University of Stuttgart
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Complex Network Systems

Structural metrics

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Winter

Types

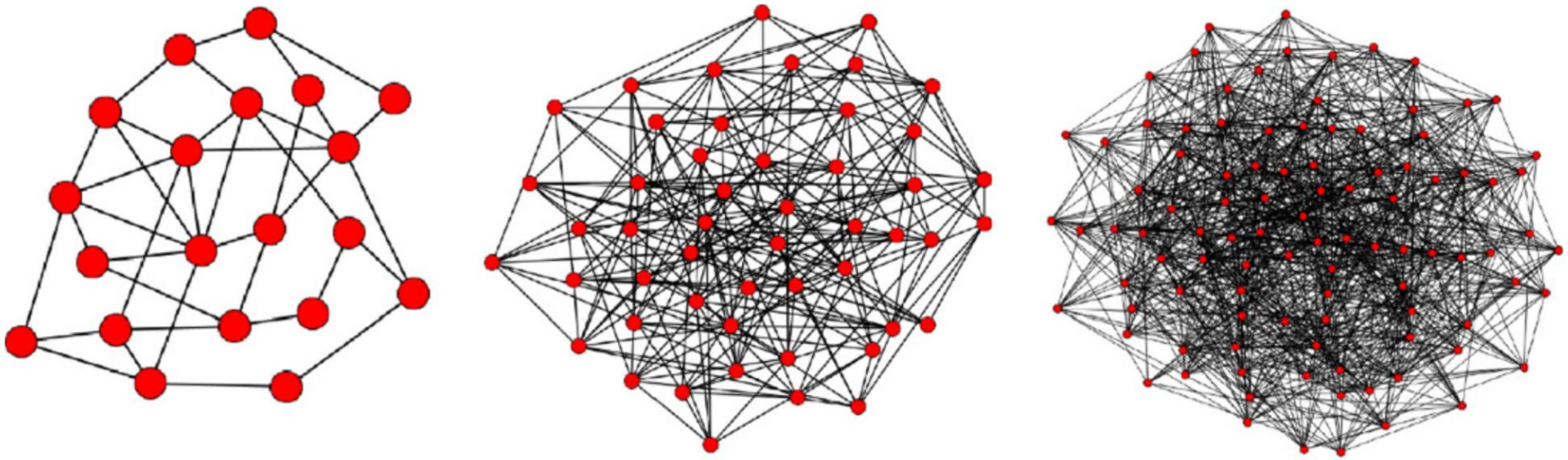
Graph-level metrics

- ✔ Size
- Density
- Paths and distances
- Neighbourhoods
- Egocentric network
- Clustering coefficient
- Transitivity
- Cores
- Cliques
- Communities

Node-level metrics

- Closeness centrality
- Betweenness centrality
- Degree centrality
- Eigenvector centrality
- Katz centrality
- PageRank

Density



20 nodes, 38 links
Density = 0.20
Links per node = 1.9

50 nodes, 245 links
Density = 0.20
Links per node = 4.9

100 nodes, 990 links
Density = 0.20
Links per node = 9.9

Hoppe, B. and Reinelt, C. (2010) Social network analysis and the evaluation of leadership networks, *The Leadership Quarterly*, 21(4), pp. 600-619.

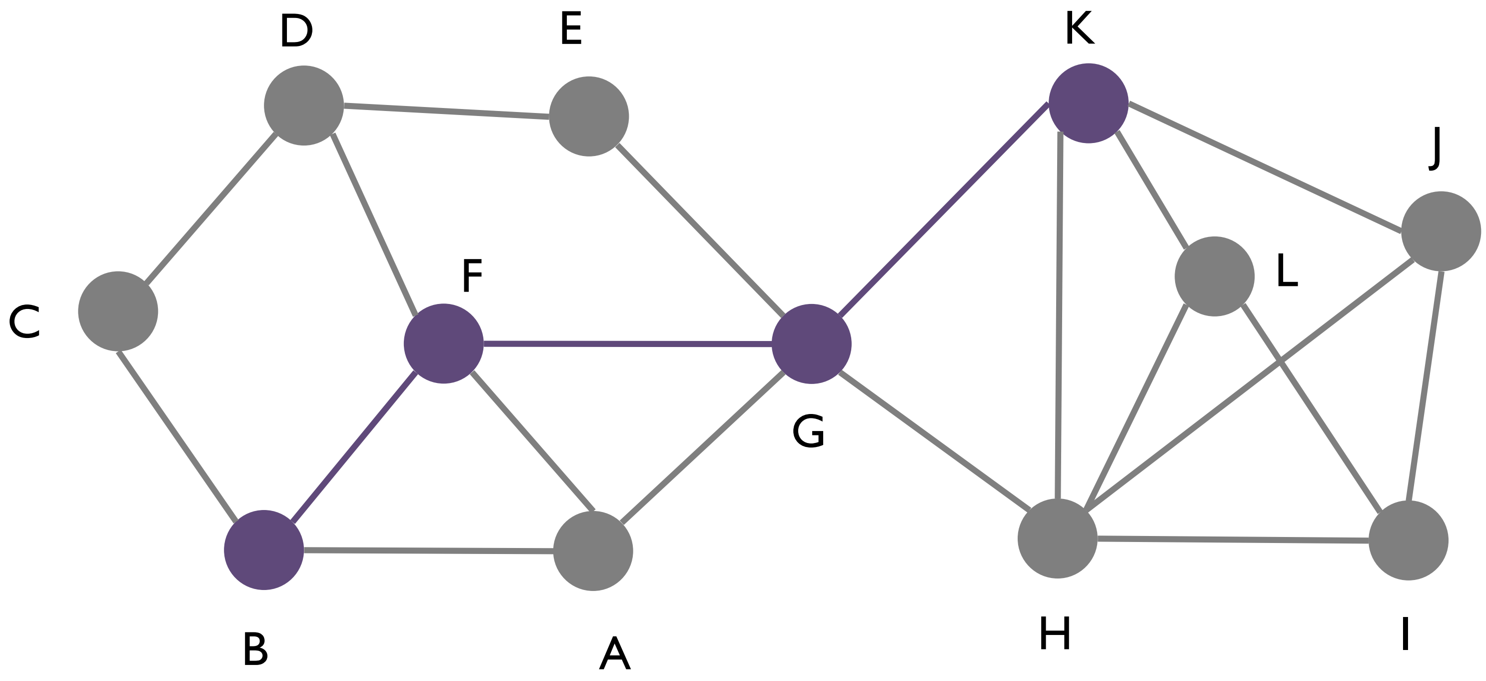
Density

- Ratio of existing edges to all possible edges
- Gives a sense of how closely knit the network is
- nx.density(G)

Paths

- Path
- Cycle
- Eulerian path
- Hamiltonian path

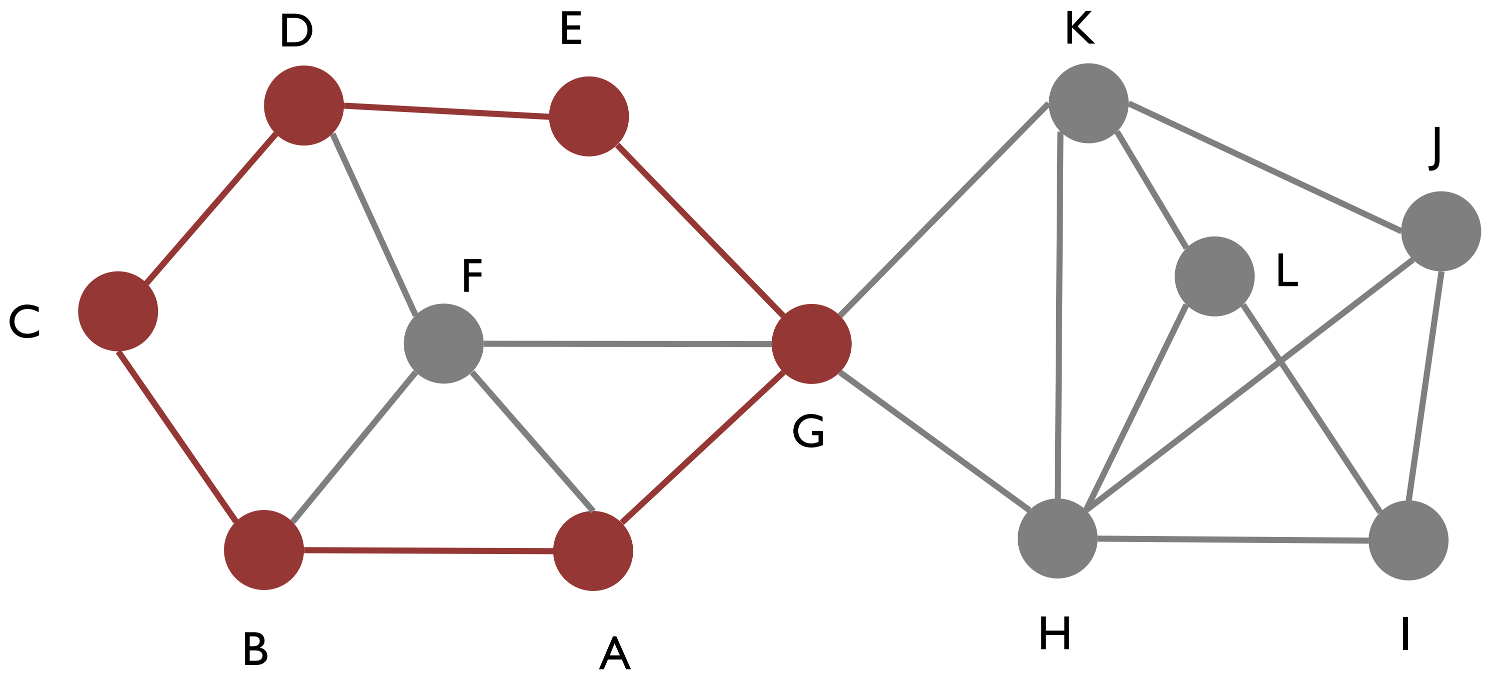
Path



$P_{BK} = ?$

$P_{BK} = \{(B,F),(F,G),(G,K)\}$

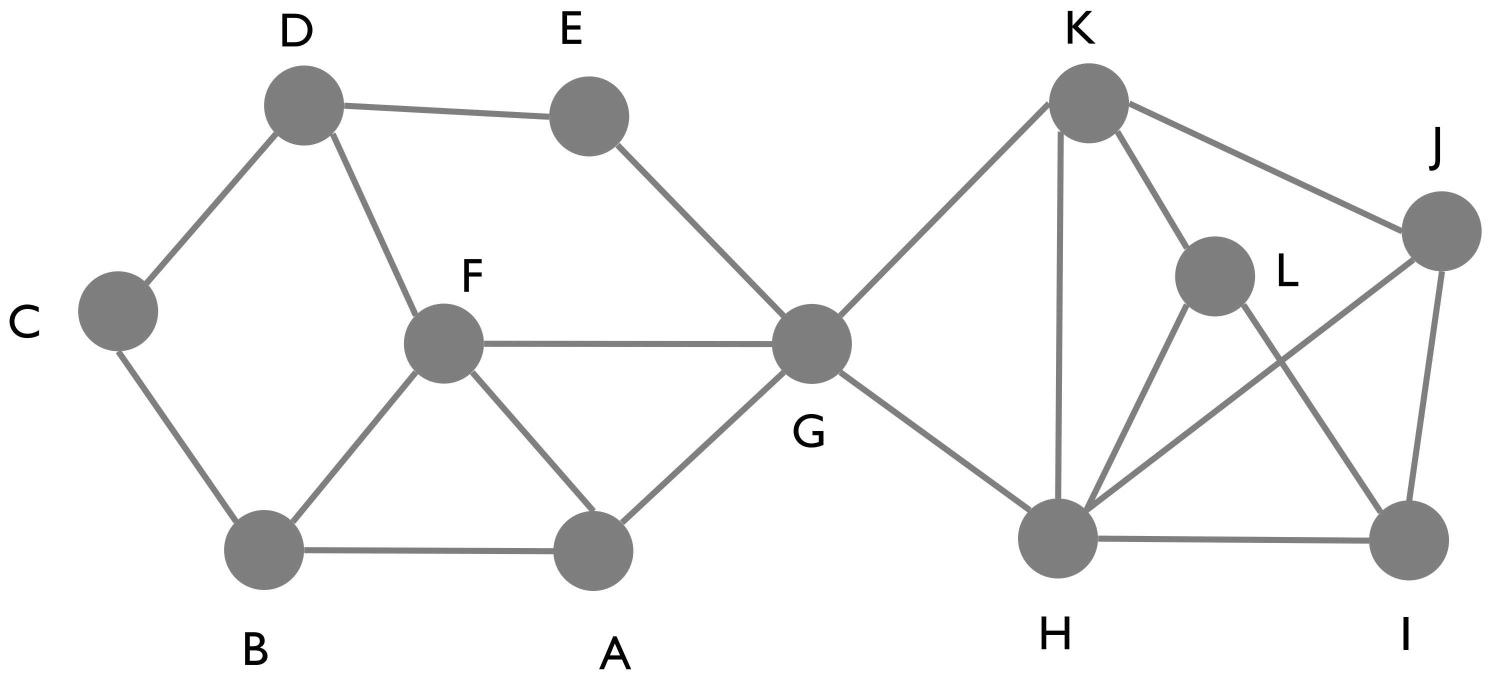
Cycle



$P_{BB} = ?$

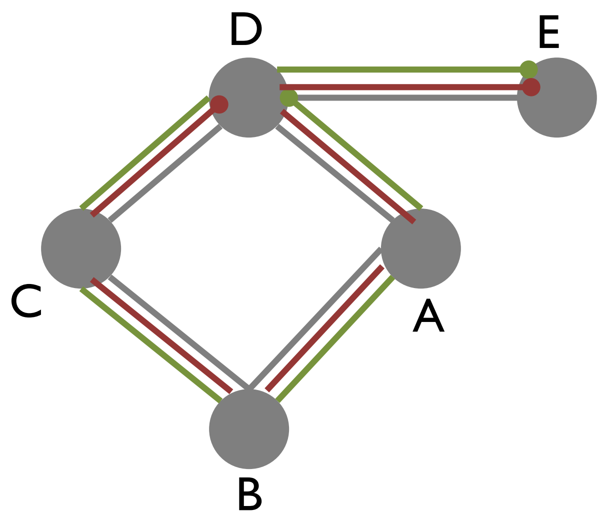
$P_{BB} = \{(B, A), (A, G), (G, E), (E, D), (D, C), (C, B)\}$

Eulerian path



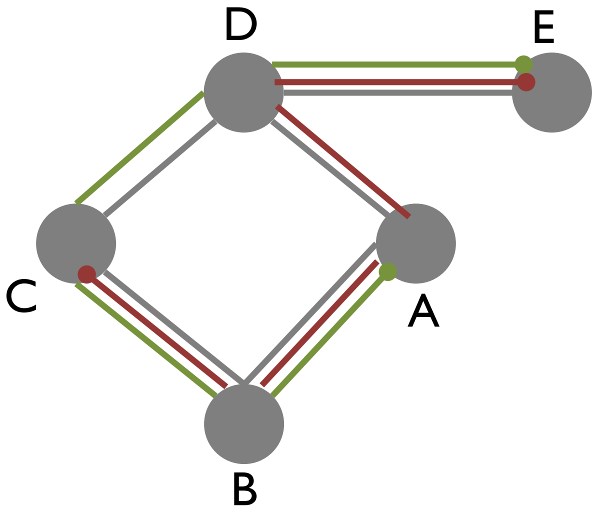
$P_{Eulerian} = ?$

Eulerian path



$P_{Eulerian}$ =?

Hamiltonian path

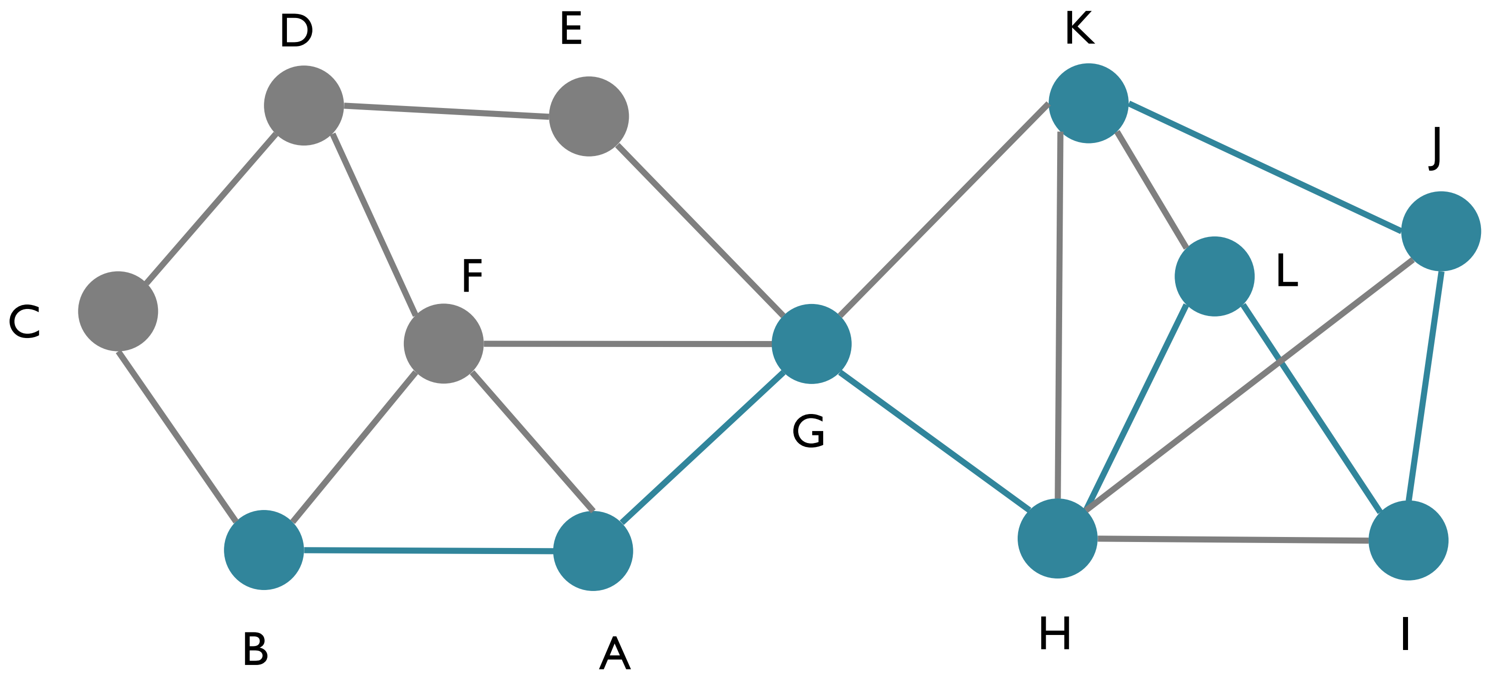


$P_{Hamiltonian} = ?$

Distance measures

- Path length
- Shortest path
- Average path length
- Eccentricity
- Diameter
- Radius
- Center
- Periphery

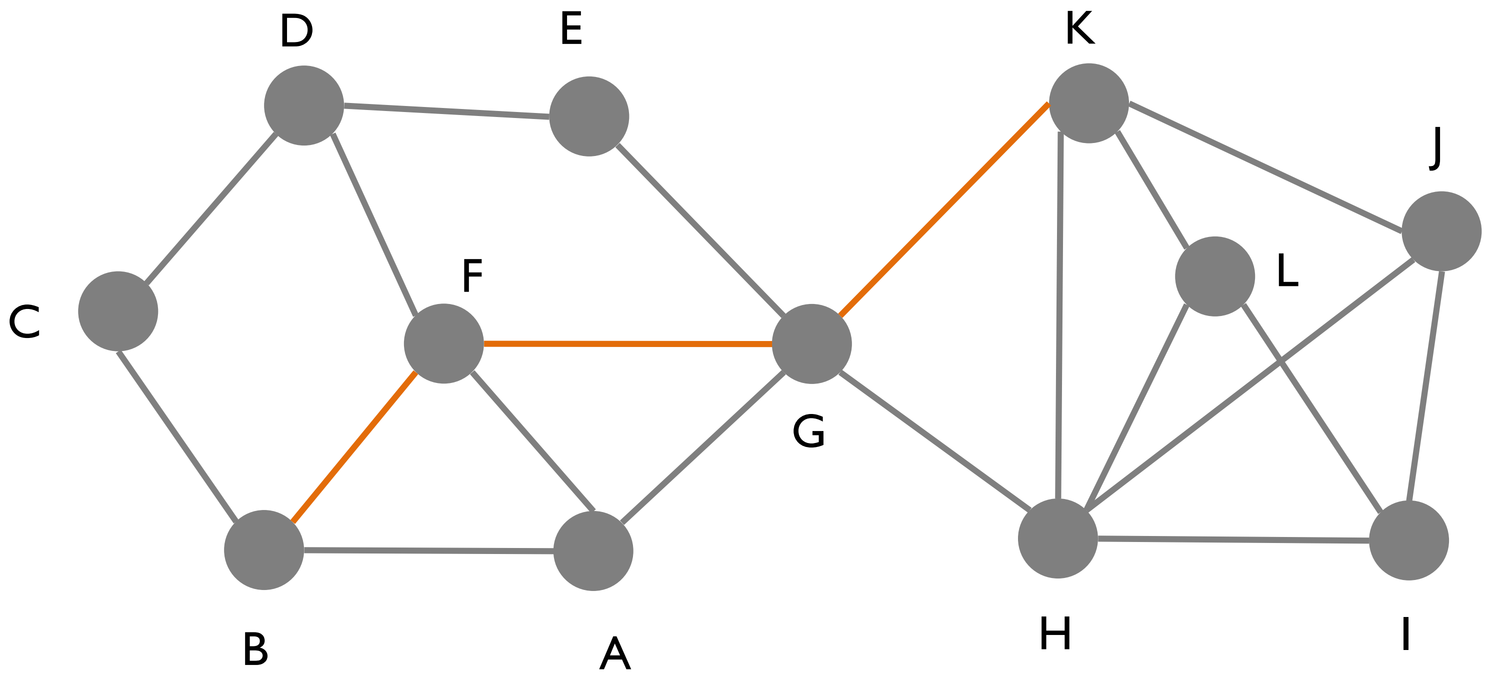
Path length



n_{BK} =?

$n_{BK} = 7$

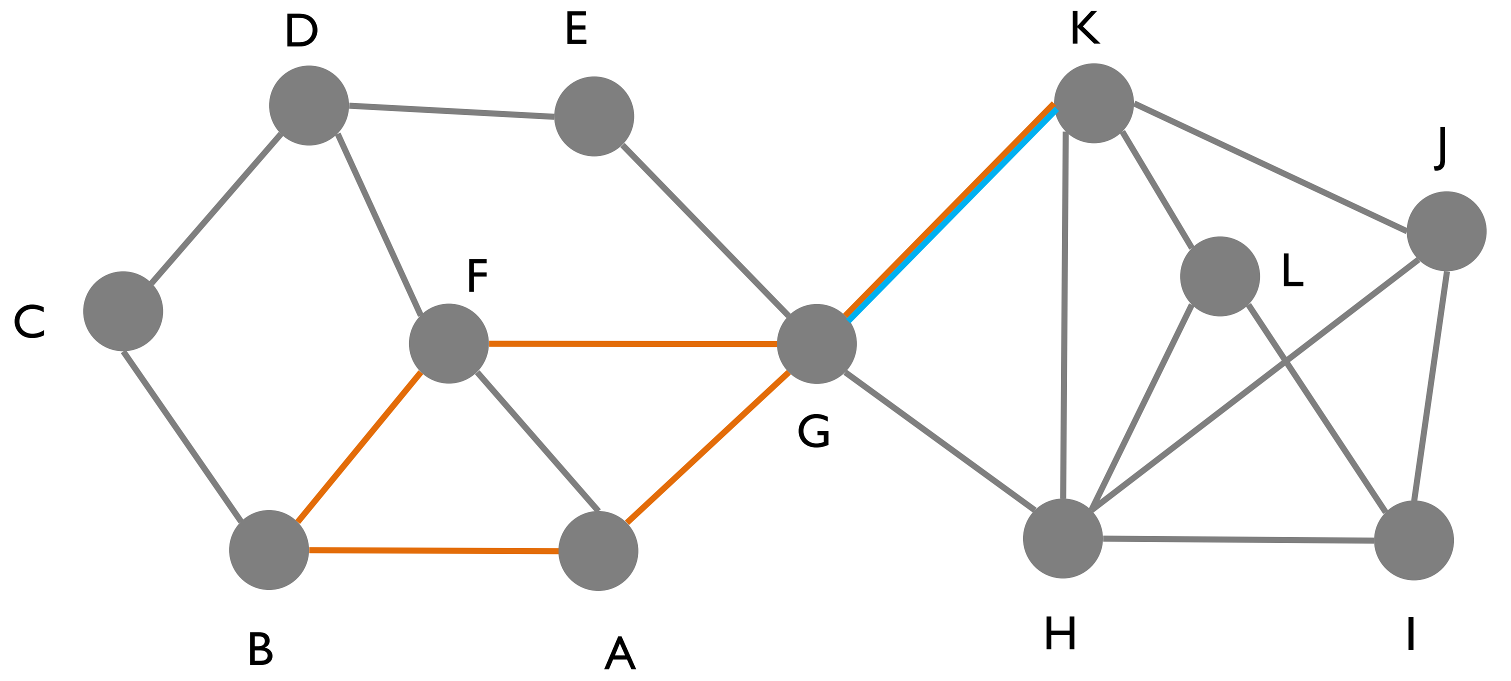
Shortest path



$d_{BK} =?$

$d_{BK} = 3$

Shortest path



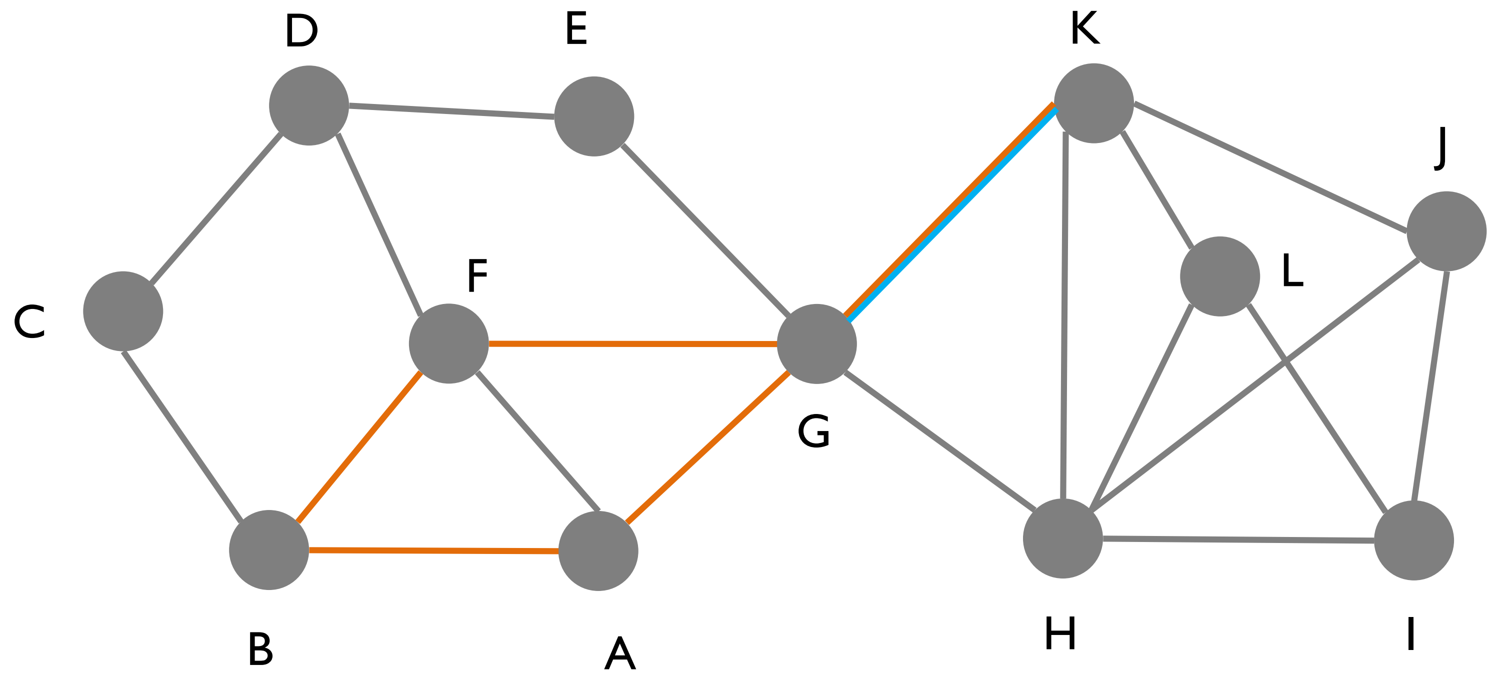
$d_{BK} =?$

$d_{BK} = 3$

Shortest path

Distance

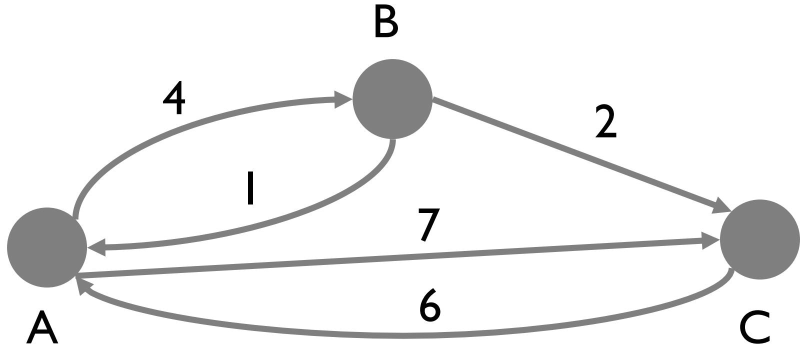
Geodesic



$d_{BK} = ?$

$d_{BK} = 3$

How to determine the distance between two vertices?



Floyd-Warshall algorithm

$$D^{(0)} = \begin{bmatrix} 0 & 4 & 7 \\ 1 & 0 & 2 \\ 6 & \infty & 0 \end{bmatrix}$$

$$D^{(1)} = \begin{bmatrix} 0 & 4 & 7 \\ 1 & 0 & 2 \\ 6 & 10 & 0 \end{bmatrix}$$

Take vertex A

$$D_{CB} = D_{CA} + D_{AB}$$
$$D_{CB} = 6 + 4 = 10$$

$$D^{(2)} = \begin{bmatrix} 0 & 4 & 6 \\ 1 & 0 & 2 \\ 6 & 10 & 0 \end{bmatrix}$$

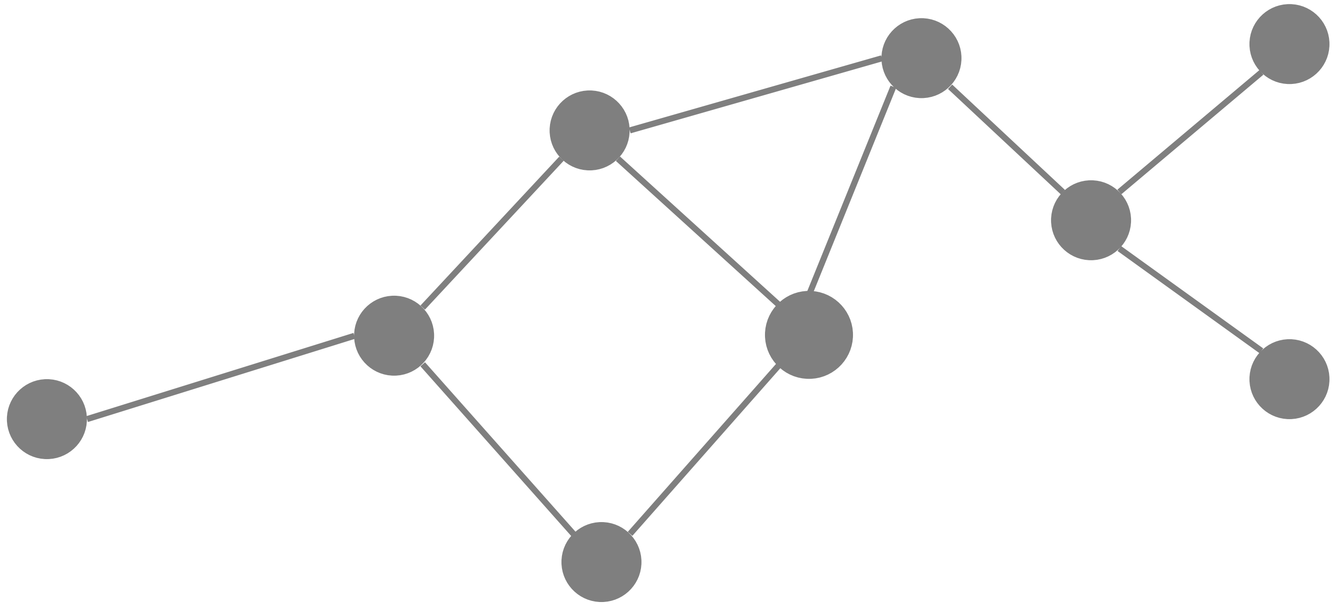
Take vertex B

$$D_{AC} = D_{AB} + D_{BC}$$
$$D_{AC} = 4 + 2 = 6$$

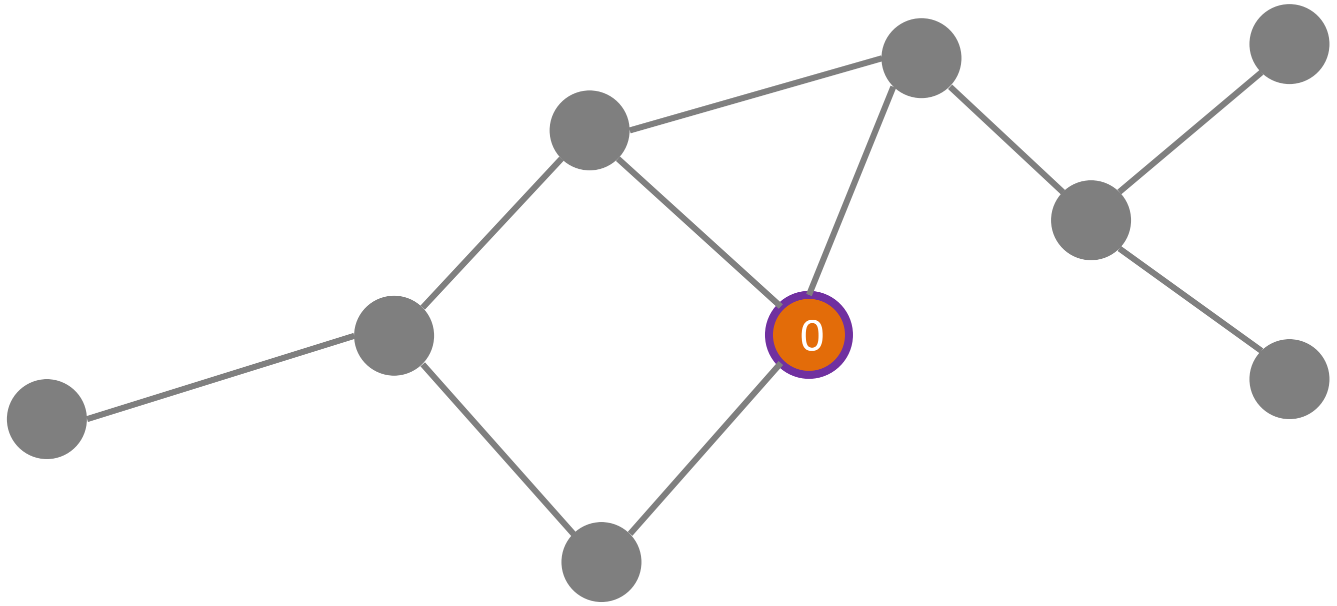
$$D^{(3)} = \begin{bmatrix} 0 & 4 & 6 \\ 1 & 0 & 2 \\ 6 & 10 & 0 \end{bmatrix}$$

Take vertex C

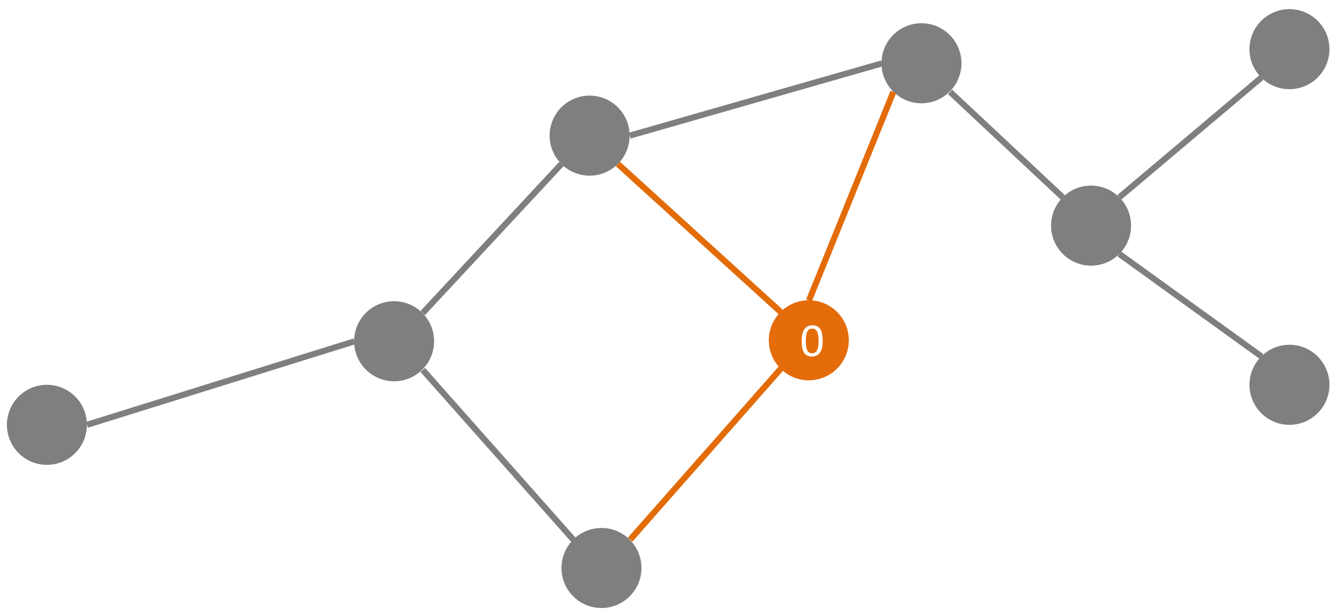
Breadth-first search



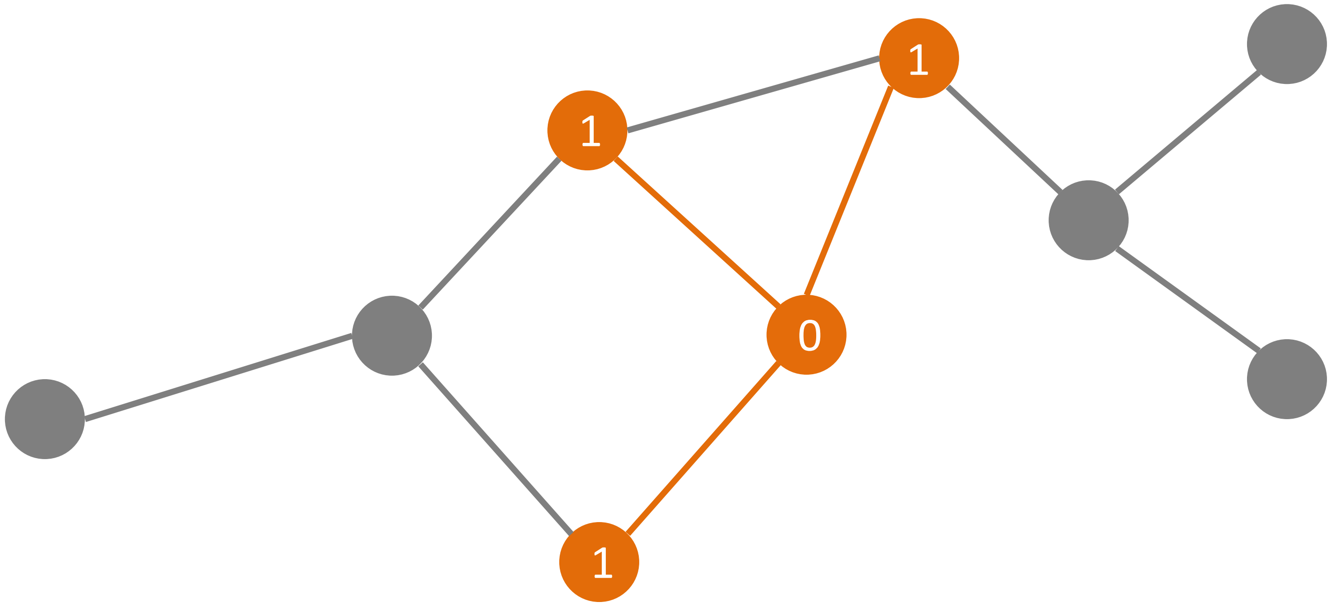
Breadth-first search



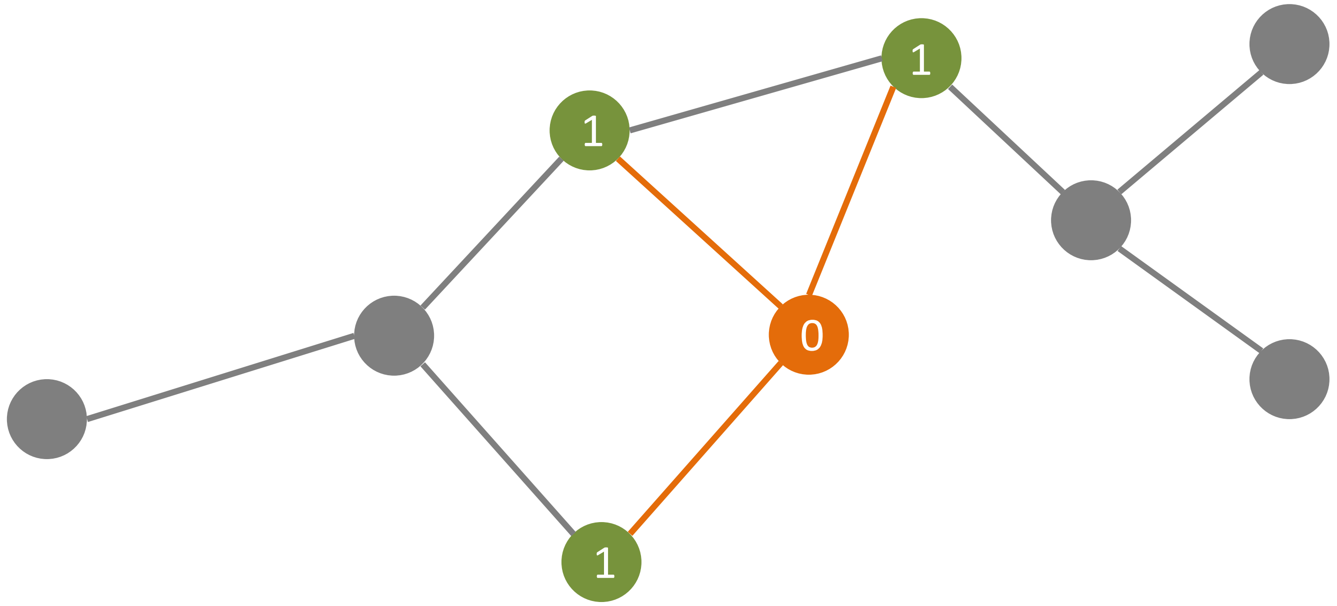
Breadth-first search



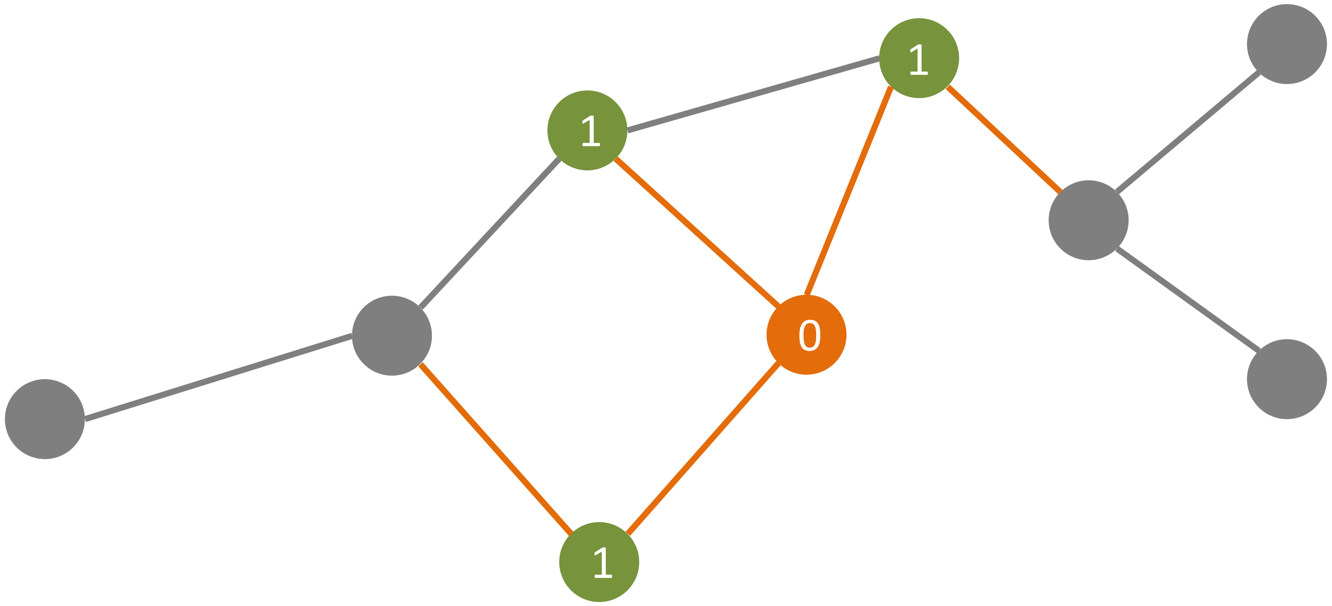
Breadth-first search



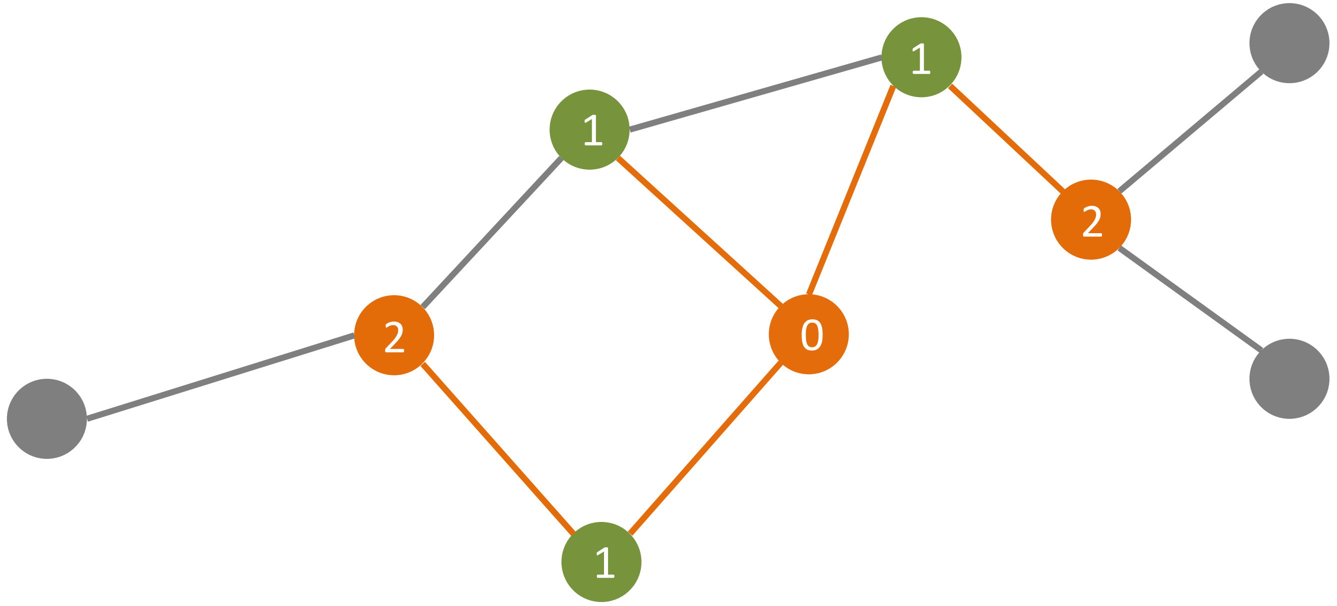
Breadth-first search



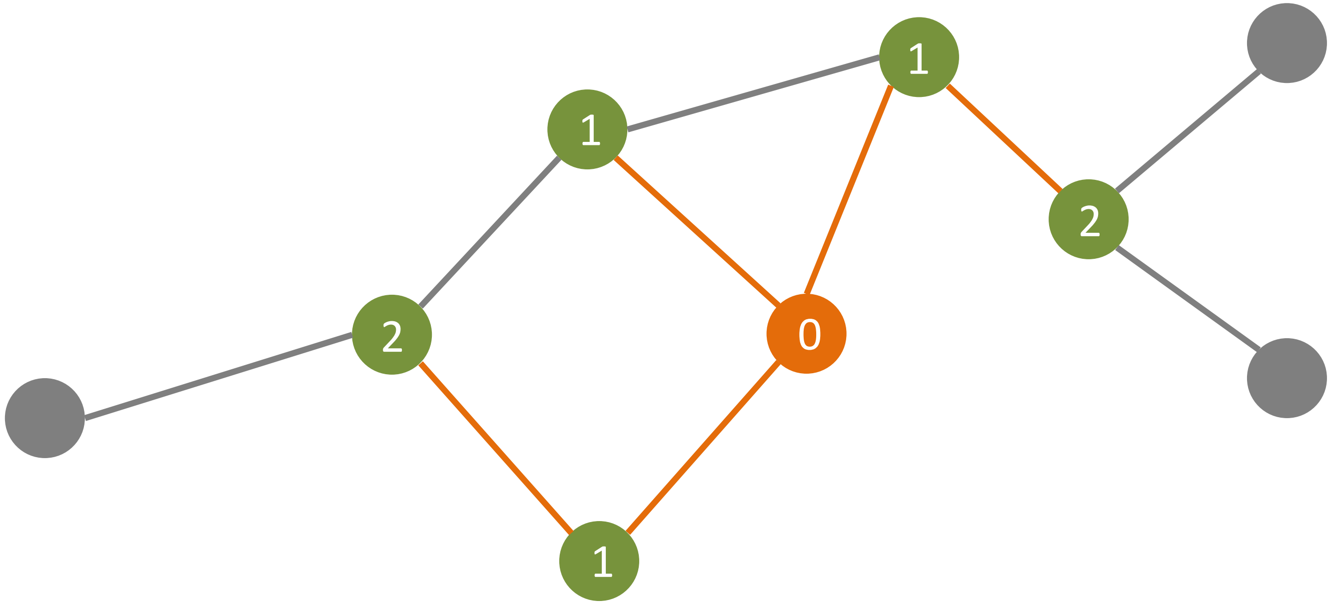
Breadth-first search



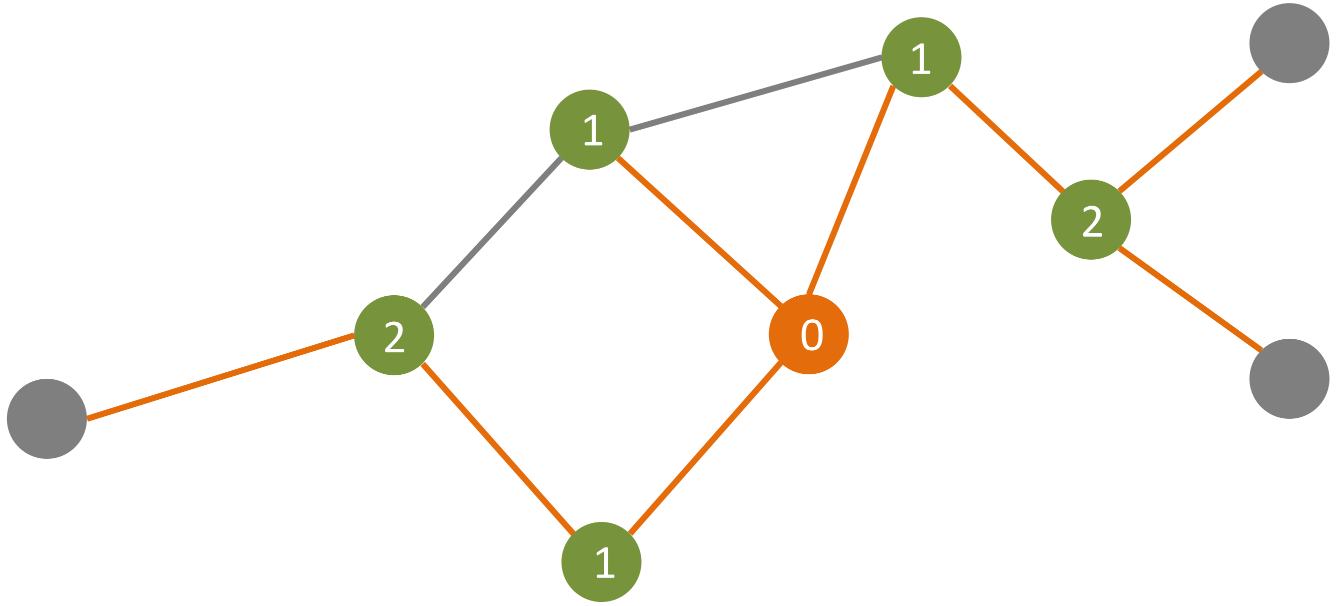
Breadth-first search



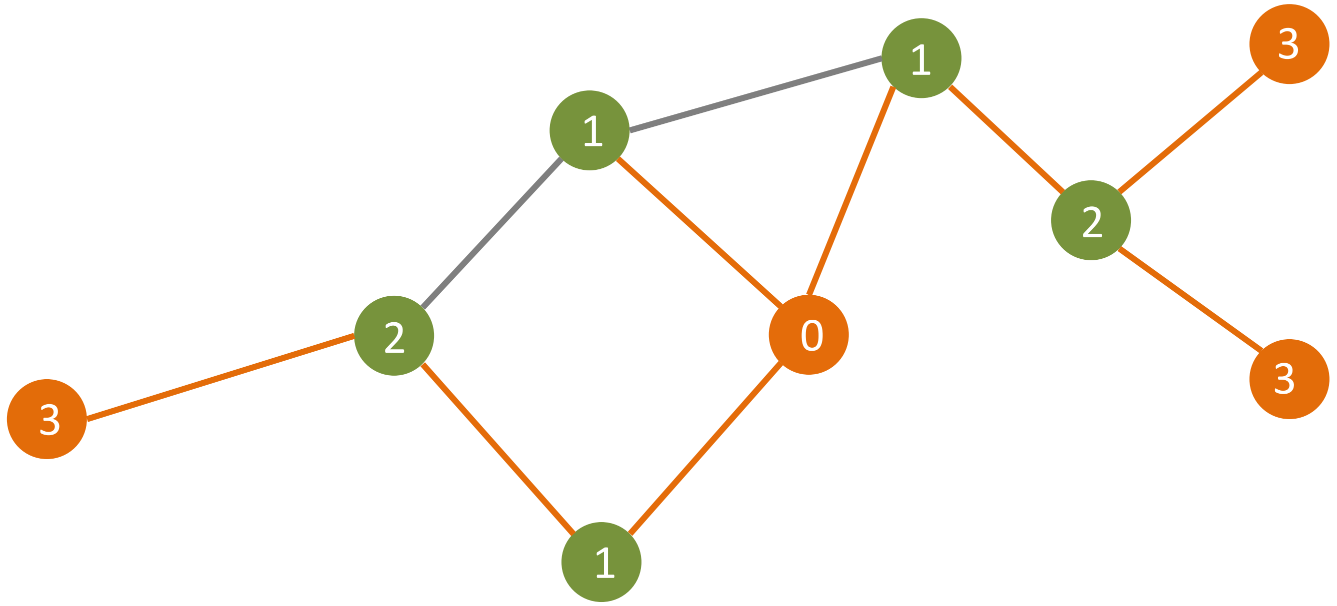
Breadth-first search



Breadth-first search

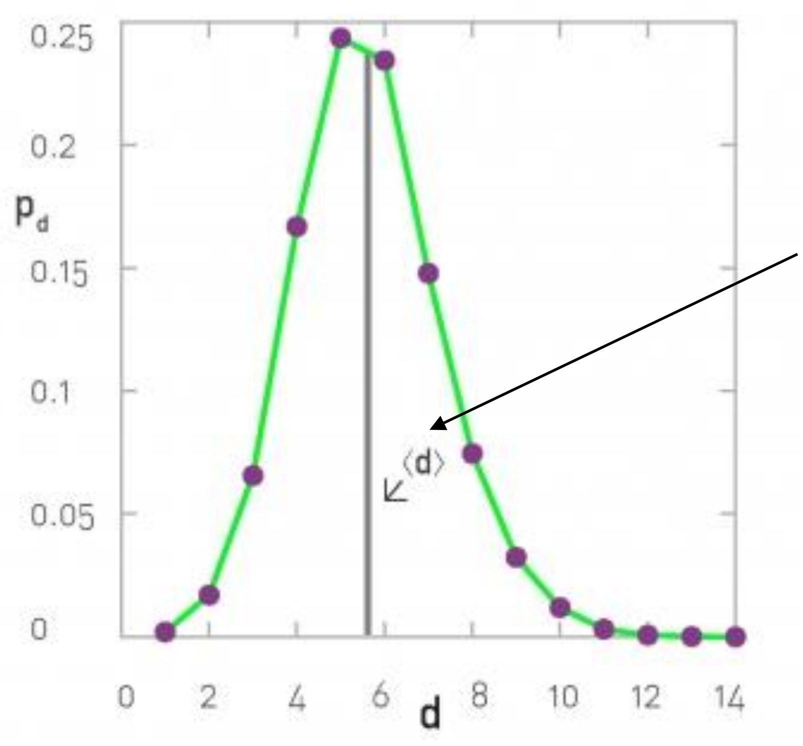


Breadth-first search



Protein-protein interaction network of yeast

distance distribution



average shortest path length

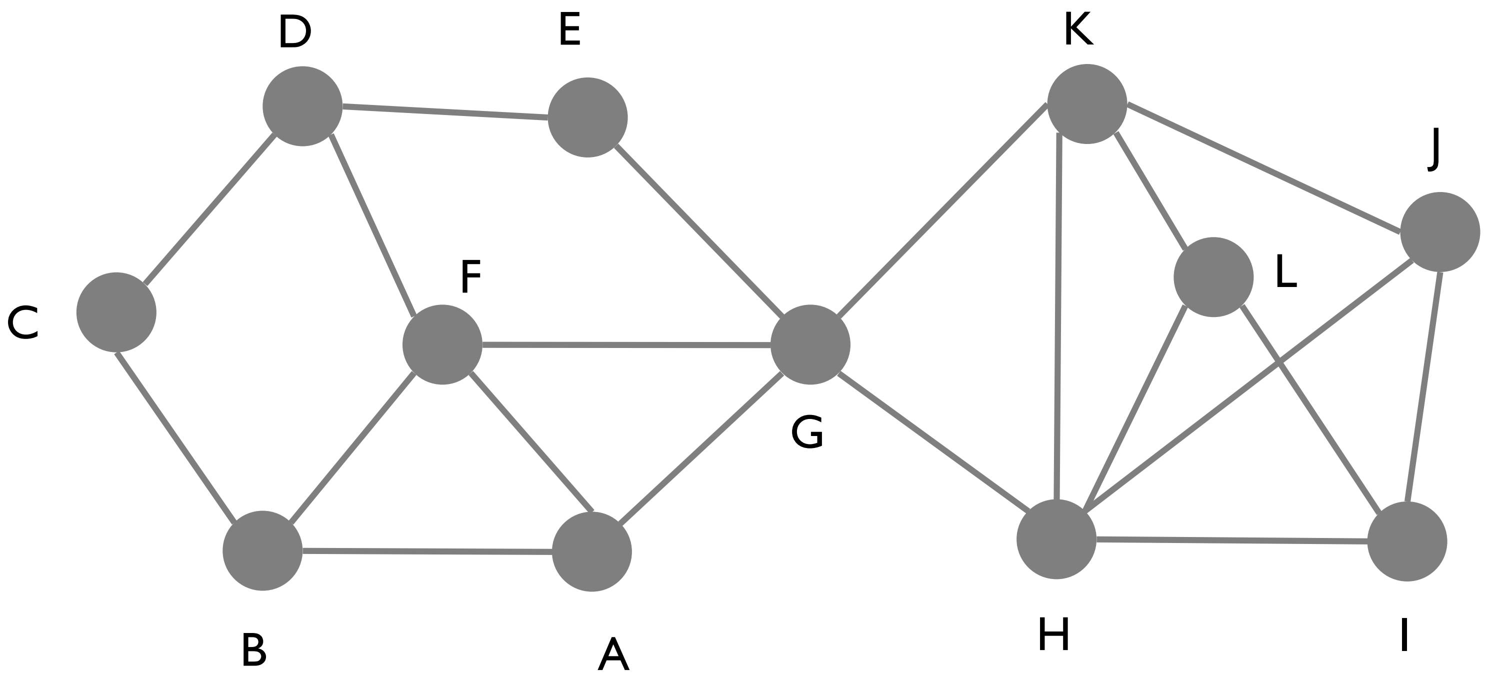
Shortest path

- Social network
  - finding friends-of-friends
  - six degree of separation

- nx.shortest_path
  - computes shortest paths given the whole graph, source node and target node

Eccentricity

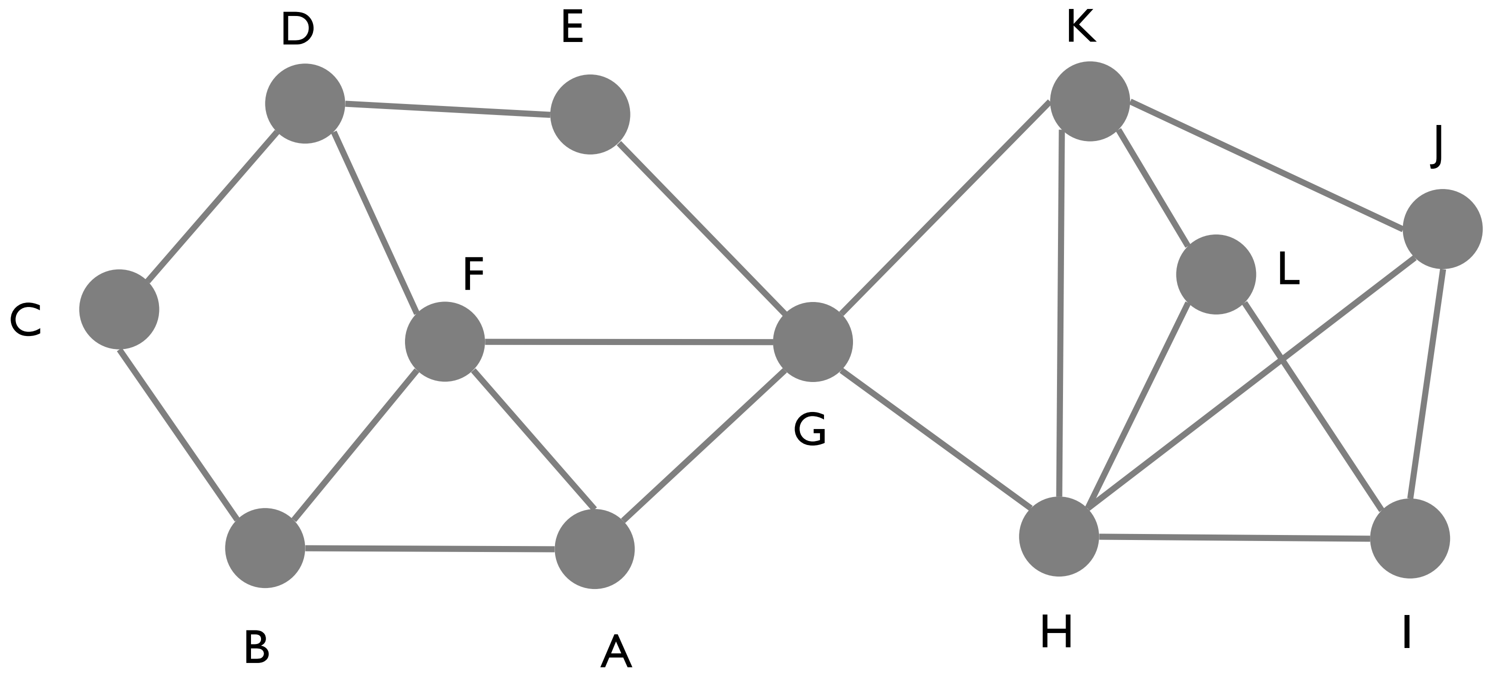
- Maximum distance from a vertex to all other vertices
- Measure of how far from the center a vertex is
- nx.eccentricity(G,node)



Eccentricity of node H?

H to I	1
H to J	1
H to K	1
H to L	1
H to G	1
H to A	2
H to B	3
H to C	4
H to D	3
H to E	2
H to F	2

Diameter



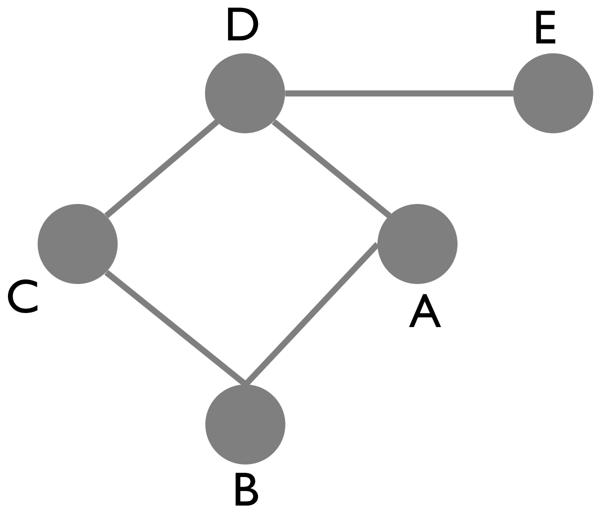
$d_{max} =?$

$d_{max} = 5$

Diameter

- Distance from one end of the network to another
  - maximum shortest path
- Gives a sense of the network's overall size
- nx.diameter(G)
  - works only if the graph *G* is connected

Average path length



$$\bar{d} = \frac{d_{AB} + d_{AC} + d_{AD} + d_{AE} + d_{BC} + d_{BD} + d_{BE} + d_{CD} + d_{CE} + d_{DE}}{10}$$

$$\bar{d} = \frac{1+2+1+2+1+2+3+1+2+1}{10} = 1.6$$

$\bar{d} =?$

Other distance measures

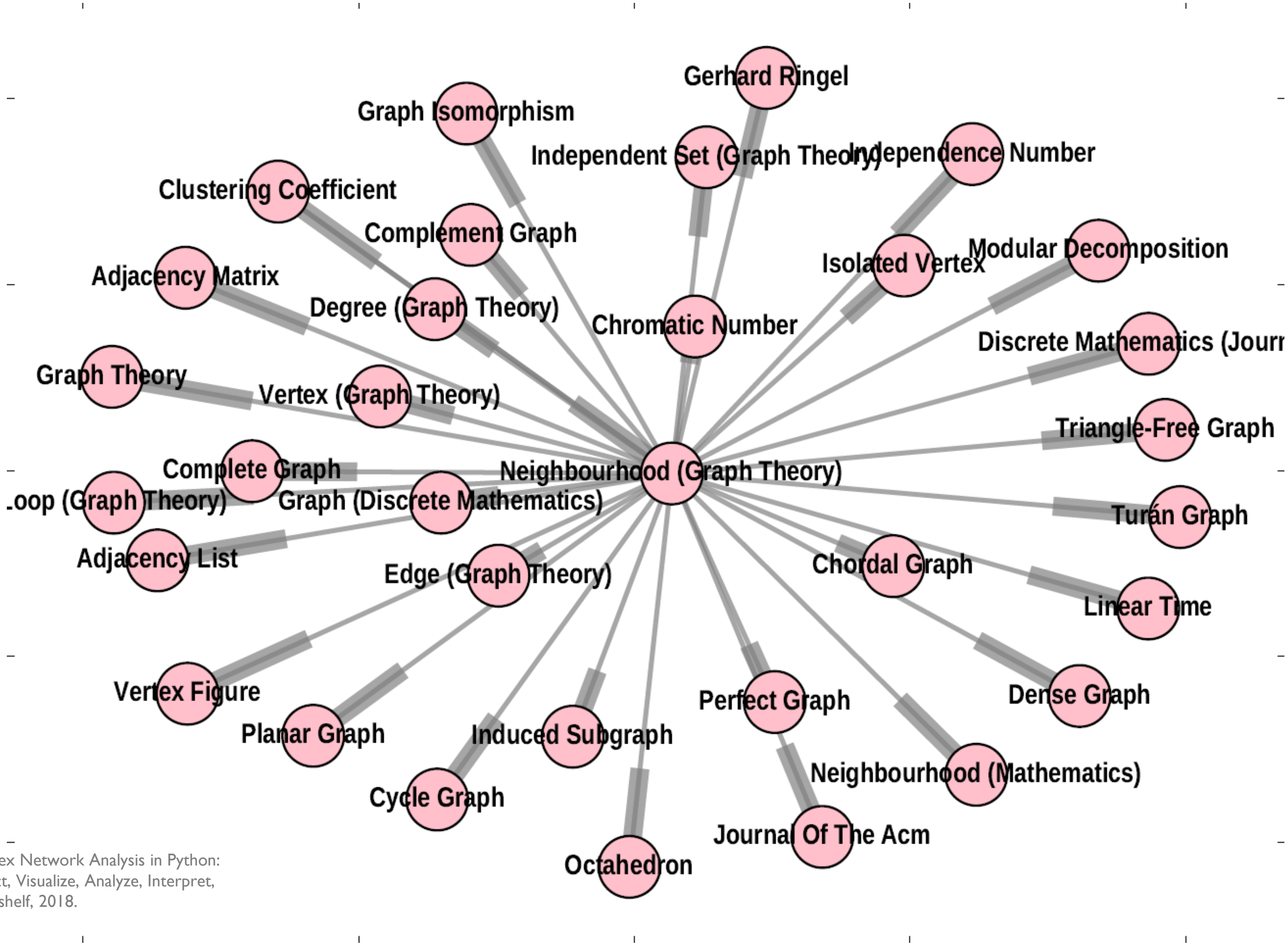
- Radius of a network
  - it is not a half of the diameter!
  - nx.radius(G)

- Center of a network
  - all nodes whose eccentricity equals the radius
  - nx.center(G)

- Periphery of a network
  - all nodes whose eccentricity equals the diameter
  - nx.periphery(G)

Connected components

- *Component* is a set of vertices reachable from one vertex (in undirected graph)
- If every pair of nodes is connected by some path, then the network is *connected*



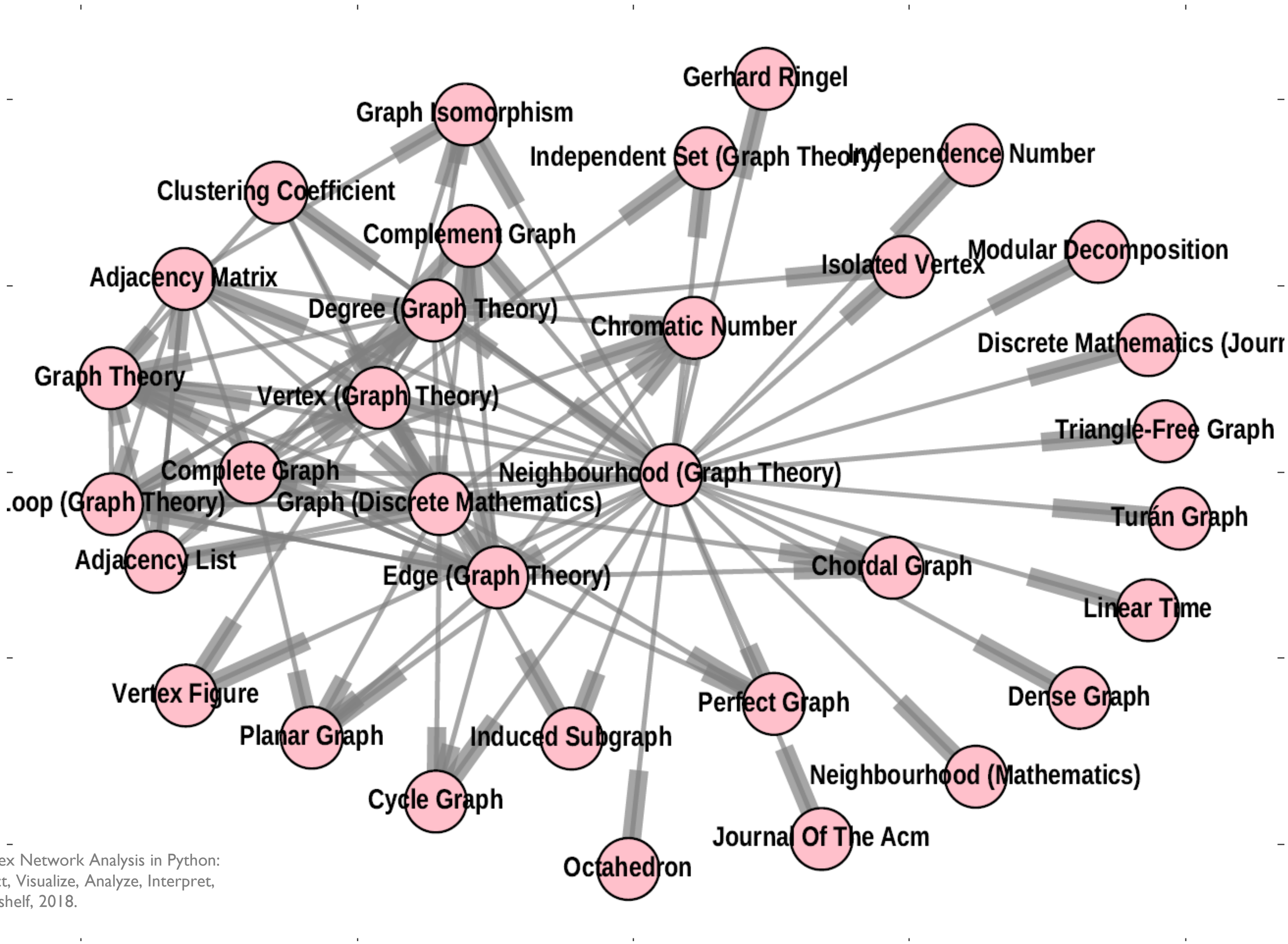
Zinoviev, D.. Complex Network Analysis in Python: Recognize, Construct, Visualize, Analyze, Interpret, The Pragmatic Bookshelf, 2018.

Neighbourhood

- Neighbourhood of a node is a set of all nodes adjacent to that node
- <u>Localised</u> properties of network graphs
- Usually convey little information

Neighbourhood

- Social network analysis pays special attention to neighbourhoods
  - relatives
  - close friends
  - colleagues
  - most significant *alters* of the *ego*



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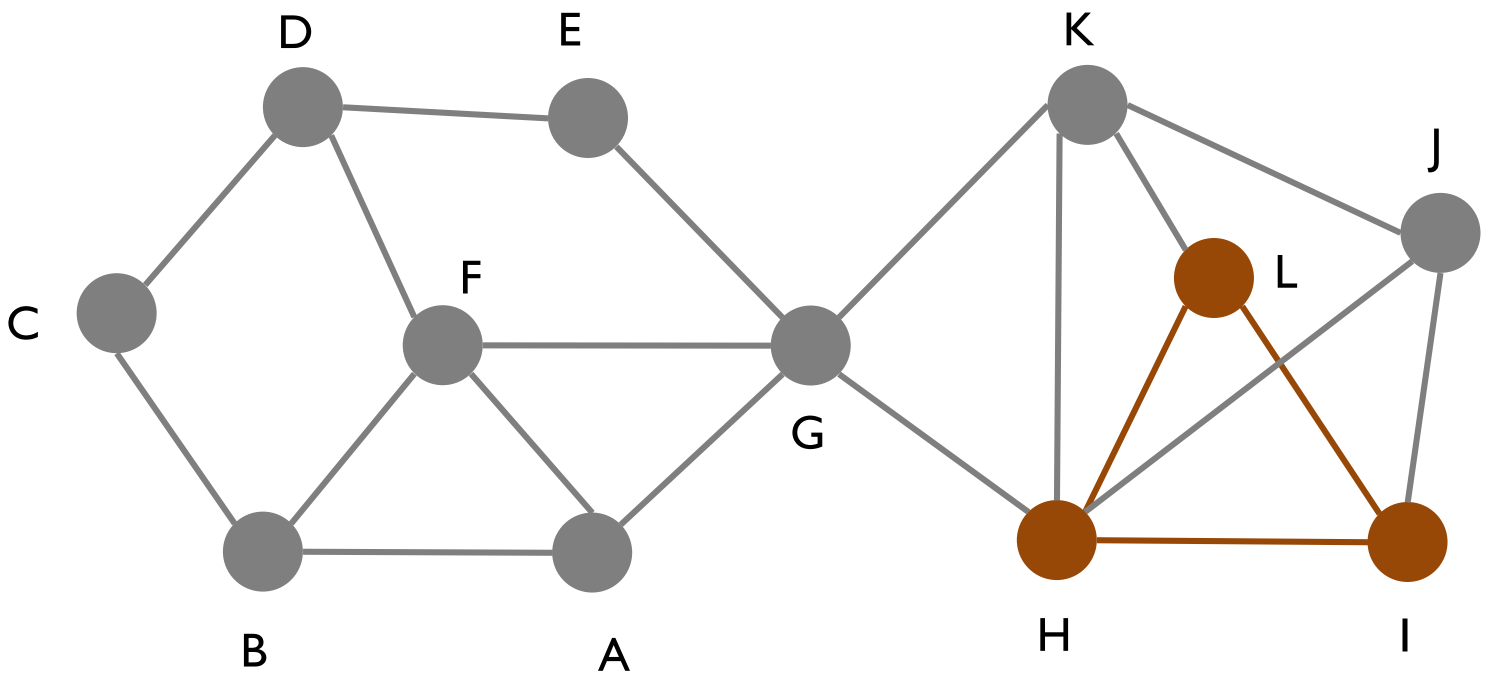
Egocentric network

- *Ego* is the central node
- *Alters* are all other nodes
- Should be much denser than a neighbourhood
  - some nodes may be connected only to the hub
  - others may form triangles that involve more neighbourhood members
- nx.ego_graph(G, node)

Triadic closure

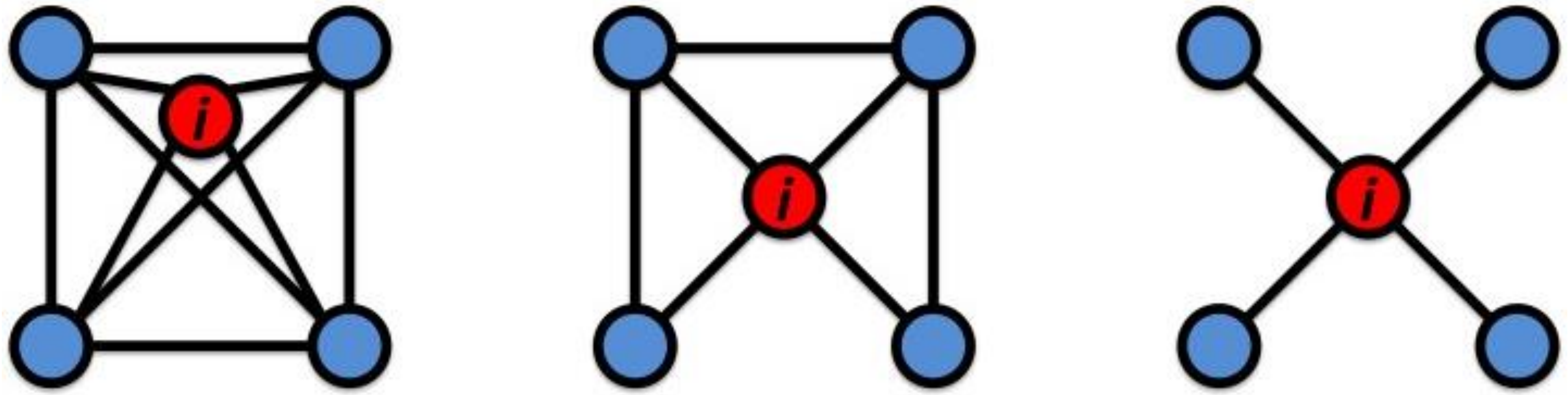


If X is connected to Y and Y to Z, then X is connected to Z

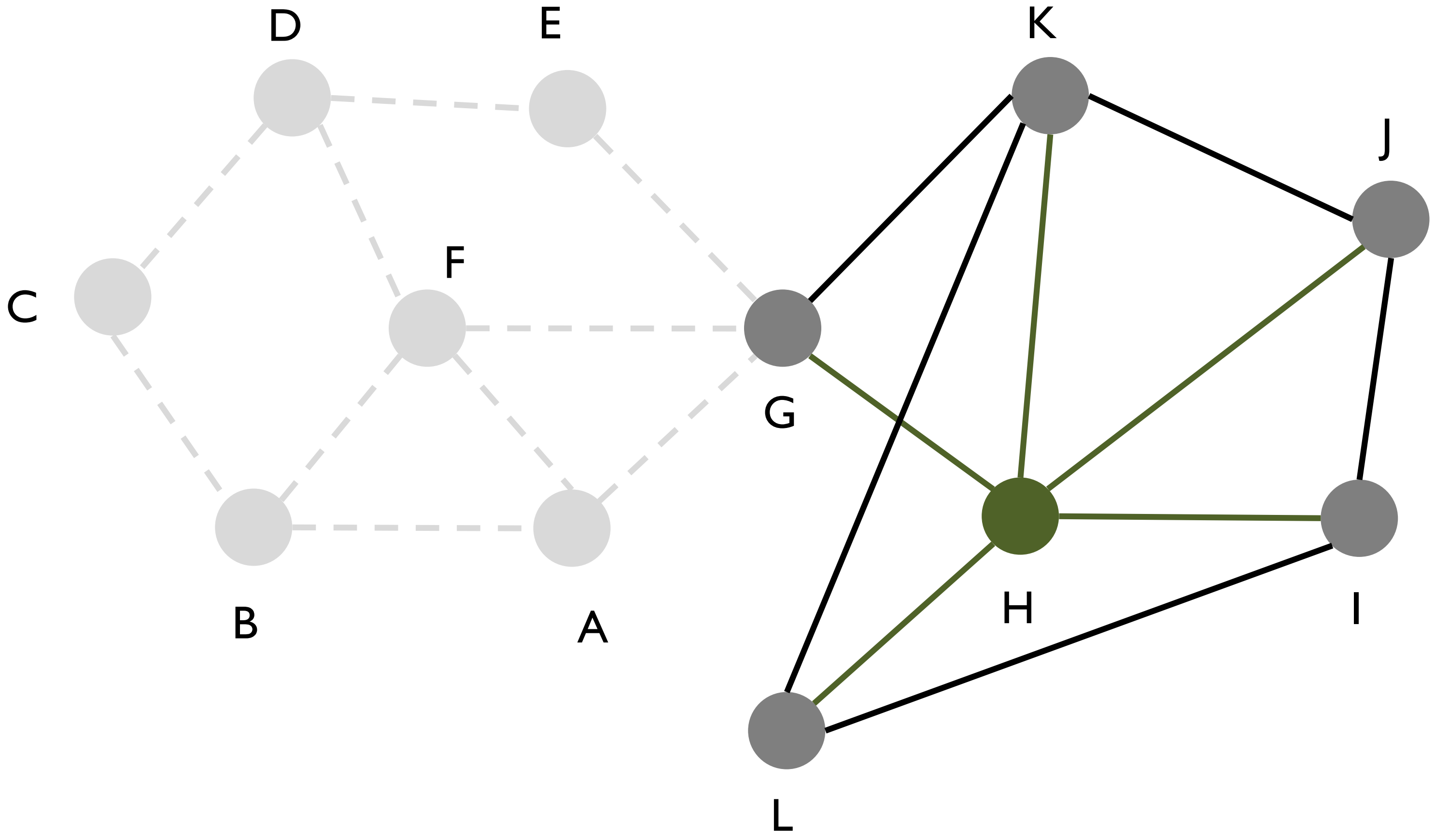
Triadic closure

- Social network
  - if two people know the same person, they are likely to know each other
  - triads or triangles essential units of social network analysis
  - number of triads can be used to find clusters and communities of individuals that all know each other

Clustering coefficient



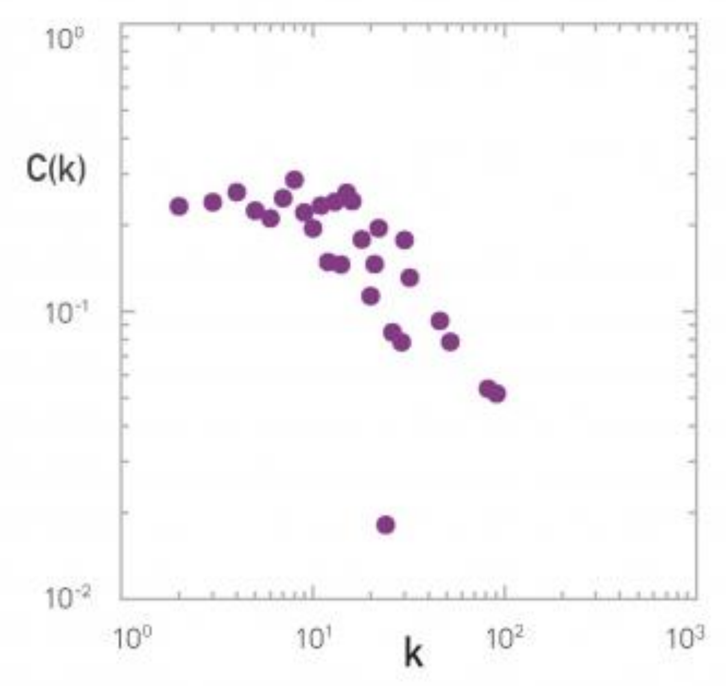
$C_i = 1$ $\quad$ $C_i = 1/2$ $\quad$ $C_i = 0$



$C_H = ?$

Protein-protein interaction network of yeast

average of the clustering coefficient of all nodes with the same degree k



Dependence of the clustering coefficient on the node's degree

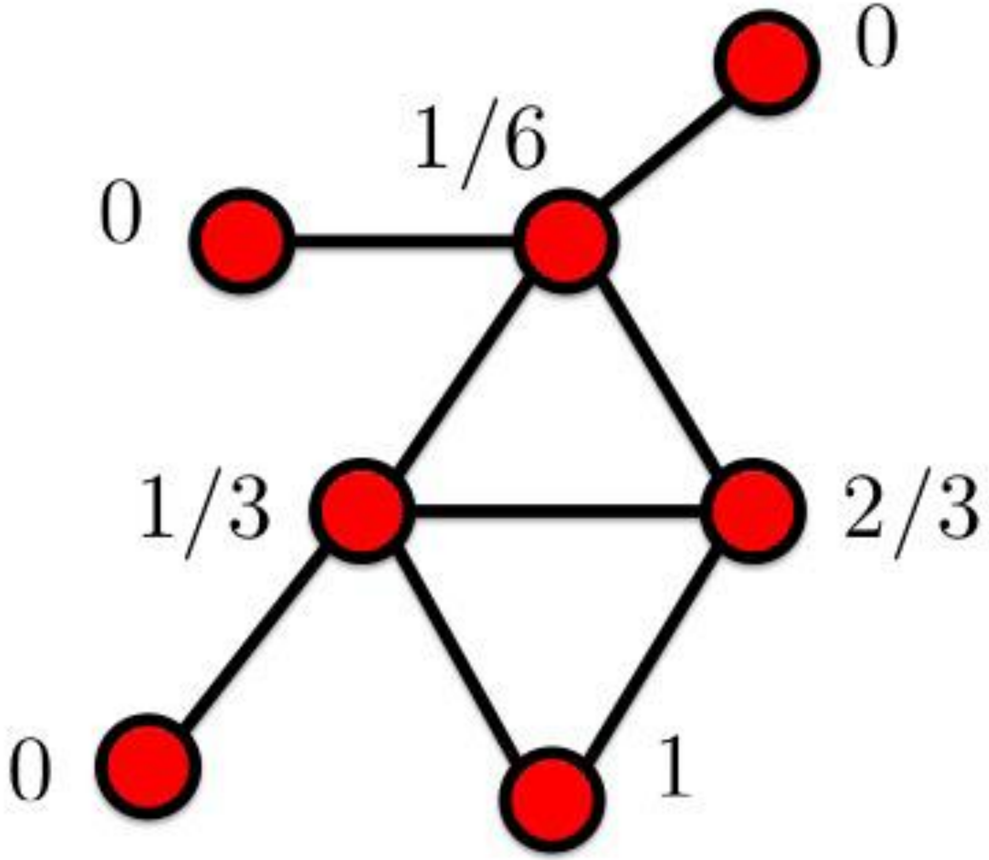
Clustering coefficient

- Fraction of all possible triangles that contain a given node and exist
- Measure of the prevalence of triangles in a network
- Measure of "stardom"
- nx.clustering(G, node)

Transitivity

$\bar{C} = ?$

$C = ?$



$$\bar{C} = \frac{13}{42} \approx 0.310$$

$$C = \frac{3}{8} = 0.375$$

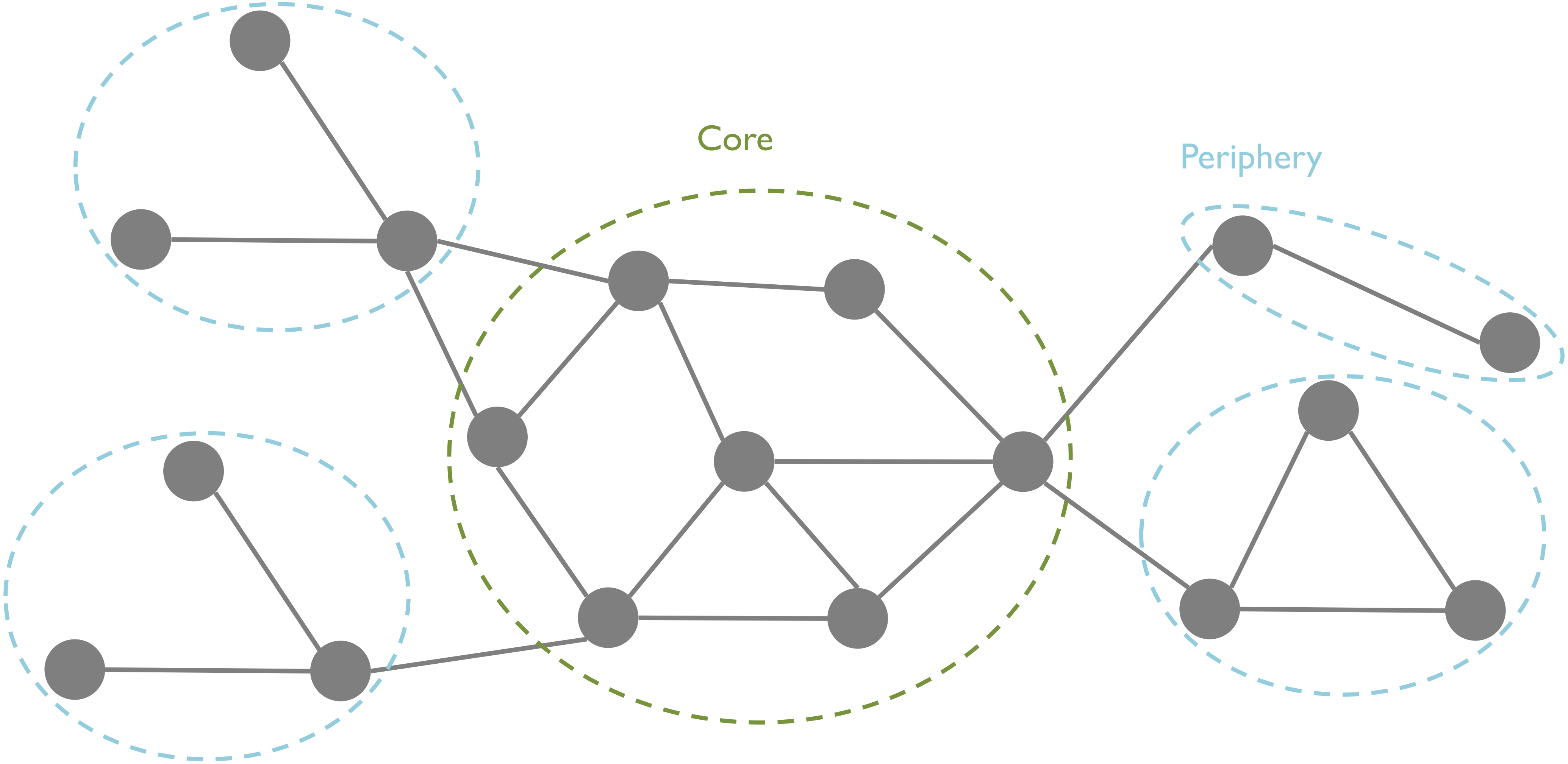
Transitivity

- Ratio of all triangles to all, closed and "open", triangles
- Expresses how interconnected a graph is in terms of a ratio of actual over possible connections
- nx.transitivity(G)

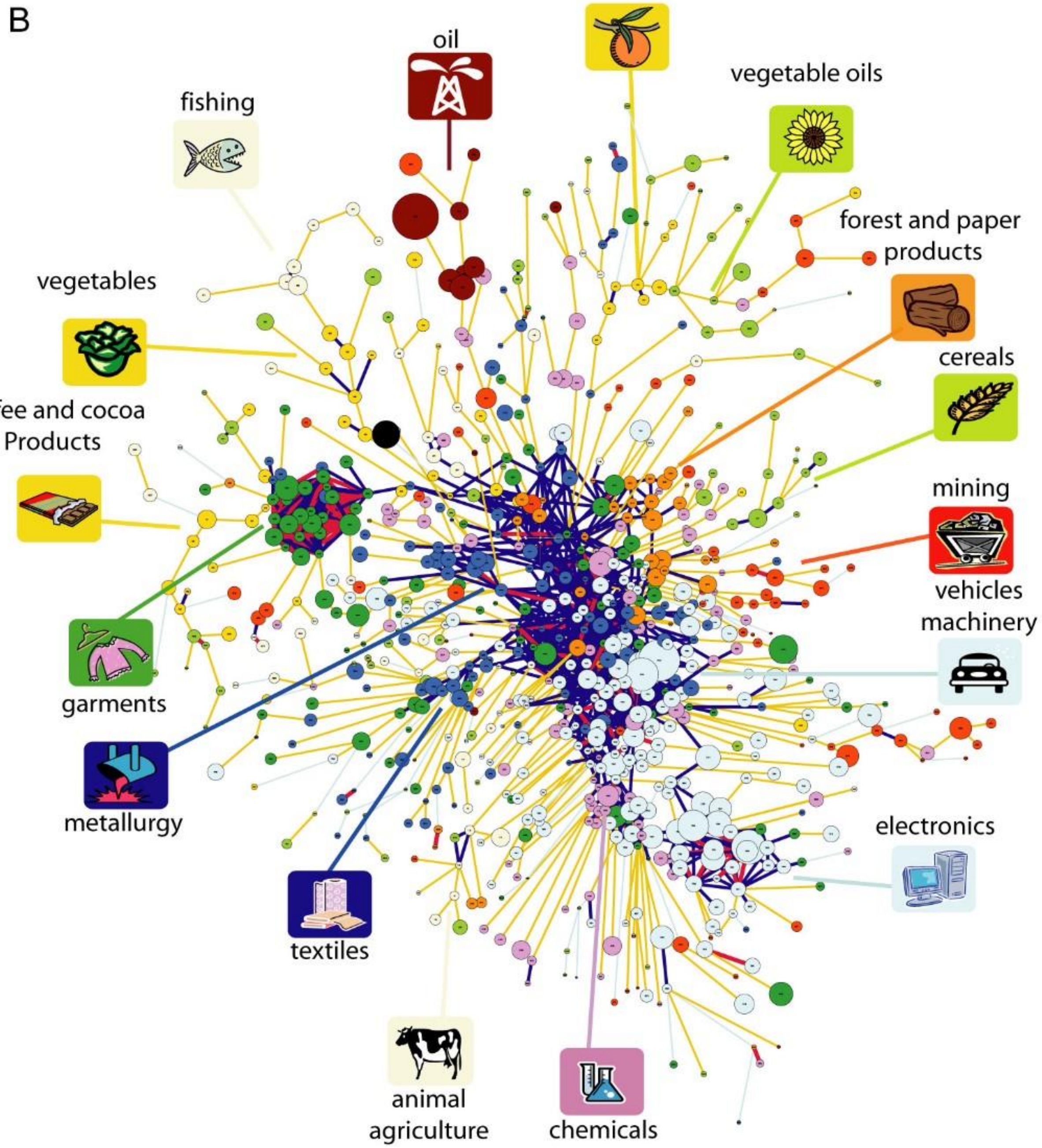
Core-peripheral analysis

- Network consists of two sets of nodes
  - Core
    - nodes that are more or less tightly interconnected
  - Periphery
    - nodes that are tightly connected to the core, but weakly or not connected at all to the other peripheral nodes

- “Hairy” appearance


Core
Periphery

Hidalgo, C. A. and Klinger, B. and Barabási, A.-L. and Hausmann, R. (2007) The Product Space Conditions the Development of Nations, Science, 317(5837), pp. 482-487.



Wang, Chaojun, Core-Periphery Trading Networks, 2016, http://dx.doi.org/10.2139/ssrn.2747117

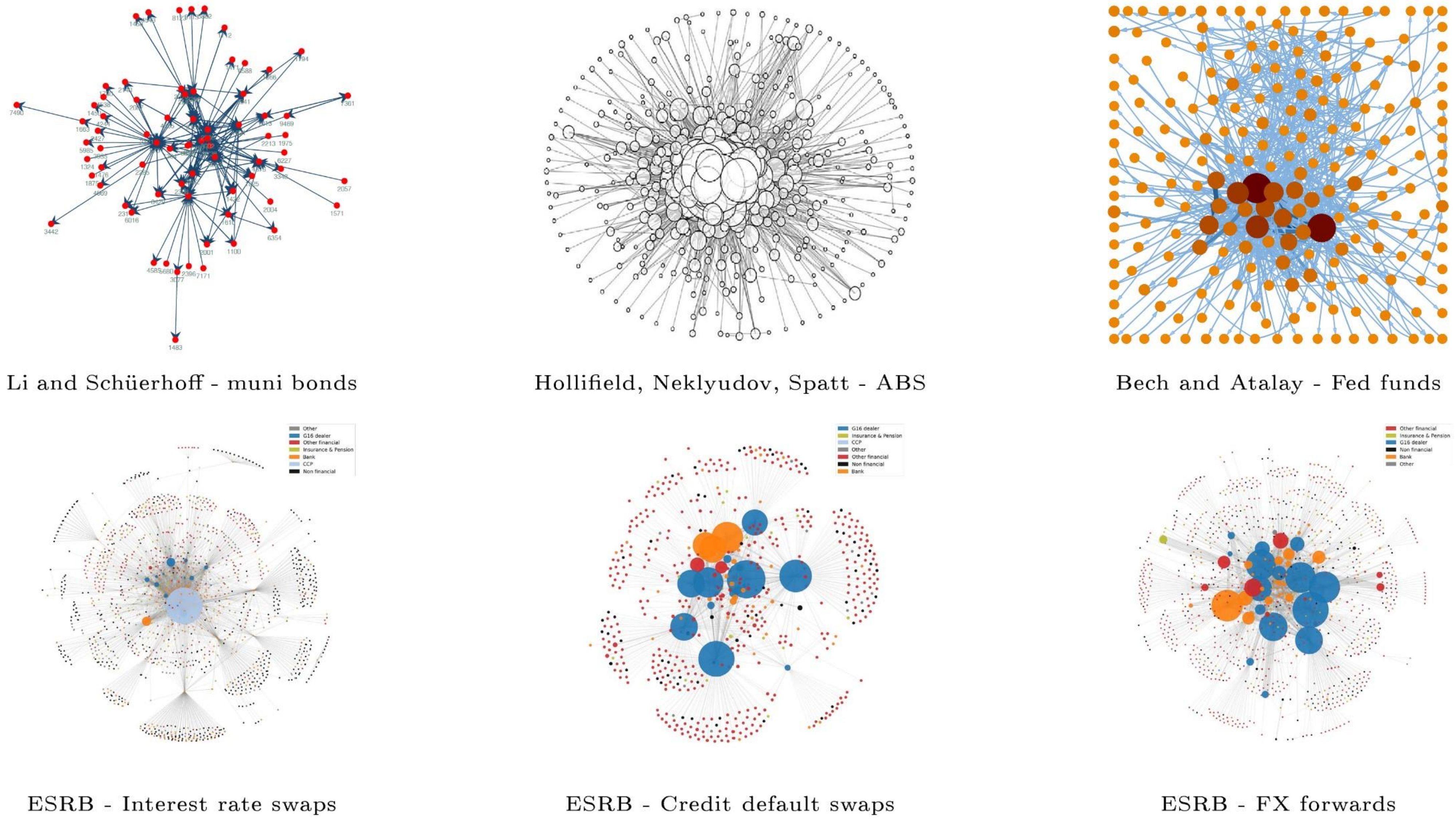


Figure 1 – Core-periphery networks in OTC markets

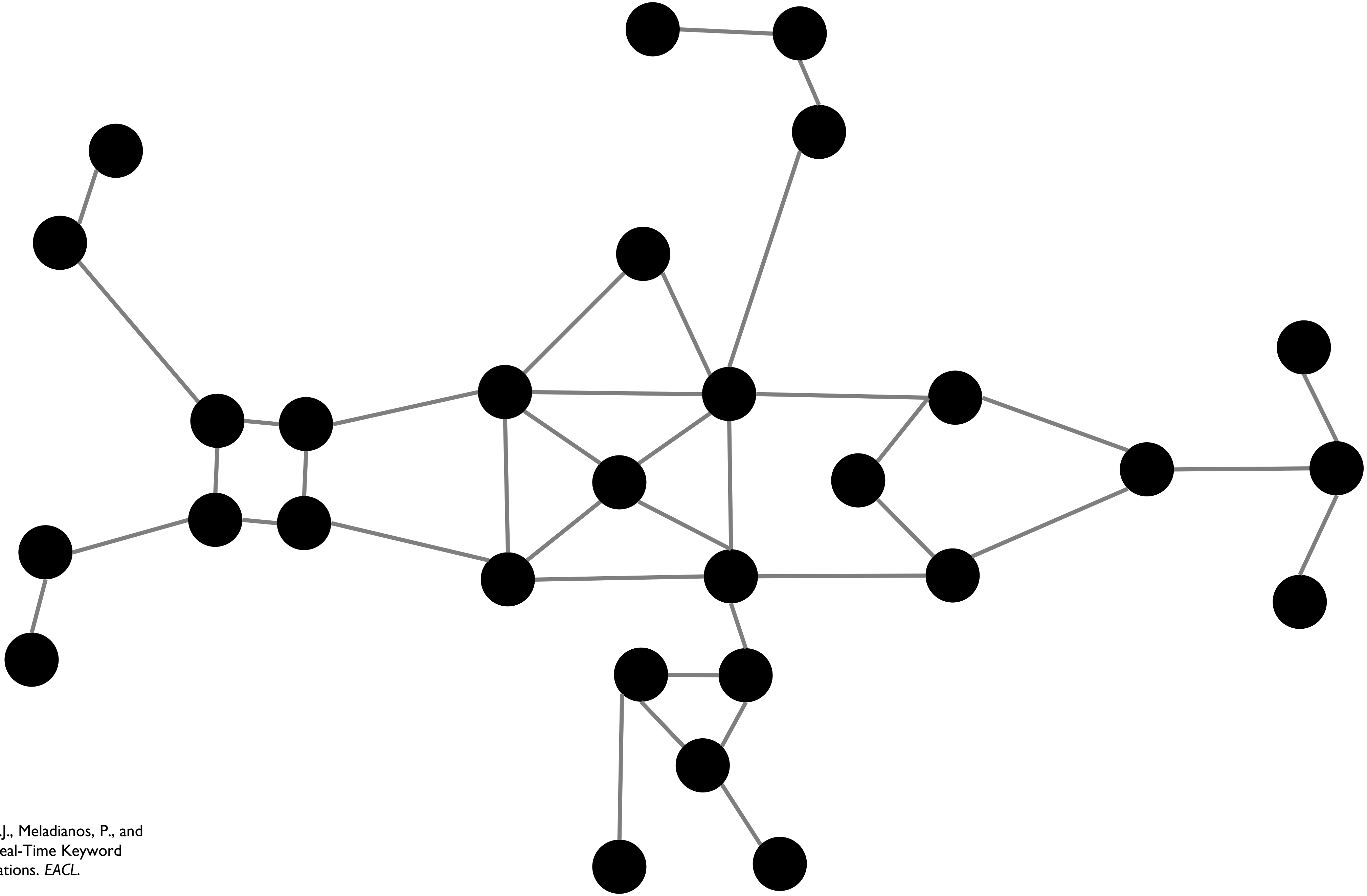
k-core

- *k*-core (*k*>*0*) is a subgraph of the original network graph such that each node in the subgraph has at least *k*-neighbours

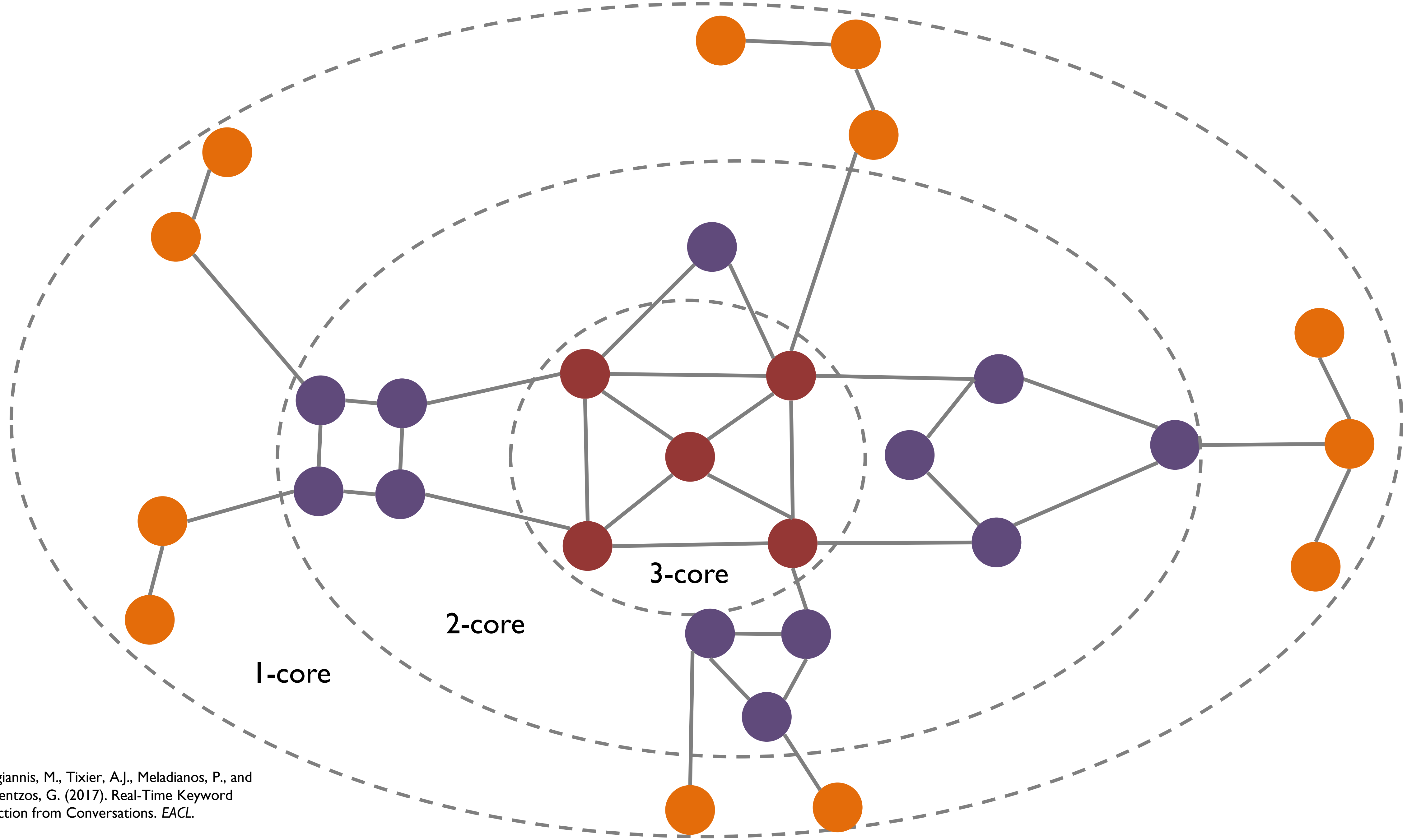
- Main core is the one with the largest possible *k*

- Used to study the clustering structure of social networks, the evolution of random graphs, applied in bioinformatics, network visualisation, Internet structure, spreading of economic crises, etc.

- nx.k_core(G)



Vazirgiannis, M., Tixier, A.J., Meladianos, P., and Nikolentzos, G. (2017). Real-Time Keyword Extraction from Conversations. *EACL*.



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Antonios Garas *et al* 2010 *New J. Phys.* **12** 113043

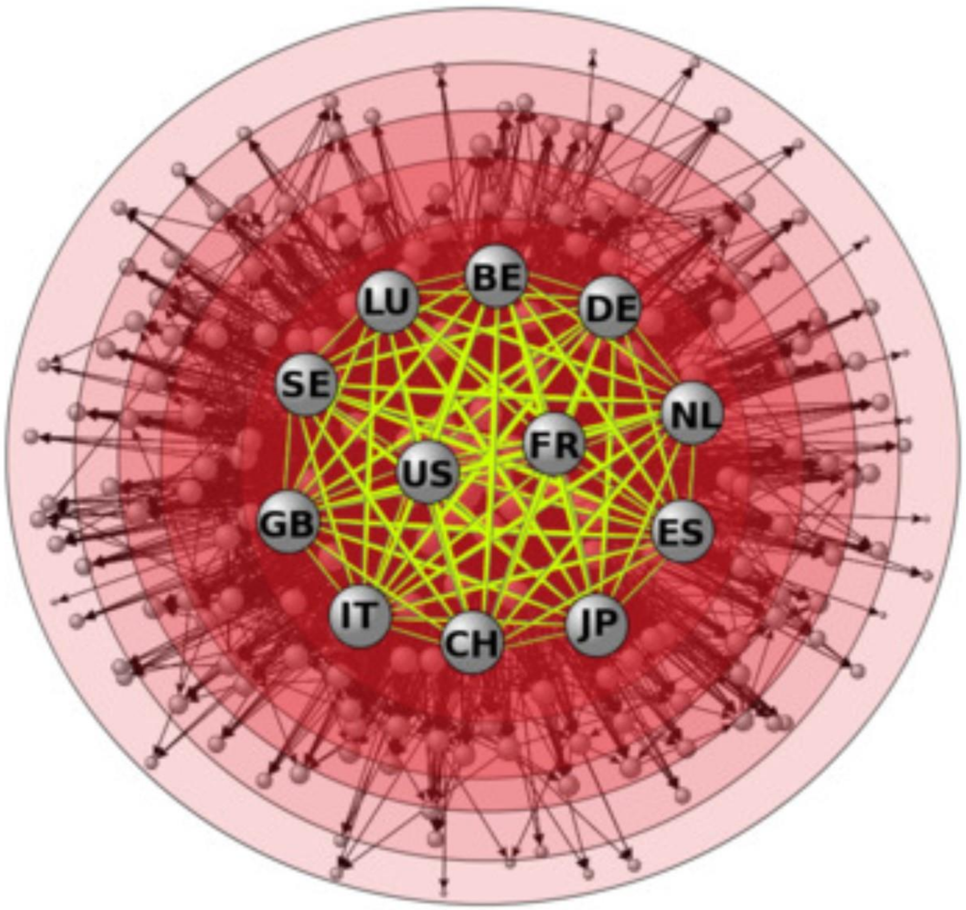


FIG. 1. **An illustration of the layered structure of the global economic network of 206 countries of the world using the large corporation subsidiary relations.**

k-core

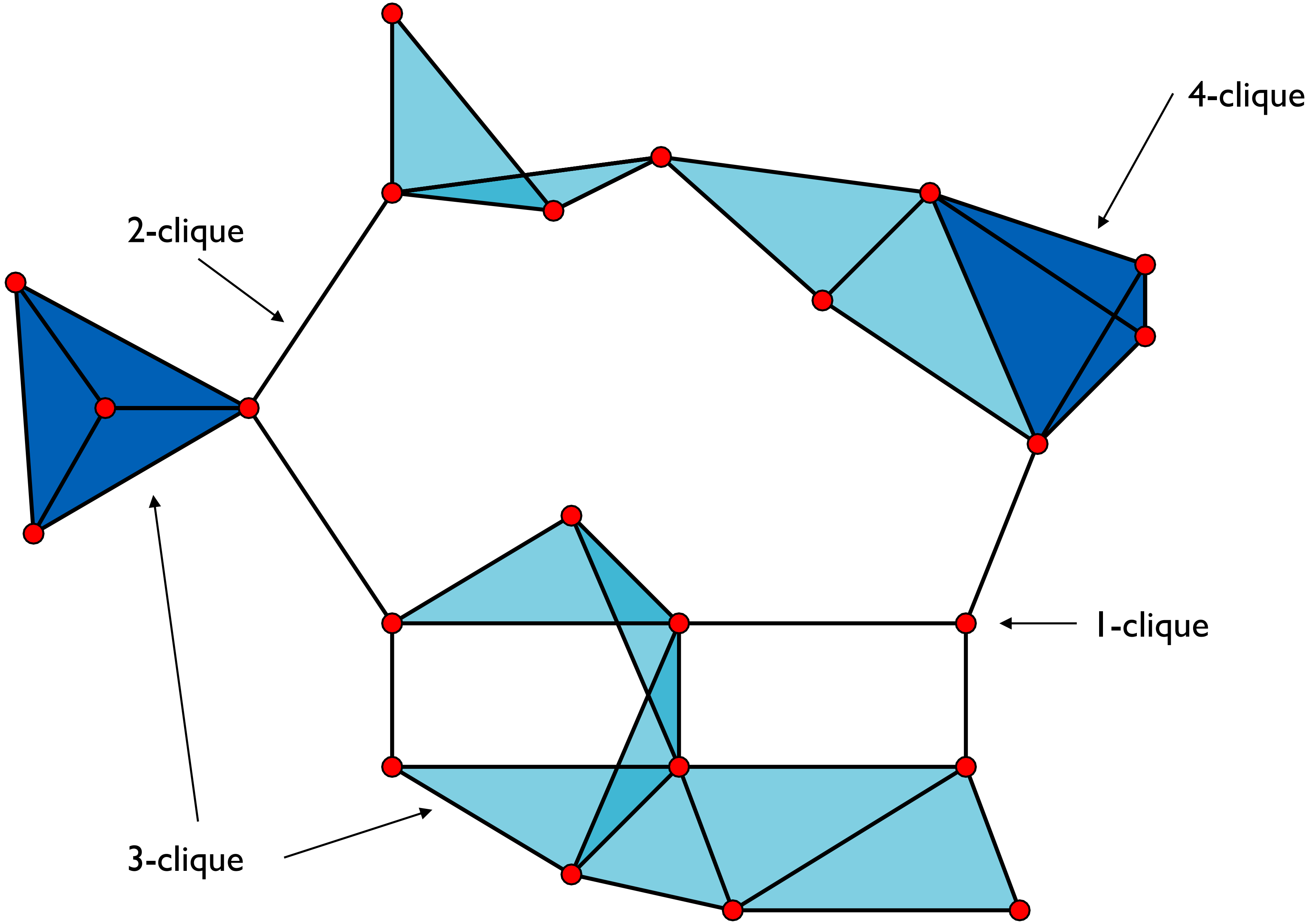
- *k*-core construction process
  - remove all nodes from the original graph that have a degree smaller than *k* and all the incident nodes*
  - remove nodes that have fewer than *k* neighbors
  - iterate until no remaining node has fewer than *k* neighbors
  - the remaining nodes form the *k*-core

*A node is incident to an edge if it is start or end of the edge. Two nodes are adjacent if they are incident to the same edge.

Cliques

- Zoom in in a search for smaller network building blocks
- *k*-clique is a subset of a *k* nodes such that each node is directly connected to each other node in the clique
  - complete subgraphs
  - the degree of a node in a *k*-clique is at least *k-1*

- nx.find_cliques(G)
- nx.enumerate_all_cliques(G)



Wikipedia, Clique (Graph Theory), November 28, 2018.

Sources

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