Class Notes
CIS 502 Analysis of Algorihtm
3-Graph Traversal

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1 Graphics Basics

- A graph G is an ordered pair of two sets (V, E).
- V is a set of vertices/points/nodes, which is always a finite set.
- E is a set of unordered pair of vertices.
- An edge is represented as (u, v). Here we abuse the notion of ordered pair to represent unordered pair.

1.1 Two representation of Graph

When we talked about graph without adjective, that mean it is a undirected graph. Suppose the number of vertices is $|V| = n$.

- A vertex is incident to an edge if the vertex is one of the two vertices the edge connects.
- If an edge (u, v) has end points u and v, we say it is an incident to vertex u and v.
- u, v are adjacent if $(u, v) \in \mathbb{E}$.
- The degree of vertex v is the number of edges incident on v.

1.1.1 Adjacency Matrix

Adjacency Matrix is an sysmetric matrix for undirected graph where

$$V_{ij} = \begin{cases} |(i,j)| & \text{, if } (i,j) \in \mathbb{E}. \\ 0 & \text{, otherwise.} \end{cases}$$

1.1.2 Adjacency List

Adjacency List is an array of size n of linked list, where i-th entry is a linked list consisting of the neighbos of vertex-i. It is default representation of graph.

Space = $O(n + m)$

1.2 Connectivity

1.2.1 Path

A path in a graph is a sequence of vertices

$$v_0 v_1 \cdots v_k$$

, such that $(v_i, v_{i+1}) \in \mathbb{E}$ for $i = 0, 1, 2, \cdots, k-1$ A simple path is a path that does not repeat vertices.

Lemma: If there is a path (u, v), there must be a simple path (u, v).

1.2.2 Cycle

A cycle in a graph is a sequence of vertices

$$v_0 v_1 \cdots v_k v_0$$

, such that $(v_i, v_{i+1}) \in \mathbb{E}$ for $i = 0, 1, 2, \cdots, k-1$ and $(v_k, v_0) \in \mathbb{E}$. All v_is are distinct.

1.2.3 Connectivity

- u, v is **connected** if there is a path between them.
- G is **connected** if $\forall u, v \in V$, there is a path between u and v.
- The **connected components** of G are maximal subset of vertices that are pairwise connected.

1.2.4 Connection is equivalence relation

Connection relation in a graph is an equivalence relation.

- Reflexive Relation (take Path of length 0)
- Symmetric Relation (reversible path)
- Transitive Relation: If a Graph has a uv path and also vw path then it will also contain uw path.

Because connnection is the equivalence relation, pairwise connected vertices form a connected component.

2 Tree

Tree is a connected acyclic graph.

2.1 Rooted tree

2.1.1 Inductive Defintion

A nice thing about Inductive defintion is it is useful for the proofs by induction.

- **Rule 1:** A graph consist of a single vertex v is a rooted tree with v as the root.
- **Rule 2:** If (T_1, r_1), (T_2, r_2), $\cdot$, (T_k, r_k) are rooted trees, then the tree (T, r) consisting of a new node r as root and edges $(r, r_1), \cdots, (r, r_k)$ is a rooted tree.

2.2 Structural induction Proof

Statement: Any tree with n nodes has $n - 1$ edges.

Since any tree can be transformed into a rooted tree, the induction can be as following:

- Statement: Any rooted tree on n nodes has $n-1$ edges.
- Base case: Single node tree with no edge. The statement is true.
- Inductive hypothesis: For a rooted tree T_r, built up from $(T_1, r_1), (T_2, r_2), \cdots, (T_n, r_n)$ using rule 2. Assume the statement is true for all the trees $T_1, T_2, \cdots, T_k$ and prove it for t.
- Inductive step:
  - Let tree T_i have n_i nodes, $i = 1, 2, \cdots, k$. Then T has $\sum_{i=1}^{k} n_i + 1$ nodes.
  - By the inductive hypothesis, T_i has $n_i - 1$ edges.
  - Total number of edges is $T = \sum_{i=1}^{k}(n_i - 1) + k = \sum_{i=1}^{k} n_i$.
  - The number of edge is one less than the number of nodes. It proofs the inductive step.

3 Traversal

Traversal: Visiting all parts of the graph.

3.1 Traversal rooted tree

3.1.1 Post-order traversal

- First traverse each of the children
- Visit the root.