf{K4}, \mathbf{K4}]$ with the following properties:*

- *it contains no ∞-chessboard;*
- *it contains a k-chessboard for every $k < \omega$.*

Then $\mathsf{Log}\,(\mathcal{C})$ is not recursively enumerable.

Clearly, Theorems 2 and 3 together imply Theorem 1. It follows from Theorem 3 that if classes $\mathcal{C}_1$ or $\mathcal{C}_2$ contain only finite transitive frames of arbitrarily large finite depth then $\mathsf{Log}\,(\mathcal{C}_1 \times \mathcal{C}_2)$ is not recursively enumerable. Here is a consequence of Theorem 2 which involves logics from the standard nomenclature (see, e.g., [5] for their syntax and semantics):

COROLLARY 3.1. *Let L_1 and L_2 be any logics from the list*

$$\mathbf{K4}, \mathbf{K4.1}, \mathbf{K4.2}, \mathbf{K4.3}, \mathbf{S4}, \mathbf{S4.1}, \mathbf{S4.2}, \mathbf{S4.3},$$
$$\mathbf{GL}, \mathbf{GL.3}, \mathbf{Grz}, \mathbf{Grz.3}, \mathsf{Log}\,(\omega, <), \mathsf{Log}\,(\omega, \leq).$$

Then both $[L_1, L_2]$ and $L_1 \times L_2$ are undecidable and lack the (abstract) fmp.

In some cases, we can even say a bit more. We remind the reader that $\mathbf{K4.3}$ is the logic of all transitive frames (W, R) that are *weakly connected*:

$$\forall x, y, z \in W \ (xRy \wedge xRz \rightarrow y = z \vee yRz \vee zRy).$$

Note that, according to [9], all normal unimodal logics containing $\mathbf{K4.3}$ are Kripke complete, and by [40], those of them that are finitely axiomatisable are decidable, but do not necessarily have the fmp.

Now consider the logic $\mathbf{DisK4.3}$ determined by all Kripke frames for $\mathbf{K4.3}$ which do not contain subframes that can be p-morphically mapped onto a two-element cluster followed by a reflexive point ②→∘ or a two-element cluster followed by an irreflexive point ②→•. In other words, a frame (W, R) for $\mathbf{K4.3}$ is a frame for $\mathbf{DisK4.3}$ iff it satisfies the following aspect of *discreteness*:

$$\begin{gathered}\text{there are no points } x_0, x_1, \dots, x_n, \dots, x_\infty \text{ in } W \text{ such that}\\ x_0 R x_1 R x_2 R \dots R x_n R \dots R x_\infty, \\ x_i \neq x_{i+1} \text{ and } \neg(x_\infty R x_i) \text{ for all } i < \omega.\end{gathered} \tag{7}$$

The logic **DisK4.3** can be axiomatised by adding to **K4.3** the (subframe) canonical formulas $\alpha(\textcircled{2}\!\to\!\bullet)$ and $\alpha(\textcircled{2}\!\to\!\circ)$ or, which is the same, the corresponding Fine's subframe formulas (for details see [10, 39, 5]).

A number of important 'linear' modal logics are extensions of **DisK4.3**, for example, $\mathsf{Log}\,(\omega,<)$, $\mathsf{Log}\,(\omega,\leq)$, **GL.3**, and **Grz.3**, where **GL.3** and **Grz.3** are the logics of Noetherian irreflexive and reflexive linear orders, respectively. We remind the reader that a frame (W,R) is *Noetherian* if it contains no infinite ascending chains $x_0Rx_1Rx_2R\dots$ where $x_i \neq x_{i+1}$. It is not hard to see that $\mathsf{Log}\,(\omega,\leq) \subseteq$ **Grz.3**. It should also be noted that each of the logics **DisK4.3**, $\mathsf{Log}\,(\omega,<)$, $\mathsf{Log}\,(\omega,\leq)$, **GL.3**, and **Grz.3** has frames containing infinite descending chains; for example, $(\{\infty\}\cup\mathbb{Z},>)$ is a frame for $\mathsf{Log}\,(\omega,<)$.

THEOREM 4. *Let L be any Kripke complete bimodal logic having an ∞-chessboard among its frames and containing $[\mathbf{K4},\mathbf{DisK4.3}]$. Then L is Π^1_1-hard.*

We will show that this result applies to a number of 'standard' product logics:

COROLLARY 4.1. *Let L_1 be like in Corollary 3.1 and*

$$L_2 \;\in\; \{\mathsf{Log}\,(\omega,<),\; \mathsf{Log}\,(\omega,\leq),\; \mathbf{GL.3},\; \mathbf{Grz.3},\; \mathbf{DisK4.3}\}.$$

Then any Kripke complete bimodal logic L in the interval

$$[L_1,L_2] \;\subseteq\; L \;\subseteq\; L_1\times L_2$$

is Π^1_1-hard. In fact, the product logics $L_1\times L_2$ are Π^1_1-complete.

We also obtain the following interesting corollary. As the commutator of two recursively axiomatisable logics is recursively axiomatisable by definition, Theorem 4 yields a number of *Kripke incomplete* commutators of Kripke complete and finitely axiomatisable logics:

COROLLARY 4.2. *Let L_1 and L_2 be like in Corollary 4.1. Then the commutator $[L_1,L_2]$ is Kripke incomplete.*

It is worth noting that if $L_2 = \mathbf{GL.3}$ then $L_1\times L_2$ is the *only* Kripke complete logic between $[L_1,L_2]$ and $L_1\times L_2$, for any Kripke complete logic L_1; for details see [14].

§4. No finite model property.

In this section we prove Theorem 2 (i). We define a formula φ_∞ such that, for any rooted frame $\mathfrak{F}$ for $[\mathbf{K4},\mathbf{K4}]$,

$$\varphi_\infty \text{ is satisfiable in } \mathfrak{F} \qquad \text{iff} \qquad \mathfrak{F} \text{ is an } \infty\text{-chessboard.} \tag{8}$$

By (6), this clearly implies that, for any logic L specified in Theorem 2, φ_∞ is L-satisfiable, but only in infinite frames for L, that is, L does not have the fmp.

The formula φ_∞ and its 'finite variant' φ_{fin} to be defined in Section 5.4 play a crucial role in all of our undecidability proofs in Section 5.

To begin with, take two propositional variables h and v, and define new modal operators by setting, for every bimodal formula ψ,

$$\text{⬘}\psi \;=\; \big[h\to\Diamond_h\big(\neg h\wedge(\psi\vee\Diamond_h\psi)\big)\big]\wedge\big[\neg h\to\Diamond_h\big(h\wedge(\psi\vee\Diamond_h\psi)\big)\big],$$

$$\text{⬗}\psi \;=\; \big[v\to\Diamond_v\big(\neg v\wedge(\psi\vee\Diamond_v\psi)\big)\big]\wedge\big[\neg v\to\Diamond_v\big(v\wedge(\psi\vee\Diamond_v\psi)\big)\big],$$

$$\text{⬒}\psi \;=\; \neg\text{⬘}\neg\psi, \qquad \text{and} \qquad \text{◨}\psi = \neg\text{⬗}\neg\psi.$$

(Similar operators were used by Spaan [35] and in [31, 11].)

Define φ_∞ to be the conjunction of the following formulas:

$$\boxminus\boxbar\big((h \vee \diamondbar h \to \boxbar h) \wedge (\neg h \vee \diamondbar\neg h \to \boxbar\neg h)\big), \tag{9}$$

$$\boxminus\boxbar\big((v \vee \diamondminus v \to \boxminus v) \wedge (\neg v \vee \diamondminus\neg v \to \boxminus\neg v)\big), \tag{10}$$

$$⬗⬘(◨\bot \wedge ⬒\bot), \tag{11}$$

$$⬒◨(◨\bot \wedge ⬒\bot \to d), \tag{12}$$

$$◨⬘(\neg d \wedge ⬒d), \tag{13}$$

$$⬒⬗(d \wedge ◨\neg d), \tag{14}$$

$$⬒◨(d \to ⬒⬗d), \tag{15}$$

$$⬒◨(\neg d \to ◨⬘\neg d). \tag{16}$$

Suppose first that φ_∞ is satisfied at the root r of a model $\mathfrak{M}$ based on a frame $\mathfrak{F} = (W, R_1, R_2)$ for $[\mathbf{K4}, \mathbf{K4}]$. Then both R_1 and R_2 are transitive, they commute and satisfy the Church–Rosser property. We show that in this case $\mathfrak{F}$ must be an ∞-chessboard, and so infinite.

Define new binary relations $\bar{R}_1 = \bar{R}_1^{\mathfrak{M}}$ and $\bar{R}_2 = \bar{R}_2^{\mathfrak{M}}$ on W by means of (2) and (3) above. By (9)–(10), $\mathfrak{F}$ satisfies **(cb1)** and **(cb2)**, and so $\bar{R}_1$ and $\bar{R}_2$ satisfy (tran), (com) and (chro). Moreover, for all $x \in W$,

$$\begin{array}{lcl} (\mathfrak{M}, x) \models ⬘\psi & \text{iff} & \exists y \in W\ (x\bar{R}_1 y \text{ and } (\mathfrak{M}, y) \models \psi), \\ (\mathfrak{M}, x) \models ⬗\psi & \text{iff} & \exists y \in W\ (x\bar{R}_2 y \text{ and } (\mathfrak{M}, y) \models \psi). \end{array}$$

We will use the following abbreviations. For every formula ψ, $\Diamond \in \{⬘, ⬗\}$ and $\Box \in \{⬒, ◨\}$, let

$$\Diamond^0\psi \quad = \quad \Box^0\psi \quad = \quad \psi$$

and, for $n < \omega$, let

$$\begin{array}{lcl} \Diamond^{n+1}\psi & = & \Diamond\Diamond^n\psi, \qquad\qquad \Box^{n+1}\psi \quad = \quad \Box\Box^n\psi, \qquad \text{and} \\ \Diamond^{=\mathbf{n}}\psi & = & \Diamond^n\psi \wedge \Box^{n+1}\neg\psi. \end{array}$$

(The last formula means ‘see ψ in n steps but not in $n+1$ steps.’)

Now it should be clear that if we define the horizontal and vertical ranks $hr(x) = hr^{\mathfrak{M}}(x)$ and $vr(x) = vr^{\mathfrak{M}}(x)$ of a point x by means of (4) and (5), then we have

$$hr(x) \quad = \quad \begin{cases} n, & \text{if } n < \omega \text{ and } (\mathfrak{M}, x) \models ⬘^{=\mathbf{n}}\top, \\ \infty, & \text{otherwise,} \end{cases}$$

$$vr(x) \quad = \quad \begin{cases} n, & \text{if } n < \omega \text{ and } (\mathfrak{M}, x) \models ⬗^{=\mathbf{n}}\top, \\ \infty, & \text{otherwise.} \end{cases}$$

The reader can readily check, using (com) and (chro), that if $x\bar{R}_1 y$ then $vr(x) = vr(y)$, and if $x\bar{R}_2 y$ then $hr(x) = hr(y)$.

Let

$$V \quad = \quad \{x \in W \mid \exists u \in W\ r\bar{R}_1 u \bar{R}_2 x\}.$$

LEMMA 1. *Suppose that* $\mathfrak{M}$ *is a model based on a rooted frame for* $[\mathbf{K4}, \mathbf{K4}]$. *If* $(\mathfrak{M}, r) \models \varphi_\infty$ *then, for all* $n < \omega$, *there exists* $x_n \in V$ *such that* $hr(x_n) = vr(x_n) = n$. (*Therefore, if* φ_∞ *is satisfiable in a rooted frame* $\mathfrak{F}$ *for* $[\mathbf{K4}, \mathbf{K4}]$ *then* $\mathfrak{F}$ *is an* ∞*-chessboard.*)

PROOF. First, we claim that the following formulas are true in $\mathfrak{M}$, for all $n < \omega$:

$$\text{⬒}\text{◨}(\neg d \to \text{⬘}\top), \tag{17}$$

$$\text{⬒}\text{◨}(d \to \text{⬒}^n \text{⬗}^n d), \tag{18}$$

$$\text{⬒}\text{◨}(\neg d \to \text{◨}^n \text{⬘}^n \neg d). \tag{19}$$

Indeed, (17) is a straightforward consequence of (12), (16) and (com). We prove (18) by induction on n. The case $n = 0$ is trivial. Suppose now that (18) holds for some n. Take some $w \in V$ with $(\mathfrak{M}, w) \models d$ and $z_1, \dots, z_n, z_{n+1}$ such that

$$w \bar{R}_1 z_1 \bar{R}_1 \dots \bar{R}_1 z_n \bar{R}_1 z_{n+1}.$$

Then $z_n \in V$ and, by IH, there are $w_1, \dots, w_n \in V$ such that

$$z_n \bar{R}_2 w_1 \bar{R}_2 \dots \bar{R}_2 w_n \qquad \text{and} \qquad (\mathfrak{M}, w_n) \models d.$$

By (chro), there are $s_1, \dots, s_n \in V$ such that $w_i \bar{R}_1 s_i$, for $i = 1, \dots, n$, and $z_{n+1} \bar{R}_2 s_1 \bar{R}_2 \dots \bar{R}_2 s_n$. Since $w_n \bar{R}_1 s_n$, it follows from (15) that there exists s_{n+1} such that

$$s_n \bar{R}_2 s_{n+1} \qquad \text{and} \qquad (\mathfrak{M}, s_{n+1}) \models d,$$

from which $(\mathfrak{M}, z_{n+1}) \models \text{⬗}^{n+1} d$. The proof of (19) is analogous, it uses (16) in place of (15).

Now we define inductively four infinite sequences

$$x_0, x_1, x_2, \dots, \quad y_0, y_1, y_2, \dots, \quad u_0, u_1, u_2, \dots \quad \text{and} \quad v_0, v_1, v_2, \dots \tag{20}$$

of points from W such that, for every $i < \omega$,

(gen1) $(\mathfrak{M}, x_i) \models d \wedge \text{◨}\neg d$,

(gen2) $(\mathfrak{M}, y_i) \models \neg d \wedge \text{⬒} d$,

(gen3) $r \bar{R}_2 u_i$, $u_i \bar{R}_1 x_i$ and $u_i \bar{R}_1 y_i$, that is, $vr(u_i) = vr(x_i) = vr(y_i)$, and

(gen4) if $i > 0$ then $r \bar{R}_1 v_i$, $v_i \bar{R}_2 x_i$ and $v_i \bar{R}_2 y_{i-1}$, that is, $hr(v_i) = hr(x_i) = hr(y_{i-1})$.

(We do not claim at this point that, say, all the x_i are distinct.)

To begin with, by (11), there are u_0, x_0 such that $r \bar{R}_2 u_0 \bar{R}_1 x_0$ and

$$(\mathfrak{M}, x_0) \models \text{⬒}\bot \wedge \text{◨}\bot. \tag{21}$$

By (12), $(\mathfrak{M}, x_0) \models d$. By (13), there is y_0 such that $u_0 \bar{R}_1 y_0$ and

$$(\mathfrak{M}, y_0) \models \neg d \wedge \text{⬒} d.$$

So **(gen1)**–**(gen3)** hold for $i = 0$.

Now suppose that, for some $n < \omega$, x_i and y_i with **(gen1)**–**(gen4)** have already been defined for all $i \leq n$. By **(gen3)** for $i = n$ and by (com), there is

v_{n+1} such that $r\bar{R}_1 v_{n+1}\bar{R}_2 y_n$. So by (14), there is x_{n+1} such that $v_{n+1}\bar{R}_2 x_{n+1}$ and

$$(\mathfrak{M}, x_{n+1}) \models d \wedge \boxbar\neg d.$$

Now again by (com), there is u_{n+1} such that $r\bar{R}_2 u_{n+1}\bar{R}_1 x_{n+1}$. So, by (13), there is y_{n+1} such that $u_{n+1}\bar{R}_1 y_{n+1}$ and

$$(\mathfrak{M}, y_{n+1}) \models \neg d \wedge \boxminus d,$$

as required (see Fig. 1). Observe that x_i and y_i are in V for all $i < \omega$.

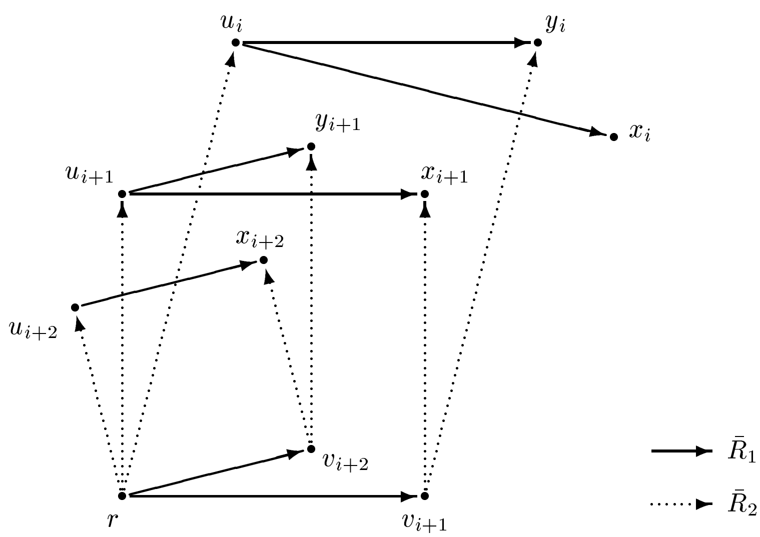

FIGURE 1. Generating the points x_i, y_i, u_i and v_i.

We claim that, for all $i, n < \omega$,

$$(\mathfrak{M}, x_i) \models \diamondminus^n\top \leftrightarrow \diamondbar^n\top, \qquad \text{that is,} \quad hr(x_i) = vr(x_i), \tag{22}$$

$$(\mathfrak{M}, y_i) \models \diamondminus^{n+1}\top \leftrightarrow \diamondbar^n\top, \quad \text{that is,} \quad hr(y_i) = vr(y_i) + 1. \tag{23}$$

Indeed, if $n = 0$ then (22) is trivial, and (23) follows from **(gen2)** and (17). So we may assume that $n > 0$.

To prove (22), suppose first that we have $(\mathfrak{M}, x_i) \models \diamondminus^n\top$. Then there is a point z such that $x_i\bar{R}_1^n z$. By **(gen1)**, $(\mathfrak{M}, x_i) \models d$. So, $(\mathfrak{M}, z) \models \diamondbar^n d$, by (18). Using (com), we find a point v such that $x_i\bar{R}_2^n v$ and $v\bar{R}_1^n u$, from which $(\mathfrak{M}, x_i) \models \diamondbar^n\top$. Conversely, suppose $(\mathfrak{M}, x_i) \models \diamondbar^n\top$, that is, there are points $z_1, \dots, z_n$ such that $x_i\bar{R}_2 z_1\bar{R}_2 \dots \bar{R}_2 z_n$. By **(gen1)**, $(\mathfrak{M}, x_i) \models \boxbar\neg d$, and so $(\mathfrak{M}, z_1) \models \neg d$. Therefore, by (19) and (17), we have $(\mathfrak{M}, z_n) \models \diamondminus^n\top$, and then obtain $(\mathfrak{M}, x_i) \models \diamondminus^n\top$ using (com).

To show (23), assume first that we have $(\mathfrak{M}, y_i) \models \diamondbar^n\top$. Then there is a point z such that $y_i\bar{R}_2^n z$. By **(gen2)**, $(\mathfrak{M}, y_i) \models \neg d$. So, by (19), $(\mathfrak{M}, z) \models \diamondminus^n\neg d$, and by (17), $(\mathfrak{M}, z) \models \diamondminus^{n+1}\top$. Now $(\mathfrak{M}, y_i) \models \diamondminus^{n+1}\top$ follows by (com). Conversely, suppose $(\mathfrak{M}, y_i) \models \blacklozenge^{n+1}\top$, that is, there are points $z_1, \dots, z_n, z_{n+1}$ such that $y_i \bar{R}_1 z_1 \bar{R}_1 \dots \bar{R}_1 z_n \bar{R}_1 z_{n+1}$. By **(gen2)**, $(\mathfrak{M}, y_i) \models \blacksquare d$, and so $(\mathfrak{M}, z_1) \models d$. Therefore, by (18), we have $(\mathfrak{M}, z_{n+1}) \models \lozenge^n\top$. And finally, using (com) we obtain $(\mathfrak{M}, y_i) \models \lozenge^n\top$.

Next, we claim that, for all $n < \omega$,

$$vr(u_n) = n, \tag{24}$$

$$hr(v_n) = n, \tag{25}$$

$$hr(x_n) = vr(x_n) \quad = \quad n. \tag{26}$$

First we prove (24) by induction on n. For $n = 0$, it follows from the definition of x_0 (see (21)) and **(gen3)**. Suppose that (24) holds for some $n < \omega$. Then

$$vr(u_{n+1}) \overset{\textbf{(gen3)}}{=} vr(x_{n+1}) \overset{(22)}{=} hr(x_{n+1}) \overset{\textbf{(gen4)}}{=}$$

$$hr(y_n) \overset{(23)}{=} vr(y_n) + 1 \overset{\textbf{(gen3)}}{=} vr(u_n) + 1 \overset{\text{(IH)}}{=} n + 1.$$

Now (25) and (26) follow from (24) and

$$hr(v_n) \overset{\textbf{(gen4)}}{=} hr(x_n) \overset{(22)}{=} vr(x_n) \overset{\textbf{(gen3)}}{=} vr(u_n),$$

as required. ⊣

Let us now prove the '$\Leftarrow$' direction of (8).

LEMMA 2. *φ_∞ is satisfiable in any ∞-chessboard.*

PROOF. We begin with some definitions. Fix some $k < \omega$ and a frame $\mathfrak{F} = (W, R_1, R_2)$ for $[\mathbf{K4}, \mathbf{K4}]$ with root r. We call a model $\mathfrak{N}$ over $\mathfrak{F}$ a *perfect k-chessboard model* if the following hold:

(a) $\mathfrak{N}$ satisfies **(cb1)** and **(cb2)**;
(b) for every point $v \in W$, if $r\bar{R}_1^{\mathfrak{N}} v$ then $hr^{\mathfrak{N}}(v)$ is finite;
(c) for every point $u \in W$, if $r\bar{R}_2^{\mathfrak{N}} u$ then $vr^{\mathfrak{N}}(u)$ is finite;
(d) for every $n < k$, there is a point $v_n \in W$ with $r\bar{R}_1^{\mathfrak{N}} v_n$ and $hr^{\mathfrak{N}}(v_n) = n$;
(e) for every $n < k$, there is a point $u_n \in W$ with $r\bar{R}_2^{\mathfrak{N}} u_n$ and $vr^{\mathfrak{N}}(u_n) = n$.

We call $\mathfrak{N}$ a *perfect ∞-chessboard model*, if (d) and (e) hold for $k = \omega$.

CLAIM 2.1. (i) *If $\mathfrak{F}$ is a k-chessboard then there is a perfect k-chessboard model based on $\mathfrak{F}$.*
(ii) *If $\mathfrak{F}$ is an ∞-chessboard then there is a perfect ∞-chessboard model based on $\mathfrak{F}$.*

PROOF OF CLAIM 2.1. (i) Take a k-chessboard $\mathfrak{F}$ with root r. Then there is a model $\mathfrak{M}$ based on $\mathfrak{F}$ that satisfies **(cb1)** and **(cb2)**, and such that there exist points x_n with $hr^{\mathfrak{M}}(x_n) = vr^{\mathfrak{M}}(x_n) = n$ for every $n \le k$. We know that $\bar{R}_1^{\mathfrak{M}}$ and $\bar{R}_2^{\mathfrak{M}}$ satisfy (tran), (com) and (chro).

We may assume that $(\mathfrak{M}, r) \models \neg h \wedge \neg v$ (if this is not the case, we change the truth-values values of h and v to the 'opposite'). Define a new model $\mathfrak{N}$ over $\mathfrak{F}$ by taking

$$(\mathfrak{N}, x) \models h \quad \text{iff} \quad (\mathfrak{M}, x) \models h \ \text{ and } \ hr^{\mathfrak{M}}(x) \text{ is finite},$$

$$(\mathfrak{N}, x) \models v \quad \text{iff} \quad (\mathfrak{M}, x) \models v \ \text{ and } \ vr^{\mathfrak{M}}(x) \text{ is finite}.$$

We show that $\mathfrak{N}$ satisfies conditions (a)–(e). Observe first that for all $x, y \in W$,

$$\text{if } xR_1y \text{ then } hr^{\mathfrak{M}}(x) \geq hr^{\mathfrak{M}}(y), \tag{27}$$

$$\text{if } xR_2y \text{ then } vr^{\mathfrak{M}}(x) \geq vr^{\mathfrak{M}}(y). \tag{28}$$

Now take a point u such that $hr^{\mathfrak{M}}(u)$ is finite. Then it follows from (27) that, for all $v \in W$, we have $u\bar{R}_1^{\mathfrak{M}}v$ iff $u\bar{R}_1^{\mathfrak{N}}v$. Similarly, if $vr^{\mathfrak{M}}(u)$ is finite then, for all $v \in W$, we have $u\bar{R}_2^{\mathfrak{M}}v$ iff $u\bar{R}_2^{\mathfrak{N}}v$. Therefore, for all $u \in W$,

$$\text{if } hr^{\mathfrak{M}}(u) \text{ is finite then } hr^{\mathfrak{N}}(u) = hr^{\mathfrak{M}}(u), \tag{29}$$

$$\text{if } vr^{\mathfrak{M}}(u) \text{ is finite then } vr^{\mathfrak{N}}(u) = vr^{\mathfrak{M}}(u). \tag{30}$$

We are now in a position to prove (a)–(e) for $\mathfrak{N}$.

(a) It is easy to see that, since $\mathfrak{M}$ satisfies **(cb1)**, R_1 and $\bar{R}_2^{\mathfrak{M}}$ are Church–Rosser and commute. Therefore, for all x, y with xR_1y, we have $vr^{\mathfrak{M}}(x) = vr^{\mathfrak{M}}(y)$, which implies **(cb1)** for $\mathfrak{N}$. The proof of **(cb2)** is similar: we use the fact that R_2 and $\bar{R}_1^{\mathfrak{M}}$ are Church–Rosser and commute.

(b) Let $r\bar{R}_1^{\mathfrak{N}}u$ and suppose that $hr^{\mathfrak{N}}(u) = \infty$. By (29), we then have $hr^{\mathfrak{M}}(u) = \infty$, and so $(\mathfrak{N}, u) \models \neg h$. Since $(\mathfrak{M}, r) \models \neg h$, we also have $(\mathfrak{N}, r) \models \neg h$. So there is a v such that rR_1vR_1u and $(\mathfrak{N}, v) \models h$. But then $hr^{\mathfrak{M}}(v)$ must be finite, contrary to vR_1u, $hr^{\mathfrak{M}}(u) = \infty$, and (27). So $hr^{\mathfrak{N}}(u) < \infty$.

(c) is similar. We use (30) and (28).

(d) Take an $n < k$. Then there is x_{n+1} such that $hr^{\mathfrak{M}}(x_{n+1}) = vr^{\mathfrak{M}}(x_{n+1}) = n+1$. We have either $x_{n+1} = r$, or rR_1x_{n+1}, or rR_2x_{n+1}, $rR_1z_{n+1}R_2x_{n+1}$. Since $\bar{R}_1^{\mathfrak{M}}$ and R_2 commute and are Church–Rosser, if two points are R_2-connected then their horizontal ranks in $\mathfrak{M}$ must be the same. So in any case we have a point z_{n+1} such that $hr^{\mathfrak{M}}(z_{n+1}) = n+1$ and either $z_{n+1} = r$ or rR_1z_{n+1}. By (29), $hr^{\mathfrak{N}}(z_{n+1}) = n+1$, and so there is u_n such that $z_{n+1}\bar{R}_1^{\mathfrak{N}}u_n$ and $hr^{\mathfrak{N}}(u_n) = n$. So we have $r\bar{R}_1^{\mathfrak{N}}u_n$ as required.

(e) is proved in the same way using (30).

(ii) If $\mathfrak{F}$ is an ∞-chessboard then the above proofs for (d) and (e) show that in fact $\mathfrak{N}$ satisfies (d) and (e) for $k = \omega$, which completes the proof of Claim 2.1. ⊣

Now suppose that $\mathfrak{F} = (W, R_1, R_1)$ is an ∞-chessboard with root r. By Claim 2.1, there is a perfect ∞-chessboard model $\mathfrak{N}$ based on $\mathfrak{F}$. Define a valuation of the propositional variable d in $\mathfrak{N}$ by taking, for all $x \in W$,

$$(\mathfrak{N}, x) \models d \qquad \text{iff} \qquad hr^{\mathfrak{N}}(x) \leq vr^{\mathfrak{N}}(x) < \infty. \tag{31}$$

We claim that $(\mathfrak{N}, r) \models \varphi_\infty$. Indeed, (9) and (10) hold because of property (a) of the perfect ∞-chessboard model $\mathfrak{N}$, and so $\bar{R}_1^{\mathfrak{N}}$ and $\bar{R}_2^{\mathfrak{N}}$ satisfy (com) and (chro). The proof for the remaining conjuncts is straightforward. We only consider (13). Take a u such that $r\bar{R}_2^{\mathfrak{N}}u$. Then, by (c), $vr^{\mathfrak{N}}(u) = n$ for some $n < \omega$. By (d), there is v_{n+1} such that $r\bar{R}_1^{\mathfrak{N}}v_{n+1}$ and $hr^{\mathfrak{N}}(v_{n+1}) = n+1$. Then, by (com) and (chro), there is y such that $u\bar{R}_1^{\mathfrak{N}}y$ and $hr^{\mathfrak{N}}(y) = n+1$. We also have $vr^{\mathfrak{N}}(y) = vr^{\mathfrak{N}}(u) = n$, and so $(\mathfrak{N}, y) \models \neg d$. On the other hand, if x is such that $y\bar{R}_1^{\mathfrak{N}}x$ then $hr^{\mathfrak{N}}(x) \leq n$ and $vr^{\mathfrak{N}}(x) = n$, from which $(\mathfrak{N}, x) \models d$, as required. ⊣

§5. Undecidability. In the proof of Lemma 1 above we saw how the formula φ_∞ ensured the existence of a sort of 'diagonal points' x_n with $hr(x_n) = vr(x_n) = n$. We will use these points to encode parts of the '$\omega \times \omega$-grid' in frames with two transitive commuting and Church–Rosser relations.

Various undecidable problems can be 'represented' on the $\omega \times \omega$-grid, say, versions of the halting problems for Turing machines, register machines, etc., Post's correspondence problem, as well as the infinite tiling (or domino) problems. In Sections 5.2 and 5.3 we show two examples: the halting problem for Turing machines and infinite tiling problems.

To prove our undecidability results, we will reduce a sufficiently complex problem for Turing machines or tilings to the satisfiability problem for the logic in question. More precisely, we will use

- non-recursively enumerable problems, viz., the non-halting problem for Turing machines or the $\omega \times \omega$ tiling problem, to obtain the general undecidability result of Theorem 2 (which covers, in particular, recursively enumerable logics like $\mathbf{K4} \times \mathbf{K4}$);
- a recursively enumerable problem whose complement is not recursively enumerable, namely, the halting problem for Turing machines, to prove non-recursive enumerability in Theorem 3;
- Σ_1^1-hard problems, viz., the non-halting problem for recurrent non-deterministic Turing machines or the recurrent tiling problem, to obtain Π_1^1-hardness in Theorem 4.

5.1. Encoding the $\omega \times \omega$-grid. The enumeration of the points of $\omega \times \omega$ we use below has been introduced in several papers dealing with undecidable multimodal logics; see, e.g., [18, 28, 31]. However, in all these cases either the language had *next-time operators* or the frames were *linear*. Here we show that one can code this enumeration even if the frames are branching (and, of course, transitive), and no next-time operators are available.

Let $pair : \omega \to \omega \times \omega$ be the function defined recursively by taking:

- $pair(0) = (0, 0)$,
- if $pair(n) = (0, j)$ then $pair(n + 1) = (j + 1, 0)$,
- otherwise, if $pair(n) = (i + 1, j)$ then $pair(n + 1) = (i, j + 1)$;

see Fig. 2. It is easy to see that $pair$ is one-one and onto. Let $\sharp : \omega \times \omega \to \omega$ denote the inverse of the function $pair$. If $pair(n)$ is not on the wall (that is, the first coordinate of $pair(n)$ is different from 0) then define $left_n$ to be the $\sharp$ of the left neighbour of $pair(n)$. The reader can readily check the following important properties of these functions, for all $n > 0$:

(t1) If neither $pair(n)$ nor $pair(n - 1)$ are on the wall then $left_n = left_{n-1} + 1$.
(t2) If $n > 1$ and $pair(n)$ is not on the wall, but $pair(n - 1)$ is on the wall, then $n > 2$, $pair(n - 2)$ is not on the wall, and $left_n = left_{n-2} + 1$.
(t3) $pair(n)$ is on the wall iff $pair(left_{n-1})$ is on the wall.
(t4) Either $pair(n)$ or $pair(n - 1)$ is not on the wall.

We will require the following propositional variables:

- grid (marking the points of the grid),
- left (a pointer from n to $left_n$ when $pair(n)$ is not on the wall),

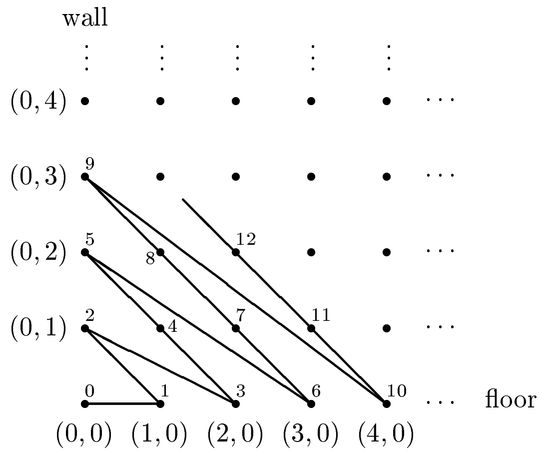

FIGURE 2. The enumeration *pair*.

- wall (marking the wall, i.e., the pairs of the form $(0, n)$).

Let φ_{grid} be the conjunction of (9), (10) and the formulas (32)–(38):

$$\boxminus\boxbar\big(\boxminus\bot \to (\mathsf{grid} \leftrightarrow \boxbar\bot)\big), \tag{32}$$

$$\boxminus\boxbar(\boxminus\bot \wedge \mathsf{grid} \to \mathsf{wall}), \tag{33}$$

$$\boxminus\boxbar(\mathsf{wall} \to \mathsf{grid}), \tag{34}$$

$$\boxminus\boxbar\big(\diamondminus\mathsf{wall} \to \boxminus(\mathsf{grid} \to \mathsf{wall})\big), \tag{35}$$

$$\boxminus\boxbar\big(\diamondminus\top \to (\mathsf{grid} \leftrightarrow \diamondbar^{=1}\diamondminus^{=1}\mathsf{grid})\big), \tag{36}$$

$$\boxminus\boxbar\big(\mathsf{grid} \wedge \diamondminus\top \to \big(\mathsf{wall} \leftrightarrow \diamondbar(\diamondminus^{=1}\mathsf{left} \wedge \diamondminus\mathsf{wall})\big)\big), \tag{37}$$

$$\boxminus\boxbar\Big[\mathsf{left} \leftrightarrow \Big((\diamondminus^{=1}\top \wedge \boxbar\bot) \vee \big(\diamondbar(\diamondminus^{=2}\mathsf{left} \wedge \diamondminus\mathsf{wall}) \wedge \diamondbar^{=1}\diamondminus^{=2}\mathsf{left}\big) \vee \big(\diamondbar(\diamondminus^{=1}\mathsf{left} \wedge \neg\diamondminus\mathsf{wall}) \wedge \diamondbar^{=1}\diamondminus^{=1}\mathsf{left}\big)\Big)\Big]. \tag{38}$$

LEMMA 3. $\varphi_\infty \wedge \varphi_{grid}$ *is satisfiable in any* ∞*-chessboard.*

PROOF. Let $\mathfrak{F} = (W, R_1, R_2)$ be an ∞-chessboard with root r. By Claim 2.1, there is a perfect ∞-chessboard model $\mathfrak{N}$ over $\mathfrak{F}$. Define a valuation of the propositional variables grid, wall and left in $\mathfrak{N}$ by taking, for all $x \in W$,

$$\begin{aligned}(\mathfrak{N}, x) \models \mathsf{grid} \quad &\text{iff} \quad hr^{\mathfrak{N}}(x) = vr^{\mathfrak{N}}(x) < \infty, \\ (\mathfrak{N}, x) \models \mathsf{wall} \quad &\text{iff} \quad hr^{\mathfrak{N}}(x) = vr^{\mathfrak{N}}(x) = \sharp(0, j) \ \text{ for some } j < \omega, \\ (\mathfrak{N}, x) \models \mathsf{left} \quad &\text{iff} \quad hr^{\mathfrak{N}}(x) = n, \ vr^{\mathfrak{N}}(x) = \mathit{left}_n \ \text{ for some } n < \omega \\ &\qquad \text{such that } \mathit{pair}(n) \text{ is not on the wall.}\end{aligned} \tag{39}$$

Then it is straightforward to check that $(\mathfrak{N}, r) \models \varphi_\infty \wedge \varphi_{grid}$. ⊣

The next lemma shows that in fact φ_{grid} 'forces' the $\omega \times \omega$-grid onto 'diagonal points of finite rank.'

LEMMA 4. *Suppose that $\mathfrak{M}$ is a model based on a rooted frame $\mathfrak{F} = (W, R_1, R_2)$ for $[\mathbf{K4}, \mathbf{K4}]$. If $(\mathfrak{M}, r) \models \varphi_{grid}$ then the following hold, for all $n, m < \omega$ and all $x \in V$ such that $hr(x) = n$ and $vr(x) = m$:*

(i) $(\mathfrak{M}, x) \models \mathsf{grid}$ *iff* $n = m$,
(ii) $(\mathfrak{M}, x) \models \text{⬘}^{=1}\mathsf{left}$ *iff* $n > 0$, *pair*$(n-1)$ *is not on the wall and* $m = \mathit{left}_{n-1}$,
(iii) $(\mathfrak{M}, x) \models \mathsf{wall}$ *iff* $n = m$ *and pair*(n) *is on the wall,*
(iv) $(\mathfrak{M}, x) \models \mathsf{left}$ *iff pair*(n) *is not on the wall and* $m = \mathit{left}_n$.

PROOF. We use the same notation as in Section 4, in particular, $\bar{R}_1 = \bar{R}_1^{\mathfrak{M}}$ and $\bar{R}_2 = \bar{R}_2^{\mathfrak{M}}$, $hr(x) = hr^{\mathfrak{M}}(x)$ and $vr(x) = vr^{\mathfrak{M}}(x)$, and

$$V \quad = \quad \{x \in W \mid \exists u \in W \; r\bar{R}_1 u \bar{R}_2 x\}.$$

The proof proceeds by induction on n. For $n = 0$, we obtain (i) by (32), (iii) by (33) and (34), and (iv) by (38).

Now take any $n > 0$ and suppose that the lemma holds for all $k < n$. Throughout, we will use the following observation. Given numbers $a, b < \omega$ and some $x \in V$ with $hr(x) = a$ and $vr(x) = b$, there exists what we call a *perfect* $a \times b$-*rectangle starting at* x, that is, there are points $x_{i,j}$ (for $i \leq a$, $j \leq b$) such that

- $x = x_{a,b}$,
- $hr(x_{i,j}) = i$ and $vr(x_{i,j}) = j$,
- $x_{i,j}\bar{R}_1 x_{k,j}$ for $i > k$, and $x_{i,j}\bar{R}_2 x_{i,k}$ for $j > k$.

Indeed, given x, take an a-long $\bar{R}_1$-path and a b-long $\bar{R}_2$-path starting from x, and then 'close them' under the Church-Rosser property.

(i) We claim that, for all $m < \omega$ and all $x \in V$ with $hr(x) = n$ and $vr(x) = m$,

$$(\mathfrak{M}, x) \models \text{⬘}^{=1}\mathsf{grid} \quad \text{iff} \quad m = n - 1. \tag{40}$$

Indeed, suppose first that $m = n - 1$. Take a perfect $n \times (n-1)$-rectangle $x_{i,j}$ ($i \leq n$, $j \leq n-1$) starting at x. Then by IH (i), $(\mathfrak{M}, x_{n-1,n-1}) \models \mathsf{grid}$, and so $(\mathfrak{M}, x) \models \text{⬘}\mathsf{grid}$. Now let u be such that $x\bar{R}_1 u$ and $(\mathfrak{M}, u) \models \mathsf{grid}$. Then we have $hr(u) = k < n$ and $vr(u) = vr(x) = n - 1 < \omega$. By IH (i), we have $k = n-1$, and so $(\mathfrak{M}, x) \not\models \text{⬘}^2\mathsf{grid}$. Conversely, suppose that $(\mathfrak{M}, x) \models \text{⬘}^{=1}\mathsf{grid}$. Then there is u such that $x\bar{R}_1 u$ and $(\mathfrak{M}, u) \models \mathsf{grid}$. We have $hr(u) = k < n$ and $vr(u) = vr(x) = m$. So $m = k$ follows, by IH (i). Now take a perfect $n \times k$-rectangle $x_{i,j}$ ($i \leq n$, $j \leq k$) starting at x. By IH (i) again, we have $(\mathfrak{M}, x_{k,k}) \models \mathsf{grid}$. Since $(\mathfrak{M}, x) \models \text{⬘}^{=1}\mathsf{grid}$ and $x = x_{n,k}\bar{R}_1 x_{k,k}$, we must have $m = k = n - 1$ as required in (40).

Our next claim is that, for all $m < \omega$ and all $x \in V$ with $hr(x) = n$ and $vr(x) = m$,

$$(\mathfrak{M}, x) \models \text{⬗}^{=1}\text{⬘}^{=1}\mathsf{grid} \quad \text{iff} \quad m = n. \tag{41}$$

Indeed, suppose first that $m = n$. Take a perfect $n \times n$-rectangle $x_{i,j}$ ($i \leq n$, $j \leq n$) starting at x. Then $(\mathfrak{M}, x_{n,n-1}) \models \text{⬘}^{=1}\mathsf{grid}$, by (40), and therefore $(\mathfrak{M}, x) \models \text{⬗}\text{⬘}^{=1}\mathsf{grid}$. Now, the fact that $(\mathfrak{M}, x) \not\models \text{⬗}^2\text{⬘}^{=1}\mathsf{grid}$ also follows from (40). Conversely, suppose that $(\mathfrak{M}, x) \models \text{⬗}^{=1}\text{⬘}^{=1}\mathsf{grid}$. Then there is u such that $x\bar{R}_2 u$ and $(\mathfrak{M}, u) \models \text{⬘}^{=1}\mathsf{grid}$. Since $hr(u) = n$, by (40) we obtain that

$vr(u) = n-1$, and so $m \ge n$. Now take a perfect $n \times m$-rectangle $x_{i,j}$ $(i \le n$, $j \le m)$ starting at x. By (40) again, $(\mathfrak{M}, x_{n,n-1}) \models \text{⬘}^{=1}\mathsf{grid}$, so $m = n$ must hold.

Now claim (i) of Lemma 4 follows from (41) and (36).

(ii) The proof is similar to the proof of (40); we only use IH (iv) in place of IH (i). In fact, we can even prove a slightly stronger claim: for all $i, m < \omega$ and all $x \in V$ with $hr(x) = n$ and $vr(x) = m$,

$$(\mathfrak{M}, x) \models \text{⬘}^{=i}\mathsf{left} \quad \text{iff} \quad n \ge i,\ pair(n-i) \text{ is not on the wall},\ m = left_{n-i}. \tag{42}$$

Indeed, suppose first that $n \ge i$, $pair(n-i)$ is not on the wall and $m = left_{n-i}$. Take a perfect $n \times left_{n-i}$-rectangle $x_{a,b}$ $(a \le n,\ b \le left_{n-i})$ starting at x. By IH (iv), $(\mathfrak{M}, x_{n-i, left_{n-i}}) \models \mathsf{left}$, and so $(\mathfrak{M}, x) \models \text{⬘}^{i}\mathsf{left}$. Now let u be such that $x\bar{R}_1 u$ and $(\mathfrak{M}, u) \models \mathsf{left}$. Then $vr(u) = vr(x) = left_{n-i}$ and $hr(u) = k < n$. By IH (iv), $pair(k)$ is not on the wall and $vr(u) = left_k$, from which $k = n-i$ follows, implying $(\mathfrak{M}, x) \not\models \text{⬘}^{i+1}\mathsf{left}$. Conversely, suppose that $(\mathfrak{M}, x) \models \text{⬘}^{=i}\mathsf{left}$. Then $n \ge i$ and there is u such that $x\bar{R}_1^i u$ and $(\mathfrak{M}, u) \models \mathsf{left}$. So we have $hr(u) = k \le n-i$ and $vr(u) = vr(x) = m$. So, by IH (iv), $pair(k)$ is not on the wall and $m = left_k$. Now take a perfect $n \times left_k$-rectangle $x_{a,b}$ $(a \le n,\ b \le left_k)$ starting at x. By IH (iv) again, we have $(\mathfrak{M}, x_{k, left_k}) \models \mathsf{left}$, and so $k = n-i$ must hold, as required in (42).

(iii) Suppose first that $n = m$ and $pair(n)$ is on the wall. Then, by **(t4)**, $pair(n-1)$ is not on the wall. By IH (i), we have $(\mathfrak{M}, x) \models \mathsf{grid}$. So by (37), it is enough to show that

$$(\mathfrak{M}, x) \models \text{⬖}(\text{⬘}^{=1}\mathsf{left} \land \text{⬘}\mathsf{wall}). \tag{43}$$

Take a perfect $n \times m$-rectangle $x_{i,j}$ $(i \le n,\ j \le m)$ starting at x. We have $(\mathfrak{M}, x_{n, left_{n-1}}) \models \text{⬘}^{=1}\mathsf{left}$, by Lemma 4 (ii). On the other hand, by **(t3)**, $pair(left_{n-1})$ is on the wall. So, by IH (iii), $(\mathfrak{M}, x_{left_{n-1}, left_{n-1}}) \models \mathsf{wall}$, and so $(\mathfrak{M}, x_{n, left_{n-1}}) \models \text{⬘}\mathsf{wall}$. Since $x\bar{R}_2 x_{n, left_{n-1}}$, we obtain (43).

Conversely, suppose that $(\mathfrak{M}, x) \models \mathsf{wall}$. By (34), we have $(\mathfrak{M}, x) \models \mathsf{grid}$, so $n = m$ follows by Lemma 4 (i). By (37), $(\mathfrak{M}, x) \models \text{⬖}(\text{⬘}^{=1}\mathsf{left} \land \text{⬘}\mathsf{wall})$. Then there is a u such that $x\bar{R}_2 u$ and $(\mathfrak{M}, u) \models \text{⬘}^{=1}\mathsf{left} \land \text{⬘}\mathsf{wall}$. By Lemma 4 (ii), $pair(n-1)$ is not on the wall and $vr(u) = left_{n-1}$. Take a perfect $n \times left_{n-1}$-rectangle $u_{i,j}$ $(i \le n,\ j \le left_{n-1})$ starting at u. By Lemma 4 (i), we have $(\mathfrak{M}, u_{left_{n-1}, left_{n-1}}) \models \mathsf{grid}$ and so, by (35), $(\mathfrak{M}, u_{left_{n-1}, left_{n-1}}) \models \mathsf{wall}$. Now by IH (iii), $pair(left_{n-1})$ is on the wall and so, by **(t3)**, $pair(n)$ is on the wall, as required.

(iv) First, we claim that, for all $i, m < \omega$ and all $x \in V$ with $hr(x) = n$ and $vr(x) = m$,

$$(\mathfrak{M}, x) \models \text{⬖}^{=1}\text{⬘}^{=i}\mathsf{left} \quad \text{iff} \quad n \ge i,\ pair(n-i) \text{ is not on the wall and } m = left_{n-i} + 1. \tag{44}$$

The proof of this claim is similar to that of (41), using (42) in place of (40), so we leave it to the reader.

Now suppose that $\mathit{pair}(n)$ is not on the wall and $m = \mathit{left}_n$. We will show how (38) can be used to deduce $(\mathfrak{M}, x) \models \mathsf{left}$. There are three cases:

Case 1: $n = 1$. Then $m = \mathit{left}_1 = 0$, and so $(\mathfrak{M}, x) \models \text{⬘}^{=1}\top \wedge \text{◨}\bot$.

Case 2: $n > 1$ and $\mathit{pair}(n-1)$ is on the wall. Then, by **(t2)**, $\mathit{pair}(n-2)$ is not on the wall and $\mathit{left}_n = \mathit{left}_{n-2} + 1$. By **(t3)**, $\mathit{pair}(\mathit{left}_{n-2})$ is on the wall. We claim that

$$(\mathfrak{M}, x) \models \text{⬗}(\text{⬘}^{=2}\mathsf{left} \wedge \text{⬘}\mathsf{wall}) \wedge \text{⬗}^{=1}\text{⬘}^{=2}\mathsf{left}.$$

Indeed, $(\mathfrak{M}, x) \models \text{⬗}^{=1}\text{⬘}^{=2}\mathsf{left}$, by (44). Take a perfect $n \times (\mathit{left}_{n-2}+1)$-rectangle $x_{i,j}$ ($i \leq n$, $j \leq \mathit{left}_{n-2} + 1$) starting at x. Then $(\mathfrak{M}, x_{\mathit{left}_{n-2}, \mathit{left}_{n-2}}) \models \mathsf{wall}$, by IH (iii). On the other hand, $(\mathfrak{M}, x_{n, \mathit{left}_{n-2}}) \models \text{⬘}^{=2}\mathsf{left}$, by (42), and so we have $(\mathfrak{M}, x_{n, \mathit{left}_{n-2}}) \models \text{⬘}^{=2}\mathsf{left} \wedge \text{⬘}\mathsf{wall}$.

Case 3: $n > 1$ and $\mathit{pair}(n-1)$ is not on the wall. Then, by **(t1)**, $\mathit{left}_n = \mathit{left}_{n-1} + 1$. By **(t3)**, $\mathit{pair}(\mathit{left}_{n-1})$ is not on the wall. We claim that

$$(\mathfrak{M}, x) \models \text{⬗}(\text{⬘}^{=1}\mathsf{left} \wedge \neg\text{⬘}\mathsf{wall}) \wedge \text{⬗}^{=1}\text{⬘}^{=1}\mathsf{left}.$$

Indeed, $(\mathfrak{M}, x) \models \text{⬗}^{=1}\text{⬘}^{=1}\mathsf{left}$, by (44). Take a perfect $n \times (\mathit{left}_{n-1} + 1)$-rectangle $x_{i,j}$ ($i \leq n$, $j \leq \mathit{left}_{n-1} + 1$) starting at x. Then we have, by IH (iii), $(\mathfrak{M}, x_{\mathit{left}_{n-1}, \mathit{left}_{n-1}}) \not\models \mathsf{wall}$. So, by (35), $(\mathfrak{M}, x_{n, \mathit{left}_{n-1}}) \models \neg\text{⬘}\mathsf{wall}$. On the other hand, $(\mathfrak{M}, x_{n, \mathit{left}_{n-1}}) \models \text{⬘}^{=1}\mathsf{left}$, by (42).

Conversely, suppose that $(\mathfrak{M}, x) \models \mathsf{left}$. By (38), there are three cases.

Case 1: $(\mathfrak{M}, x) \models \text{⬘}^{=1}\top \wedge \text{◨}\bot$. Then $n = 1$, $m = 0 = \mathit{left}_1$, and $\mathit{pair}(1)$ is not on the wall.

Case 2: $(\mathfrak{M}, x) \models \text{⬗}(\text{⬘}^{=2}\mathsf{left} \wedge \text{⬘}\mathsf{wall}) \wedge \text{⬗}^{=1}\text{⬘}^{=2}\mathsf{left}$. By (44), we have that $\mathit{pair}(n-2)$ is not on the wall and $m = \mathit{left}_{n-2} + 1$. Take a point u such that $x\bar{R}_2u$ and $(\mathfrak{M}, u) \models \text{⬘}^{=2}\mathsf{left} \wedge \text{⬘}\mathsf{wall}$. By (42), $\mathit{vr}(u) = \mathit{left}_{n-2}$. Take a perfect $n \times \mathit{left}_{n-2}$-rectangle $u_{i,j}$ ($i \leq n$, $j \leq \mathit{left}_{n-2}$) starting at u. By Lemma 4 (i), $(\mathfrak{M}, u_{\mathit{left}_{n-2}, \mathit{left}_{n-2}}) \models \mathsf{grid}$ and so, by (35) and $(\mathfrak{M}, u) \models \text{⬘}\mathsf{wall}$, $(\mathfrak{M}, u_{\mathit{left}_{n-2}, \mathit{left}_{n-2}}) \models \mathsf{wall}$. Now by IH (iii), $\mathit{pair}(\mathit{left}_{n-2})$ is on the wall and so, by **(t3)**, $\mathit{pair}(n-1)$ is on the wall. By **(t4)**, $\mathit{pair}(n)$ is not on the wall. Finally, by **(t2)**, $\mathit{left}_n = \mathit{left}_{n-2} + 1$ as required.

Case 3: $(\mathfrak{M}, x) \models \text{⬗}(\text{⬘}^{=1}\mathsf{left} \wedge \neg\text{⬘}\mathsf{wall}) \wedge \text{⬗}^{=1}\text{⬘}^{=1}\mathsf{left}$. By (44), $\mathit{pair}(n-1)$ is not on the wall and $m = \mathit{left}_{n-1} + 1$. Take a point u such that $x\bar{R}_2u$ and $(\mathfrak{M}, u) \models \text{⬘}^{=1}\mathsf{left} \wedge \neg\text{⬘}\mathsf{wall}$. By (42), $\mathit{vr}(u) = \mathit{left}_{n-1}$. Take a perfect $n \times \mathit{left}_{n-1}$-rectangle $u_{i,j}$ ($i \leq n$, $j \leq \mathit{left}_{n-1}$) starting at u. Since $(\mathfrak{M}, u) \models \neg\text{⬘}\mathsf{wall}$, we have $(\mathfrak{M}, u_{\mathit{left}_{n-1}, \mathit{left}_{n-1}}) \not\models \mathsf{wall}$. So, by IH (iii), $\mathit{pair}(\mathit{left}_{n-1})$ is not on the wall and so, by **(t3)**, $\mathit{pair}(n)$ is not on the wall either. Finally, by **(t1)**, $\mathit{left}_n = \mathit{left}_{n-1} + 1$ as required.

This completes the proof of Lemma 4. ⊣

5.2. Encoding Turing machines. A (one-tape deterministic) *Turing machine* M has a finite tape alphabet T (including B, the *blank symbol*, and £, the 'left-end marker'), a finite set Q of states, with q_0 being the initial state and q_1 the halting state, and a *transition function* ϱ given as follows. For every $q \in Q - \{q_1\}$ and every $X \in T$, the value of $\varrho(q, X)$ is a pair (p, Y), where

- $p \in Q$ is the next state;

- either $Y \in T - \{\pounds\}$ (Y is the symbol to be written in the cell being scanned—it replaces the symbol that was there before), or $Y \in \{\mathsf{L}, \mathsf{R}\}$ (Y is the direction, left or right, in which the head moves, with L and R being fresh symbols).

We can always assume that M is such that its head never moves left of its initial position (say, by postulating that $\varrho(q, \pounds) = (p, \mathsf{R})$ always holds). Starting from an all-blank tape with the head scanning the cell next to $\pounds$, at each step there are only finitely many non-blank cells, so we can represent a *configuration* of M as an infinite sequence of the form

$$\varkappa \;=\; (\pounds, X_1, \dots, X_{n-1}, (q, X_n), X_{n+1}, \dots, X_m, B, B, \dots),$$

where $q \in Q$ is the current state, $\pounds, X_1, \dots, X_m$ is the non-blank part of the current tape description, and the head is scanning the nth cell. For example, the initial configuration $\varkappa_0$ of M looks as follows:

$$\varkappa_0 \;=\; (\pounds, (q_0, B), B, B, \dots).$$

Starting with $\varkappa_0$ and using the transition function ϱ, we define in the standard way the unique sequence of configurations $\varkappa_0, \varkappa_1, \dots$ of M which is called the *computation* of M. Let H_M denote the number of configurations in this computation (that is, $H_M < \omega$ if M eventually stops, and $H_M = \omega$ if it does not). Observe that in $\varkappa_n$ the head cannot be further to the right than the $n+1$st cell.

Now, given a Turing machine M, we define a bimodal formula φ_M as follows. Let

$$A = T \cup (Q \times T).$$

Slightly abusing notation, for every $s \in A$, we introduce a propositional variable s (in particular, we treat $(q, X) \in Q \times T$ as a single variable in this context). Then φ_M is the conjunction of the formulas:

$$\Box_v \Box_h (\mathsf{grid} \leftrightarrow \bigvee_{s \in A} s), \tag{45}$$

$$\Box_v \Box_h \bigwedge_{s \neq s' \in A} \neg (s \wedge s'), \tag{46}$$

$$\Box_v \Box_h (\Box_v \bot \wedge \Box_h \bot \to \pounds), \tag{47}$$

$$\Box_v \Box_h \big(\Diamond_v^{=1} \top \wedge \Diamond_h^{=1} \top \to (q_0, B) \big), \tag{48}$$

$$\Box_v \Box_h (\Diamond_h^{=1} \Diamond_v^{=1} \mathsf{wall} \wedge \Diamond_v \Diamond_v \top \to B), \tag{49}$$

$$\bigwedge_{\substack{\delta(q,X)=(p,\mathsf{L}) \\ Z \in T}} \Box_v \Box_h \Big(\mathsf{grid} \wedge \Diamond_h^{=1} \Diamond_v^{=1} \big((q, X) \wedge \Diamond_h (\mathsf{left} \wedge \Diamond_v Z) \big) \to (p, Z) \Big), \tag{50}$$

$$\bigwedge_{\substack{\delta(q,X)=(p,Y) \\ Y \neq \mathsf{L},\ Z \in T}} \Box_v \Box_h \Big(\mathsf{grid} \wedge \Diamond_h^{=1} \Diamond_v^{=1} \big((q, X) \wedge \Diamond_h (\mathsf{left} \wedge \Diamond_v Z) \big) \to Z \Big), \tag{51}$$

$$\bigwedge_{\substack{\delta(q,X)=(p,Y) \\ Y \in T}} \Box_v \Box_h \Big(\mathsf{grid} \wedge \Diamond_h \big(\Diamond_v^{=1} \mathsf{left} \wedge \Diamond_v (q, X) \big) \to (p, Y) \Big), \tag{52}$$

$$\bigwedge_{\substack{\delta(q,X)=(p,Y)\\ Y\notin T}} \text{⬒}\text{◨}\Big(\mathsf{grid} \wedge \text{⬗}\big(\text{⬘}^{=1}\mathsf{left} \wedge \text{⬘}(q,X)\big) \to X\Big), \tag{53}$$

$$\bigwedge_{\substack{\delta(q,X)=(p,\mathsf{R})\\ Z\in T}} \text{⬒}\text{◨}\Big(\mathsf{grid} \wedge \text{⬗}\Big(\text{⬘}^{=1}\mathsf{left} \wedge \text{⬘}\big(Z \wedge \text{⬗}(\mathsf{left} \wedge \text{⬘}(q,X))\big)\Big) \to (p,Z)\Big), \tag{54}$$

$$\bigwedge_{\substack{\delta(q,X)=(p,Y)\\ Y\neq\mathsf{R},\ Z\in T}} \text{⬒}\text{◨}\Big(\mathsf{grid} \wedge \text{⬗}\Big(\text{⬘}^{=1}\mathsf{left} \wedge \text{⬘}\big(Z \wedge \text{⬗}(\mathsf{left} \wedge \text{⬘}(q,X))\big)\Big) \to Z\Big), \tag{55}$$

$$\bigwedge_{X,Y,Z\in T} \text{⬒}\text{◨}\Big[\mathsf{grid} \wedge \text{⬗}^{=1}\text{⬘}^{=1}\Big(Z \wedge \text{⬗}\Big(\mathsf{left} \wedge \text{⬘}\big(Y \wedge \big(\mathsf{wall} \vee \text{⬗}(\mathsf{left} \wedge \text{⬘}X)\big)\big)\Big)\Big) \to Y\Big]. \tag{56}$$

LEMMA 5. *$\varphi_\infty \wedge \varphi_{grid} \wedge \varphi_M$ is satisfiable in any ∞-chessboard.*

PROOF. Let $\mathfrak{F} = (W, R_1, R_2)$ be an ∞-chessboard with root r. Take the model $\mathfrak{N}$ over $\mathfrak{F}$ defined in the proof of Lemma 3. As is shown there, $(\mathfrak{N}, r) \models \varphi_\infty \wedge \varphi_{grid}$. Define a valuation of the propositional variables $s \in A$ in $\mathfrak{N}$ by taking, for all $x \in W$,

$$(\mathfrak{N}, x) \models s \quad \text{iff} \quad hr^{\mathfrak{N}}(x) = vr^{\mathfrak{N}}(x) = \sharp(i,j) \ \text{ for some } i,j < \omega \\ \text{such that the } i\text{th symbol in } \varkappa_{\min(j,H_M-1)} \text{ is } s. \tag{57}$$

Then it is straightforward to check that $(\mathfrak{N}, r) \models \varphi_M$. ⊣

The next lemma shows that in fact φ_M 'forces' the consecutive configurations $\varkappa_0, \varkappa_1, \dots$ of the computation of M on the consecutive horizontal lines of the $\omega \times H_M$-grid (starting from the line $(0,0), (1,0), (2,0), \dots$):

LEMMA 6. *Suppose that $\mathfrak{M}$ is a model based on a frame $\mathfrak{F} = (W, R_1, R_2)$ for* $[\mathbf{K4}, \mathbf{K4}]$ *with root r. If $(\mathfrak{M}, r) \models \varphi_{grid} \wedge \varphi_M$ then, for all $s \in A$, all $n < \omega$ such that $pair(n) = (i,j)$ and $j < H_M$, and all $x \in V$ such that $hr(x) = vr(x) = n$,*

$$(\mathfrak{M}, x) \models s \quad \textit{iff} \quad \textit{the } i\textit{th symbol of the configuration } \varkappa_j \textit{ is } s. \tag{58}$$

PROOF. As before we use the notation of Section 4. The proof proceeds by induction on n. For $n = 0$, (58) follows from (47) and (46).

Suppose that $n > 0$ is such that $\mathrm{pair}(n) = (i,j)$, $j < H_M$, and (58) holds for all $k < n$. Take an $x \in V$ with $hr(x) = vr(x) = n$. If $\mathrm{pair}(n)$ is on the floor then (58) holds by (48), (49) and (46). So suppose that $\mathrm{pair}(n)$ is not on the floor, that is, $j > 0$. Then $\sharp(i+1, j-1) = n-1$, $\sharp(i, j-1) = \mathit{left}_{n-1}$ and, if $i > 0$, $\sharp(i-1, j-1) = \mathit{left}_{\mathit{left}_{n-1}}$. Let $s_i \in A$ denote the ith symbol of the configuration $\varkappa_{j-1}$. Take a perfect $n \times n$-rectangle $x_{i,j}$ ($i \le n$, $j \le n$) starting at x. By the induction hypothesis we then have

$$(\mathfrak{M}, x_{n-1,n-1}) \models s_{i+1}, \qquad (\mathfrak{M}, x_{\mathit{left}_{n-1},\mathit{left}_{n-1}}) \models s_i \tag{59}$$
$$\text{and, if } i > 0, \quad (\mathfrak{M}, x_{\mathit{left}_{\mathit{left}_{n-1}},\mathit{left}_{\mathit{left}_{n-1}}}) \models s_{i-1}.$$

Let $h < \omega$ be such that the head is scanning the hth cell of κ_{j-1}. There are four cases:

Case 1: $h = i + 1$, that is, $s_{i+1} = (q, X)$ for some $q \in Q$, $X \in T$. Then, by (59), (41), (45), and Lemma 4 (i) and (iv),

$$(\mathfrak{M}, x) \models \mathsf{grid} \wedge \text{⬗}^{=1}\text{⬘}^{=1}\big((q, X) \wedge \text{⬗}(\mathsf{left} \wedge \text{⬘} s_i)\big).$$

Now one can use either (50) and (46), or (51) and (46) (depending on the value of $\delta(q, X)$) to obtain (58), as required.

Case 2: $h = i$. This case is similar to Case 1: we only use (52) or (53) in place of (50) and (51).

Case 3: $h = i - 1$. This time we use (54) or (55).

Case 4: $h \neq i - 1, i, i + 1$. In this case we use (56). $\dashv$

5.3. Encoding tilings. A *tile type* is a 4-tuple of colours

$$t = (\mathit{left}(t), \mathit{right}(t), \mathit{up}(t), \mathit{down}(t)).$$

For a *finite* set Θ of tile types and a subset $X \subseteq \omega \times \omega$, we say that Θ *tiles* X if there exists a function (called a *tiling*) τ from X to Θ such that, for all $(i, j) \in X$,

- if $(i, j + 1) \in X$ then $\mathit{up}(\tau(i, j)) = \mathit{down}(\tau(i, j + 1))$ and
- if $(i + 1, j) \in X$ then $\mathit{right}(\tau(i, j)) = \mathit{left}(\tau(i + 1, j))$.

Given a finite set Θ of tile types, we introduce a propositional variable t, for every $t \in \Theta$. Let φ_Θ be the conjunction of the following formulas:

$$\text{⬒}\text{◨}(\mathsf{grid} \leftrightarrow \bigvee_{t \in \Theta} t), \tag{60}$$

$$\text{⬒}\text{◨} \bigwedge_{t \neq t' \in \Theta} \neg(t \wedge t'), \tag{61}$$

$$\text{⬒}\text{◨} \bigwedge_{\substack{t, t' \in \Theta \\ \mathit{up}(t') \neq \mathit{down}(t)}} \big(t \to \text{◨}(\text{⬘}^{=1}\mathsf{left} \to \neg\text{⬘} t')\big), \tag{62}$$

$$\text{⬒}\text{◨} \bigwedge_{\substack{t, t' \in \Theta \\ \mathit{right}(t') \neq \mathit{left}(t)}} \big(t \to \text{◨}(\mathsf{left} \to \neg\text{⬘} t')\big). \tag{63}$$

Lemma 7. *Suppose that Θ tiles $\omega \times \omega$. Then $\varphi_\infty \wedge \varphi_{grid} \wedge \varphi_\Theta$ is satisfiable in any ∞-chessboard.*

Proof. Let $\mathfrak{F}$ be an ∞-chessboard with root r. Take a model $\mathfrak{N}$ over $\mathfrak{F}$ as in the proof of Lemma 3. Then, as is shown in the proof of Lemma 3, $(\mathfrak{N}, r) \models \varphi_\infty \wedge \varphi_{grid}$ holds.

Fix some tiling $\tau : \omega \times \omega \to \Theta$. Define a valuation of the propositional variables $t \in \Theta$ in $\mathfrak{N}$ by taking, for all $x \in W$,

$$(\mathfrak{N}, x) \models t \quad \text{iff} \quad \mathit{hr}^{\mathfrak{N}}(x) = \mathit{vr}^{\mathfrak{N}}(x) = \sharp(i, j) \ \text{ for some } i, j < \omega \text{ with } \tau(i, j) = t.$$

Then it is straightforward to check that $(\mathfrak{N}, r) \models \varphi_\Theta$. $\dashv$

For every $n < \omega$, let

$$\mathit{plane}_n = \{(i, j) \mid \sharp(i, j) \leq n\}.$$

LEMMA 8. *Suppose that a model $\mathfrak{M}$ is based on a frame for* $[\mathbf{K4}, \mathbf{K4}]$ *with root r and that $(\mathfrak{M}, r) \models \varphi_{grid} \wedge \varphi_{\Theta}$. Then, for every $n < \omega$, every $x \in V$ such that $hr(x) = vr(x) = n$, and every perfect $n \times n$-rectangle $x_{i,j}$ $(i \leq n,\, j \leq n)$ starting at x, the function $\tau : plane_n \to \Theta$ defined by*

$$\tau(i,j) = t \qquad \textit{iff} \qquad (\mathfrak{M}, x_{\sharp(i,j),\sharp(i,j)}) \models t$$

is a tiling of $plane_n$.

PROOF. The proof is by induction on n. For $n = 0$ the statement is obvious. Suppose that $n > 0$ and the statement of the lemma holds for all $k < n$. Take a perfect $n \times n$-rectangle $x_{i,j}$ $(i \leq n,\, j \leq n)$ starting at x. Since $left_n$ (if $pair(n)$ is not on the wall) and $left_{n-1}$ (if $pair(n)$ is not on the floor) are both smaller than n, the statement holds by IH, Lemma 4, (62) and (63). ⊣

5.4. Proofs of Theorems 2–4. We are now in a position to prove the results of Section 3. As we already saw, Theorem 1 is an immediate consequence of Theorems 2 and 3.

PROOF OF THEOREM 2. Item (i), the lack of the fmp, was proved in Section 4. Here we give two different proofs of undecidability, one using Turing machines, and another using tilings.

Let L be as specified in the formulation of Theorem 2. First we reduce the undecidable *non-halting problem for Turing machines* (see, e.g., [21]) to the satisfiability problem for L. To this end, given a Turing machine M, define a formula Φ_M to be the conjunction of the formulas φ_∞, φ_{grid}, φ_M introduced above, and

$$\text{⬒}\text{◨} \bigwedge_{X \in T} \neg(q_1, X). \tag{64}$$

We claim that

Φ_M is L-satisfiable iff M does not stop having started from an all-blank tape.

Suppose first that Φ_M is satisfied in a model $\mathfrak{M}$ for L. As $[\mathbf{K4}, \mathbf{K4}] \subseteq L$ and $[\mathbf{K4}, \mathbf{K4}]$ is Kripke complete, we may assume that the underlying frame of $\mathfrak{M}$ is a frame for $[\mathbf{K4}, \mathbf{K4}]$. Suppose that M eventually stops. Then $H_M < \omega$ and there is $i < \omega$ such that the ith symbol of $\varkappa_{H_M - 1}$ is (q_1, X), for some $X \in T$. Let $n = pair(i, H_M - 1)$. By Lemma 1, there is some $x \in V$ such that $hr(x) = vr(x) = n$. So by Lemma 6, $(\mathfrak{M}, x) \models (q_1, X)$, contrary to (64).

Now suppose that M does not stop having started from an all-blank tape. By assumption, L has an ∞-chessboard $\mathfrak{F}$ with root r among its frames. Take the model $\mathfrak{N}$ over $\mathfrak{F}$ defined in the proof of Lemma 5. As is shown there, $(\mathfrak{N}, r) \models \varphi_\infty \wedge \varphi_{grid} \wedge \varphi_M$. It is straightforward to see that (64) also holds at r in $\mathfrak{N}$.

Our second proof uses tilings. We reduce the following undecidable (see [37, 4]) $\omega \times \omega$*-tiling problem* to the satisfiability problem for L: given a finite set Θ of tile types, decide whether Θ can tile $\omega \times \omega$.

Indeed, using Lemma 8, it is straightforward to show that if $\varphi_\infty \wedge \varphi_{grid} \wedge \varphi_\Theta$ is L-satisfiable then Θ tiles $plane_n$, for all $n < \omega$. A standard compactness argument (or König's lemma) shows that if a given finite set Θ of tile types tiles $plane_n$ for every $n < \omega$, then it actually tiles the whole $\omega \times \omega$-grid.

On the other hand, since L has an ∞-chessboard $\mathfrak{F}$ among its frames, if Θ tiles $\omega \times \omega$, then $\varphi_\infty \wedge \varphi_{grid} \wedge \varphi_\Theta$ is L-satisfiable, by Lemma 7.

Both proofs above show that L must be undecidable. ⊣

Proof of Theorem 3. Now we deal with the logic $\mathsf{Log}\,(\mathcal{C})$ such that $\mathcal{C}$ contains a k-chessboard for every $k < \omega$, but no ∞-chessboard. This time we reduce the (undecidable, but recursively enumerable) *halting problem for Turing machines* to the satisfiability problem for $\mathsf{Log}\,(\mathcal{C})$. To this end, given a Turing machine M, define a formula φ_{fin} in the same way as φ_∞ but with the 'generating' conjuncts (13) and (14) replaced by their 'relativised' versions

$$◨\big(\neg ⬘ \bigvee_{X \in T}(q_1, X) \to ⬘(\neg d \wedge ⬒ d)\big), \tag{65}$$

$$⬒\big(\neg ⬖ \bigvee_{X \in T}(q_1, X) \to ⬖(d \wedge ◨\neg d)\big), \tag{66}$$

and with two extra conjuncts

$$\bigwedge_{X \in T} ⬒\big(⬖(q_1, X) \to ◨(\mathsf{grid} \to (q_1, X))\big), \tag{67}$$

$$\bigwedge_{X \in T} ◨\big(⬘(q_1, X) \to ⬒(\mathsf{grid} \to (q_1, X))\big) \tag{68}$$

added. Let Ψ_M be the conjunction of φ_{fin}, φ_{grid} and φ_M. We claim that

Ψ_M is $\mathsf{Log}\,(\mathcal{C})$-satisfiable iff M stops having started from an all-blank tape.

Suppose first that Ψ_M is satisfied at the root r of a model $\mathfrak{M}$ that is based on a frame $\mathfrak{F} = (W, R_1, R_2)$ from $\mathcal{C}$. Then both R_1 and R_2 are transitive, they commute and are Church–Rosser. Define $\bar{R}_1^{\mathfrak{M}}$ and $\bar{R}_2^{\mathfrak{M}}$ as in (2) and (3), and the horizontal and vertical ranks of points as in (4) and (5). Then **(cb1)** and **(cb2)** are satisfied by (9) and (10), and so $\bar{R}_1^{\mathfrak{M}}$ and $\bar{R}_2^{\mathfrak{M}}$ satisfy (tran), (com) and (chro).

Using (65) and (66), we start to 'generate' the points x_n, u_n and v_n in the same way as in the proof of Lemma 1 (see (20) and Fig. 1). We claim that there is $N < \omega$ such that

$$\text{either} \quad (\mathfrak{M}, u_N) \models ⬘ \bigvee_{X \in T}(q_1, X) \quad \text{or} \quad (\mathfrak{M}, v_N) \models ⬖ \bigvee_{X \in T}(q_1, X). \tag{69}$$

For suppose this is not the case. Then φ_{fin} generates the x_n, u_n and v_n for all $n < \omega$ in the same way as φ_∞ did. So, as the proof of Lemma 1 shows, we have points x_n with $hr^{\mathfrak{M}}(x_n) = vr^{\mathfrak{M}}(x_n) = n$, for every $n < \omega$. Therefore, $\mathfrak{F}$ is an ∞-chessboard, which is a contradiction since $\mathcal{C}$ does not contain such frames.

So let $N < \omega$ be the smallest number such that (69) holds. Suppose, for example, that $(\mathfrak{M}, u_N) \models ⬘(q_1, X)$ for some $X \in T$. (Note that by (45)–(47) and (68), we have $N > 0$.) Then the points $x_0, \dots, x_N$ and $u_0, \dots, u_N$ are generated like in the proof of Lemma 1. As $hr^{\mathfrak{M}}(x_N) = vr^{\mathfrak{M}}(x_N) = N$ by (26), Lemma 4 (i) implies that $(\mathfrak{M}, x_N) \models \mathsf{grid}$. As $u_N \bar{R}_1^{\mathfrak{M}} x_N$, $(\mathfrak{M}, x_N) \models (q_1, X)$ follows by (68). Let $pair(N) = (i, j)$. By Lemma 6, the ith symbol in $\varkappa_j$ is (q_1, X), and so M must stop no later than in j steps. The case when $(\mathfrak{M}, v_N) \models ⬖(q_1, X)$ is similar; we have to use (67) in place of (68).

Now suppose that M stops having started from an all-blank tape, that is, $H_M < \omega$. As we know, L has a k-chessboard $\mathfrak{F}$ with root r among its frames, for some $k \geq H_M$. By Claim 2.1, there is a perfect k-chessboard model $\mathfrak{N}$ based on $\mathfrak{F}$. Define a valuation of the propositional variable d in $\mathfrak{N}$ as in (31). Extend this model to the 'grid' and 'Turing machine variables' as in (39) and (57). Then $(\mathfrak{N}, r) \models \varphi_{grid} \wedge \varphi_M$. A proof similar to that of Lemma 2 shows that $(\mathfrak{N}, r) \models \varphi_{fin}$ also holds. Moreover, it is not hard to see that (67) and (68) hold at r in $\mathfrak{N}$ as well. ⊣

To prove Theorem 4 with the help of Turing machines, one should find a suitable Σ_1^1-hard problem. A *non-deterministic* Turing machine M is called *recurrent* if, having started from the all-blank tape, it has a computation that never halts and reenters the initial state q_0 infinitely often. It is known (see, e.g., [19]) that the problem 'given a non-deterministic Turing machine M, decide whether it is recurrent' is Σ_1^1-complete. By appropriately modifying the formulas above, it is not difficult to reduce this problem to the satisfiability problems for the logics mentioned in Theorem 4. However, the formulas become even more complex than before, so below we give a (more transparent) proof with the help of a recurrent tiling problem instead.

Proof of Theorem 4. The following *recurrent* tiling problem is known to be Σ_1^1-complete [17]: given a finite set Θ of tile types and a $t_0 \in \Theta$, decide whether Θ tiles the $\omega \times \omega$-grid in such a way that t_0 occurs infinitely often on the wall.

So suppose that Θ and some $t_0 \in \Theta$ are given. Define Ψ_{Θ,t_0} to be the conjunction of φ_∞, φ_{grid}, φ_Θ, and the formulas

$$◨⬘\mathsf{recc}, \tag{70}$$

$$⬒◨(\mathsf{recc} \to \neg ⬗\mathsf{grid}), \tag{71}$$

$$⬒\big(⬗\mathsf{recc} \to ⬗(\mathsf{wall} \wedge t_0)\big), \tag{72}$$

$$\bigwedge_{t \in \Theta} ⬒\big(⬗ t \to ◨(\mathsf{grid} \to t)\big), \tag{73}$$

$$\bigwedge_{t \in \Theta} ◨\big(⬘ t \to ⬒(\mathsf{grid} \to t)\big). \tag{74}$$

Now let L be as specified in the formulation of the theorem. We claim that

$$\Psi_{\Theta,t_0} \text{ is } L\text{-satisfiable} \quad \text{iff} \quad \Theta \text{ tiles } \omega \times \omega \text{ with } t_0 \text{ occurring infinitely often on the wall.} \tag{75}$$

Suppose first that Ψ_{Θ,t_0} is satisfied at the root r of a model $\mathfrak{M}$ for L. Since L is Kripke complete, we may assume that $\mathfrak{M}$ is based on a frame $\mathfrak{F} = (W, R_1, R_2)$ for L. In particular, $\mathfrak{F}$ is a frame for $[\mathbf{K4}, \mathbf{DisK4.3}]$. Then both R_1 and R_2 are transitive, they commute and are Church–Rosser. We also know that R_2 is weakly connected and satisfies (7). Define the relations $\bar{R}_1 = \bar{R}_1^{\mathfrak{M}}$ and $\bar{R}_2 = \bar{R}_2^{\mathfrak{M}}$ as in (2) and (3). Then they satisfy (tran), (com) and (chro). Moreover, since $\bar{R}_2 \subseteq R_2$ and R_2 satisfies (7), $\bar{R}_2$ satisfies (7) as well.

Note that $\bar{R}_2$ is not necessarily weakly connected. However, it always has the following property:

CLAIM 4.1. *For all $x, y, z \in W$, if $x\bar{R}_2 y$, $x\bar{R}_2 z$ and $vr(y) > vr(z)$ then $y\bar{R}_2 z$.*

PROOF OF CLAIM 4.1. Clearly, it is enough to show that if x, y, z are such that $x\bar{R}_2 y$ and $x\bar{R}_2 z$ but neither $z\bar{R}_2 y$ nor $y\bar{R}_2 z$ hold, then $vr(y) = vr(z)$. This statement is an immediate consequence of the following property:

$$\forall xyz\ \big(x\bar{R}_2 y \wedge x\bar{R}_2 z \wedge \neg y\bar{R}_2 z \wedge \neg z\bar{R}_2 y \longrightarrow \forall w\ (y\bar{R}_2 w \leftrightarrow z\bar{R}_2 w)\big).$$

The case of $y = z$ is obvious. So suppose $y \neq z$. Since $x\bar{R}_2 y$ and $x\bar{R}_2 z$, but the points y and z do not $\bar{R}_2$-see each other, they must have the same 'vertical colour,' say, v. Now suppose that $y\bar{R}_2 w$. Then there is some u with vertical colour $\neg v$ such that $yR_2 u$ and either $u = w$ or $uR_2 w$. Since R_2 is weakly connected, we have either $yR_2 z$ or $zR_2 y$. If $zR_2 y$ then $z\bar{R}_2 w$ follows by transitivity. So suppose $yR_2 z$. Then either $uR_2 z$ or $zR_2 u$. We cannot have $uR_2 z$, because $y\bar{R}_2 z$ does not hold, so $zR_2 u$. Therefore, $z\bar{R}_2 w$. ⊣

Our next observation is that, for every $x \in W$,

$$\text{if } vr(x) \neq 0 \text{ then there is } u \text{ such that } x\bar{R}_2 u \text{ and } vr(u) = 0. \tag{76}$$

Indeed, if $x = r$ then (76) follows from (11) and (chro). If $r\bar{R}_1 x$ then take a u with $r\bar{R}_2 u$ and $vr(u) = 0$, which gives (76) by (chro). If $r\bar{R}_2 x$ then take a u with $r\bar{R}_2 u$ and $vr(u) = 0$. Then we have $x\bar{R}_2 u$ by Claim 4.1. Finally, if $r\bar{R}_1 z\bar{R}_2 x$ for some z then take a u with $z\bar{R}_2 u$ and $vr(u) = 0$. Then again $x\bar{R}_2 u$ follows from Claim 4.1.

Next, we show that

$$\bar{R}_2 \text{ is irreflexive.} \tag{77}$$

Suppose otherwise, that is, there is $x \in W$ with $x\bar{R}_2 x$. Then there is y such that $xR_2 yR_2 x$ and the 'v-colours' of x and y are different, i.e., $x \neq y$. By (76), there is u such that $x\bar{R}_2 u$ and $vr(u) = 0$, and so $uR_2 x$ cannot hold. But then we arrive to a contradiction with the property (7) of R_2 because $xR_2 yR_2 xR_2 \dots R_2 u$.

CLAIM 4.2. *For every $x \in V$ with $(\mathfrak{M}, x) \models \mathsf{grid}$, there is $n < \omega$ such that $vr(x) = n$.*

PROOF OF CLAIM 4.2. If $(\mathfrak{M}, x) \models \boxminus\bot$ then the claim holds by (32). Now let $x_0 = x$. Starting from x_0, we construct a sequence $x_0, x_1, \dots$ as follows. Suppose that $(\mathfrak{M}, x_n) \not\models \boxminus\bot$. Then, by (36), we have $(\mathfrak{M}, x_n) \models \Diamond^{=1}\Diamond^{=1}\mathsf{grid}$, and so there are points y_{n+1} and x_{n+1} such that

- $x_n\bar{R}_2 y_{n+1}\bar{R}_1 x_{n+1}$,
- there is no point z such that $x_n\bar{R}_2 z\bar{R}_2 y_{n+1}$,
- $(\mathfrak{M}, x_{n+1}) \models \mathsf{grid}$.

Moreover, if we let $u_0 = x_0$ and $u_1 = y_1$ and use (com) then, for each $n > 0$ such that $(\mathfrak{M}, x_n) \not\models \boxminus\bot$, we have points u_n such that $u_n\bar{R}_2 u_{n+1}\bar{R}_1 y_{n+1}$. We claim that there is some $n < \omega$ such that $(\mathfrak{M}, x_n) \models \boxminus\bot$. Suppose otherwise. Then we have the points u_n for all $n < \omega$. By (77), $u_n \neq u_{n+1}$ for all $n < \omega$. By (76), there is u_∞ such that $x_0\bar{R}_2 u_\infty$ and $vr(u_\infty) = 0$. So, by Claim 4.1 and the fact that $x_0\bar{R}_2 u_n$, we have $u_n\bar{R}_2 u_\infty$ for all $n < \omega$. But this is impossible in view of the property (7) of $\bar{R}_2$.

So let $n < \omega$ be such that $(\mathfrak{M}, x_n) \models ⬒\bot$ holds. Then $(\mathfrak{M}, x_n) \models ◨\bot$ follows from (32), and so $\mathit{vr}(x_n) = 0$. We claim that for all $i \leq n$,

$$\mathit{vr}(x_{n-i}) = i.$$

The proof is by induction on i. The basis of induction has been shown above. So suppose that our claim holds for every j with $j < i \leq n$, and take x_{n-i}. Then $x_{n-i}\bar{R}_2 y_{n-i+1}\bar{R}_1 x_{n-i+1}$. By IH, $\mathit{vr}(x_{n-i+1}) = i-1$, and so $\mathit{vr}(y_{n-i+1}) = i-1$ as well. Suppose that $\mathit{vr}(x_{n-i}) > i$. Then there is w such that $x_{n-i}\bar{R}_2 w$ and $\mathit{vr}(w) > i-1$. So, by Claim 4.1, $w\bar{R}_2 y_{n-i+1}$. Since there is no point z such that $x_{n-i}\bar{R}_2 z \bar{R}_2 y_{n-i+1}$, we arrive to a contradiction. Therefore, $\mathit{vr}(x_{n-i}) = i$. $\dashv$

Claim 4.3. *For every $n < \omega$ there exist $m \geq n$, $m < \omega$, and $x \in V$ such that $\mathit{hr}(x) = \mathit{vr}(x) = m$ and $(\mathfrak{M}, x) \models \mathsf{wall} \wedge t_0$.*

Proof of Claim 4.3. Fix an $n < \omega$. Since $(\mathfrak{M}, r) \models \varphi_\infty$, there exists u_n such that $r\bar{R}_2 u_n$ and $\mathit{vr}(u_n) = n$ (see **(gen3)** and (24) in the proof of Lemma 1). By (70), there is w such that $u_n\bar{R}_1 w$ and $(\mathfrak{M}, w) \models \mathsf{recc}$. So $\mathit{vr}(w) = n$ as well. By (com), there is v such that $r\bar{R}_1 v \bar{R}_2 w$. By (72), there is z such that $v\bar{R}_2 z$ and $(\mathfrak{M}, z) \models \mathsf{wall} \wedge t_0$. Then, by (60), $(\mathfrak{M}, z) \models \mathsf{grid}$. So, by Claim 4.2, we have $\mathit{vr}(z) = m$ for some $m < \omega$. By (71), $w\bar{R}_2 z$ cannot hold. So it follows from Claim 4.1 that $m = \mathit{vr}(z) \geq \mathit{vr}(w) = n$.

We can show now that there exists $x \in V$ such that $\mathit{hr}(x) = \mathit{vr}(x) = m$ and $(\mathfrak{M}, x) \models \mathsf{wall} \wedge t_0$. By (com), there is u such that $r\bar{R}_2 u \bar{R}_1 z$ and $\mathit{vr}(u) = m$. In view of **(gen4)** and (25), there is a point v_m such that $r\bar{R}_1 v_m$ and $\mathit{hr}(v_m) = m$. By (chro), there is x such that $u\bar{R}_1 x$ and $v_m\bar{R}_2 x$, and so $\mathit{hr}(x) = \mathit{vr}(x) = m$. Finally, we obtain $(\mathfrak{M}, x) \models \mathsf{grid}$ by Lemma 4 (i), and $(\mathfrak{M}, x) \models \mathsf{wall} \wedge t_0$ by (35) and (74). $\dashv$

Claim 4.4. *For all $n < m < \omega$ and $x \in V$ with $\mathit{hr}(x) = \mathit{vr}(x) = n$, and for every perfect $n \times n$-rectangle $x_{i,j}$ $(i, j \leq n)$ starting at x, there exist a $y \in V$ with $\mathit{hr}(y) = \mathit{vr}(y) = m$ and a perfect $m \times m$-rectangle $y_{i,j}$ $(i, j \leq m)$ starting at y such that,*

$$\text{for every } i \leq n \text{ and every } t \in \Theta,\ \ (\mathfrak{M}, x_{i,i}) \models t \quad \text{iff} \quad (\mathfrak{M}, y_{i,i}) \models t. \tag{78}$$

Proof of Claim 4.4. Take some $n < m < \omega$, x and a perfect rectangle starting at x as specified above. Let u be such that $r\bar{R}_2 u \bar{R}_1 x$. Then $\mathit{vr}(u) = n$. By Lemma 1, there are points u_m and x_m such that $r\bar{R}_2 u_m \bar{R}_1 x_m$ and $\mathit{vr}(u_m) = \mathit{vr}(x_m) = \mathit{hr}(x_m) = m$. So there are points $u_{m-1}, u_{m-2}, \dots, u_{n+1}$ such that $\mathit{vr}(u_i) = i$ and

$$u_m \bar{R}_2 u_{m-1} \bar{R}_2 u_{m-2} \bar{R}_2 \dots \bar{R}_2 u_{n+1},$$

and points $y_{m-1,m}, y_{m-2,m}, \dots, y_{n,m}$ such that $\mathit{hr}(y_{i,m}) = i$ and

$$x_m \bar{R}_1 y_{m-1,m} \bar{R}_1 y_{m-2,m} \bar{R}_1 \dots \bar{R}_1 y_{n,m}.$$

By Claim 4.1, we also have $u_{n+1}\bar{R}_2 u$. By (chro), there are points $y_{n,m-1}, y_{n,m-2}, \dots, y_{n,n}$ such that $\mathit{vr}(y_{n,i}) = i$ and

$$y_{n,m} \bar{R}_2 y_{n,m-1} \bar{R}_2 y_{n,m-2} \bar{R}_2 \dots \bar{R}_2 y_{n,n}$$

and $u\bar{R}_1 y_{n,n}$.

We claim that if we choose y to be x_m and take any perfect $m \times m$-rectangle starting at y that contains the points $y_{i,m}$ $(m-1 \leq i \leq n)$ and $y_{n,j}$ $(m-1 \leq j \leq n)$ above, then (78) is satisfied. Indeed, first let $i = n$. Then $(\mathfrak{M}, y_{n,n}) \models \mathsf{grid}$ by Lemma 4 (i), and so (78) holds by $x = x_{n,n}$, $u\bar{R}_2x$, $u\bar{R}_2y_{n,n}$ and (74). Now fix some $i < n$, and suppose that, say, $(\mathfrak{M}, x_{i,i}) \models t$, for some $t \in \Theta$. Then $x_{n,n}\bar{R}_1x_{i,n}\bar{R}_2x_{i,i}$ and $y_{n,n}\bar{R}_1y_{i,n}\bar{R}_2y_{i,i}$. By (com), there are u_x and u_y such that $u\bar{R}_2u_x$, $u\bar{R}_2u_y$, $u_x\bar{R}_1x_{i,i}$, $u_y\bar{R}_1y_{i,i}$, and so $vr(u_x) = vr(u_y) = i$. By (chro), there is w such that $u_x\bar{R}_1w$ and $y_{i,n}\bar{R}_2w$, so $hr(w) = vr(w) = i$. By Lemma 4 (i), we have $(\mathfrak{M}, w) \models \mathsf{grid}$. Then $(\mathfrak{M}, y_{i,i}) \models t$ follows from (73), (74) and Lemma 4 (i). ⊣

Claims 4.3, 4.4 and Lemma 8 imply, with the help of König's lemma, that there is a tiling of $\omega \times \omega$ with t_0 occurring infinitely often on the wall, as required.

Now let us prove the '$\Leftarrow$' direction of (75). Take a recurrent tiling of $\omega \times \omega$. By assumption, L has an ∞-chessboard $\mathfrak{F}$ with root r among its frames. Define a model $\mathfrak{N}$ over $\mathfrak{F}$ as in the proof of Lemma 7. As is shown in that proof, $(\mathfrak{N}, r) \models \varphi_\infty \wedge \varphi_{grid} \wedge \varphi_\Theta$. Then, for all x in $\mathfrak{F}$, define

$$(\mathfrak{N}, x) \models \mathsf{recc} \quad \text{iff} \quad \text{there is } z \text{ such that } (\mathfrak{N}, z) \models \mathsf{wall} \wedge t_0 \text{ and either } x = z \text{ or } z\bar{R}_2^{\mathfrak{N}}x.$$

It is not hard to see that (70)–(74) are also satisfied at r in $\mathfrak{N}$. ⊣

Proof of Corollary 4.1. Let L_1, L_2 and L be as specified in the formulation of the corollary. Then we know that L has a frame that is a product of two rooted linear orders each of which contains an infinite descending chain of distinct points.

We show that such a frame is an ∞-chessboard. Let $\mathfrak{F}_1 = (W_1, <_1)$ and $\mathfrak{F}_2 = (W_2, <_2)$ be two rooted linear orders with infinite descending chains

$$x_0 \gneq_1 x_1 \gneq_1 x_2 \gneq_1 \dots \qquad \text{and} \qquad y_0 \gneq_2 y_1 \gneq_2 y_2 \gneq_2 \dots$$

of points from W_1 and W_2, respectively. Define a valuation $\mathfrak{V}$ in $\mathfrak{F}_1 \times \mathfrak{F}_2$ by taking:

$$\begin{aligned} \mathfrak{V}(h) &= \{(x,y) \mid x_0 \leq_1 x\} \cup \{(x,y) \mid x_n \leq_1 x <_1 x_{n-1},\ 0 < n < \omega,\ n \text{ is even}\}, \\ \mathfrak{V}(v) &= \{(x,y) \mid y_0 \leq_2 y\} \cup \{(x,y) \mid y_n \leq_2 x <_2 y_{n-1},\ 0 < n < \omega,\ n \text{ is even}\}, \end{aligned}$$

and let $\mathfrak{M} = (\mathfrak{F}_1 \times \mathfrak{F}_2, \mathfrak{V})$. It is not hard to see that for all (x, y) in $\mathfrak{F}_1 \times \mathfrak{F}_2$, and for all $n < \omega$,

$$\begin{aligned} hr^{\mathfrak{M}}(x,y) = n \quad &\text{iff} \quad \text{either } n = 0 \text{ and } x_0 \leq_1 x, \text{ or } x_n \leq_1 x <_1 x_{n-1}, \\ vr^{\mathfrak{M}}(x,y) = n \quad &\text{iff} \quad \text{either } n = 0 \text{ and } y_0 \leq_2 y, \text{ or } y_n \leq_2 y <_2 y_{n-1}. \end{aligned}$$

It follows that $\mathfrak{F}_1 \times \mathfrak{F}_2$ is an ∞-chessboard. Therefore, by Theorem 4, L is Π^1_1-hard.

For the Π^1_1 upper bound, it is readily seen by a step-by-step argument that, for each of the listed pairs L_1 and L_2, their product $L_1 \times L_2$ is determined by products of countable L_1- and L_2-frames. Now, a Kripke model $\mathfrak{M}$ over such a frame can be selected with universal second-order quantification. Once $\mathfrak{M}$ is selected, the check that $\mathfrak{M} \models \varphi$ is first-order. ⊣

§6. Discussion. We conclude this paper with a few remarks on related results and further research.

The undecidability theorems presented in this paper are optimal in the sense that all 'natural' logics containing $[\mathbf{K4}, \mathbf{K4}]$ and having no frames of arbitrary finite or infinite depth are in fact decidable. They are not optimal, however, in the sense that

(a) some logics determined by products of linear frames are known to be of higher complexity than it follows from the results of this paper, and
(b) for some of the discussed logics the exact complexity is still unknown.

Let us first discuss (a). Interesting examples are the logics

$$\mathsf{Log}\,((\omega,<)\times(\omega,<)) \qquad \text{and} \qquad \mathsf{Log}\,\{(\omega,<)\times\mathfrak{F} \mid \mathfrak{F}\models \mathbf{K4.3}\}$$

which are shown to be Π^1_1-hard in [35, 31] and [11, Theorem 7.12]. (In the context of this paper these logics are covered by Theorem 3 which 'only' shows that they are not recursively enumerable.) Note that the Π^1_1-complete logics

$$\mathsf{Log}\,(\omega,<)\times\mathsf{Log}\,(\omega,<) \qquad \text{and} \qquad \mathsf{Log}\,((\omega,<)\times(\omega,<))$$

are different because, for instance, $(\{\infty\}\cup\mathbb{Z},>)\times(\{\infty\}\cup\mathbb{Z},>)$ is a frame for $\mathsf{Log}\,(\omega,<)\times\mathsf{Log}\,(\omega,<)$ and it is an ∞-chessboard satisfying φ_∞, while the frame $(\omega,<)\times(\omega,<)$ is not an ∞-chessboard, and so φ_∞ is not $\mathsf{Log}\,((\omega,<)\times(\omega,<))$-satisfiable. The proofs of Π^1_1-hardness of these logics uses the same enumeration of the $\omega\times\omega$ grid as in Section 5.1. The difference is that if both components are linear then one can also write a formula that generates the diagonal 'forwards,' as opposed to our φ_∞ that does it 'backwards.' For more examples and details the reader is referred to [11].

As concerns (b), we note first that we have obtained Π^1_1-completeness results only for 'transitive' products where one component is a 'linear discrete' modal logic. The exact complexity of undecidable product logics like $\mathbf{K4}\times\mathbf{GL}$ or $\mathbf{Grz}\times\mathbf{Grz}$ remains unknown. However, we conjecture that there are logics of much higher complexity than Π^1_1 satisfying the conditions of Theorem 4 and that this can be proved using the technique of Thomason [36].

Because of the extremely high computational complexity of product logics, an interesting and promising direction of research is to consider various relativisations of the product construction. In the extreme, when arbitrary relativisations are allowed, we may end up with the fusion of the combined modal logics [26]. On the other hand, it is shown in [16] that 'expanding domain' relativisations of product logics with transitive frames can be decidable, though not in primitive recursive time. In particular, bimodal logics interpreted in two-dimensional structures are decidable, if one component—call it the flow of time—is a finite linear order (or a finite transitive tree) and the other component is composed of transitive trees (or partial orders/quasi-orders/finite linear orders) expanding over the time. As we saw in this paper, none of these logics is decidable when interpreted in models with constant domains. Further, [23] presents an investigation of expanding domain relativisations along $(\omega,<)$ of products with $\mathsf{Log}\,(\omega,<)$ and shows that, for example, the expanding domain relativisations of

$\mathsf{Log}\,(\omega, <) \times \mathbf{K4}$, $\mathsf{Log}\,(\omega, <) \times \mathbf{S4}$ and $\mathsf{Log}\,(\omega, <) \times \mathbf{S4.3}$ are undecidable. It remains open whether the expanding domain relativisations of products of 'branching or non-discrete transitive' logics like $\mathbf{S4} \times \mathbf{S4}$ or $\mathbf{S4.3} \times \mathbf{K4}$ are decidable.

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