CS182: Introduction to Machine Learning – Decision Trees

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Q & A:

How do these in-class polls work?

- Scan QR Code for accessing the polls
- Answer all poll questions **during lecture for full credit** or **within 24 hours for half credit**
- You have 8 free “poll points” for the semester that will excuse you from all polls from a single lecture; you cannot use more than 3 poll points consecutively.

Poll Question 1:

Which of the following did you bring to class today? Select all that apply

A. A smartphone

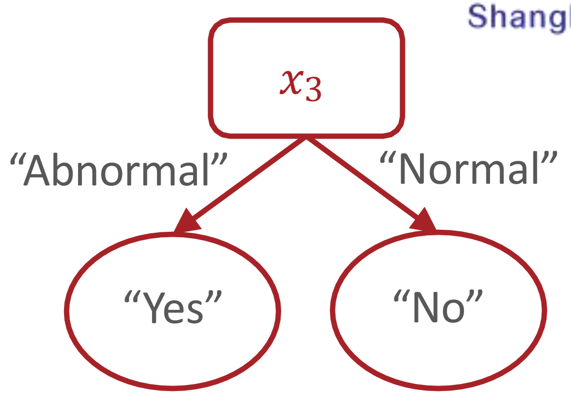
B. A computer

C. A smart watch

D. No device

Recall: Decision Stump

x_1 Family History	x_2 Resting Blood Pressure	x_3 Cholesterol	y Heart Disease?
Yes	Low	Normal	No
No	Medium	Normal	No
No	Low	Abnormal	Yes
Yes	Medium	Normal	Yes
Yes	High	Abnormal	Yes



Recall: Decision Stump Questions

1. How can we pick which feature to split on?

2. Why stop at just one feature? **Don't!**

   a) If we split on more than one feature, how do we decide the order to spilt on?

From Decision Stump …

x_1 Family History	x_2 Resting Blood Pressure	x_3 Cholesterol	y Heart Disease?
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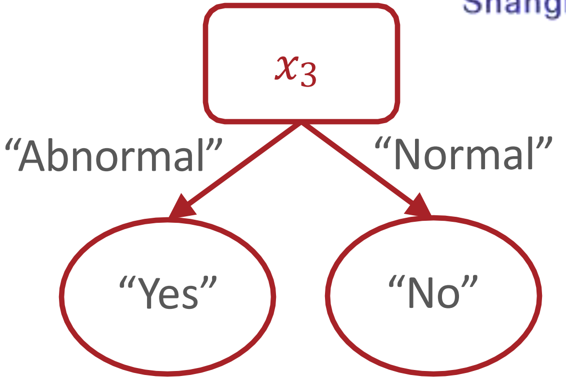

From Decision Stump to Decision Tree

x_1 Family History	x_2 Resting Blood Pressure	x_3 Cholesterol	y Heart Disease?
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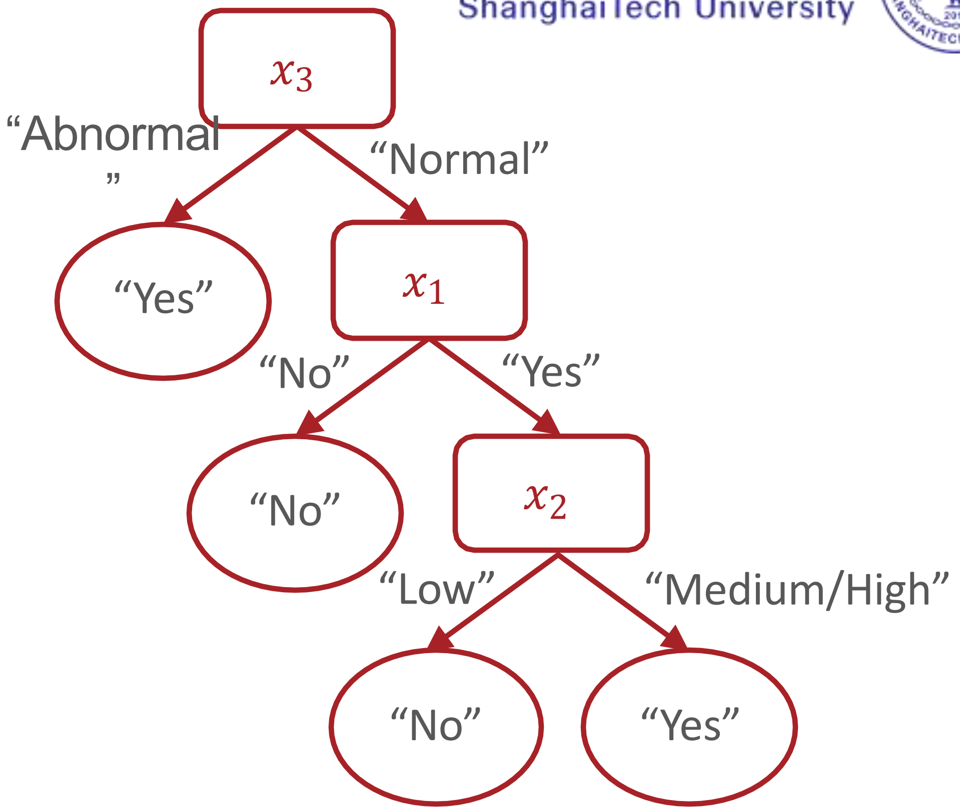

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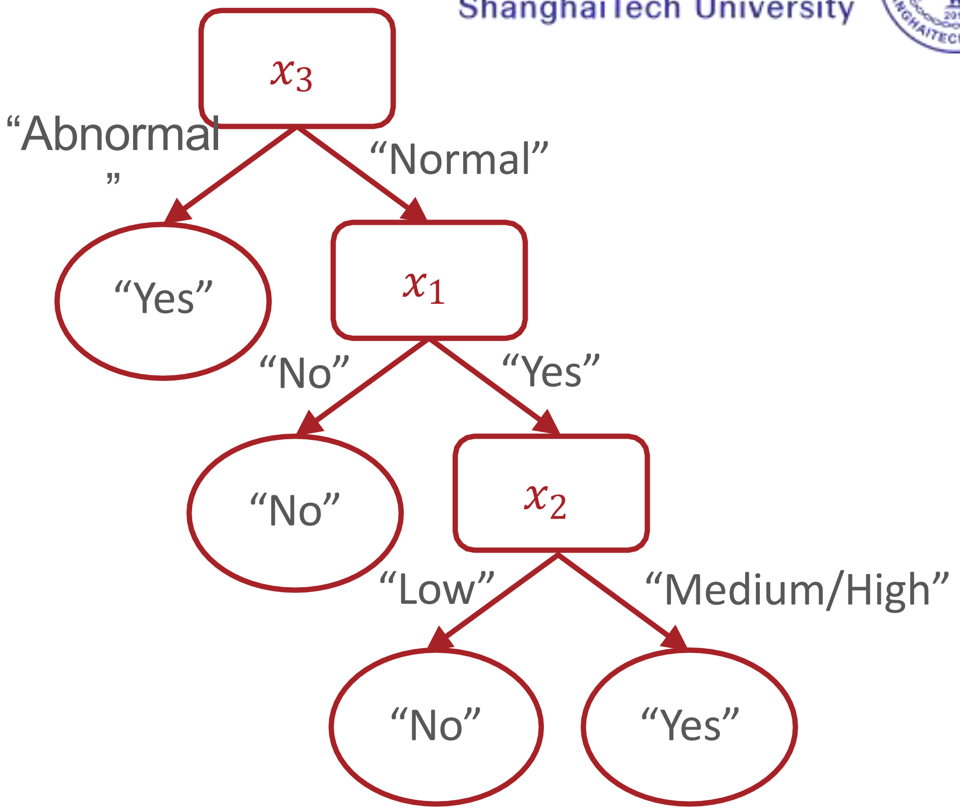

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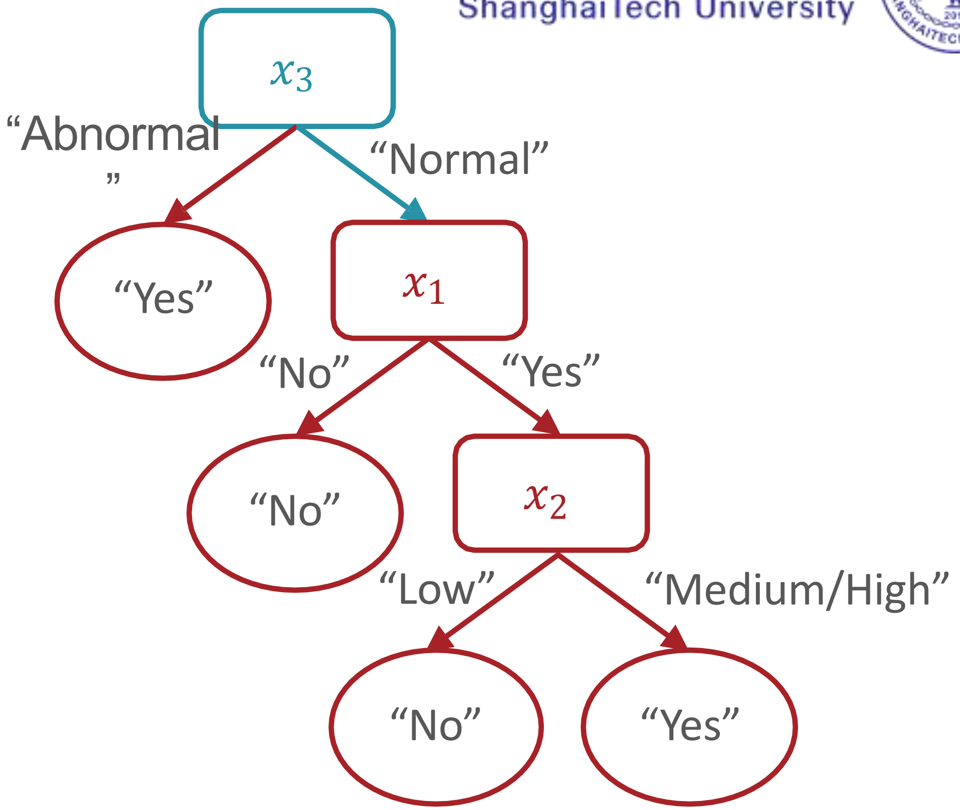

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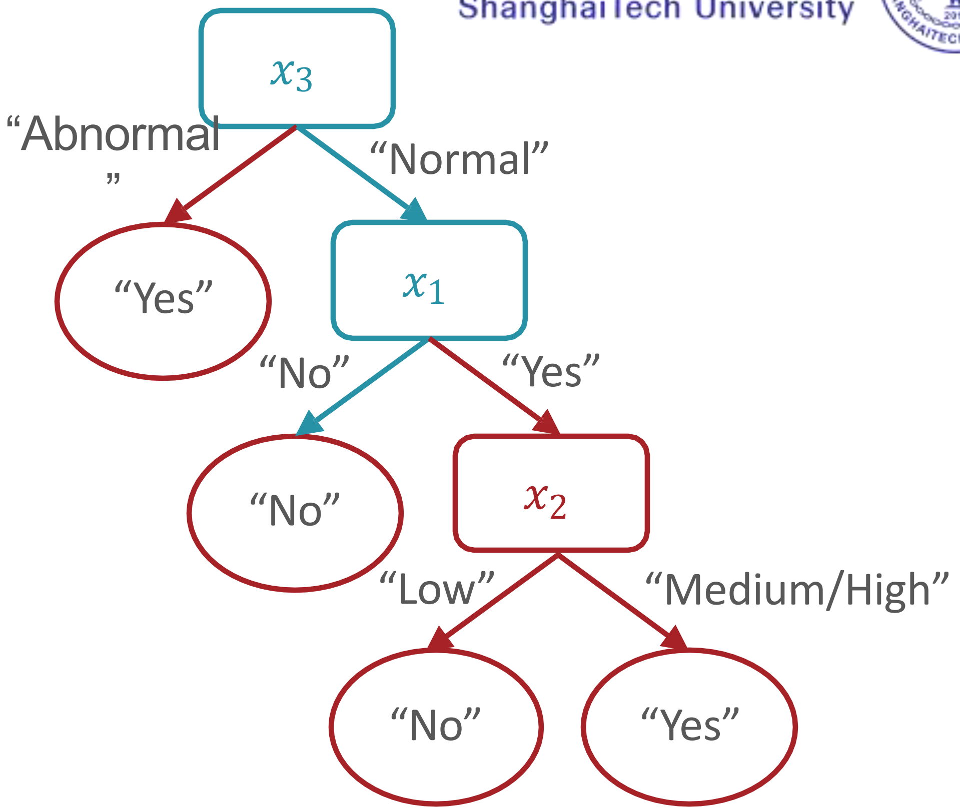

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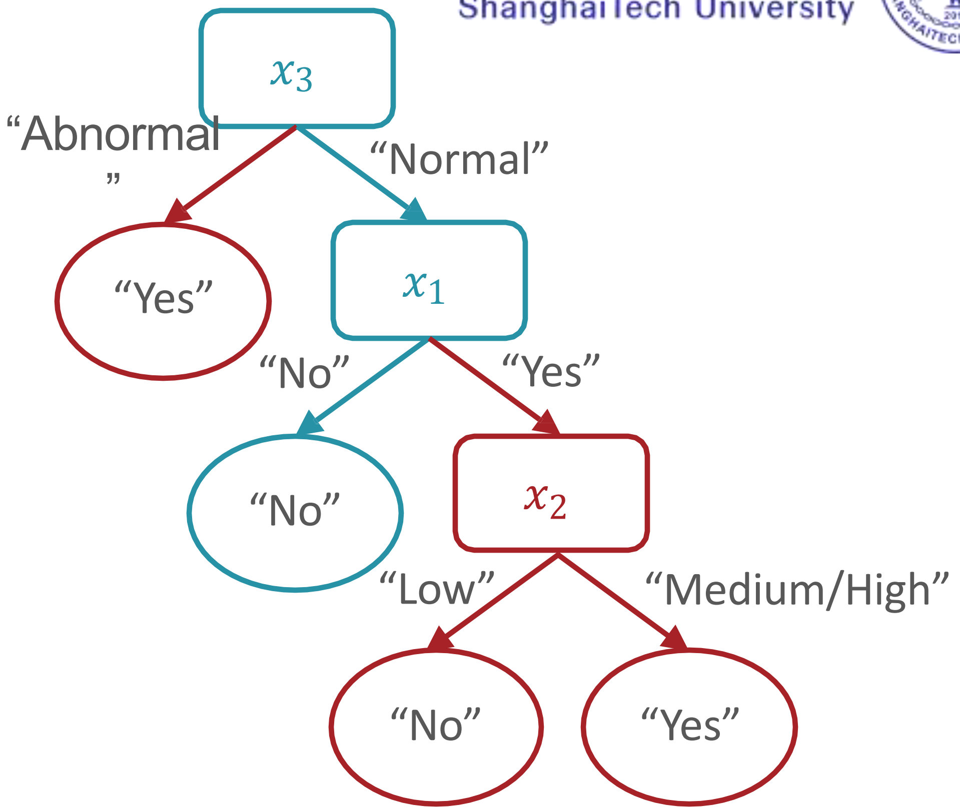

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Decision Tree: Pseudocode

def $h(\boldsymbol{x}')$:

- walk from root node to a leaf node

while(true):

if current node is internal (non-leaf):

check the associated attribute, x_d

go down branch according to x'_d

if current node is a leaf node:

return label stored at that leaf

Decision Tree Questions

1. How can we pick which feature to split on?

2. Why stop at just one feature?

   a) If we split on more than one feature, how do we decide the order to spilt on?

Splitting Criterion

- A **splitting criterion** is a function that measures how good or useful splitting on a particular feature is *for a specified dataset*
- Idea: when deciding which feature to split on, use the one that optimizes the splitting criterion

Training Error Rate as a Splitting Criterion

x_1 Family History	x_2 Resting Blood Pressure	x_3 Cholesterol	y Heart Disease?
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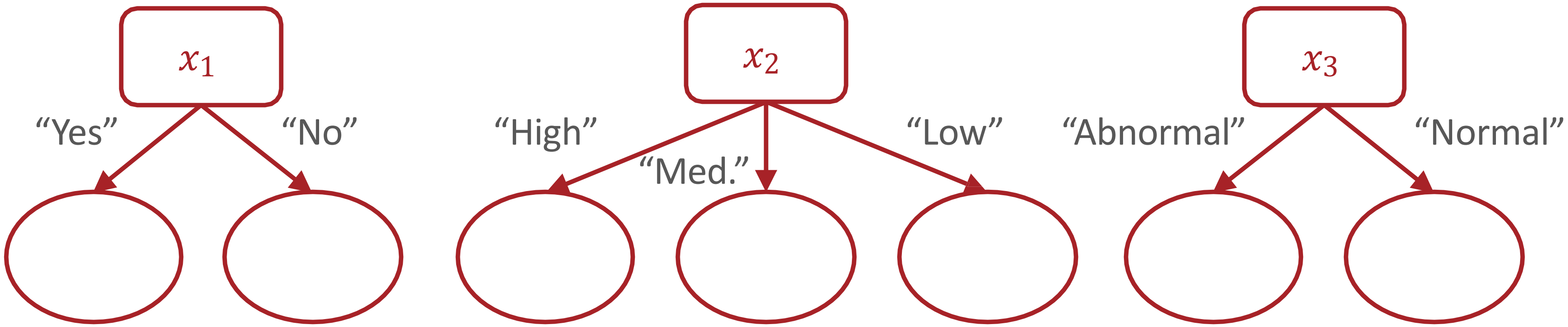

Training error rate:

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Training error rate:

Poll Question 2:

- Which feature would you split on using training error rate as the splitting criterion?

x_1	x_2	y
1	0	0
1	0	0
1	0	1
1	0	1
1	1	1
1	1	1
1	1	1
1	1	1

A. x_1

B. x_2

C. Either x_1 or x_2

D. Neither x_1 nor x_2

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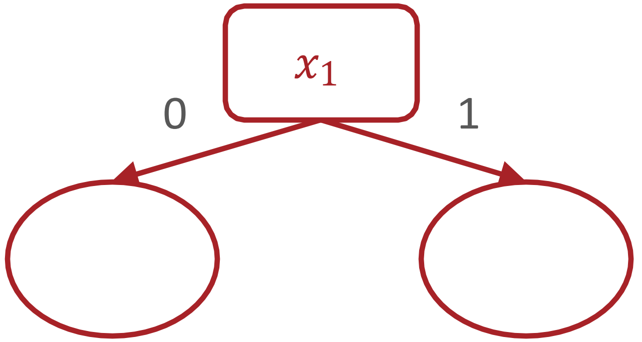

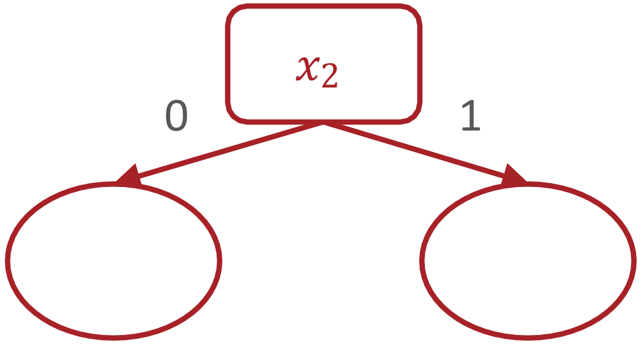

Training error rate:

Splitting Criterion

- A **splitting criterion** is a function that measures how good or useful splitting on a particular feature is *for a specified dataset*
- Idea: when deciding which feature to split on, use the one that optimizes the splitting criterion
- Potential splitting criteria:
  - Training error rate (minimize)
  - Gini impurity (minimize) → CART algorithm
  - Mutual information (maximize) → ID3 algorithm

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  - Gini impurity (minimize) → CART algorithm
  - **Mutual information** (maximize) → ID3 algorithm

Entropy

- The **entropy** of a *random variable* describes the uncertainty of its outcome: the higher the entropy, the less certain we are about what the outcome will be.

$$H(X) = -\sum_{v \in V(X)} P(X = v) \log_2\big(P(X = v)\big)$$

where X is a (discrete) random variable

$V(X)$ is the set of possible values X can take on

Entropy

- The **entropy** of a *set* describes how uniform or pure it is: the higher the entropy, the more impure or "mixed-up" the set is

$$H(S) = -\sum_{v \in V(S)} \frac{|S_v|}{|S|} \log_2\left(\frac{|S_v|}{|S|}\right)$$

where S is a collection of values,

$V(S)$ is the set of unique values in S

S_v is the collection of elements in S with value v

- If all the elements in S are the same, then

$$\mathrm{H}(S) = -1\log_2(1) = 0$$

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where S is a collection of values,

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S_v is the collection of elements in S with value v

- If S is split fifty-fifty between two values, then

$$\mathrm{H}(S) = -\frac{1}{2}\log_2\left(\frac{1}{2}\right) - \frac{1}{2}\log_2\left(\frac{1}{2}\right) = -\log_2\left(\frac{1}{2}\right) = 1$$

Mutual Information

- The **mutual information** between *two random variables* describes how much clarity knowing the value of one random variables provides about the other

$$I(Y;X) = H(Y) - H(Y|X)$$

$$= H(Y) - \sum_{v \in V(X)} P(X = v) H(Y|X = v)$$

where X and Y are random variables

$V(X)$ is the set of possible values X can take on

$H(Y|X = v)$ is the conditional entropy of Y given $X = v$

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Mutual Information

- The **mutual information** between *a feature and the label* describes how much clarity knowing the feature provides about the label

$$I(y; x_d) = H(y) - H(y|x_d)$$
$$= H(y) - \sum_{v \in V(x_d)} f_v \left(H(Y_{x_d=v}) \right)$$

where x_d is a feature and y is the set of all labels

$V(x_d)$ is the set of possible values x_d can take on

f_v is the fraction of data points where $x_d = v$

$Y_{x_d=v}$ is the set of all labels where $x_d = v$

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Mutual Information Example

x_d	y
1	1
1	1
0	0
0	0

$$I(x_d, Y) = H(Y) - \sum_{v \in V(x_d)} (f_v) \left(H(Y_{x_d = v}) \right)$$

$$= 1 - \frac{1}{2} H(Y_{x_d=0}) - \frac{1}{2} H(Y_{x_d=1})$$

$$= 1 - \frac{1}{2}(0) - \frac{1}{2}(0) = 1$$

Mutual Information Example

x_d	y
1	1
0	1
1	0
0	0

$$I(x_d, Y) = H(Y) - \sum_{v \in V(x_d)} (f_v)\left(H\left(Y_{x_d=v}\right)\right)$$

$$= 1 - \frac{1}{2} H\left(Y_{x_d=0}\right) - \frac{1}{2} H\left(Y_{x_d=1}\right)$$

$$= 1 - \frac{1}{2}(1) - \frac{1}{2}(1) = 0$$

Poll Question 3:

Which feature would you split on using mutual information as the splitting criterion?

x_1	x_2	y
1	0	0
1	0	0
1	0	1
1	0	1
1	1	1
1	1	1
1	1	1
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A. x_1

B. x_2

C. Either x_1 or x_2

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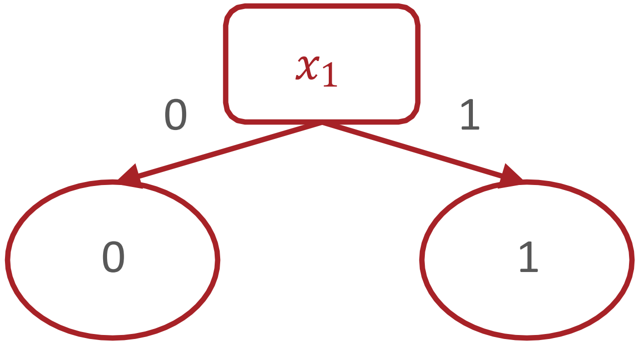
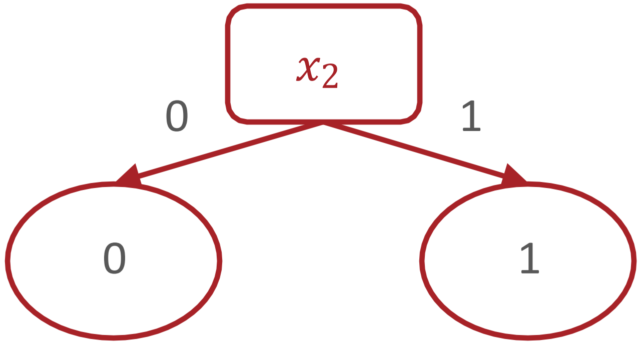


Mutual Information: 0



Mutual Information: $H(Y) - \frac{1}{2}H(Y_{x_2=0}) - \frac{1}{2}H(Y_{x_2=1})$

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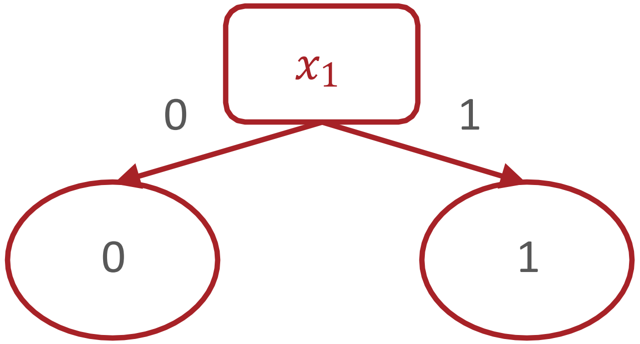
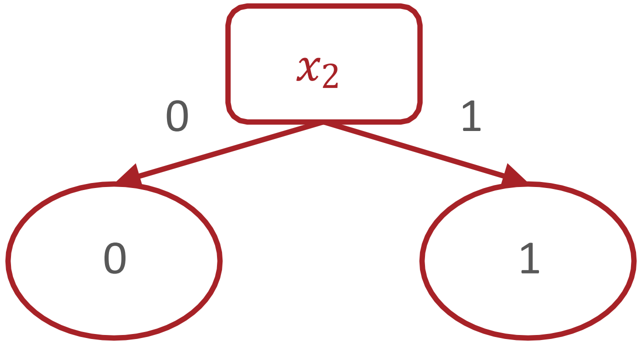


Mutual Information: 0



Mutual Information: $-\frac{2}{8}\log_2\frac{2}{8} - \frac{6}{8}\log_2\frac{6}{8} - \frac{1}{2}(1) - \frac{1}{2}(0) \approx 0.31$

Decision Tree: Questions

1. How can we pick which feature to split on?

2. Why stop at just one feature?

    a) If we split on more than one feature, how do we decide the order to spilt on?

Decision Tree: Pseudocode

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```
def train(𝒟):
    store root = tree_recurse(𝒟)
def tree_recurse(𝒟′):
    q = new node()
    base case – if (SOME CONDITION):
    recursion – else:
        find best attribute to split on, x_d
        q.split = x_d
        for v in V(x_d), all possible values of x_d:
```

$$\mathcal{D}_v = \left\{ \left(x^{(n)}, y^{(n)} \right) \in \mathcal{D} \mid x_d^{(n)} = v \right\}$$

```
            q.children(v) = tree_recurse(𝒟_v)
    return q
```

Decision Tree: Pseudocode

```
def train(𝒟):
    store root = tree_recurse(𝒟)
def tree_recurse(𝒟′):
    q = new node()
    base case – if (𝒟′ is empty OR
        all labels in 𝒟′ are the same OR
        all features in 𝒟′ are identical OR
        some other stopping criterion):
            q.label = majority_vote(𝒟′)
    recursion – else:
    return q
```