Chapter 9 马尔可夫链

△ 前提：下一步之状态（即下一步之分布列）只与当前这一步状态相关

即：$P(\underset{\text{下一步}}{X_{n+1}} = \underset{\text{具体状态}}{j} \mid X_n = i, X_{n-1} = i_{n-1}, \cdots X_0 = i_0)$

$= P(X_{n+1} = j \mid X_n = i)$

问题简化成：该步状态为 i，则下一步状态为 j 之概率？（与具体第 n 步无关）

$= q_{ij}$

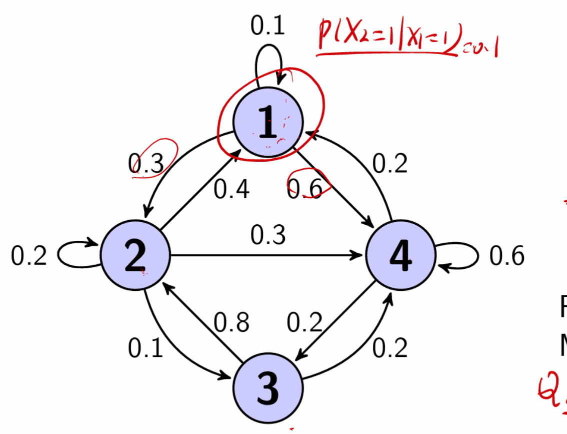
△ Transition Matrix:

$$Q = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1m} \\ \vdots & \ddots & & \vdots \\ q_{m1} & \cdots & & q_{mm} \end{bmatrix}$$

Graphical and Matrix Form of Markov Chain

State-transition diagram

$P(X_2=1 \mid X_1=1) = 0.1$



$\sum_{j=1}^{M} P(X_{n+1}=j \mid X_n=i) = 1$

$\Leftrightarrow \sum_{j=1}^{M} P(X_{n+1}=j, X_n=i) = P(X_n=i)$

$$Q = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0.1 & 0.3 & 0 & 0.6 \\ 2 & 0.4 & 0.2 & 0.1 & 0.3 \\ 3 & 0 & 0.8 & 0 & 0.2 \\ 4 & 0.2 & 0 & 0.2 & 0.6 \end{array}$$

stochastic Matrix

Row sum = 1 in Transition Matrix.

$Q_{ij} = P(X_{n+1}=j \mid X_n=i)$

$\tilde{P}(\cdot) = P(\cdot \mid X_n=i)$

性质：矩阵每行和为1

（第 i 行表示：当前状态为 i 之条件下，下步状态之分布列 → 和为1）

△ 基本计算

○ 当前状态为 i，n 步后为 j 之概率 $q_{ij}^{(n)}$

$= P(X_n = j \mid X_0 = i)$

$n=2$: $q_{ij}^{(2)} = \sum_k q_{ik} \cdot q_{kj} = Q^2(i,j)$ → 死去之矩阵乘法运算法则

n 很：

$q_{ij}^{(n)} = q_{ij}^{m+n} = \sum_k q_{ik}^{(m)} q_{kj}^{(n)} = Q^n(i,j)$

○ 若给定初始状态分布列（初始状态不再是某特定值）

init: $\vec{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_M)$

则 n 步后之状态分布列为：$\vec{\alpha} Q^n$

n 步后状态为 j 之概率 $P(X_n = j) = (\vec{\alpha} Q^n)_j$

Ex 1:

Example

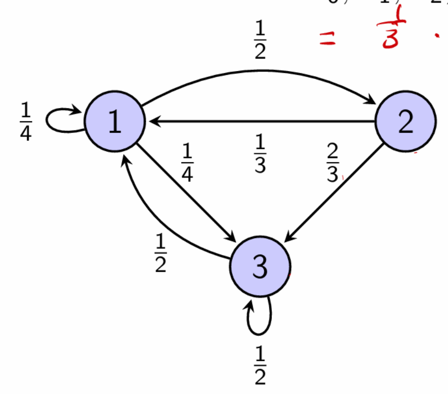
$P(X_0=1, X_1=2, X_2=3)$

chain's rule: $P(X_0=1) \cdot P(X_1=2 \mid X_0=1) \cdot P(X_2=3 \mid X_1=2, X_0=1)$

$= P(X_0=1) \cdot q_{1,2} \cdot q_{2,3}$ （$P(X_2=3 \mid X_1=2)$）

$= \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{9}$

Given a Markov chain $X_0, X_1, X_2, \ldots$ with state space $\mathcal{S} = \{1,2,3\}$.



$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 2 & \frac{1}{3} & 0 & \frac{2}{3} \\ 3 & \frac{1}{2} & 0 & \frac{1}{2} \end{array}$$

- If $P(X_0 = 1) = \frac{1}{3}$, find $P(X_0 = 1, X_1 = 2, X_2 = 3)$.

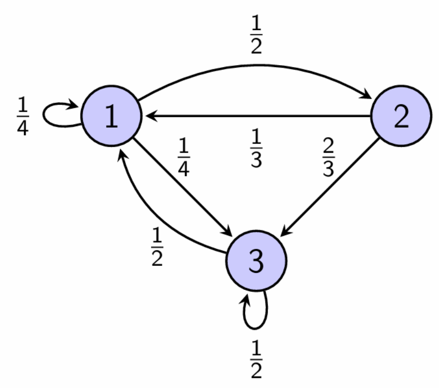
Ex 2:

Example

$E[X_2 \mid X_0=1] = \sum_{j=1}^{3} j \cdot P(X_2=j \mid X_0=1)$

$= \sum_{j=1}^{3} j \cdot q_{1,j}^{(2)}$

Given a Markov chain $X_0, X_1, X_2, \ldots$ with state space $\mathcal{S} = \{1,2,3\}$.



$= 1 \cdot q_{1,1}^{(2)} + 2 \cdot q_{1,2}^{(2)} + 3 \cdot q_{1,3}^{(2)}$

$= 1 \cdot \frac{17}{48} + 2 \cdot \frac{1}{8} + 3 \cdot \frac{25}{48}$

$= \frac{104}{48}$

$= \frac{13}{6}$

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 2 & \frac{1}{3} & 0 & \frac{2}{3} \\ 3 & \frac{1}{2} & 0 & \frac{1}{2} \end{array}$$

- Find $E(X_2 \mid X_0 = 1)$.

△ 状态之相关性质

○ 常返态(Recurrent)与瞬过态(Transient)

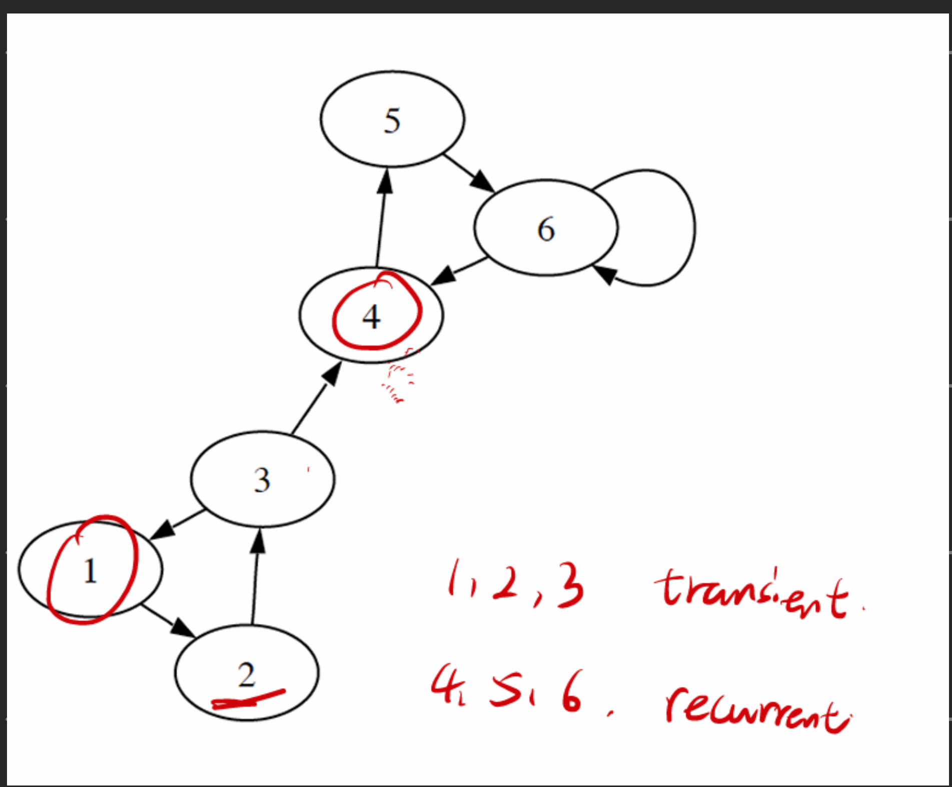
Recurrent: $\sum_{n=1}^{\infty} P_{ii} \to +\infty$

↳ 从该点出发能回到该点

↳ 回来之概率一定为1

Transient: $\sum_{n=1}^{\infty} P_{ii} < +\infty$

↳ 从该点出去后，只要有任何可能回不到该点，则一定为 Transient.



1 2 3: 有可能出去后进入 4 5 6 循环 (回不来了) → Transient

只要有可能就是Transient

○ 可约(Reducible)不可约(Irreducible)

Irreducible: 任何一对状态 i, j, 都可有限步内到达 (强连通有向图)

⇒ 对于 i, j, ∃ n 使 $Q^n_{ij} > 0$ ($p^{(n)}_{ij} > 0$)

否则即为可约

Theo: 不可约 Markov chain ⇒ 所有状态一定都为 Recurrent (显然成立)

○ 周期性

Tips: 周期/非周期只针对不可约链! 马尔科夫链之

但状态之 period 无所谓

状态i之周期为d(i): 所有从i到i路径长之gcd

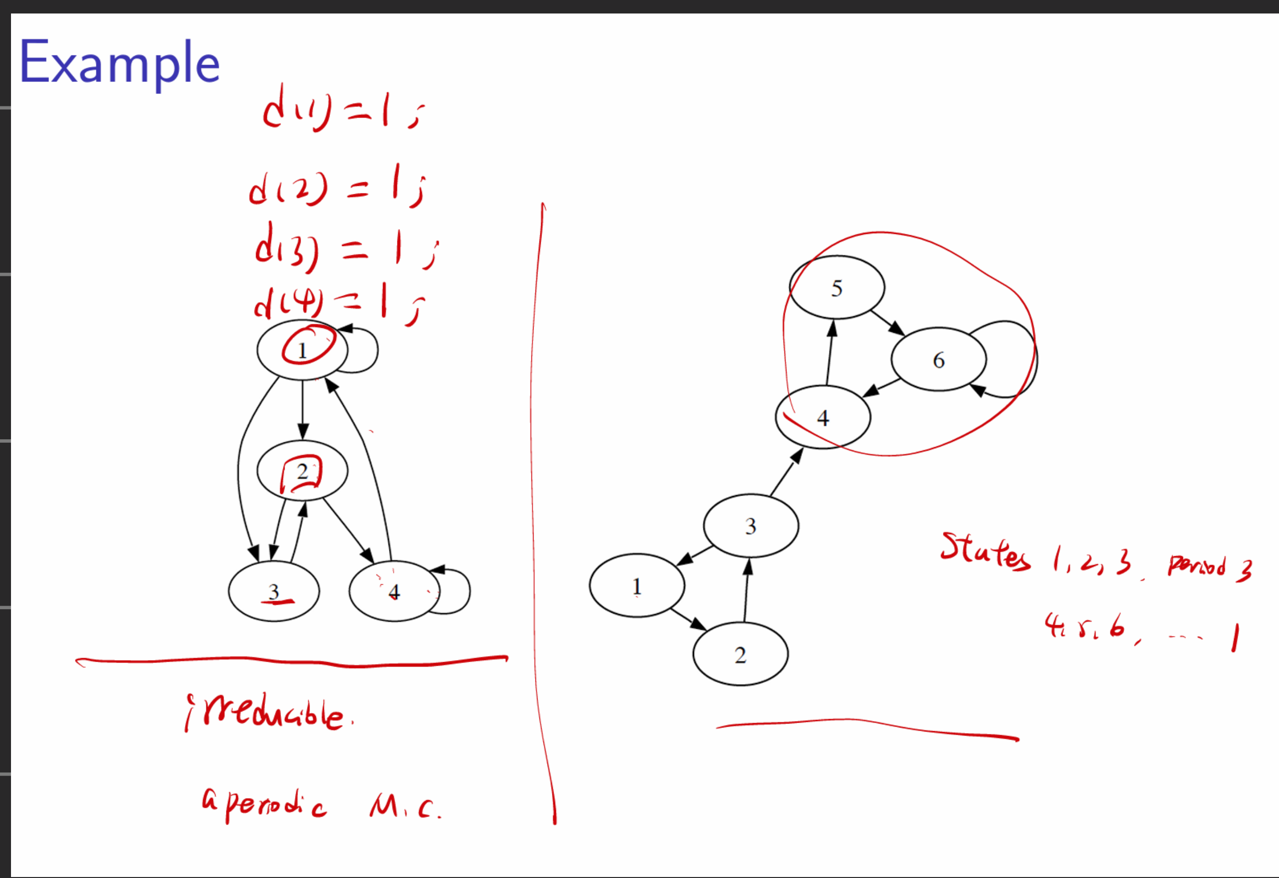
(如果回不来，则 d(i) = +∞)

$\Rightarrow \gcd\{n>0 : Q^n_{ii} > 0\}$

d(i) = 1 : aperiodic

Markov chain 周期性:

- *不可约, 所有状态 d(i) > 1 ⇒ periodic
- *不可约, 所有状态 d(i) = 1 ⇒ aperiodic



△ 平稳分布

设第n步状态分布列为 $\vec{\pi}_n$

即 $\pi_n^{(i)} = P(X_n = i)$

则 $\pi_{n+1}^{(j)} = \sum_i \pi_n^{(i)} q_{ij}$

$\Rightarrow \vec{\pi}_{n+1} = \vec{\pi}_n Q$

平稳分布 ⇒ $S_j = \sum S_i q_{ij}$

即 $\vec{s} = \vec{s} Q$

对于简单低阶之Q, 可直接用 求出 $\vec{s}$

性质

1. 不可约马尔可夫链 ⇒ 有且仅有一个 $\vec{s}$，且 $s_i > 0$

2. 不可约、非周期马尔可夫链 ⇒ 对于所有初状态分布 $\vec{\pi_0}$，都有 $\vec{\pi_n} \xrightarrow[n\to+\infty]{} \vec{s}$

$$Q^n \xrightarrow[n\to+\infty]{} \begin{bmatrix} \vec{s} \\ \vec{s} \\ \vec{s} \\ \vdots \\ \vec{s} \end{bmatrix} \qquad (\vec{\pi_n} = \vec{\pi_0} Q^n \to \vec{s})$$

3. 双随机(Double Stochastic)矩阵
   行、列和均为1，
   则 $\exists\, \vec{s} = (\frac{1}{n}, \frac{1}{n}, \cdots \frac{1}{n})$

△ 可逆性 → 也称 Detailed Balanced (细致平稳)

$\exists\, \vec{s} = (s_1 \cdots s_n)$ 满足 $\sum s_i = 1$

有 $s_i q_{ij} = s_j q_{ji}$

即：在某种分布下，当前为 i 下步为 j 之概率

= 当前为 j 下步为 i 之概率

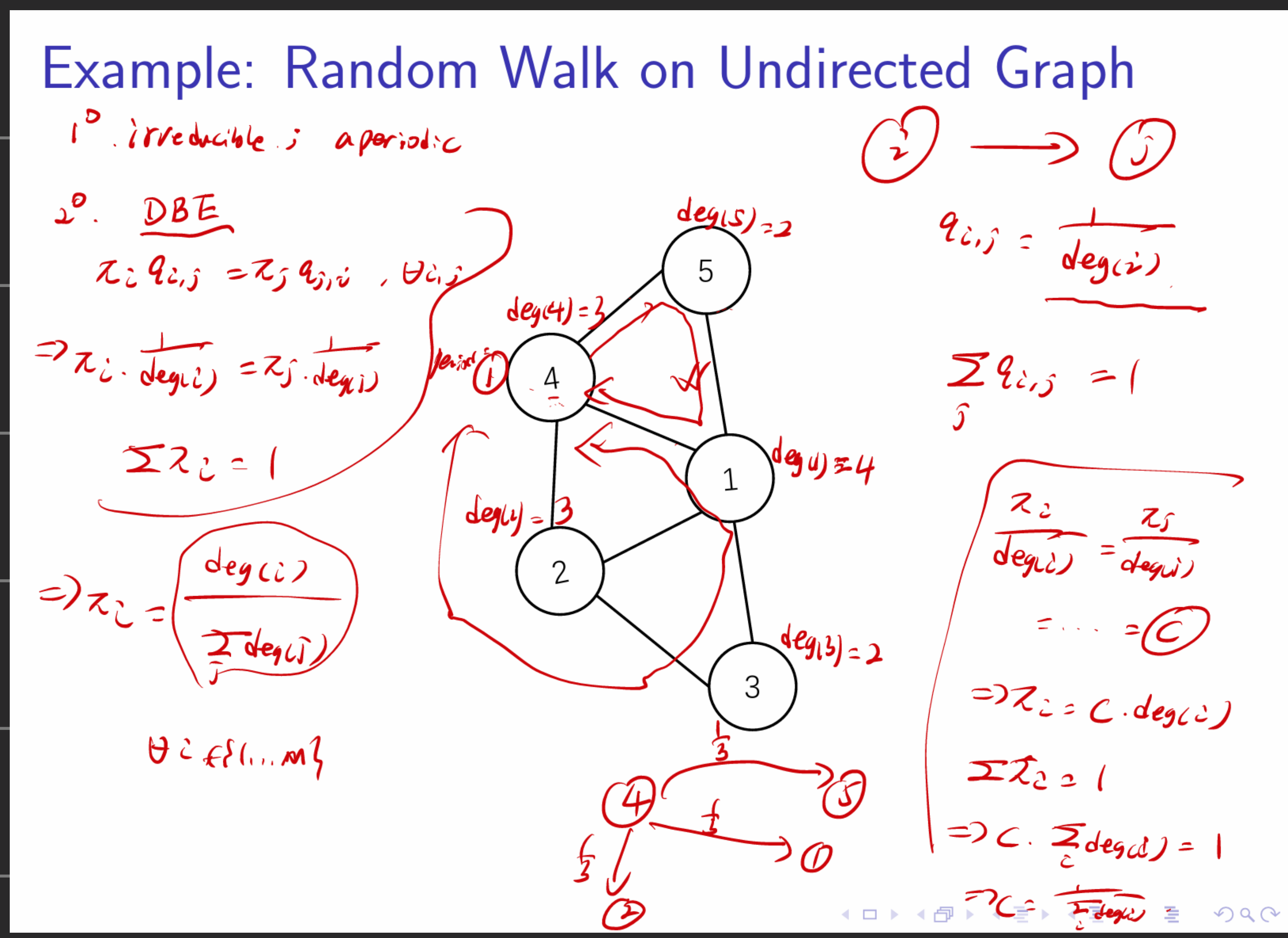
⇒ 则称为 Reversible with respect to $\vec{s}$

性质

1. 对于不可约马尔可夫链：

若 $\exists\, \vec{\pi_0}$ 使其可逆，则 $\vec{\pi_0}$ 也为唯一之平稳分布

Ex：若 Q 为对称矩阵，则 $\exists$ 唯一之 $\vec{s} = (\frac{1}{n}, \frac{1}{n} \cdots \frac{1}{n})$，使其可逆&平稳



显然不可约&非周期

可逆性：$\pi_i q_{ij} = \pi_j q_{ji}$

$$\Rightarrow \begin{cases} \frac{\pi_i}{deg(i)} = \frac{\pi_j}{deg(j)} \\ \sum \pi_i = 1 \end{cases}$$

$$\Rightarrow \pi_i \propto deg(i)$$

$$\vec{\pi} = (\frac{4}{14}, \frac{3}{14}, \frac{2}{14}, \frac{3}{14}, \frac{2}{14})$$

↳ 可逆分布&唯一之平稳分布

<<好题精选>>

Problem 6

A fair coin is flipped repeatedly. We use H to denote "Head appeared" and T to denote the "Tail appeared".

(a) What is the expected number of flips until the pattern HTHT is observed?

(b) What is the expected number of flips until the pattern THTT is observed?

(c) What is the probability that pattern HTHT is observed earlier than THTT?

(a) Denote $t(\cdot)$ as the time for transferring from the current state to the ending state. The state space is $\{H, T, HT, HTH, HTHT\}$, and the transition relationship between is can be demonstrated as follows:

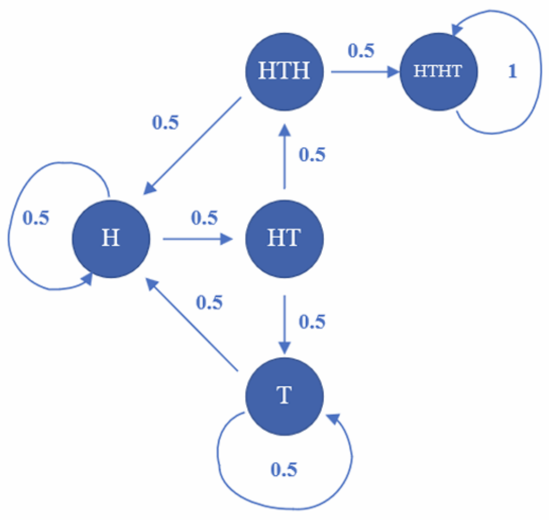


Figure 1: 6(1)

Then the expectation of step numbers for transferring from one state to the ending state can be listed as follows:

$$\begin{cases} E(t(H)) = \frac{1}{2}E(t(H)) + \frac{1}{2}E(t(HT)) + 1 \\ E(t(T)) = \frac{1}{2}E(t(T)) + \frac{1}{2}E(t(H)) + 1 \\ E(t(HT)) = \frac{1}{2}E(t(T)) + \frac{1}{2}E(t(HTH)) + 1 \\ E(t(HTH)) = \frac{1}{2}E(t(HTHT)) + \frac{1}{2}E(t(H)) + 1 \\ E(t(HTHT)) = 0 \end{cases}$$

Then we can obtain that $E(t(H)) = 18$ and $E(t(T)) = 20$. Therefore, the expected numbers of flips from starting is $\frac{1}{2}E(H) + \frac{1}{2}E(T) + 1 = 20$.

思路:

① 将各种可能之中途状态列举出来

(包括H、T初始状态，与中间态HT、HTH、

HTHT) 到达该状态则只转移至自身.

② 列转移方程即可

方程推得有无问题？

由 $\pi^{(j)}_{n+1} = \sum p_{ij} \pi^{(i)}_n$,

应该有 $E(t(H)) = \frac{1}{2}E(t(H)) + \frac{1}{2}E(t(T)) + \frac{1}{2}E(t(HTH)) + 1$

(由H之上一步推导而来，而非H之下一步推导

而来(?))

其他方程同理

(b) 同理

(c) The state space in this problem can be conclude by $\{H, HT, HTH, HTHT, T, TH, THT, THTT\}$, and the relationship between each state can be demonstrated as follows:

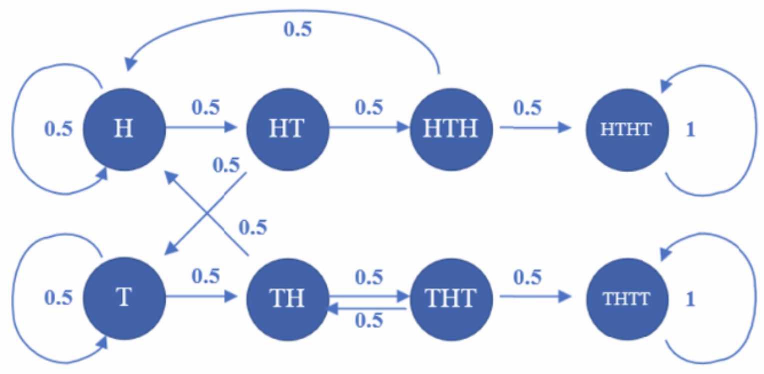


Figure 3: 6(3)

The relationship between the probability of each state finally ends up in $HTHT$ can be listed as follows:

$$\begin{cases} P(HTHT) = 1 \\ P(THTT) = 0 \\ P(HTH) = \frac{1}{2}P(HTHT) + \frac{1}{2}P(H) \\ P(THT) = \frac{1}{2}P(THTT) + \frac{1}{2}P(TH) \\ P(HT) = \frac{1}{2}P(HTH) + \frac{1}{2}P(T) \\ P(TH) = \frac{1}{2}P(THT) + \frac{1}{2}P(H) \\ P(H) = \frac{1}{2}P(H) + \frac{1}{2}P(HT) \\ P(T) = \frac{1}{2}P(T) + \frac{1}{2}P(TH). \end{cases} \tag{5}$$

Finally we can get that $P(H) = \frac{5}{7}$ and $P(T) = \frac{4}{7}$. Thus the probability of pattern $HTHT$ observed earlier than $THTT$ is $\frac{1}{2}P(H) + \frac{1}{2}P(T) = \frac{9}{14}$ by assuming the same initial state of entering states H and T.

○ 转移方程:

同理如题

也有问题(?)