CS182: Introduction to Machine Learning – Learning Theory (Finite Case & Infinite Case)

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Finite Case

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Statistical Learning Theory Model

1. Data points are generated i.i.d. from some *unknown* distribution

$$\boldsymbol{x}^{(n)} \sim p^*(\boldsymbol{x})$$

2. Labels are generated from some *unknown* function

$$y^{(n)} = c^*\left(\boldsymbol{x}^{(n)}\right)$$

3. The learning algorithm chooses the hypothesis (or classifier) with lowest *training* error rate from a specified hypothesis set, $\mathcal{H}$

4. Goal: return a hypothesis (or classifier) with low *true* error rate

Types of Error

- True error rate
  - Actual quantity of interest in machine learning
  - How well your hypothesis will perform on average across all possible data points
- Test error rate
  - Used to evaluate hypothesis performance
  - Good estimate of your hypothesis's true error
- Validation error rate
  - Used to set hypothesis hyperparameters
  - Slightly "optimistic" estimate of your hypothesis's true error
- Training error rate
  - Used to set model parameters
  - Very "optimistic" estimate of your hypothesis's true error

Types of Risk (a.k.a. Error)

- Expected risk of a hypothesis h (a.k.a. true error)

$$R(h) = P_{\boldsymbol{x} \sim p^*}\left(c^*(\boldsymbol{x}) \neq h(\boldsymbol{x})\right)$$

- Empirical risk of a hypothesis h (a.k.a. training error)

$$\begin{aligned}\hat{R}(h) &= P_{\boldsymbol{x} \sim \mathcal{D}}\left(c^*(\boldsymbol{x}) \neq h(\boldsymbol{x})\right) \\ &= \frac{1}{N}\sum_{n=1}^{N} \mathbb{1}\left(c^*\left(\boldsymbol{x}^{(n)}\right) \neq h\left(\boldsymbol{x}^{(n)}\right)\right) \\ &= \frac{1}{N}\sum_{n=1}^{N} \mathbb{1}\left(y^{(n)} \neq h\left(\boldsymbol{x}^{(n)}\right)\right)\end{aligned}$$

where $\mathcal{D} = \left\{\left(\boldsymbol{x}^{(n)}, y^{(n)}\right)\right\}_{n=1}^{N}$ is the training data set and $\boldsymbol{x} \sim \mathcal{D}$ denotes a point sampled uniformly at random from $\mathcal{D}$

Three Hypotheses of Interest

1. The *true function*, c^*

2. The *expected risk minimizer*,

$$h^* = \underset{h \in \mathcal{H}}{\operatorname{argmin}} R(h)$$

3. The *empirical risk minimizer*,

$$\hat{h} = \underset{h \in \mathcal{H}}{\operatorname{argmin}} \hat{R}(\mathrm{h})$$

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Poll Question 1: Which of the following are *always* true?

A. $c^* = h^*$
B. $c^* = \hat{h}$
C. $h^* = \hat{h}$
D. $c^* = h^* = \hat{h}$
E. None of the above
F. TOXIC

- The *true function*, c^*
- The *expected risk minimizer*,

$$h^* = \underset{h \in \mathcal{H}}{\operatorname{argmin}} R(h)$$

- The *empirical risk minimizer*,

$$\hat{h} = \underset{h \in \mathcal{H}}{\operatorname{argmin}} \hat{R}(\mathrm{h})$$

Key Question

- Given a hypothesis with zero/low training error, what can we say about its true error?

PAC Learning

- PAC = <u>P</u>robably <u>A</u>pproximately <u>C</u>orrect

- PAC Criterion:

$$P\left(\left|R(h) - \hat{R}(h)\right| \epsilon \geq\right) \mathcal{H} \ni h \forall \delta - 1 \leq$$

for some ϵ (difference between expected and empirical risk) and δ (probability of “failure”)

  - We want the PAC criterion to be satisfied for $\mathcal{H}$ with small values of ϵ and δ

Sample Complexity

- The sample complexity of an algorithm/hypothesis set, $\mathcal{H}$, is the number of labelled training data points needed to satisfy the PAC criterion for some δ and ϵ
- Four cases
  - Realizable vs. Agnostic
    - Realizable → $c^* \in \mathcal{H}$
    - Agnostic → c^* might or might not be in $\mathcal{H}$
  - Finite vs. Infinite
    - Finite → $|\mathcal{H}| < \infty$
    - Infinite → $|\mathcal{H}| = \infty$

Theorem 1: Finite, Realizable Case

- For a finite hypothesis set $\mathcal{H}$ s.t. $c^* \in \mathcal{H}$ and arbitrary distribution p^*, if the number of labelled training data points satisfies

$$M \geq \frac{1}{\epsilon}\left(\ln(|\mathcal{H}|) + \ln\left(\frac{1}{\delta}\right)\right)$$

then with probability at least $1-\delta$, all $h \in \mathcal{H}$ with $\hat{R}(h) = 0$ have $R(h) \leq \epsilon$

Proof of Theorem 1: Finite, Realizable Case

1. Assume there are K "bad" hypotheses in $\mathcal{H}$, i.e., $h_1, h_2, \ldots, h_K$ that all have $R(h_k) > \epsilon$
2. Pick one bad hypothesis, h_k
   A. Probability that h_k correctly classifies the first training data point $< 1 - \epsilon$
   B. Probability that h_k correctly classifies all M training data points $< (1 - \epsilon)^M$
3. Probability that at least one bad hypothesis correctly classifies all M training data points =

$$P(h_1 \text{ correctly classifies all } M \text{ training data points} \cup h_2 \text{ correctly classifies all } M \text{ training data points} \cup \\ \vdots \\ \cup\, h_K \text{ correctly classifies all } M \text{ training data points})$$

Proof of Theorem 1: Finite, Realizable Case

$$P(h_1 \text{ correctly classifies all } M \text{ training data points} \cup$$
$$h_2 \text{ correctly classifies all } M \text{ training data points} \cup$$
$$\vdots$$
$$\cup\, h_K \text{ correctly classifies all } M \text{ training data points})$$

$$\leq \sum_{k=1}^{K} P(h_k \text{ correctly classifies all } M \text{ training data points})$$

by the union bound: $P\,(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\leq P(A) + P(B)$$

Proof of Theorem 1: Finite, Realizable Case

$$\sum_{k=1}^{K} P(h_k \text{ correctly classifies all } M \text{ training data points})$$

$$< k(1-\epsilon)^M \leq |\mathcal{H}|(1-\epsilon)^M$$

because $k \leq |\mathcal{H}|$

3. Probability that at least one bad hypothesis correctly classifies all M training data points $\leq |\mathcal{H}|(1-\epsilon)^M$
4. Using the fact that $1 - x \leq \exp(-x) \ \forall x$,
   $$|\mathcal{H}|(1-\epsilon)^M \leq |\mathcal{H}| \exp(-\epsilon)^M = |\mathcal{H}| \exp(-M\epsilon)$$
5. Probability that at least one bad hypothesis correctly classifies all M training data points $\leq |\mathcal{H}| \exp(-M\epsilon)$, which we want to be *low*, i.e., $|\mathcal{H}| \exp(-M\epsilon) \leq \delta$

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Proof of Theorem 1: Finite, Realizable Case

$$|\mathcal{H}|\exp(-M\epsilon) \le \delta \rightarrow \exp(-M\epsilon) \le \frac{\delta}{|\mathcal{H}|}$$

$$\rightarrow -M\epsilon \le \ln\left(\frac{\delta}{|\mathcal{H}|}\right)$$

$$\rightarrow M \ge \frac{1}{\epsilon}\left(-\ln\left(\frac{\delta}{|\mathcal{H}|}\right)\right)$$

$$\rightarrow M \ge \frac{1}{\epsilon}\left(\ln\left(\frac{|\mathcal{H}|}{\delta}\right)\right)$$

$$\rightarrow M \ge \frac{1}{\epsilon}\left(\ln(|\mathcal{H}|) + \ln\left(\frac{1}{\delta}\right)\right)$$

Proof of Theorem 1: Finite, Realizable Case

6. Given $M \geq \frac{1}{\epsilon}\left(\ln(|\mathcal{H}|) + \ln\left(\frac{1}{\delta}\right)\right)$ labelled training data points, the probability that $\exists$ a bad hypothesis $h_k \in \mathcal{H}$ with $R(h_k) > \epsilon$ and $\hat{R}(h_k) = 0$ is $\leq \delta$

$$\Updownarrow$$

Given $M \geq \frac{1}{\epsilon}\left(\ln(|\mathcal{H}|) + \ln\left(\frac{1}{\delta}\right)\right)$ labelled training data points, the probability that all hypotheses $h_k \in \mathcal{H}$ with $R(h_k) > \epsilon$ have $\hat{R}(h_k) > 0$ is $\geq 1 - \delta$

Aside: Proof by Contrapositive

- The contrapositive of a statement $A \Rightarrow B$ is $\neg B \Rightarrow \neg A$
- A statement and its contrapositive are logically equivalent, i.e., $A \Rightarrow B$ means that $\neg B \Rightarrow \neg A$
- Example: "it's raining $\Rightarrow$ Henry brings am umbrella"

  is the same as saying

  "Henry didn't bring an umbrella $\Rightarrow$ it's not raining "

Proof of Theorem 1: Finite, Realizable Case

7. Given $M \geq \frac{1}{\epsilon}\left(\ln(|\mathcal{H}|) + \ln\left(\frac{1}{\delta}\right)\right)$ labelled training data points, the probability that all hypotheses $h_k \in \mathcal{H}$ with $R(h_k) > \epsilon$ have $\hat{R}(h_k) > 0$ is $\geq 1 - \delta$

$$\Updownarrow$$

Given $M \geq \frac{1}{\epsilon}\left(\ln(|\mathcal{H}|) + \ln\left(\frac{1}{\delta}\right)\right)$ labelled training data points, the probability that all hypotheses $h_k \in \mathcal{H}$ with $\hat{R}(h_k) = 0$ have $R(h_k) \leq \epsilon$ is $\geq 1 - \delta$

(proof by contrapositive)

Theorem 1: Finite, Realizable Case

- For a finite hypothesis set $\mathcal{H}$ s.t. $c^* \in \mathcal{H}$ and arbitrary distribution p^*, if the number of labelled training data points satisfies

$$M \geq \frac{1}{\epsilon}\left(\ln(|\mathcal{H}|) + \ln\left(\frac{1}{\delta}\right)\right)$$

then with probability at least $1 - \delta$, all $h \in \mathcal{H}$ with $\hat{R}(h) = 0$ have $R(h) \leq \epsilon$

Making the bound tight and solving for ϵ gives...

Statistical Learning Theory Corollary

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- For a finite hypothesis set $\mathcal{H}$ s.t. $c^* \in \mathcal{H}$ and arbitrary distribution p^*, given a training data set S s.t. $|S| = M$, all $h \in \mathcal{H}$ with $\hat{R}(h) = 0$ have

$$R(h) \leq \frac{1}{M}\left(\ln(|\mathcal{H}|) + \ln\left(\frac{1}{\delta}\right)\right)$$

with probability at least $1 - \delta$.

Theorem 2: Finite, Agnostic Case

- For a finite hypothesis set $\mathcal{H}$ and arbitrary distribution p^*, if the number of labelled training data points satisfies

$$M \geq \frac{1}{2\epsilon^2}\left(\ln(|\mathcal{H}|) + \ln\left(\frac{2}{\delta}\right)\right)$$

then with probability at least $1 - \delta$, all $h \in \mathcal{H}$ satisfy

$\left|R(h) - \hat{R}(h)\right| \leq \epsilon$

- Bound is inversely quadratic in ϵ, e.g., halving ϵ means we need four times as many labelled training data points

- Again, making the bound tight and solving for ϵ gives…

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Statistical Learning Theory Corollary

- For a finite hypothesis set $\mathcal{H}$ and arbitrary distribution p^*, given a training data set S s.t. $|S| = M$, all $h \in \mathcal{H}$ have

$$R(h) \leq \hat{R}(h) + \sqrt{\frac{1}{2M}\left(\ln(|\mathcal{H}|) + \ln\left(\frac{2}{\delta}\right)\right)}$$

with probability at least $1 - \delta$.

Infinite Case

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What happens when $|\mathcal{H}| = \infty$?

- For a finite hypothesis set $\mathcal{H}$ and arbitrary distribution p^*, given a training data set S s.t. $|S| = M$, all $h \in \mathcal{H}$ have

$$R(h) \leq \hat{R}(h) + \sqrt{\frac{1}{2M}\left(\ln(|\mathcal{H}|) + \ln\left(\frac{2}{\delta}\right)\right)}$$

with probability at least $1 - \delta$.

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with probability at least $1 - \delta$.

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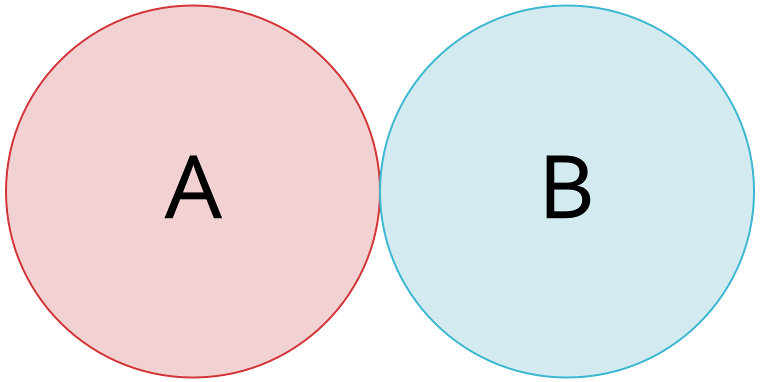
The Union Bound...

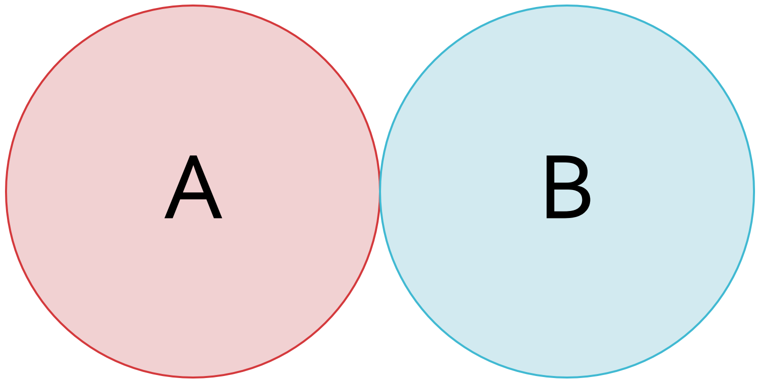
$$P\{A \cup B\} \leq P\{A\} + P\{B\}$$



The Union Bound is Bad

$$P\{A \cup B\} \leq P\{A\} + P\{B\}$$

$$P\{A \cup B\} = P\{A\} + P\{B\} - P\{A \cap B\}$$

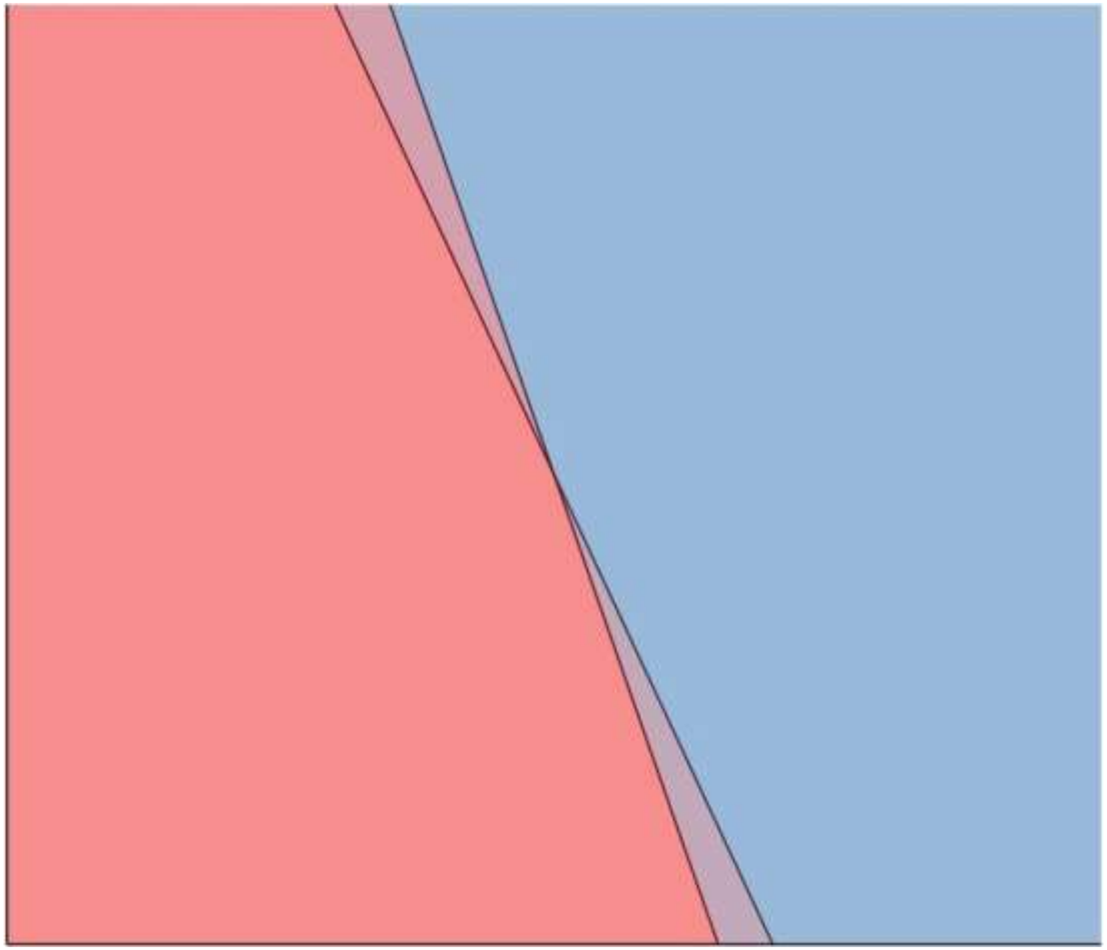


Intuition

If two hypotheses $h_1, h_2 \in \mathcal{H}$ are very similar, then the events

- "h_1 is consistent with the first m training data points"
- "h_2 is consistent with the first m training data points"

will overlap a lot!



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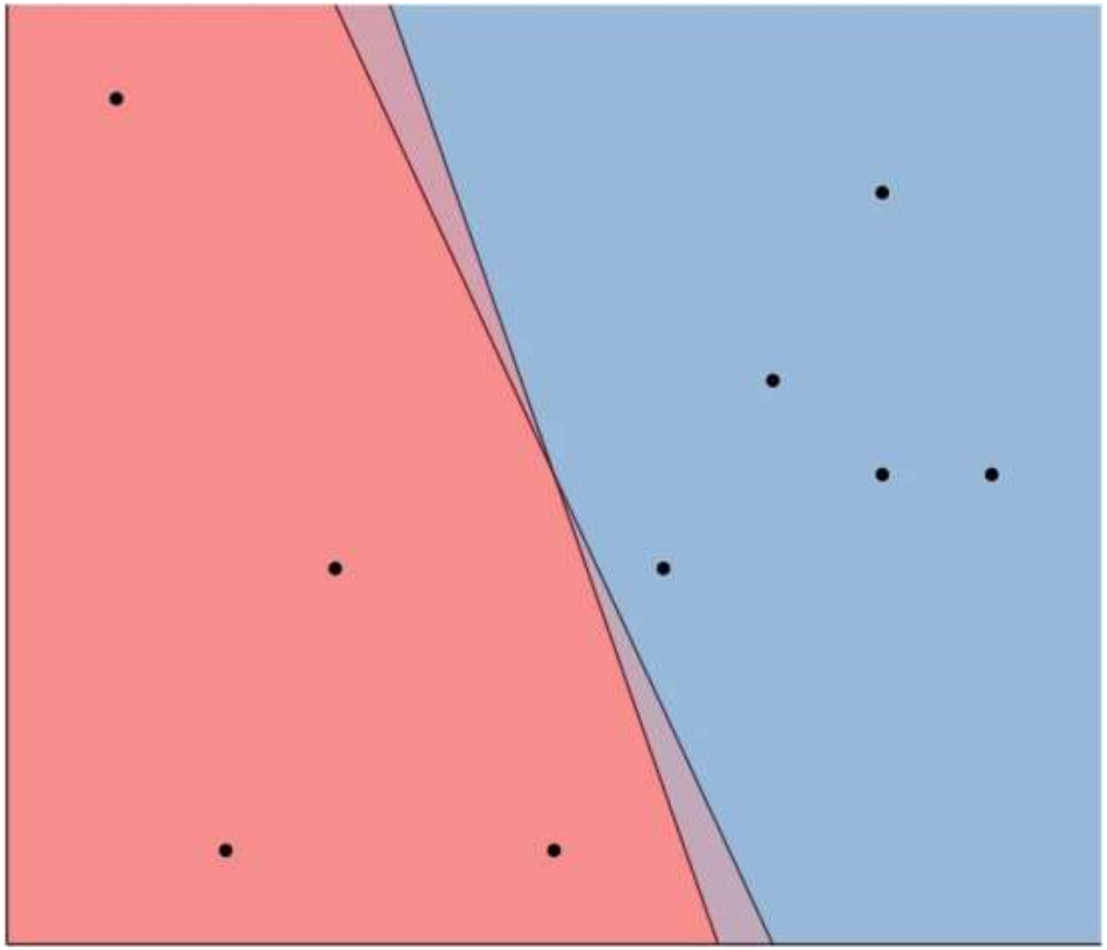


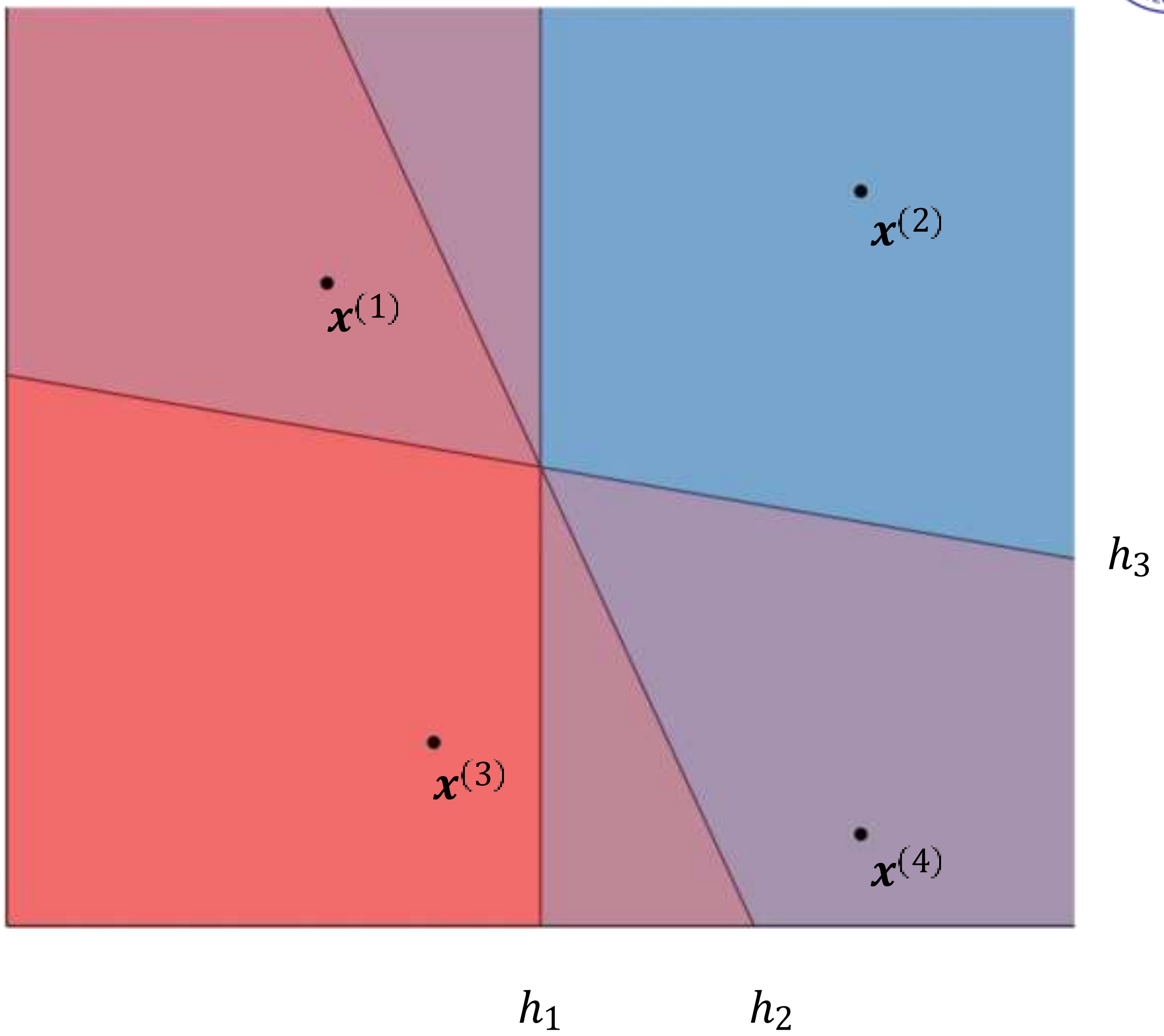
Labellings

- Given some finite set of data points $S = \left(\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(M)}\right)$ and some hypothesis $h \in \mathcal{H}$, applying h to each point in S results in a **labelling**
  - $\left(h\left(\boldsymbol{x}^{(1)}\right), \dots, h\left(\boldsymbol{x}^{(M)}\right)\right)$ is a vector of M +1's and -1's
- Given $S = \left(\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(M)}\right)$, each hypothesis in $\mathcal{H}$ induces a labelling but not necessarily a unique labelling
  - The set of labellings induced by $\mathcal{H}$ on S is

$$\mathcal{H}(S) = \left\{\left(h\left(\boldsymbol{x}^{(1)}\right), \dots, h\left(\boldsymbol{x}^{(M)}\right)\right) \,\middle|\, h \in \mathcal{H}\right\}$$

Example: Labellings

$\mathcal{H} = \{h_1, h_2, h_3\}$



Example: Labellings

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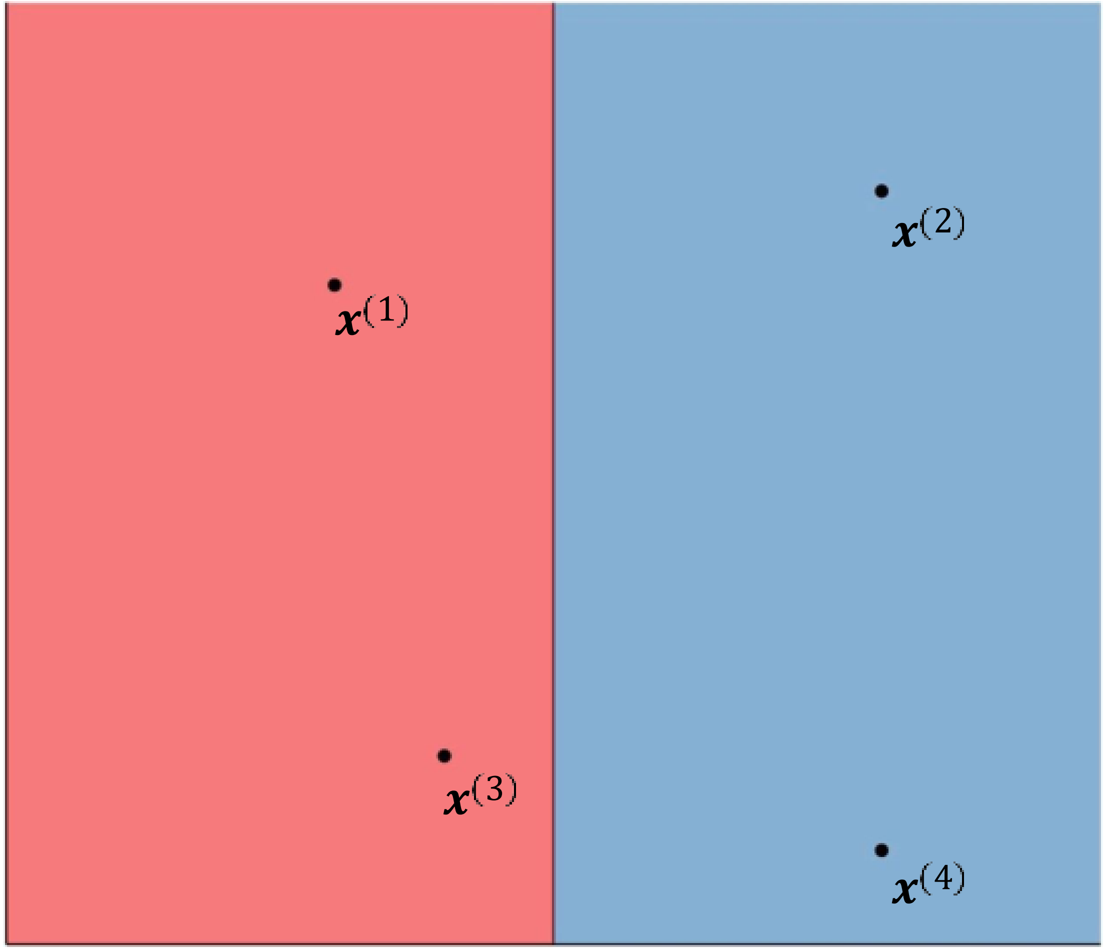
$$\mathcal{H} = \{h_1, h_2, h_3\}$$

$$\left(h_1(\boldsymbol{x}^{(1)}), h_1(\boldsymbol{x}^{(2)}), h_1(\boldsymbol{x}^{(3)}), h_1(\boldsymbol{x}^{(4)})\right)$$
$$= (-1, +1, -1, +1)$$

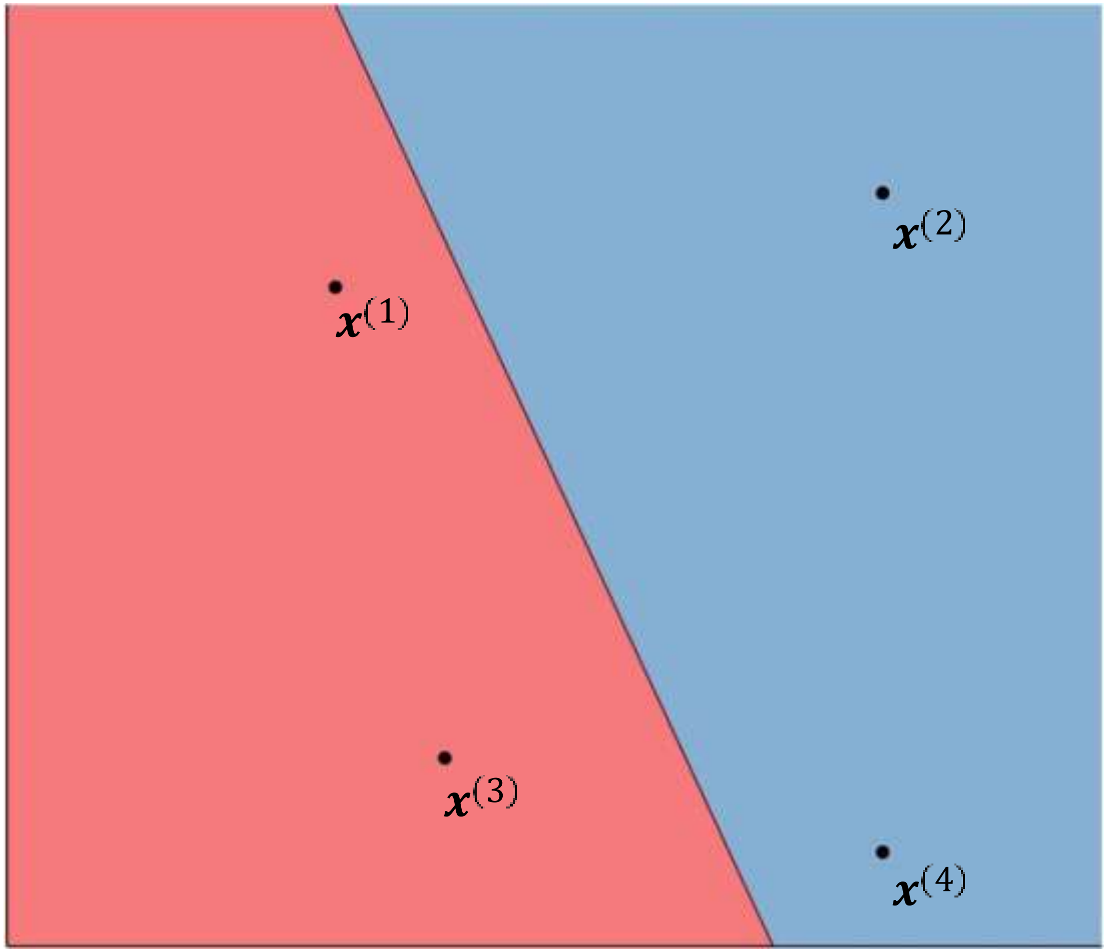


h_1

Example: Labellings

$\mathcal{H} = \{h_1, h_2, h_3\}$

$$\left(h_2\left(\boldsymbol{x}^{(1)}\right), h_2\left(\boldsymbol{x}^{(2)}\right), h_2\left(\boldsymbol{x}^{(3)}\right), h_2\left(\boldsymbol{x}^{(4)}\right)\right)$$
$$= (-1, +1, -1, +1)$$

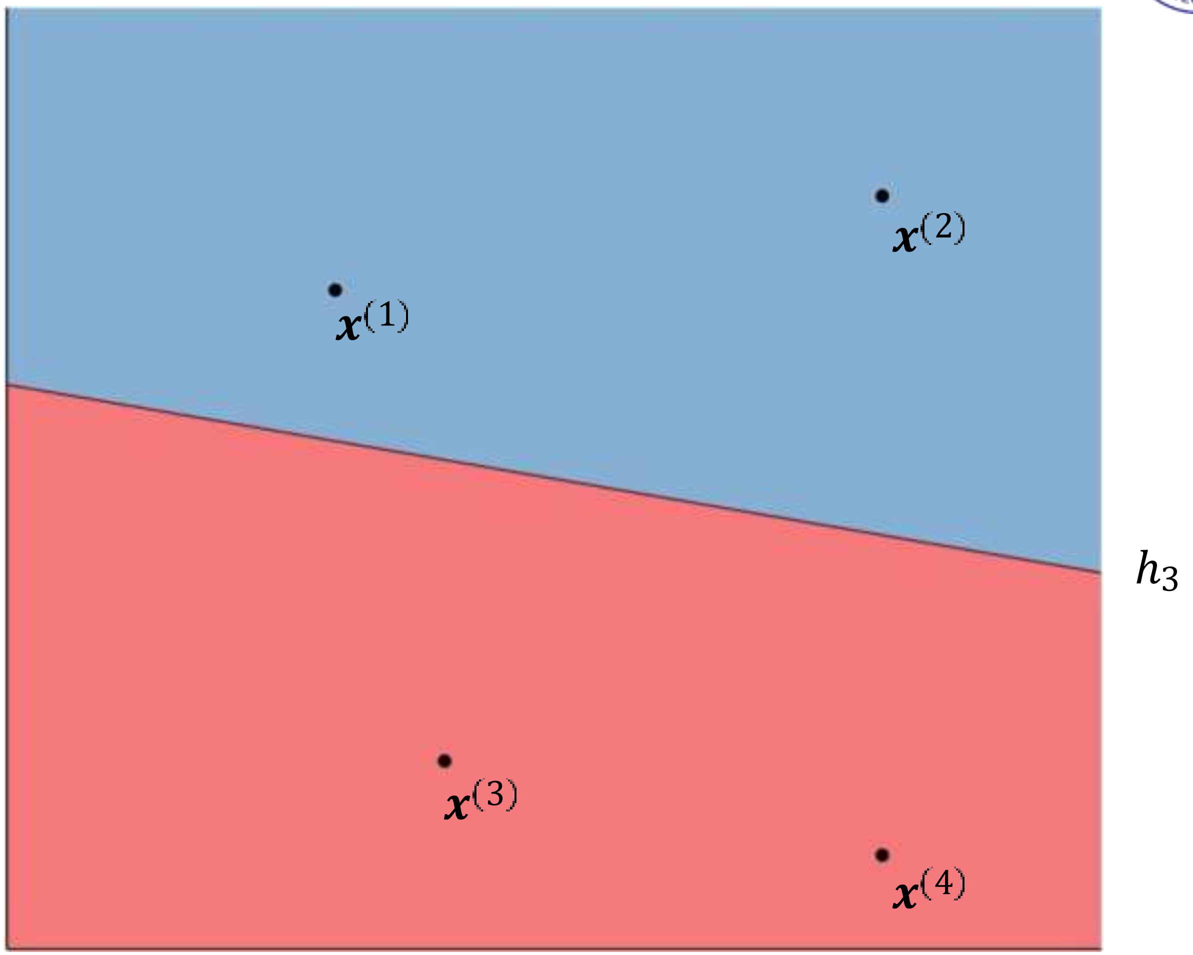


h_2

Example: Labellings

$$\mathcal{H} = \{h_1, h_2, h_3\}$$

$$\left(h_3\left(\boldsymbol{x}^{(1)}\right), h_3\left(\boldsymbol{x}^{(2)}\right), h_3\left(\boldsymbol{x}^{(3)}\right), h_3\left(\boldsymbol{x}^{(4)}\right)\right)$$
$$= (+1, +1, -1, -1)$$



Example: Labellings

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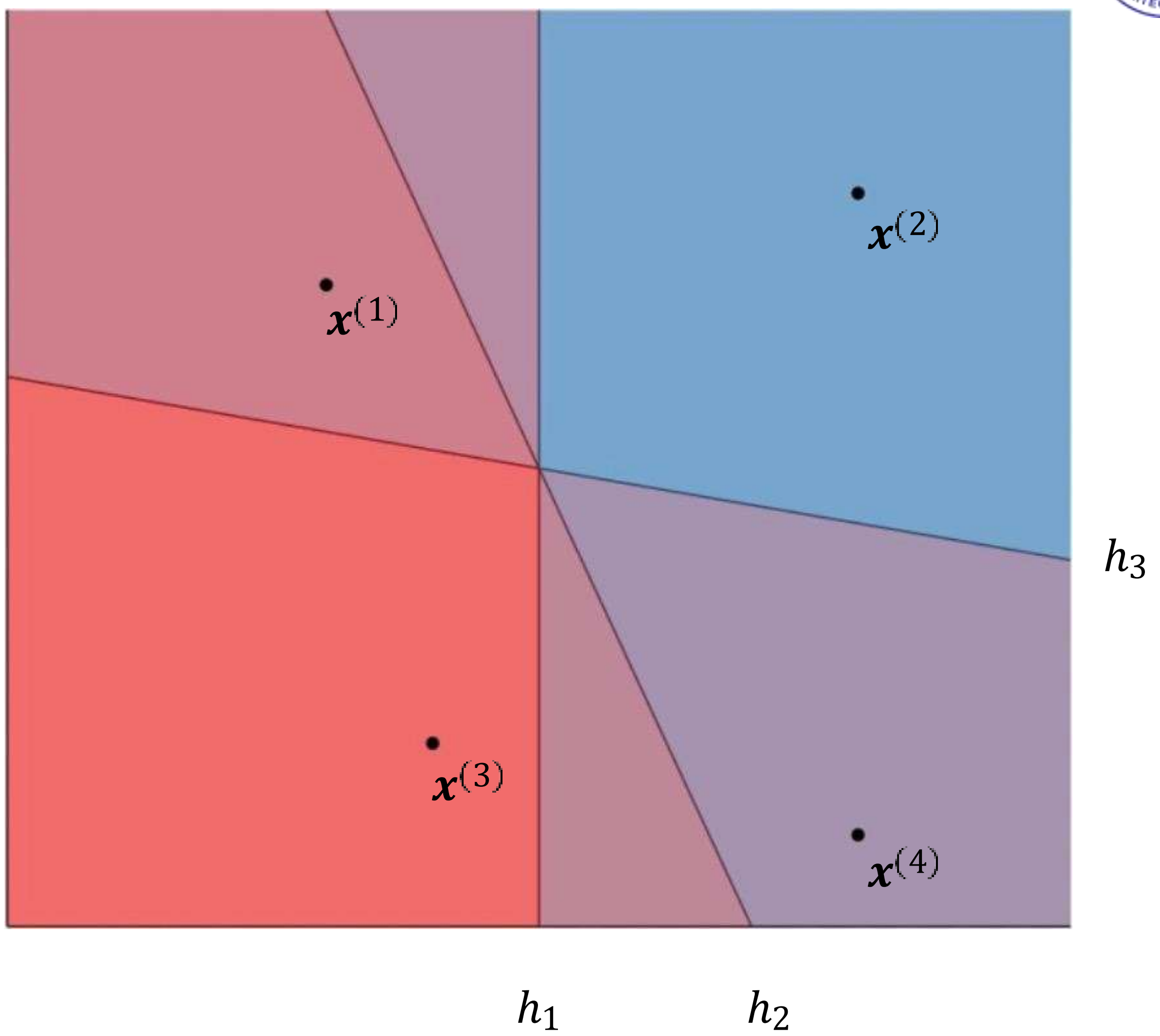
$\mathcal{H} = \{h_1, h_2, h_3\}$

$\mathcal{H}(S)$
$= \{(+1, +1, -1, -1), (-1, +1, -1, +1)\}$

$|\mathcal{H}(S)| = 2$



Example: Labellings

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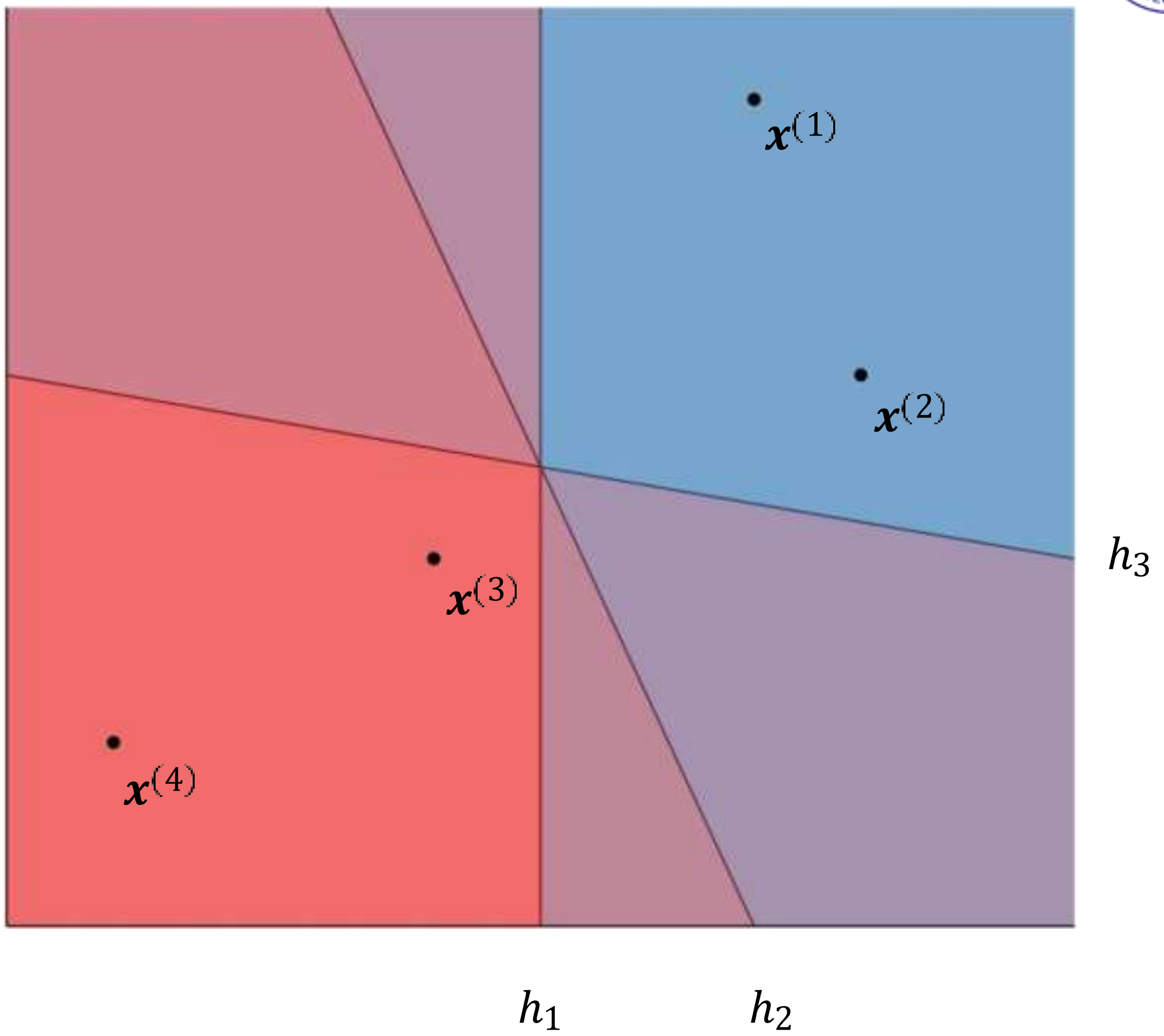
$$\mathcal{H} = \{h_1, h_2, h_3\}$$

$$\mathcal{H}(S) = \{(+1, +1, -1, -1)\}$$

$$|\mathcal{H}(S)| = 1$$

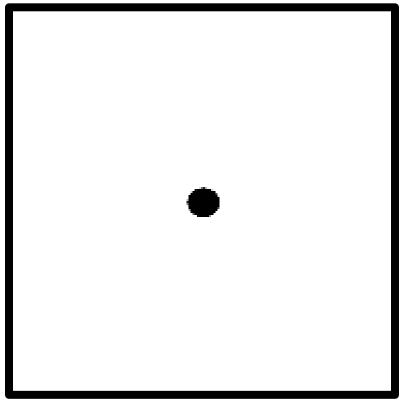


VC-Dimension

- $\mathcal{H}(S)$ is the set of all labellings induced by $\mathcal{H}$ on S
  - If $|S| = M$, then $|\mathcal{H}(S)| \leq 2^M$
  - $\mathcal{H}$ **shatters** S if $|\mathcal{H}(S)| = 2^M$
- The **VC-dimension** of $\mathcal{H}$, $VC(\mathcal{H})$, is the size of the largest set S that can be shattered by $\mathcal{H}$.
  - If $\mathcal{H}$ can shatter arbitrarily large finite sets, then $d_{VC}(\mathcal{H}) = \infty$
- To prove that $VC(\mathcal{H}) = d$, you need to show
  1. $\exists$ some set of d data points that $\mathcal{H}$ can shatter and
  2. $\nexists$ a set of $d+1$ data points that $\mathcal{H}$ can shatter

VC-Dimension: Example

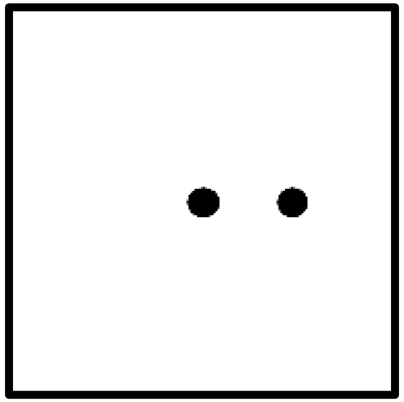
- $\boldsymbol{x} \in \mathbb{R}^2$ and $\mathcal{H}$ = all 2-dimensional linear separators
- What is $VC(\mathcal{H})$?
  - Can $\mathcal{H}$ shatter some set of 1 point?



S

VC-Dimension: Example

- $\boldsymbol{x} \in \mathbb{R}^2$ and $\mathcal{H}$ = all 2-dimensional linear separators
- What is $VC(\mathcal{H})$?
  - Can $\mathcal{H}$ shatter some set of 1 point?
  - Can $\mathcal{H}$ shatter some set of 2 points?



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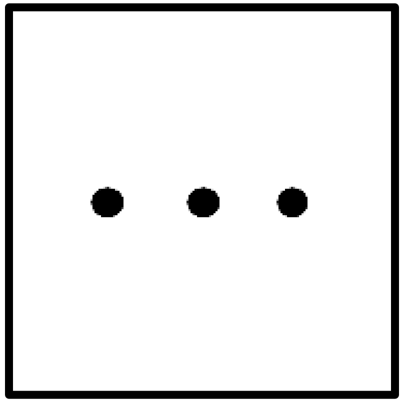
VC-Dimension: Example

- $\boldsymbol{x} \in \mathbb{R}^2$ and $\mathcal{H} =$ all 2-dimensional linear separators
- What is $VC(\mathcal{H})$?
  - Can $\mathcal{H}$ shatter some set of 1 point?
  - Can $\mathcal{H}$ shatter some set of 2 points?
  - Can $\mathcal{H}$ shatter some set of 3 points?



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VC-Dimension: Example

- $\boldsymbol{x} \in \mathbb{R}^2$ and $\mathcal{H}$ = all 2-dimensional linear separators
- What is $VC(\mathcal{H})$?
  - Can $\mathcal{H}$ shatter some set of 1 point?
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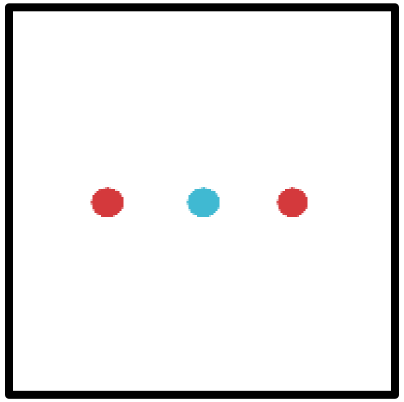
  - Can $\mathcal{H}$ shatter some set of 3 points?



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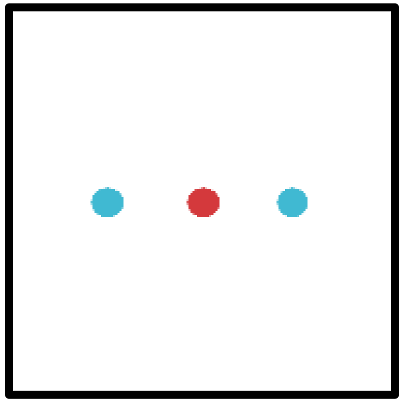
VC-Dimension: Example

- $\boldsymbol{x} \in \mathbb{R}^2$ and $\mathcal{H}$ = all 2-dimensional linear separators
- What is $VC(\mathcal{H})$?
  - Can $\mathcal{H}$ shatter some set of 1 point?
  - Can $\mathcal{H}$ shatter some set of 2 points?
  - Can $\mathcal{H}$ shatter some set of 3 points?



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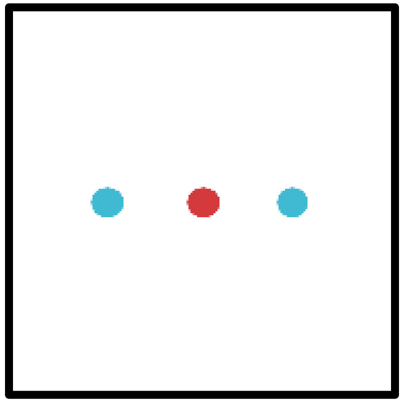
VC-Dimension: Example

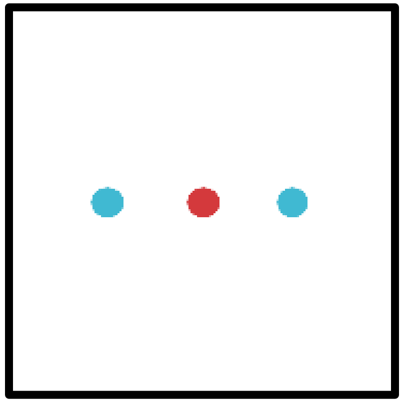
- $\boldsymbol{x} \in \mathbb{R}^2$ and $\mathcal{H}$ = all 2-dimensional linear separators
- What is $VC(\mathcal{H})$?
  - Can $\mathcal{H}$ shatter some set of 1 point?
  - Can $\mathcal{H}$ shatter some set of 2 points?
  - Can $\mathcal{H}$ shatter ***some*** set of 3 points?



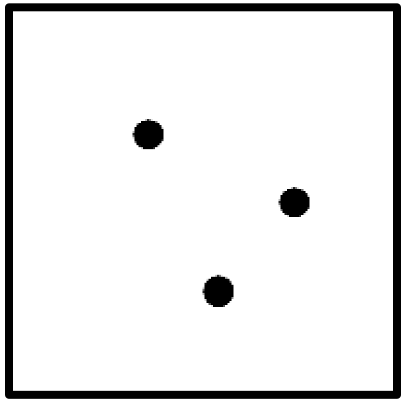
S

VC-Dimension: Example

- $\boldsymbol{x} \in \mathbb{R}^2$ and $\mathcal{H}$ = all 2-dimensional linear separators
- What is $VC(\mathcal{H})$?
  - Can $\mathcal{H}$ shatter some set of 1 point?
  - Can $\mathcal{H}$ shatter some set of 2 points?
  - Can $\mathcal{H}$ shatter some set of 3 points?



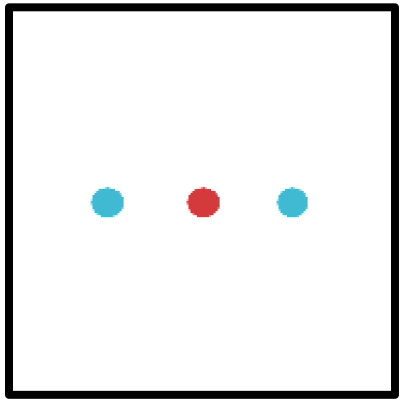
S_1



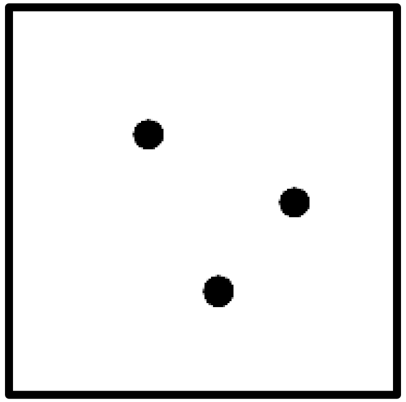
S_2

VC-Dimension: Example

- $\boldsymbol{x} \in \mathbb{R}^2$ and $\mathcal{H}$ = all 2-dimensional linear separators
- What is $VC(\mathcal{H})$?
  - Can $\mathcal{H}$ shatter some set of 1 point?
  - Can $\mathcal{H}$ shatter some set of 2 points?
  - Can $\mathcal{H}$ shatter some set of 3 points?



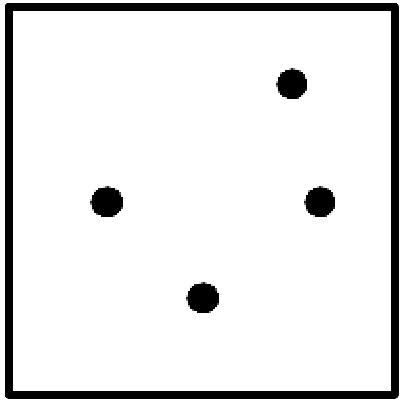
$|\mathcal{H}(S_1)| = 6$



$|\mathcal{H}(S_2)| = 8$

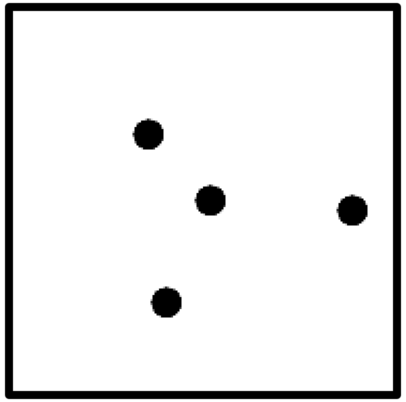
VC-Dimension: Example

- $\boldsymbol{x} \in \mathbb{R}^2$ and $\mathcal{H}$ = all 2-dimensional linear separators
- What is $VC(\mathcal{H})$?
  - Can $\mathcal{H}$ shatter some set of 1 point?
  - Can $\mathcal{H}$ shatter some set of 2 points?
  - Can $\mathcal{H}$ shatter some set of 3 points?
  - Can $\mathcal{H}$ shatter some set of 4 points?



S_1

All points on the convex hull

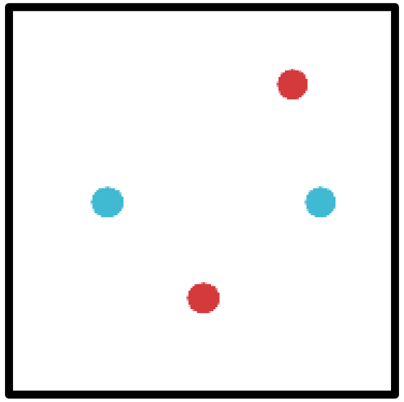


S_2

At least one point inside the convex hull

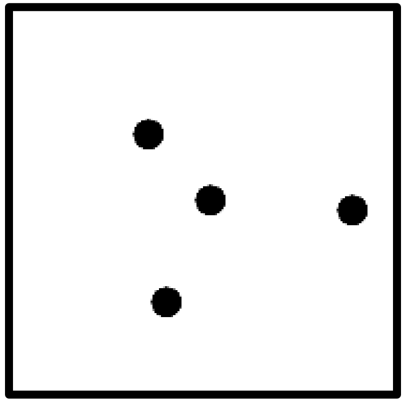
VC-Dimension: Example

- $\boldsymbol{x} \in \mathbb{R}^2$ and $\mathcal{H}$ = all 2-dimensional linear separators
- What is $VC(\mathcal{H})$?
  - Can $\mathcal{H}$ shatter some set of 1 point?
  - Can $\mathcal{H}$ shatter some set of 2 points?
  - Can $\mathcal{H}$ shatter some set of 3 points?
  - Can $\mathcal{H}$ shatter some set of 4 points?



S_1

All points on the convex hull

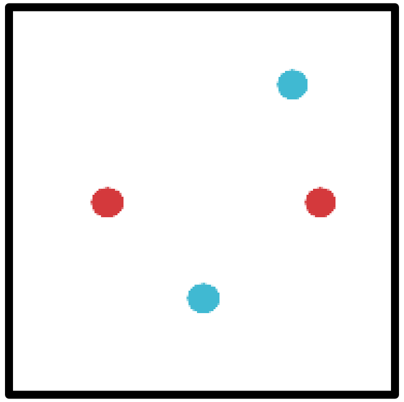


S_2

At least one point inside the convex hull

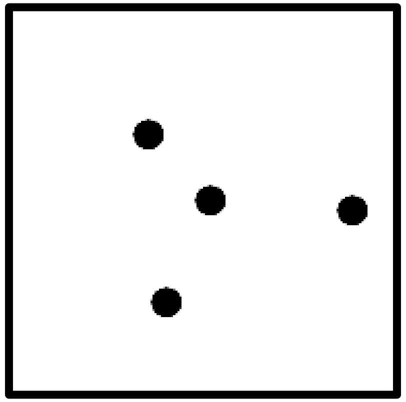
VC-Dimension: Example

- $\boldsymbol{x} \in \mathbb{R}^2$ and $\mathcal{H}$ = all 2-dimensional linear separators
- What is $VC(\mathcal{H})$?
  - Can $\mathcal{H}$ shatter some set of 1 point?
  - Can $\mathcal{H}$ shatter some set of 2 points?
  - Can $\mathcal{H}$ shatter some set of 3 points?
  - Can $\mathcal{H}$ shatter some set of 4 points?



S_1

All points on the convex hull



S_2

At least one point inside the convex hull

VC-Dimension: Example

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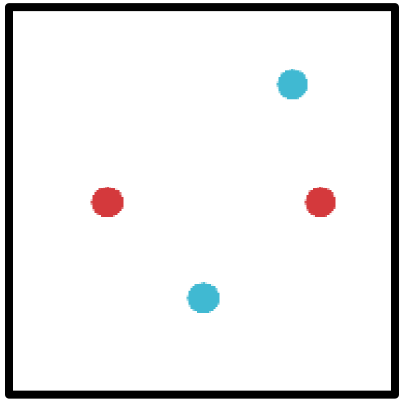
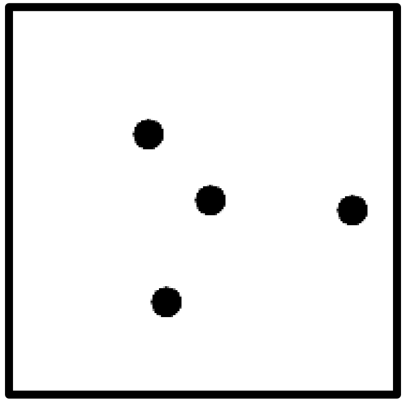
  - Can $\mathcal{H}$ shatter some set of 2 points?
  - Can $\mathcal{H}$ shatter some set of 3 points?
  - Can $\mathcal{H}$ shatter some set of 4 points?



$|\mathcal{H}(S_1)| = 14$

All points on the convex hull



S_2

At least one point inside the convex hull

VC-Dimension: Example

- $\boldsymbol{x} \in \mathbb{R}^2$ and $\mathcal{H}$ = all 2-dimensional linear separators
- What is $VC(\mathcal{H})$?
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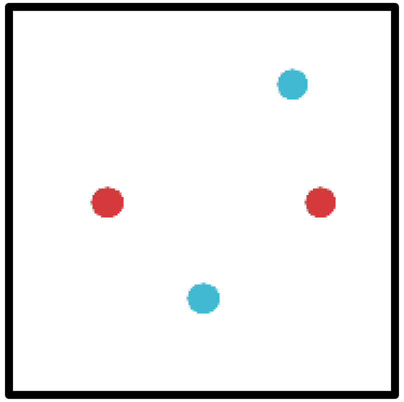
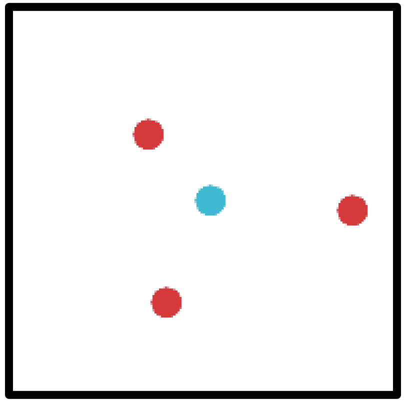
  - Can $\mathcal{H}$ shatter some set of 2 points?
  - Can $\mathcal{H}$ shatter some set of 3 points?
  - Can $\mathcal{H}$ shatter some set of 4 points?



$|\mathcal{H}(S_1)| = 14$

All points on the convex hull

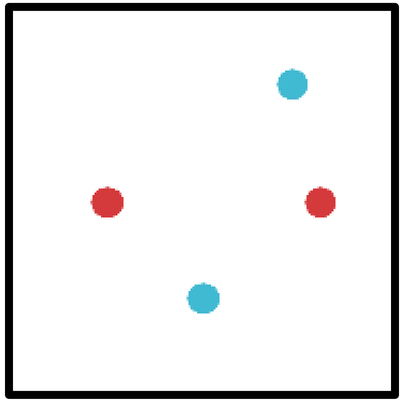


S_2

At least one point inside the convex hull

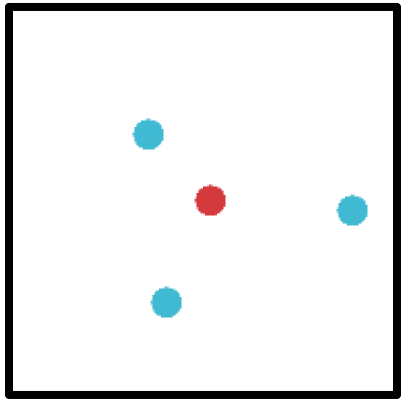
VC-Dimension: Example

- $\boldsymbol{x} \in \mathbb{R}^2$ and $\mathcal{H}$ = all 2-dimensional linear separators
- What is $VC(\mathcal{H})$?
  - Can $\mathcal{H}$ shatter some set of 1 point?
  - Can $\mathcal{H}$ shatter some set of 2 points?
  - Can $\mathcal{H}$ shatter some set of 3 points?
  - Can $\mathcal{H}$ shatter some set of 4 points?



$|\mathcal{H}(S_1)| = 14$

All points on the convex hull

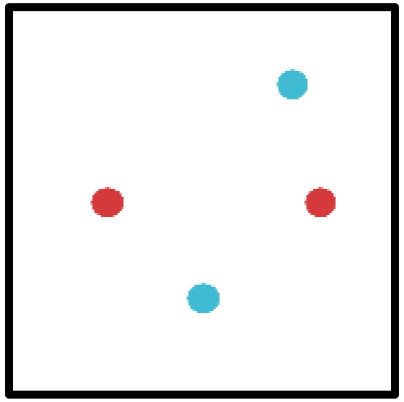


S_2

At least one point inside the convex hull

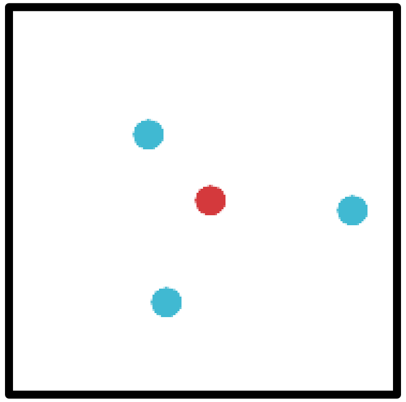
VC-Dimension: Example

- $\boldsymbol{x} \in \mathbb{R}^2$ and $\mathcal{H}$ = all 2-dimensional linear separators
- What is $VC(\mathcal{H})$?
  - Can $\mathcal{H}$ shatter some set of 1 point?
  - Can $\mathcal{H}$ shatter some set of 2 points?
  - Can $\mathcal{H}$ shatter some set of 3 points?
  - Can $\mathcal{H}$ shatter some set of 4 points?



$|\mathcal{H}(S_1)| = 14$

All points on the convex hull

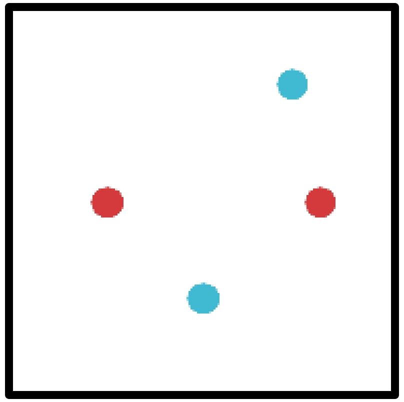


$|\mathcal{H}(S_2)| = 14$

At least one point inside the convex hull

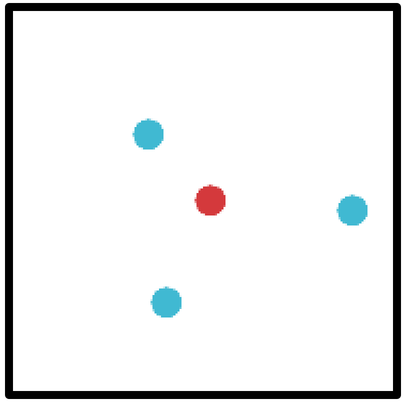
VC-Dimension: Example

- $\boldsymbol{x} \in \mathbb{R}^2$ and $\mathcal{H}$ = all 2-dimensional linear separators
- $VC(\mathcal{H}) = 3$
  - Can $\mathcal{H}$ shatter some set of 1 point?
  - Can $\mathcal{H}$ shatter some set of 2 points?
  - Can $\mathcal{H}$ shatter some set of 3 points?
  - Can $\mathcal{H}$ shatter some set of 4 points?



$|\mathcal{H}(S_1)| = 14$

All points on the convex hull



$|\mathcal{H}(S_2)| = 14$

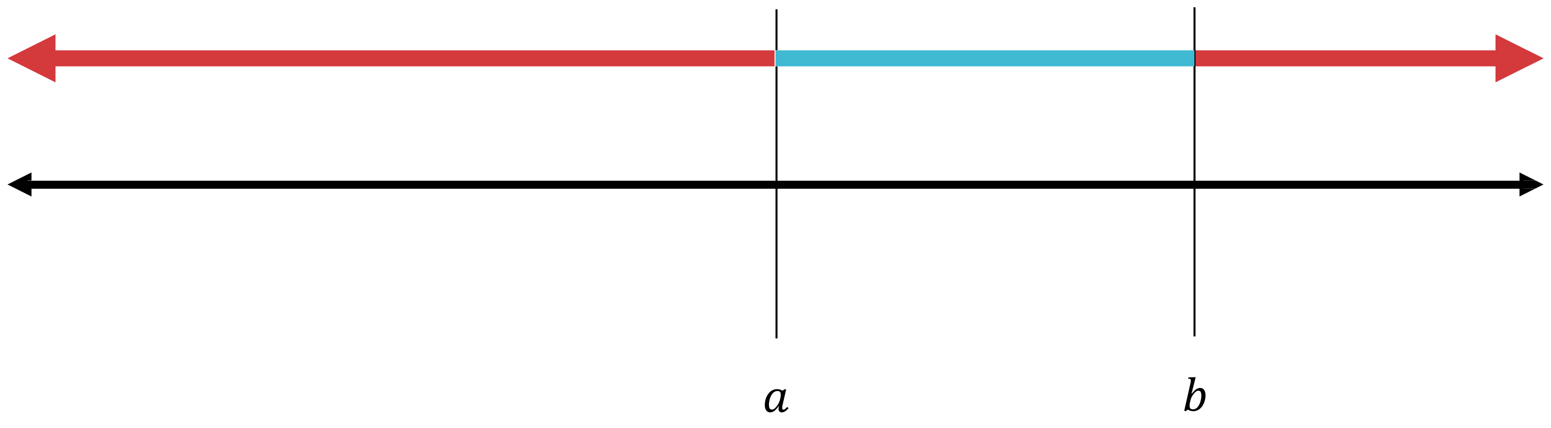
At least one point inside the convex hull

VC-Dimension: Example

- $\boldsymbol{x} \in \mathbb{R}^d$ and $\mathcal{H} =$ all d-dimensional linear separators
- $VC(\mathcal{H}) = d + 1$

VC-Dimension: Example

- $x \in \mathbb{R}$ and $\mathcal{H}$ = all 1-dimensional positive intervals

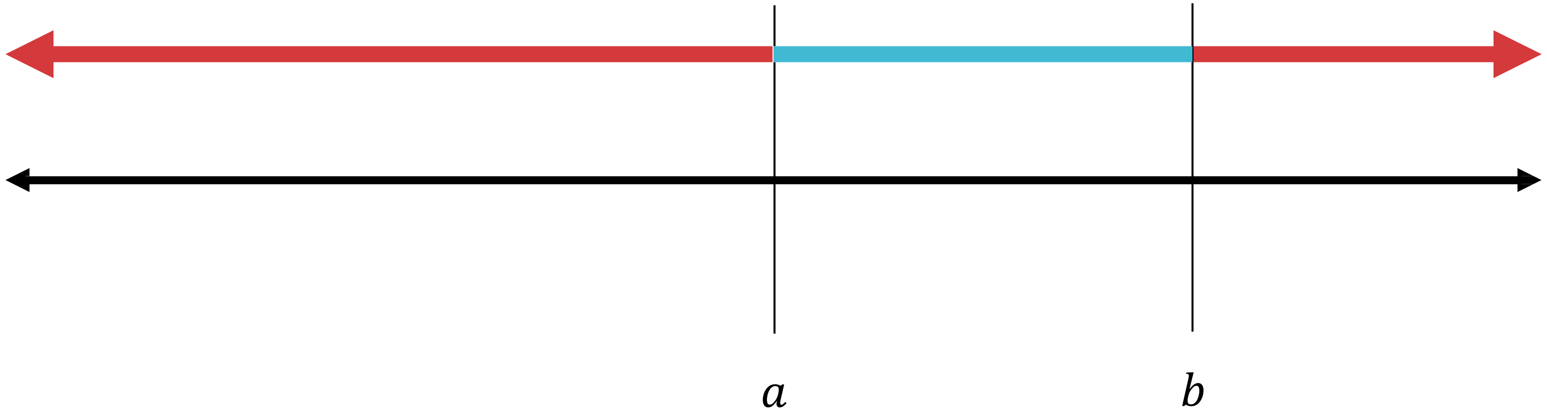


Poll Question 1:

What is $VC(\mathcal{H})$?

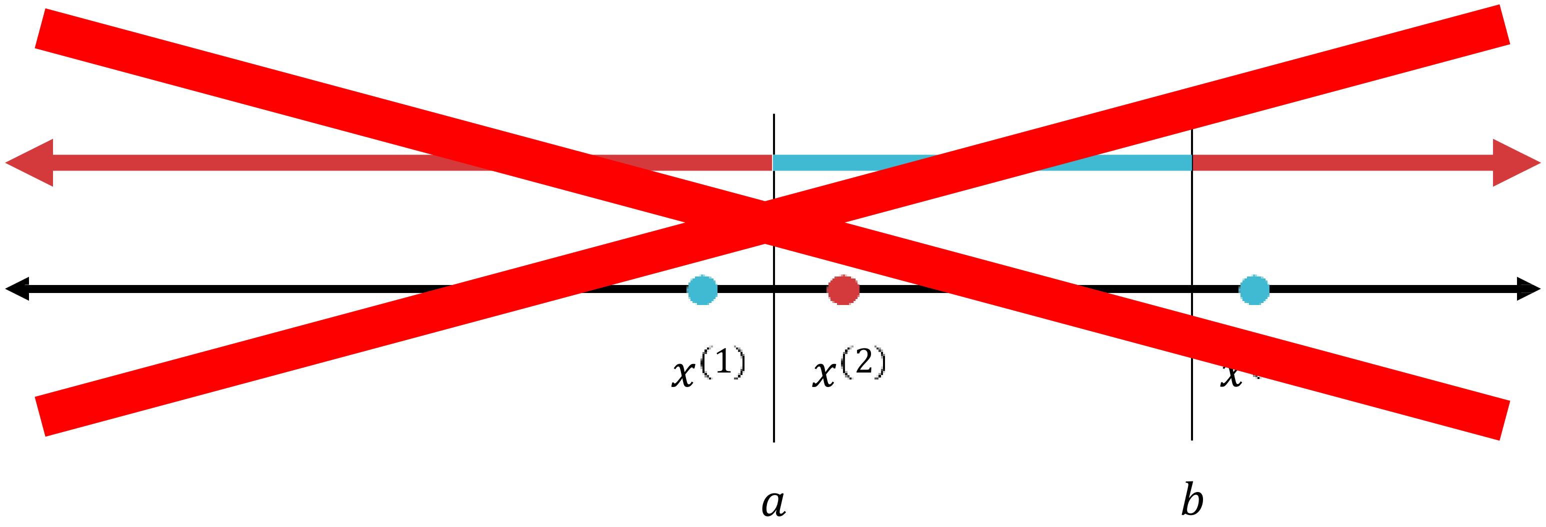
A. 0
B. 1
C. 1.5 **(TOXIC)**
D. 2
E. 3

- $x \in \mathbb{R}$ and $\mathcal{H}$ = all 1-dimensional positive intervals



VC-Dimension: Example

- $x \in \mathbb{R}$ and $\mathcal{H}$ = all 1-dimensional positive intervals



- $VC(\mathcal{H}) = 2$

Theorem 3: Vapnik-Chervonenkis (VC)-Bound

- Infinite, realizable case: for any hypothesis set $\mathcal{H}$ and distribution p^*, if the number of labelled training data points satisfies

$$M = O\left(\frac{1}{\epsilon}\left(VC(\mathcal{H})\log\left(\frac{1}{\epsilon}\right) + \log\left(\frac{1}{\delta}\right)\right)\right)$$

then with probability at least $1-\delta$, all $h \in \mathcal{H}$ with $\hat{R}(h) = 0$ have $R(h) \leq \epsilon$

Statistical Learning Theory Corollary 3

- Infinite, realizable case: for any hypothesis set $\mathcal{H}$ and distribution p^*, given a training data set S s.t. $|S| = M$, all $h \in \mathcal{H}$ with $\hat{R}(h) = 0$ have

$$R(h) \leq O\left(\frac{1}{M}\left(VC(\mathcal{H}) \log\left(\frac{M}{VC(\mathcal{H})}\right) + \log\left(\frac{1}{\delta}\right)\right)\right)$$

with probability at least $1 - \delta$.

Theorem 4: Vapnik-Chervonenkis (VC)-Bound

- Infinite, agnostic case: for any hypothesis set $\mathcal{H}$ and distribution p^*, if the number of labelled training data points satisfies

$$M = O\left(\frac{1}{\epsilon^2}\left(VC(\mathcal{H}) + \log\left(\frac{1}{\delta}\right)\right)\right)$$

then with probability at least $1 - \delta$, all $h \in \mathcal{H}$ have $\left|R(h) - \hat{R}(h)\right| \leq \epsilon$

Statistical Learning Theory Corollary 4

- Infinite, agnostic case: for any hypothesis set $\mathcal{H}$ and distribution p^*, given a training data set S s.t. $|S| = M$, all $h \in \mathcal{H}$ have

$$R(h) \leq \hat{R}(h) + O\left(\sqrt{\frac{1}{M}\left(VC(\mathcal{H}) + \log\left(\frac{1}{\delta}\right)\right)}\right)$$

with probability at least $1 - \delta$.

Approximation Generalization Tradeoff

How well does h generalize?

$$R(h) \leq \underbrace{\hat{R}(h)}_{\text{How well does } h \text{ approximate } c^*?} + O\left(\overbrace{\sqrt{\frac{1}{M}\left(VC(\mathcal{H}) + \log\left(\frac{1}{\delta}\right)\right)}}^{\text{How well does } h \text{ generalize?}}\right)$$

How well does h approximate c^*?

Approximation Generalization Tradeoff

Increases as $VC(\mathcal{H})$ increases

$$R(h) \le \underbrace{\hat{R}(h)}_{\text{Decreases as } VC(\mathcal{H}) \text{ increases}} + O\left(\overbrace{\sqrt{\frac{1}{M}\left(VC(\mathcal{H}) + \log\left(\frac{1}{\delta}\right)\right)}}^{\text{Increases as } VC(\mathcal{H}) \text{ increases}}\right)$$

Decreases as $VC(\mathcal{H})$ increases

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Learning Theory Learning Objectives

You should be able to…

- Identify the properties of a learning setting and assumptions required to ensure low generalization error
- Distinguish true error, train error, test error
- Define PAC and explain what it means to be approximately correct and what occurs with high probability
- Apply sample complexity bounds to real-world machine learning examples