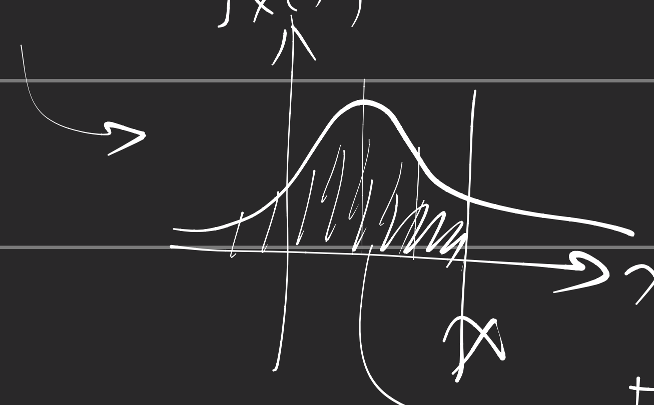
Chapter 5

PDF CDF

$f_X(x) \to \boxed{F_X(x)} = P(X \le x) = \int_{-\infty}^{x} f_X(x)\,dx$

$f_X(x)$



$F_X(x)$

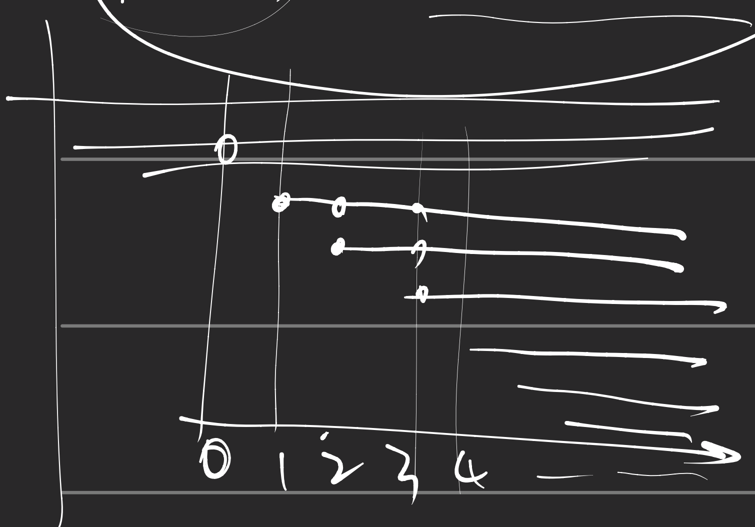
$E(X) = \int_{-\infty}^{+\infty} x f_X(x)\,dx$

$E(N) = \sum n \cdot P(N=n)$ 定义

Survival Function

$G(n) = P(N > n)$

$E(X) = \sum G(n) \quad (n \ge 0)$



$E(X) = \int_0^{+\infty} G(x)\,dx$

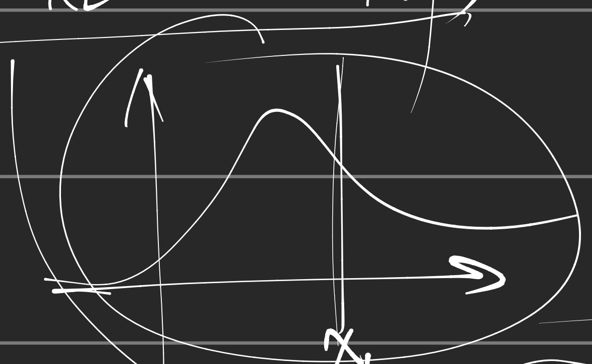
LOTUS

$E(g(N)) = \sum g(n) P(N=n)$

$E(g(x)) = \int_{-\infty}^{+\infty} g(x) f_X(x)\,dx$

$X_1 \,--\, X_n$ i.i.d. $X<Y<Z$

$P(X_1 < X_2 -- < X_n)$ $P(X_1 = X_2) > 0$

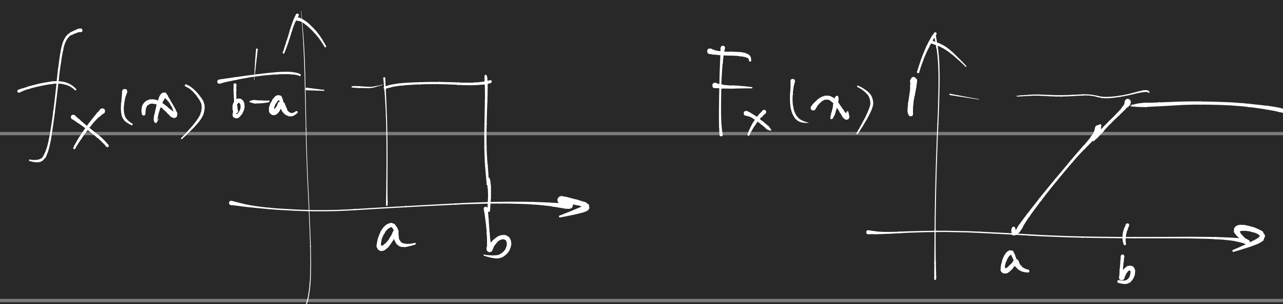


$x_2 = x_1, = 0$

$P(X_1 = X_2) = 0 \Rightarrow P(X_1, X_2 -- X_n) = 1$

互不相等 $\frac{1}{n!}$

$f(x) =$ $X \sim Unif(a,b)$



$X_1 -- X_n$ i.i.d $\sim Unif(0,1)$

$Y = \min(X_1, --, X_n)$ $E(Y)$

$E(Y) = \int_{-\infty}^{+\infty} y\, \boxed{f_Y(y)}\,dy.$

$Y = \min(\quad)$

$E(Y) = \int_0^{+\infty} \boxed{G(y)}\,dy$

$\hookrightarrow P(Y > y)$

$\min(X_1 -- X_n) > y.$

$P(X_1, X_2 -- X_n > y)$ X_i

$= P(X_1 > y)\, P(X_2 > y) -- P(X_n > y)$

$= p^n(X_1 > y)$

$X \sim Pois(\lambda)$ λ

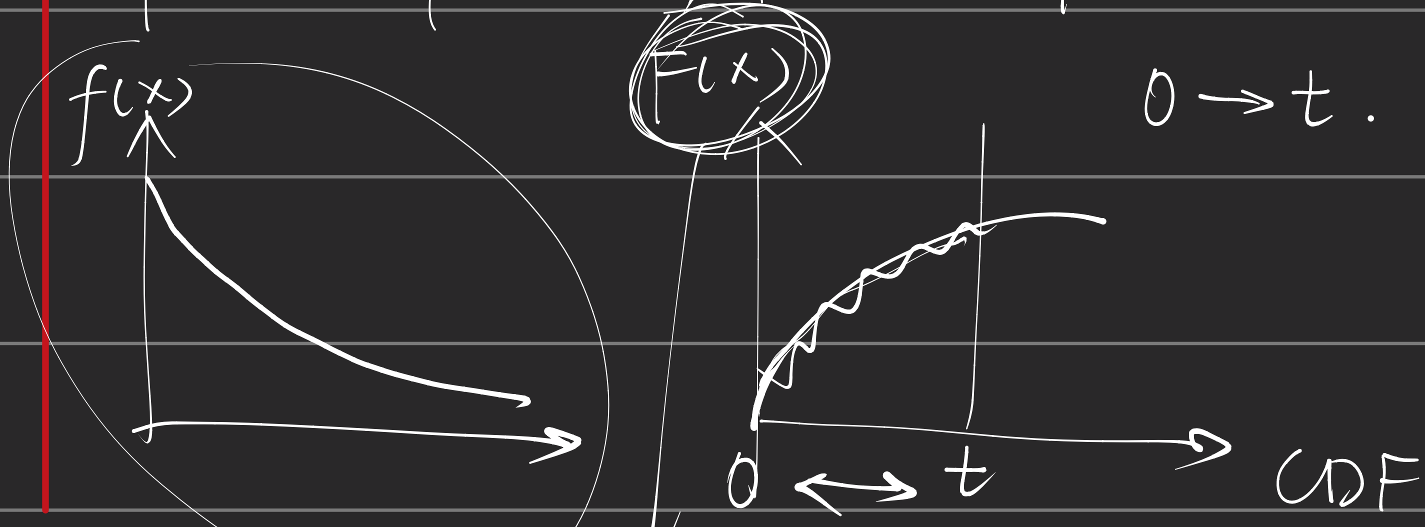
$X \sim Expo(\lambda) \Rightarrow \lambda$

$t \quad t\lambda$

$f(x) = \lambda e^{-\lambda x}, x > 0$

$F(x) = 1 - e^{-\lambda x}, x > 0$ $F(x)$

$0 \to t.$



$\lambda.$ $=1$ $\boxed{X}$ FS.

无记忆性.

$P(X \geq s+t \mid X \geq s) = P(X \geq t)$

$s+t \quad s$

t

$r(t) = \lim_{dt \to 0} \frac{P[X \in (t, t+dt) \mid X \geq t]}{dt} \to dt \cdot f(t)$

$= \lim_{dt \to 0} \frac{1}{dt} \frac{P[X \in (t, t+dt)]}{P(X \geq t)} \to 1 - F(t)$

$r(t) = \frac{f(t)}{1-F(t)} \quad f(t)$

$r(t) = \lambda \qquad t. \to \lambda \quad f(t)$

$\lambda \qquad X_1 \ \lambda_1, X_2 \lambda_2$

$X_1, \dots, X_n$ 独立 L

$X_i \sim Expo(\lambda_j)$

$L = \min(X_1, \dots, X_n)$ CDF PDF

$L \sim Expo(\lambda_1 + \dots + \lambda_n)$ CDF

$F_{X_i}(x) = 1 - e^{-\lambda_i x}$

$F_L(l) = P(L \leq l)$

$= P(\min(\ ) \leq l) = 1 - P(\min(\ ) \geq l)$

$= 1 - P(X_1 \geq l) P(X_2 \geq l) \dots$

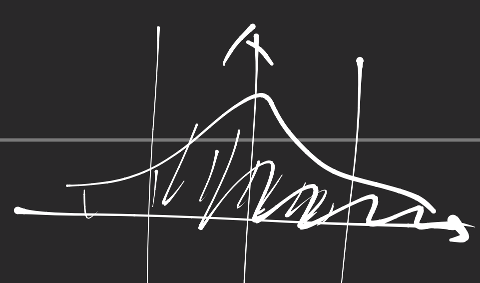
$(1 - F_{X_1}(x))(\quad)(\quad)$

$=$

$\varphi(z) \to N(0,1) \qquad \phi(z) \to N(0,1)$

$\varphi(z) = \varphi(-z)$

$\phi(z) = 1 - \phi(-z)$



$Z \sim N(0, \sigma)$

$-Z \sim N(0, \sigma)$

$Z \sim N(0,1) \to \mu = 0 \quad \sigma^2 = 1$

$\to N(\mu, \sigma^2) \to \sigma, \to \mu$

$X = \mu + \sigma Z$

$F_X(x) = P(X \leq x)$

$= P(\mu + \sigma Z \leq x) = P\left(Z \leq \frac{x-\mu}{\sigma}\right)$

$= \phi\left(\frac{x-\mu}{\sigma}\right)$

$f_X(x) = \varphi\left(\frac{x-\mu}{\sigma}\right) \frac{1}{\sigma}$

$X_1 \dots X_n$ i.i.d $\mu \ \sigma$

$\overline{X_n} = \frac{1}{n} \sum X_i \qquad n \to \infty \qquad nE(X_i)$

$E(\overline{X_n}) = E\left(\frac{1}{n} \sum X_i\right) = \frac{1}{n} E\left(\sum X_i\right) = \mu$

$Var(\overline{X_n}) = \frac{1}{n^2} Var\left(\sum X_i\right) \to n Var(X_i)$

$Var(X+Y) = Var(X) + Var(Y)$

$= \frac{1}{n} Var(X_i)$

$\sigma = \frac{1}{\sqrt{n}} \sigma \to N\left(\mu, \frac{1}{n} \sigma^2\right)$

$Z = N(0,1)$

$\boxed{\mu + \frac{\sigma}{\sqrt{n}} Z}$

$\overline{X_n} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

$n\overline{X_n} \sim N(n\mu, n\sigma^2)$

$\to \sum X_i \qquad n \to \infty$

$Y \sim Bin(n, p) \qquad n \to \infty$

$\to Y_1 + Y_2 + \dots + Y_n$

$nY_1 \qquad Y_i \sim Bern(p)$

μ, σ

矩母函数.

$\boxed{E(e^{tX})} = \boxed{\sum e^{tx} P(X=x)}$

离散 连续 $= \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx$

$X \sim Bern(p)$

$\boxed{M(t)} = E(e^{tX}) = e^{t0}P(X=0) + e^{t1}P(X=1)$

$= 1-p + pe^t$

$U \sim Unif(a,b)$

$M(t) = E(e^{tU}) = \int_a^b \frac{1}{b-a} e^{tu}\,du$

$X+Y$. 独立 多元独立变量

$\boxed{M_{X+Y}(t) = M_X(t) M_Y(t)}$

$= E[e^{t(X+Y)}] = E[e^{tX} \cdot e^{tY}]$

$= E(e^{tX}) E(e^{tY})$ $\quad E(e^{tX})$

$\hookrightarrow M_X(t) \quad \hookrightarrow M_Y(t) \quad = e^{tE(X)}$

$Bin(n,p) = \quad X = \sum X_i$

$N(0,1)$

$\boxed{Y = a+bX} \to N(\mu, \sigma^2)$

$M_Y(t) = E(e^{t(a+bX)}) = E(e^{at} e^{btX})$

$= e^{at} E(e^{btX})$

$\hookrightarrow M_X(bt)$

$Z \sim N(0,1) \quad M_Z(t) = e^{\frac{1}{2}t^2}$

$X = \mu + \sigma Z \quad M_X(t) = e^{\mu t} M_Z(\sigma t)$

$X_1 \sim N(\mu_1, \sigma_1)$

$X_2 \sim N(\mu_2, \sigma_2)$

$X_1 + X_2$

$M_{X_1+X_2}(t) = M_{X_1}(t) M_{X_2}(t)$

$e^{\mu t} e^{\frac{1}{2}\sigma^2 t^2}$

$M_{X_1+X_2}(t) = e^{(\mu_1+\mu_2)t} e^{\frac{1}{2}(\sigma_1^2+\sigma_2^2)t^2}$

$N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$

$= \sqrt{\sigma_1^2+\sigma_2^2}$

$$\boxed{E(X^n) = M^{(n)}(0)}$$

MGF

XY. $g(X,Y)$ Hint(———)

$|E(XY)| \le \sqrt{E(X^2)E(Y^2)}$

X: $X\ge 0$ 求 $P(X=0)$ 的上限

$X = X\,I(X>0)$ $\quad E(I)$

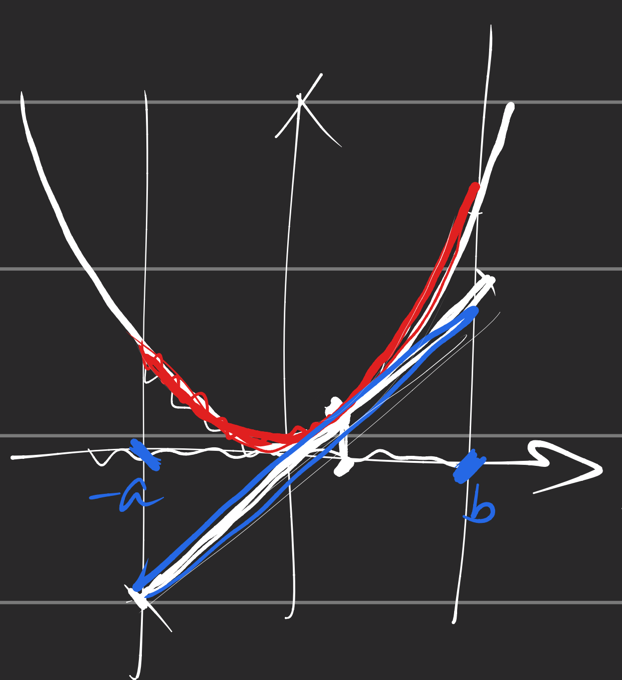
$|E(X)| = |E(X\,I(X>0))|^2 \le \sqrt{E(X^2)\,E(I^2(X>0))}$

$|E(X)| = |E(X\,I(X>0))|^2 \le \sqrt{E(X^2)\,E(I^2(X>0))}$ $\to P(X>0)$

$E^2(X) \le E(X^2)\,P(X>0)$

$P(X>0) \ge \frac{E^2(X)}{E(X^2)}$

$P(X=0) = 1-P(X>0) \le$



琴生不等式

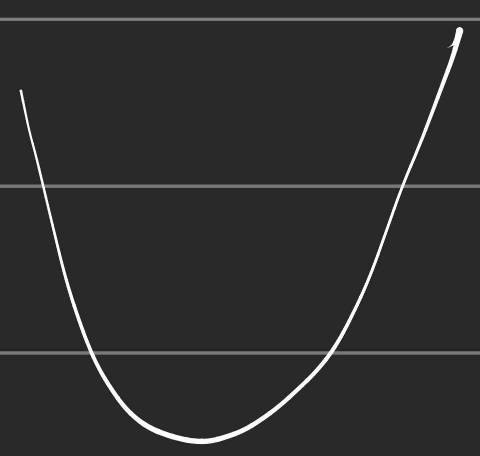
$Y = X^2 = g(X)$

$E(X)$

$g(E(X))$

$E(X) = \frac{-a+b}{2}$

$E(g(X))$



convex 凸

$E(g(X)) \ge g(E(X))$

Convey $E(g(X)) \le g(E(X))$

凹

$g(X) = a+bX$

熵. $a_1 \downarrow p_1$, $a_2 \downarrow p_2$; $a_1, a_2, \cdots, a_n \downarrow p_1, p_2, \cdots, p_n$

$p_1\ p_2\ \cdots\ p_n \quad (p_1+\cdots+p_n=1)$

$H(X) = \sum_{j=1}^{n} p_j \log_2\left(\frac{1}{p_j}\right)$

$p_1=p_2=\cdots=p_n=\frac{1}{n}$. max.

Kullback–Leibler 距离.

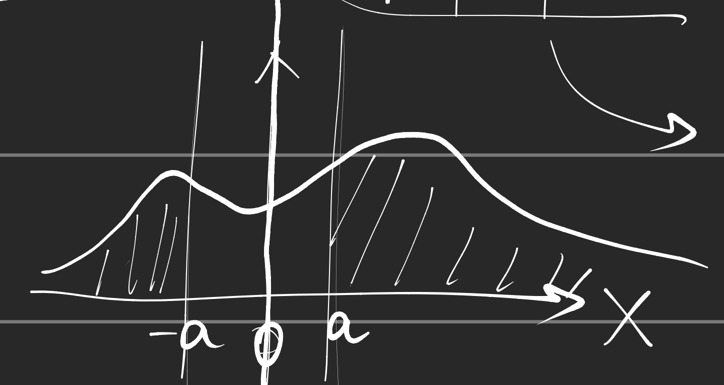
$\vec{p}=(p_1, p_2, \ldots, p_n) \quad \vec{r}=(r_1, \ldots, r_n)$

$D(\vec{p},\vec{r}) = \sum p_j\log_2\left(\frac{1}{r_j}\right) - \sum p_j \log_2\left(\frac{1}{p_j}\right)$ ≥ 0

Markov 尾概率.

Markov $P(|X|\ge a) \le \frac{E|X|}{a}$ 上界.

$\to P\left(\frac{|X|}{a}\ge 1\right)$ 上偏界 X



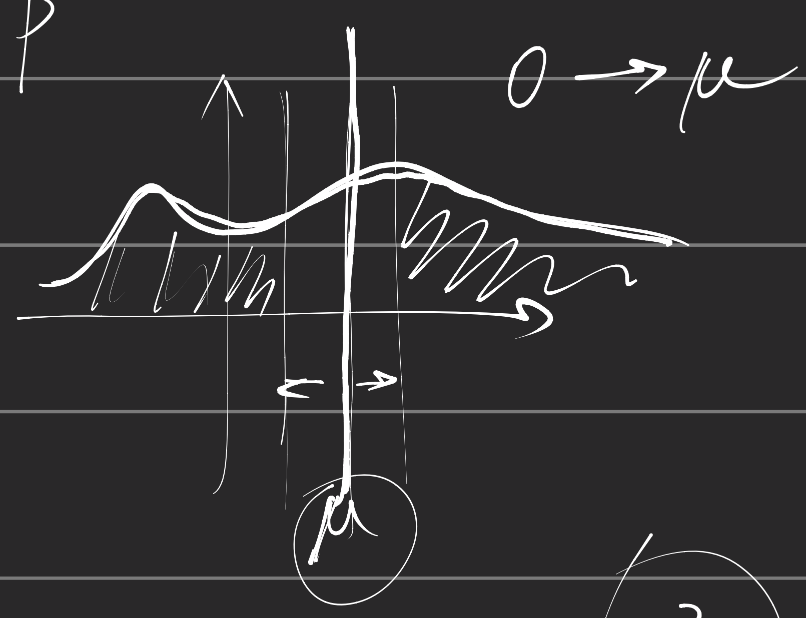
$Y = \frac{|X|}{a}$

$P(Y\ge 1) \le E(Y)$

$E(I(Y\ge 1)) \le E(Y) \quad (Y\ge 0)$

$P(Y\ge 1) \le E(Y)$

P $\quad 0 \to \mu$

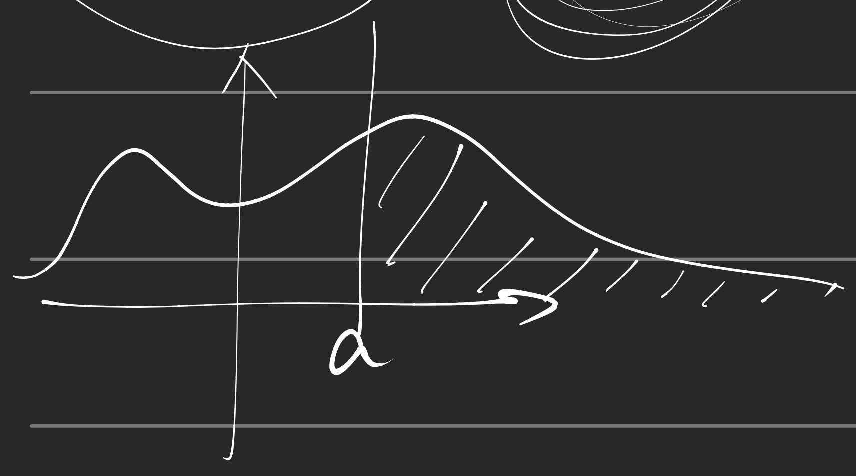


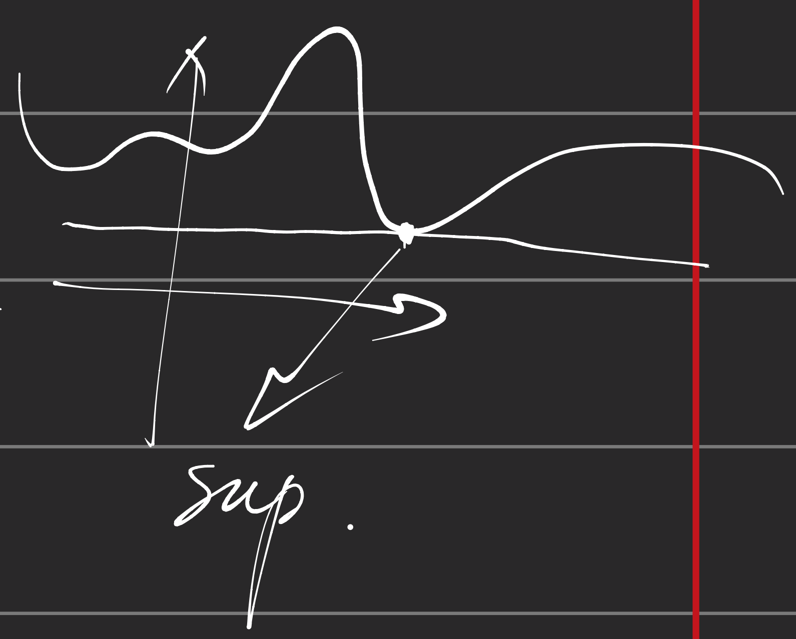
$P(|X-\mu| \le a) \le \frac{\sigma^2}{a^2} \to X = \mu, \sigma^2$

$P((X-\mu)^2 \le a^2) \le \frac{E(X-\mu)^2}{a^2} \to \sigma^2$

$t>0$

$$P(X \geq a) \leq \frac{E(e^{tX})}{e^{ta}} \qquad f(t)$$





$$P(X \leq a) \Rightarrow P(tX \geq ta) \quad (t<0)$$

$$\Rightarrow P(e^{tX} \geq e^{ta}) \leq \frac{E[e^{tX}]}{e^{ta}}$$

$t<0$

$$P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| > \varepsilon\right) \leq 2e^{-\frac{2n\varepsilon^2}{(b-a)^2}}$$

$n \to +\infty$

$$P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| > \varepsilon\right) \to 0$$