Imaging Science
Assignment H3 (Theoretical)

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1. Linear Filters

(a) Two-dimensional binomial filters can be generated by using two one-dimensional binomial filters in a separable fashion

$$(\frac{1}{16})\begin{bmatrix}1 & 4 & 6 & 4 & 1\end{bmatrix} * (\frac{1}{16})\begin{bmatrix}1\\4\\6\\4\\1\end{bmatrix} = (\frac{1}{256})\begin{bmatrix}1 & 4 & 6 & 4 & 1\\4 & 16 & 24 & 16 & 4\\6 & 24 & 36 & 24 & 6\\4 & 16 & 24 & 16 & 4\\1 & 4 & 6 & 4 & 1\end{bmatrix}$$

(b) Highpass filter is difference between identity and lowpass

$$\begin{bmatrix}0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\end{bmatrix} - (\frac{1}{256})\begin{bmatrix}1 & 4 & 6 & 4 & 1\\4 & 16 & 24 & 16 & 4\\6 & 24 & 36 & 24 & 6\\4 & 16 & 24 & 16 & 4\\1 & 4 & 6 & 4 & 1\end{bmatrix} = -\frac{1}{256}\begin{bmatrix}1 & 4 & 6 & 4 & 1\\4 & 16 & 24 & 16 & 4\\6 & 24 & -220 & 24 & 6\\4 & 16 & 24 & 16 & 4\\1 & 4 & 6 & 4 & 1\end{bmatrix}$$

(c) Another binomial filter

$$(\frac{1}{4})\begin{bmatrix}1 & 2 & 1\end{bmatrix} * (\frac{1}{4})\begin{bmatrix}1\\2\\1\end{bmatrix} = (\frac{1}{16})\begin{bmatrix}1 & 2 & 1\\2 & 4 & 2\\1 & 2 & 1\end{bmatrix}$$

Bandpass filter

$$\frac{1}{256}\begin{bmatrix}1 & 4 & 6 & 4 & 1\\4 & 16 & 24 & 16 & 4\\6 & 24 & 36 & 24 & 6\\4 & 16 & 24 & 16 & 4\\1 & 4 & 6 & 4 & 1\end{bmatrix} - (\frac{1}{256})\begin{bmatrix}0 & 0 & 0 & 0 & 0\\0 & 16 & 32 & 16 & 0\\0 & 32 & 64 & 32 & 0\\0 & 16 & 32 & 16 & 0\\0 & 0 & 0 & 0 & 0\end{bmatrix} = (\frac{1}{256})\begin{bmatrix}1 & 4 & 6 & 4 & 1\\4 & 0 & -8 & 0 & 4\\6 & -8 & -28 & -8 & 6\\4 & 0 & -8 & 0 & 4\\1 & 4 & 6 & 4 & 1\end{bmatrix}$$

(d) The stencil would be

$$\begin{bmatrix}-1 & -3 & -4 & -3 & -1\\-1 & 15 & 32 & 15 & -1\\-1 & -3 & -4 & -3 & -1\end{bmatrix}$$

2. Stencils
(a) x-direction: lowpass, y-direction: highpass & 2nd derivative
(b) x-direction: highpass & 1st derivative, y-direction: highpass & 1st derivative
(c) x-direction: highpass, y-direction: lowpass

3. First Order 1-D Derivative Filters
   a. We want to approximate the first order derivative $f'(x)$ in pixel i using the values f_{i-2}, f_{i-1}, f_i, f_{i+1}, and f_{i+2}. Firstly, we find the Taylor expansion around these pixel values:

$$f_{i-2} = f_i - 2h \cdot f_i' + \frac{4h^2}{2} f_i'' - \frac{8h^3}{6} f_i''' + \frac{16h^4}{24} f_i'''' - \frac{32h^5}{120} f_i''''' + O\left(h^6\right)$$

$$f_{i-1} = f_i - h \cdot f_i' + \frac{h^2}{2} f_i'' - \frac{h^3}{6} f_i''' + \frac{h^4}{24} f_i'''' - \frac{h^5}{120} f_i''''' + O\left(h^6\right)$$

$$f_i = f_i$$

$$f_{i+1} = f_i + h \cdot f_i' + \frac{h^2}{2} f_i'' + \frac{h^3}{6} f_i''' + \frac{h^4}{24} f_i'''' + \frac{h^5}{120} f_i''''' + O\left(h^6\right)$$

$$f_{i+2} = f_i + 2h \cdot f_i' + \frac{4h^2}{2} f_i'' + \frac{8h^3}{6} f_i''' + \frac{16h^4}{24} f_i'''' + \frac{32h^5}{120} f_i''''' + O\left(h^6\right)$$

Since we would like to approximate $f'(x)$, we now want to assign weights/coefficients to each pixel such that the linear combination of them approximates $f'(x)$:

$$\alpha_{-2} f_{i-2} + \alpha_{-1} f_{i-1} + \alpha_0 f_i + \alpha_1 f_{i+1} + \alpha_2 f_{i+2} = 0 \cdot f_i + 1 \cdot f_i' + 0 \cdot f_i'' + 0 \cdot f_i''' + 0 \cdot f_i''''$$

To find the coefficients, we substitute the pixel values with the corresponding Taylor approximation and arrange the terms to get the following equation:

$$= \left(\alpha_{-2} + \alpha_{-1} + \alpha_0 + \alpha_1 + \alpha_2\right) f_i + h\left(-2\alpha_{-2} - \alpha_{-1} + \alpha_1 + 2\alpha_2\right) f_i' +$$
$$h^2\left(\frac{4}{2}\alpha_{-2} + \frac{1}{2}\alpha_{-1} + \frac{1}{2}\alpha_1 + \frac{4}{2}\alpha_2\right) f_i'' + h^3\left(\frac{-8}{6}\alpha_{-2} - \frac{1}{6}\alpha_{-1} + \frac{1}{6}\alpha_1 + \frac{8}{6}\alpha_2\right) f_i'''$$
$$+ h^4\left(\frac{16}{24}\alpha_{-2} + \frac{1}{24}\alpha_{-1} + \frac{1}{24}\alpha_1 + \frac{16}{24}\alpha_2\right) f_i'''' + O\left(\left(\alpha_{-2} + \alpha_{-1} + \alpha_1 + \alpha_2\right) h^5\right)$$

Comparing the coefficients gives us the following linear equations (in matrix form):

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ \frac{4}{2} & \frac{1}{2} & 0 & \frac{1}{2} & \frac{4}{2} \\ -\frac{8}{6} & -\frac{1}{6} & 0 & \frac{1}{6} & \frac{8}{6} \\ \frac{16}{24} & \frac{1}{24} & 0 & \frac{1}{24} & \frac{16}{24} \end{bmatrix} \begin{bmatrix} \alpha_{-2} \\ \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/h \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system of linear equations e.g. with the Gauss elimination method yields:

$\alpha_{-2} = \frac{1}{12h}$; $\alpha_{-1} = \frac{-2}{3h}$; $\alpha_0 = 0$; $\alpha_1 = \frac{2}{3h}$; $\alpha_2 = \frac{-1}{12h}$

b. From the approximation, we can infer the weights/coefficients as:

$\alpha_{-2} = \frac{1}{12h}$; $\alpha_{-1} = \frac{-8}{12h}$; $\alpha_0 = 0$; $\alpha_1 = \frac{8}{12h}$; $\alpha_2 = \frac{-1}{12h}$

We can then evaluate the approximation by plugging in the weights into the following equation and find the leading error term:

$$\frac{1}{12h} f_{i-2} - \frac{8}{12h} f_{i-1} + \frac{8}{12h} f_{i+1} - \frac{1}{12h} f_{i+2} =$$
$$\underbrace{0 \cdot f_i + 1 \cdot f_i^{(1)} + 0 \cdot f_i^{(2)} + 0 \cdot f_i^{(3)} + 0 \cdot f_i^{(4)}}_{\text{per construction}} + h^5 \left(\frac{-32}{120} \alpha_{-2} - \frac{1}{120} \alpha_{-1} + \frac{1}{120} \alpha_1 + \frac{32}{120} \alpha_2 \right) f_i^{(5)}$$
$$+ h^6 \left(\frac{64}{720} \alpha_{-2} + \frac{1}{720} \alpha_{-1} + \frac{1}{720} \alpha_1 + \frac{64}{720} \alpha_2 \right) f_i^{(6)} + O(h^6)$$

due to symmetry of coefficients, the sixth term will cancel out and we get a leading error term from the fifth term of $O(h^4)$, so the consistency order is 4.

4. Mean and Median Filtering

2	3	7	7	8
8	2	3	7	7
3	3	3	7	7
8	7	7	9	9
7	1	7	7	7

(a) Mean filter

$$mean\ filter = \begin{bmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \end{bmatrix}$$

$$\begin{bmatrix} \frac{32}{9} & \frac{11}{3} & \frac{38}{9} & \frac{52}{9} & \frac{61}{9} \\ \frac{34}{9} & \frac{34}{9} & \frac{42}{9} & \frac{56}{9} & \frac{65}{9} \\ \frac{49}{9} & \frac{44}{9} & \frac{16}{3} & \frac{59}{9} & \frac{23}{3} \\ \frac{40}{9} & \frac{46}{9} & \frac{17}{3} & 7 & \frac{23}{3} \\ \frac{46}{9} & \frac{52}{9} & \frac{53}{9} & \frac{67}{9} & \frac{32}{3} \end{bmatrix}$$

(b)

Median filter = take the median of mask as output

$$\begin{bmatrix} 2 & 3 & 3 & 7 & 7 \\ 3 & 3 & 3 & 7 & 7 \\ 7 & 3 & 7 & 7 & 7 \\ 3 & 7 & 7 & 7 & 7 \\ 7 & 7 & 7 & 7 & 7 \end{bmatrix}$$

(c) Opening:

$f \circ B := (f \ominus B) \oplus B$

$$(f \ominus B) = \begin{bmatrix} 2 & 2 & 2 & 3 & 7 \\ 2 & 2 & 2 & 3 & 7 \\ 2 & 2 & 2 & 3 & 7 \\ 1 & 1 & 1 & 3 & 7 \\ 1 & 1 & 1 & 7 & 7 \end{bmatrix}$$

$$(f \circ B) = \begin{bmatrix} 2 & 2 & 3 & 7 & 7 \\ 2 & 2 & 3 & 7 & 7 \\ 2 & 2 & 3 & 7 & 7 \\ 2 & 2 & 7 & 7 & 7 \\ 1 & 1 & 7 & 7 & 7 \end{bmatrix}$$

Closing:

$f \square B := (f \oplus B) \ominus B$

Applying dilation $(f \oplus B)$ to the image yields:

$$(f \oplus B) = \begin{bmatrix} 8 & 8 & 7 & 8 & 8 \\ 8 & 8 & 7 & 8 & 8 \\ 8 & 8 & 9 & 9 & 9 \\ 8 & 8 & 9 & 9 & 9 \\ 8 & 8 & 9 & 9 & 9 \end{bmatrix}$$

Applying erosion after that gives:

$$f \cdot B = \begin{bmatrix} 8 & 7 & 7 & 7 & 8 \\ 8 & 7 & 7 & 7 & 8 \\ 8 & 7 & 7 & 7 & 8 \\ 8 & 8 & 8 & 9 & 9 \\ 8 & 8 & 8 & 9 & 9 \end{bmatrix}$$

5. Morphological Operations
   i. Given a 1-D signal $\boldsymbol{f}$ = $(...,0,0,0,0,1,1,1,1,...)^T$ and a 3 x 3 structuring element B, applying the following operations gives:
      - $A_B(f) := (f \oplus B) - f$
        $A_B(f) = (...,0,0,0,1,1,1,1,1,...)^T - (...,0,0,0,0,1,1,1,1,...)^T$
        $A_B(f) = (...,0,0,0,1,0,0,0,0,...)^T$

      - $B_B(f) := f - (f \ominus B)$
        $B_B(f) = (...,0,0,0,0,1,1,1,1,...)^T - (...,0,0,0,0,0,1,1,1,...)^T$
        $B_B(f) = (...,0,0,0,0,1,0,0,0,...)^T$

      - $C_B(f) := A_B(f) - B_B(f)$
        $C_B(f) = (...,0,0,0,1,0,0,0,0,...)^T - (...,0,0,0,0,1,0,0,0,...)^T$
        $C_B(f) = (...,0,0,0,1,-1,0,0,0,...)^T$

      - $D_B(f) := (f \oplus B) - (f \ominus B)$
        $D_B(f) = (...,0,0,0,1,1,1,1,1,...)^T - (...,0,0,0,0,0,1,1,1,...)^T$
        $D_B(f) = (...,0,0,0,1,1,0,0,0,...)^T$
   ii. The operation $A_B(f)$ is almost similar to a Black Top Hat filter, $B_B(f)$ to a White Top Hat filter, and $D_B(f)$ to a Selfdual Top Hat filter. The operation $C_B(f)$ seems to also be a high-pass filter.