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Imaging Science
Assignment H1 (Theoretical)

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1. *Discrete Convolution* 4/6

Convolve it three times with itself

(1) $g_i = f_i$

position i	-1	0	1
value $g_i = f_i$	0	$\frac{1}{2}$	$\frac{1}{2}$

$w_i = f_i$

position i	-1	0	1
value $w_i = f_i$	0	$\frac{1}{2}$	$\frac{1}{2}$

result $(g * w)_i = (f * f)_i$

position i	-1	0	1	2
value $(f * f)_i$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Convolution can move the bounds of the signal!

(2) $g_i = (f * f)_i$

position i	-1	0	1
value $g_i = f_i$	0	$\frac{1}{4}$	$\frac{1}{2}$

$w_i = f_i$

position i	-1	0	1
value $w_i = f_i$	0	$\frac{1}{2}$	$\frac{1}{2}$

result $(g * w)_i = ((f * f) * f)_i$

position i	-1	0	1	2	3
value $((f * f) * f)_i$	0	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(3) $g_i = ((f * f) * f)_i$

position i	-1	0	1

value $g_i = ((f * f) * f)_i$	0	$\frac{1}{8}$	$\frac{3}{8}$

$w_i = f_i$

position i	-1	0	1
value $w_i = f_i$	0	$\frac{1}{2}$	$\frac{1}{2}$

result $(g * w)_i = (((f * f) * f) * f)_i$

position i	-1	0	1	2	3	4
value $(((f * f) * f) * f)_i$	0	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

2. Properties of the Convolution 6/6

Proof of the properties possessed by a discrete convolution in 1-D:

a. Linearity

$$(\alpha \cdot f + \beta \cdot g) * w = \sum_{k=-\infty}^{\infty} (\alpha \cdot f_{i-k} + \beta \cdot g_{i-k}) \cdot w_k$$

$$= \sum_{k=-\infty}^{\infty} \alpha \cdot f_{i-k} \cdot w_k + \beta \cdot g_{i-k} \cdot w_k$$

$$= \alpha \sum_{k=-\infty}^{\infty} f_{i-k} \cdot w_k + \beta \sum_{k=-\infty}^{\infty} g_{i-k} \cdot w_k$$

$$= \alpha \cdot (f * w) + \beta \cdot (g * w)$$

b. Commutativity

$$f * w = \sum_{k=-\infty}^{\infty} f_{i-k} \cdot w_k$$

Let $k = i - n$,

$k = \infty \rightarrow n = -\infty$ and $k = -\infty \rightarrow n = \infty$

then by substituting this into the above equation we can rewrite it as follows:

$$f * w = \sum_{n=-\infty}^{\infty} f_n \cdot w_{i-n}$$

$$\sum_{n=-\infty}^{\infty} w_{i-n} \cdot f_n = w * f$$

Hence, $f * w = w * f$

c. Identity

If $f * e = f$, then e must be the delta function δ. Convolving a signal with the delta function passes all the input signal without changes, thus making the delta function the identity for convolution.

In discrete case:

$$e_i = \begin{cases} 1\ (i = 1) \\ 0\ (else) \end{cases}$$

References:
https://www.analog.com/media/en/technical-documentation/dsp-book/dsp_book_Ch7.pdf
http://www.songho.ca/dsp/convolution/convolution_commutative.html

3. Continuous Fourier Transform of a Discrete Filter 3/6

Utilizing the translation property of the continuous Fourier Transform,

$$f(x - x_0, y - y_0) \Leftrightarrow F(u, v)e^{-j2\pi(x_0u/M + y_0v/N)}$$

$$f\left(x - x_0,\ y - y_o\right) \Leftrightarrow F[f](u,\ v)\ e^{-i2\pi\left(x_0 u + y_o v\right)}$$

We can then compute $F[g](u, v)$ as:

$$\frac{1}{8}F[f](u,v)\left(-e^{-i2\pi(u+v)} + e^{-i2\pi(-u+v)} - 2e^{-i2\pi(u)} + 2e^{-i2\pi(-u)} - e^{-i2\pi(u-v)} + e^{-i2\pi(-u-v)}\right)$$

$$= \frac{1}{8}F[f](u,v)\left(-4e^{-i2\pi u} + 4e^{i2\pi u}\right)$$

$$= \frac{4}{8}\left(-e^{-i2\pi u} + e^{i2\pi u}\right)F[f](u,v)$$

$$= (i\sin(2\pi u))\ F[f](u,v)$$