Homework 1

Due: 23:59 (CST), Mar 7, 2025
Full Mark: 50 points, 6 problems including 1 bonus.

Read all the instructions below carefully before you start working on the assignment, and before you make a submission.

- You are required to write down all the major steps towards making your conclusions; otherwise you may obtain limited points of the problem.
- Write your homework in English; otherwise you will get no points of this homework.
- Any form of plagiarism will lead to 0 point of this homework.
- If there are any descriptions unclear, we recommend you to ask on Piazza.

Problem 1

(Classical Probability Model *5'+5'***)**

a) A fair six-sided die is rolled 6 times. What is the probability of obtaining at least one repeated value?

b) A *norepeatword* is a sequence of at least one (and possibly all) of the usual 26 letters $a, b, c, ..., z$, with no repetitions allowed. For example, "course" is a *norepeatword*, but "statistics" is not. Order matters, so "course" is not the same as "source." A *norepeatword* is chosen randomly, with all *norepeatword*s equally likely. Show that the probability that it uses all 26 letters is very close to $1/e$.

Problem 2

(Story Proof *5'+5'+4'***)**

a) Show that for all positive integers n and k with $n \geq k$,

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k},$$

doing this in two ways: (i) algebraically and (ii) with a story, giving an interpretation for why both sides count the same thing.
Hint: Imagine $n+1$ people, with one of them pre-designated as "president".

b) Show using a story proof that

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1},$$

where n and k are positive integers with $n \geq k$. This is called the hockey stick identity.
Hint: Imagine arranging a group of people by age, and then think about the oldest person in a chosen subgroup.

c) Suppose that a large pack of Haribo gummi bears can have anywhere between 30 and 50 gummi bears. There are 5 flavors: pineapple (clear), raspberry (red), orange (orange), strawberry (green, mysteriously), and lemon (yellow). How many possibilities (cases) are there for the composition of such a pack of gummi bears? You can leave your answer in terms of a couple binomial coefficients, but not a sum of lots of binomial coefficients.

Problem 3

(Geometric Probability Model: Stick *8'***)**

You get a stick and break it randomly into three pieces simultaneously. What is the probability that you can make a triangle using such three pieces?

Problem 4

(Geometric Probability Model: Needle *8'*)

There are infinitely many parallel lines on a plane, with a spacing of L. There is a needle of length l where $l \leq L$. Suppose the needle falls on the plane in an equiprobable fashion (it is on the plane horizontally, not vertically), what is the probability of it intersecting at least one of these parallel lines?

Problem 5

(Coupon Collection *7'+3'***)**

If each box of a brand of crispy instant noodle contains a coupon, and there are 114 different types of coupons.

a) Given $n \geq 114$, what is the probability that buying n boxes can collect all 114 types of coupons? You may search about the Stirling number of the second kind. **Hint:** There are two recommended methods to solve this problem.

b) When such probability is no less than 80%, what is the minimum number of n? What about 90%? You may write code about this, and you should attach a graph to demonstrate this.

Problem 6

(Bonus *10'***)**

There are 100 passengers lined up to board an airplane with 100 seats (with each seat assigned to one of the passengers). The first passenger in line crazily decides to sit in a randomly chosen seat (with all seats equally likely). Each subsequent passenger takes his or her assigned seat if available, and otherwise sits in a random available seat. What is the probability that the last passenger in line gets to sit in his or her assigned seat?
Hint: Call the seat assigned to the j-th passenger in line "seat j" (regardless of whether the airline calls it seat 23A or whatever). What are the possibilities for which seats are available to the last passenger in line, and what is the probability of each of these possibilities?