University of Stuttgart
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Complex Network Systems

Scale-free networks

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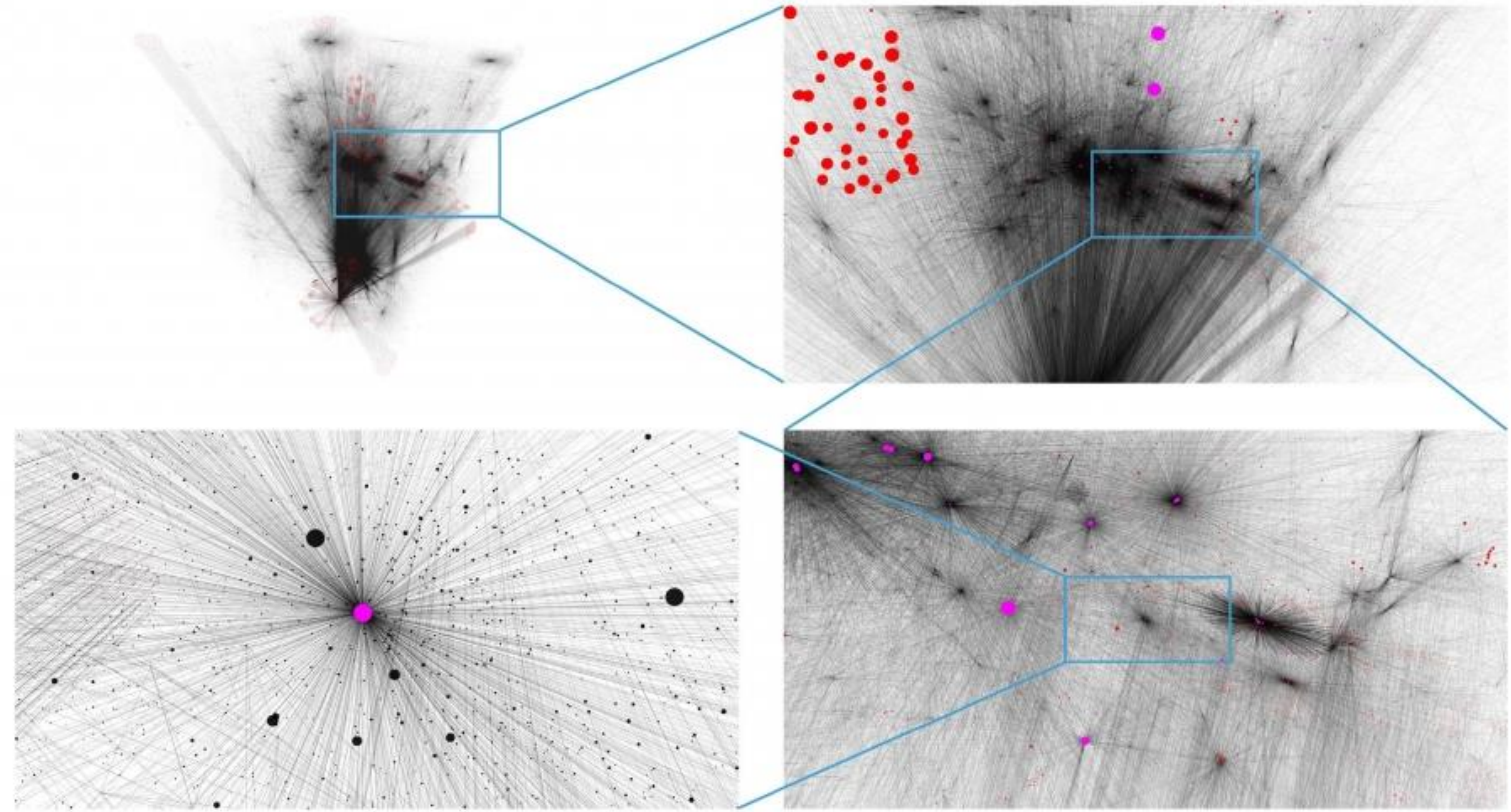
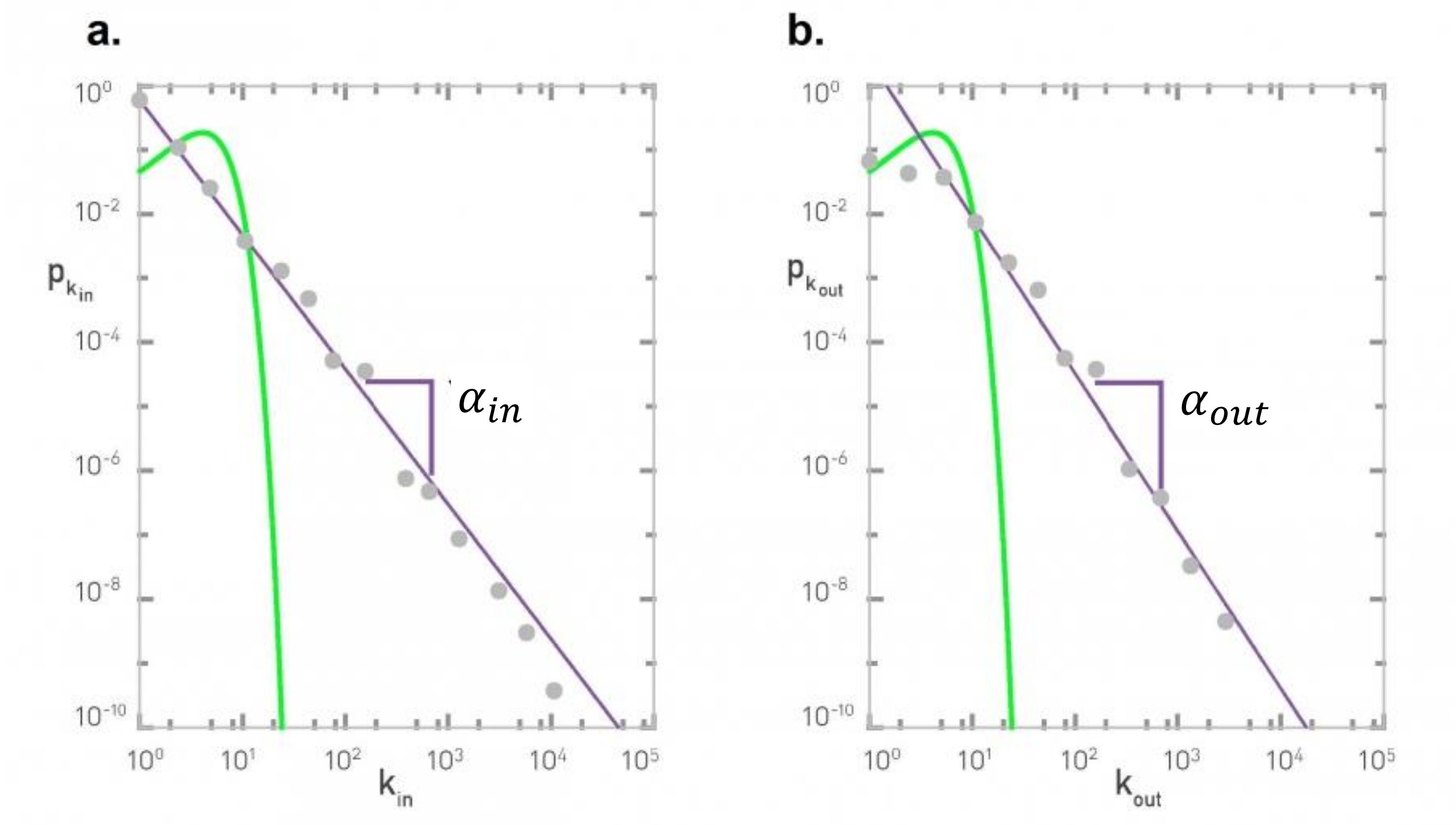
Winter

Courtesy of M. Martino



Degree distribution of WWW



Power-law degree distribution

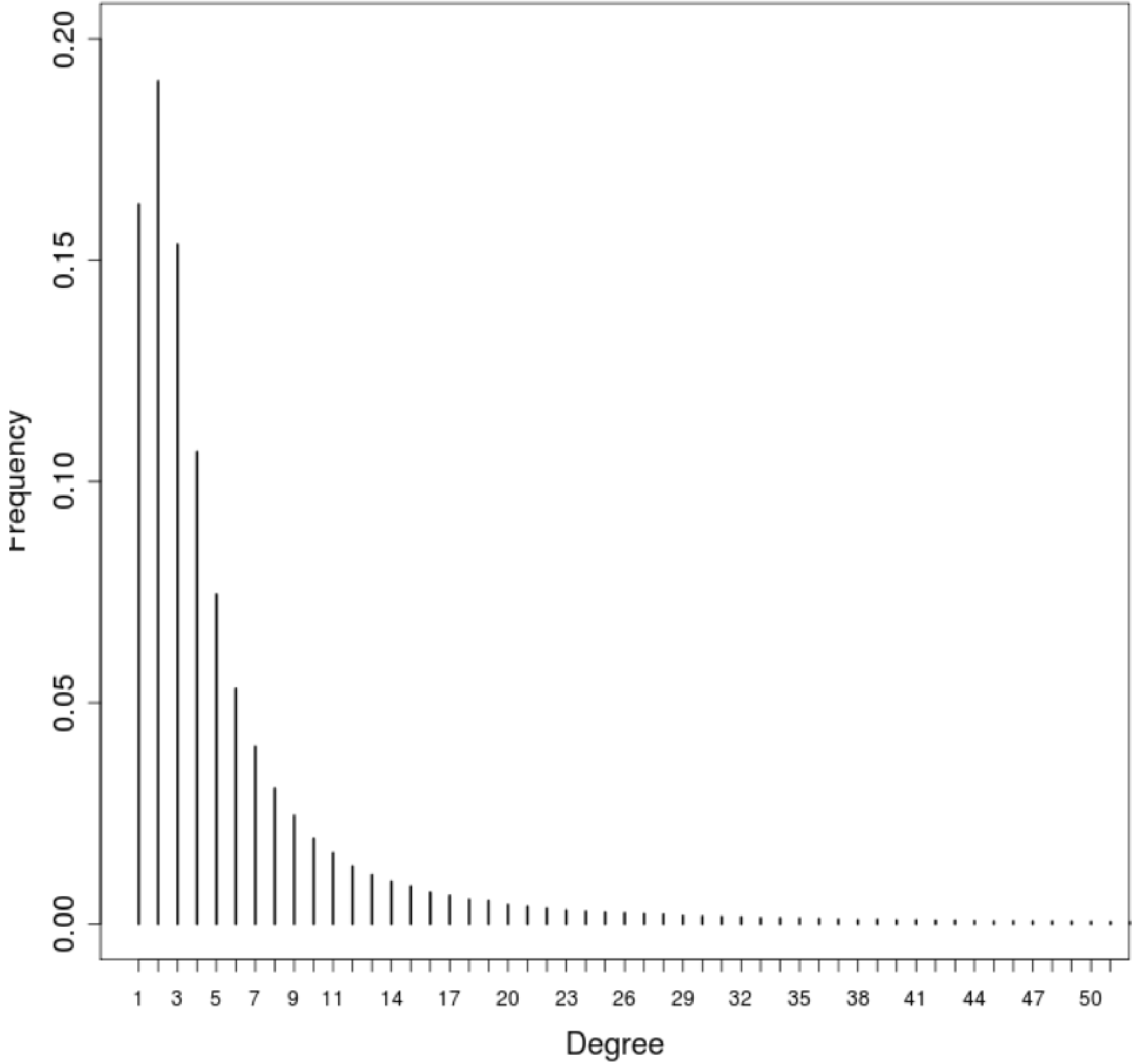
$$P(k) = Ck^{-\alpha}$$

normalisation constant

degree exponent

Power-law degree exponents

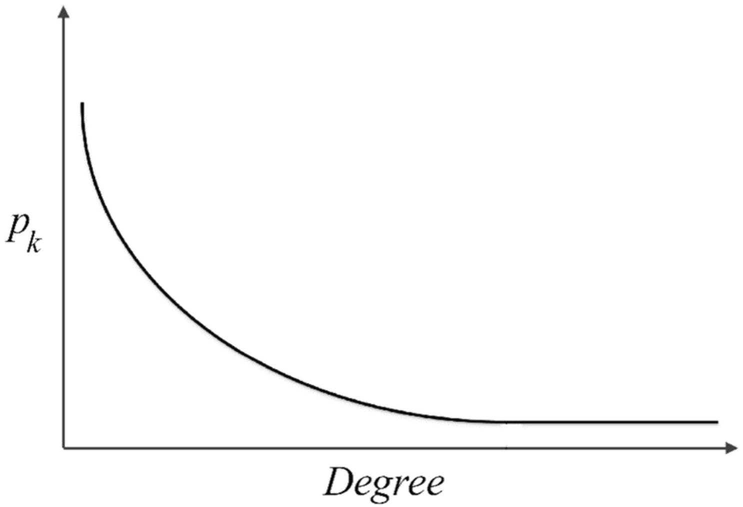
- Typical $2 \leq \alpha \leq 3$
  - **Web graph**
    - $\alpha_{in} = 2.1, \alpha_{out} = 2.4$ [Broder et al. 2000]
  - **Autonomous systems**
    - $\alpha = 2.4$ [Faloutsos 1999]
  - **Actor collaborations**
    - $\alpha = 2.3$ [Barabási-Albert 2000]
  - **Citations to papers**
    - $\alpha \approx 3$ [Redner 1998]
  - **Online social networks**
    - $\alpha \approx 2$ [Leskovec et al. 2007]



Tail contains 1098 highly collaborative scholars (0.16% of all authors)

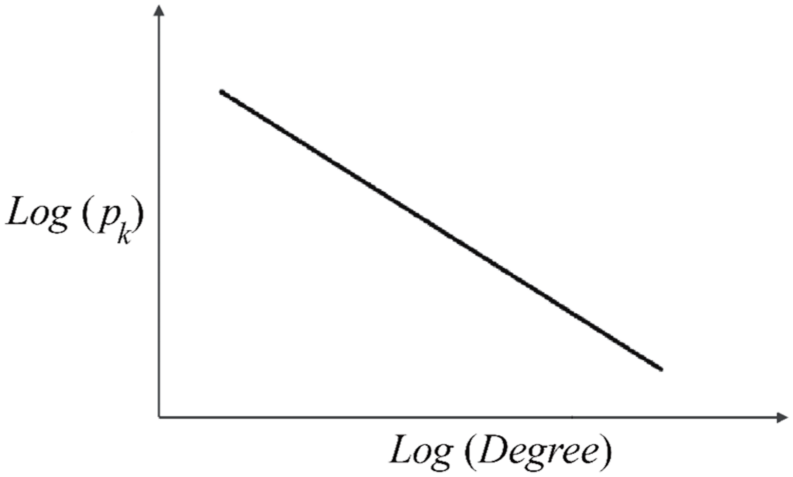
Starting degree: 111
$\alpha = 4.4$
$p = 0.11$

Typical shape



(a) Power-Law Degree Distribution

log-log plot



(b) Log-Log Plot of Power-Law Degree Distribution

Power-law distribution: Test

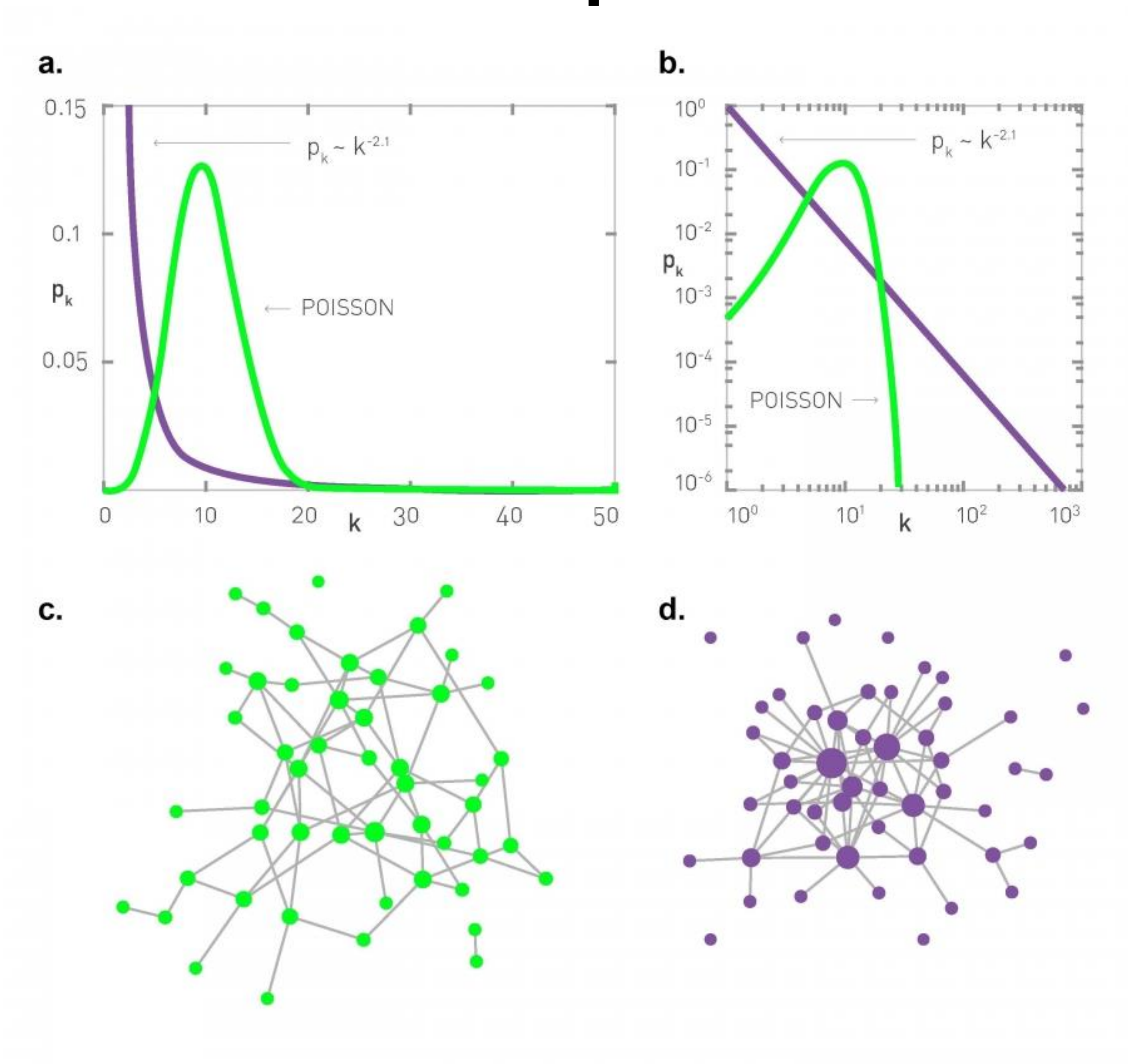
- Test whether a network follows a power-law distribution
  1. Choose a popularity measure and compute it for the whole network
     - e.g., number of friends for all nodes
  2. Compute p_k
     - Fraction of individuals having popularity k
  3. Plot a log-log graph, where the x-axis represents $\log(k)$ and the y-axis represents $\log(p_k)$
  4. Observe if there is a straight line. If yes, a power-law distribution exists

Scale-free networks

A scale-free network is a network whose degree distribution follows a power law

HUBS

Poisson vs power-law distributions



Poisson vs power-law distributions

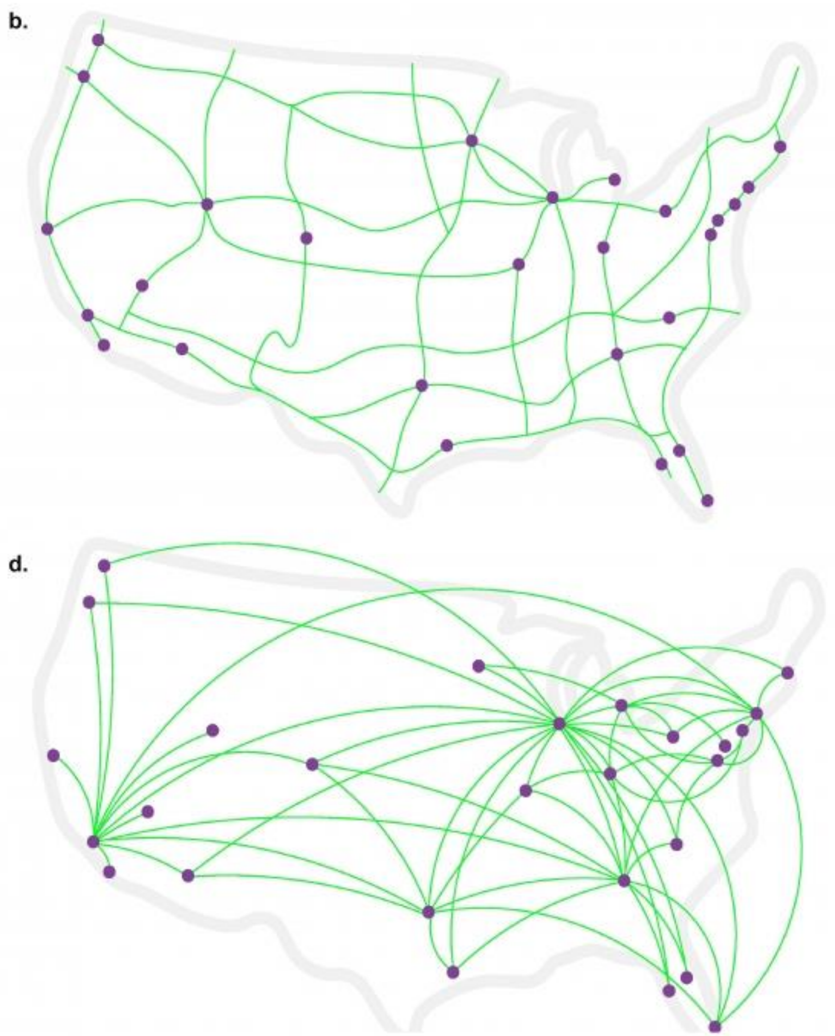
Let us use the WWW to illustrate the properties of the high-*k regime*

The probability to have a node with $k = 100$ *is*

- *About* $P(100) \cong 10^{-30}$ *in a Poisson distribution*
- *About if* $P(100) \cong 10^{-4}$ *if* $P(k)$ *follows a power law*
- *Consequently, if the WWW were to be a random network, according to the Poisson prediction, we would expect* $N_{k>100} \cong 10^{-18}$*, or none.*
- *For a power-law degree distribution, we expect about* $N_{k>100} = 10^{9}$

A.-L. Barabási. *Linked: The New Science of Networks*. Plume, New York, 2002.

Random vs scale-free networks



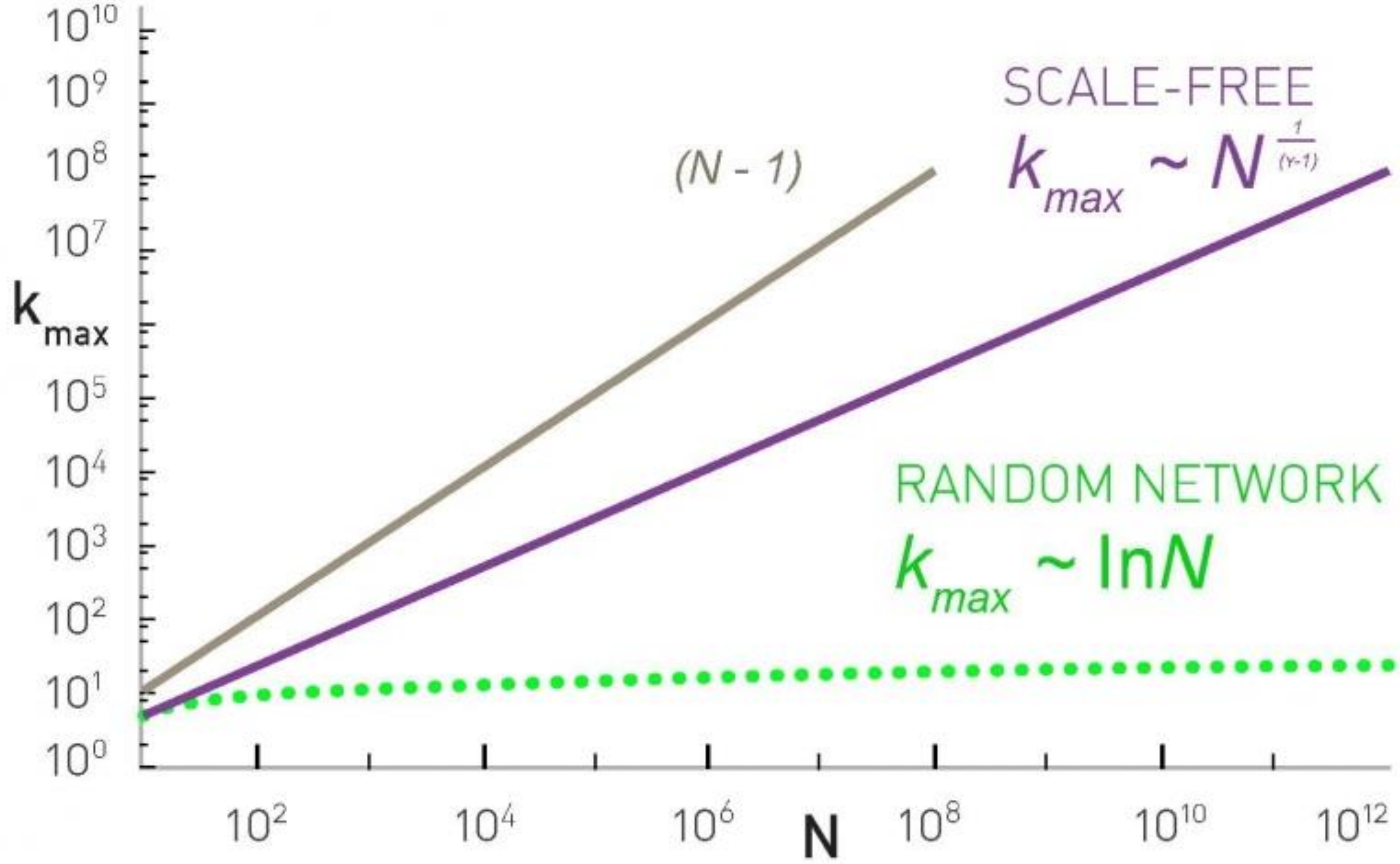
Hubs are large in scale-free networks

Expected maximum degree, k_{max}

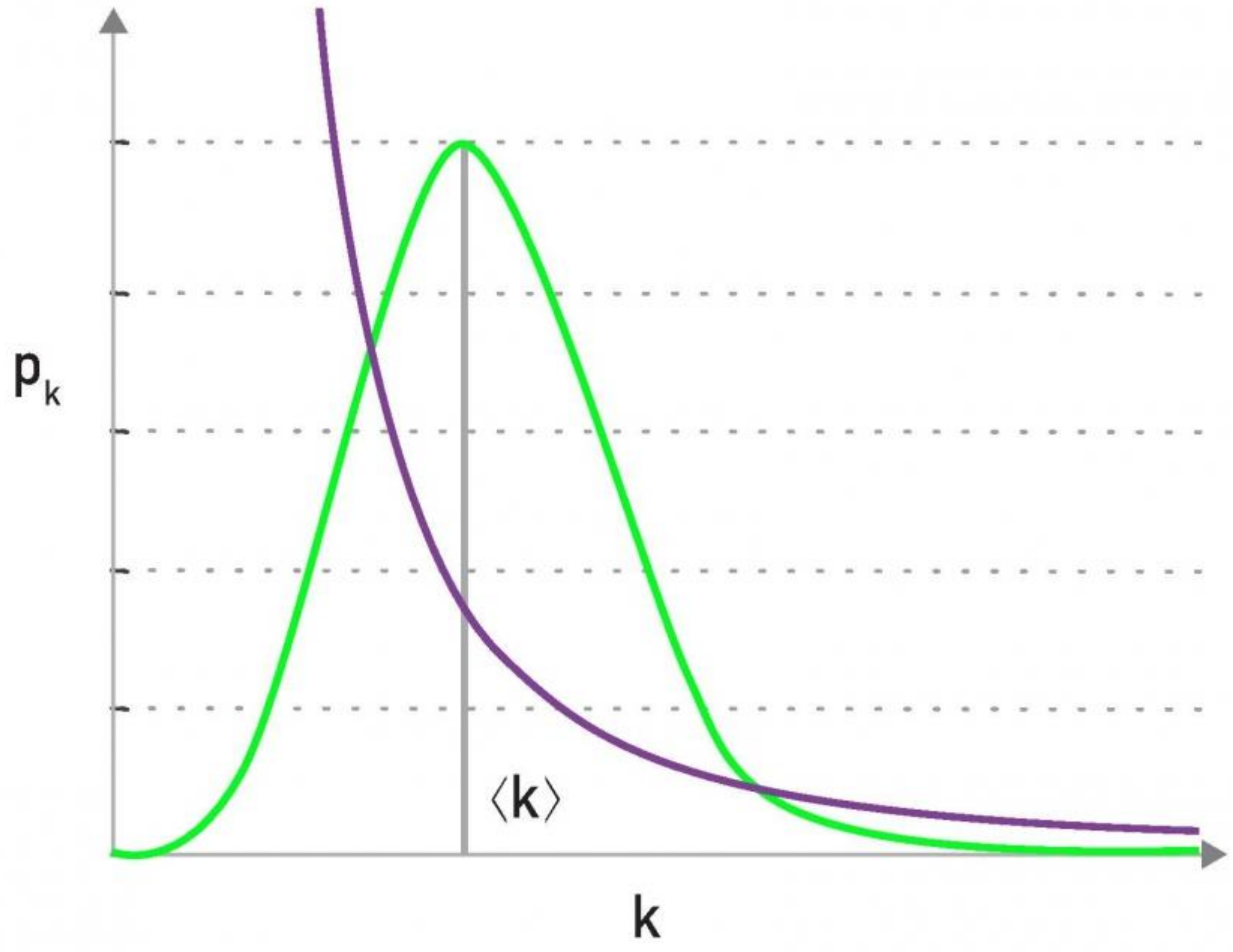
$$k_{max} = k_{min} N^{\frac{1}{\alpha-1}}$$

- k_{max} increases with the size of the network
  - → the larger a system is, the larger its biggest hub
- For $\alpha > 2$, k_{max} increases slower than N
  → the largest hub will contain a decreasing fraction of links as N increases
- For $\alpha = 2$, $k_{max} \sim N$.
  - → The size of the biggest hub is $O(N)$
- For $\alpha < 2$, k_{max} increases faster than N: condensation phenomena
  - → the largest hub will grab an increasing fraction of links. Anomaly!

Hubs are large in scale-free networks



Hubs are large in scale-free networks



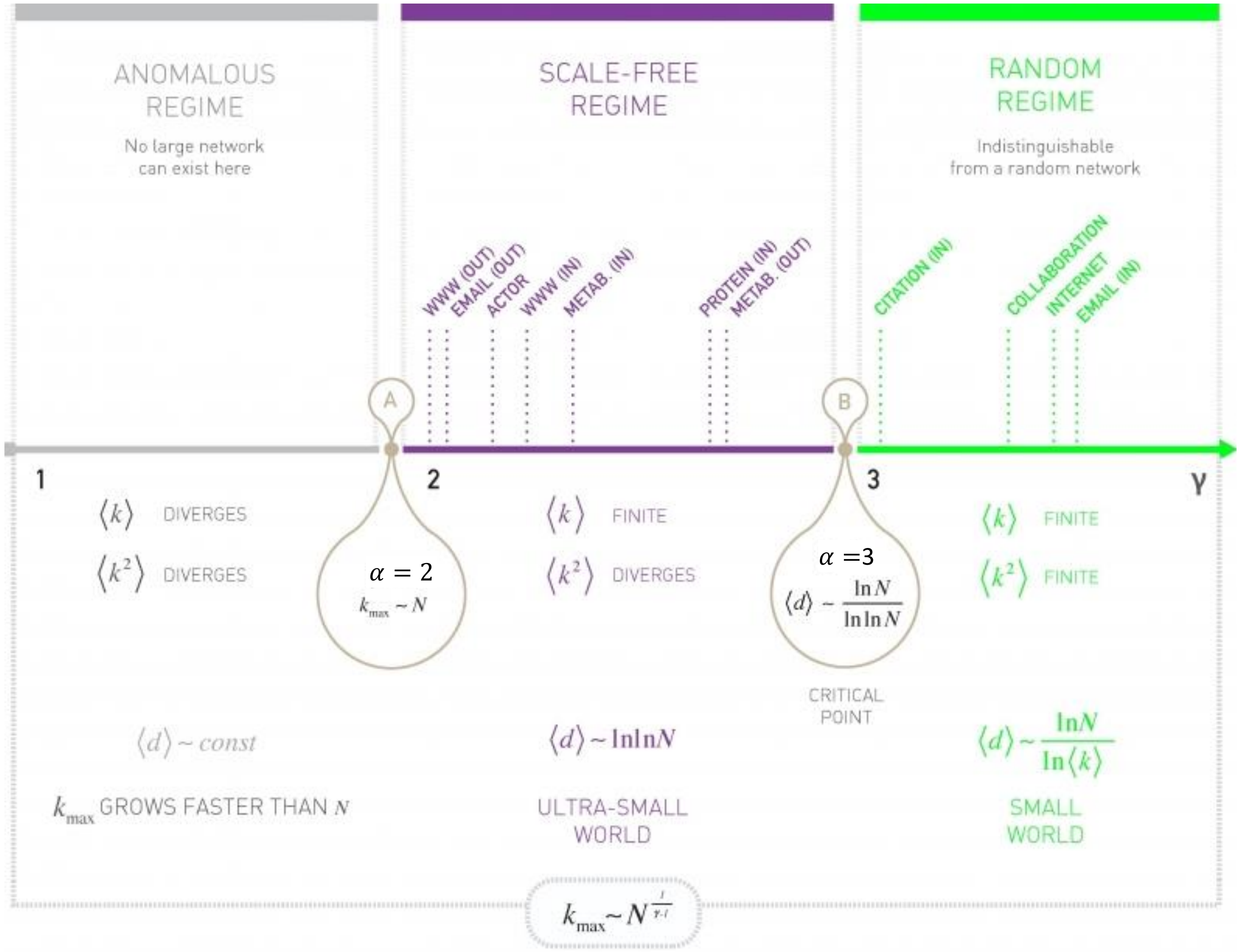
Ultra-small property

- Shrinks the average path lengths. Therefore most scale-free networks of practical interest are not only “small” but are “ultra-small”. This is a consequence of the hubs, that act as bridges between many small degree nodes.

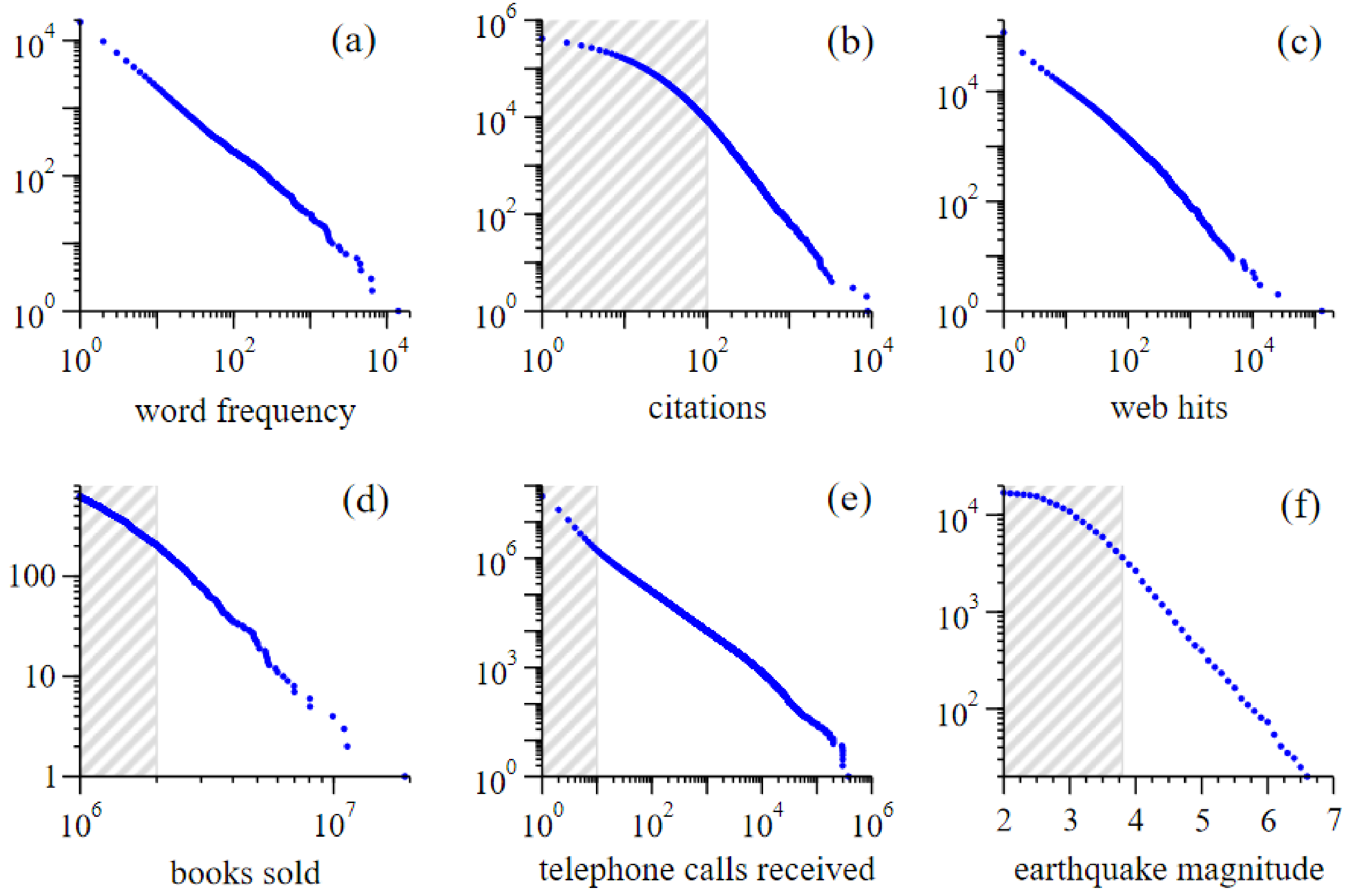
- Changes the dependence of $\bar{d}$ on the system size. The smaller is α, the shorter are the distances between the nodes.

- Only for $\alpha > 3$ we recover the $\ln N$ dependence, the signature of the small-world property characterising random networks.

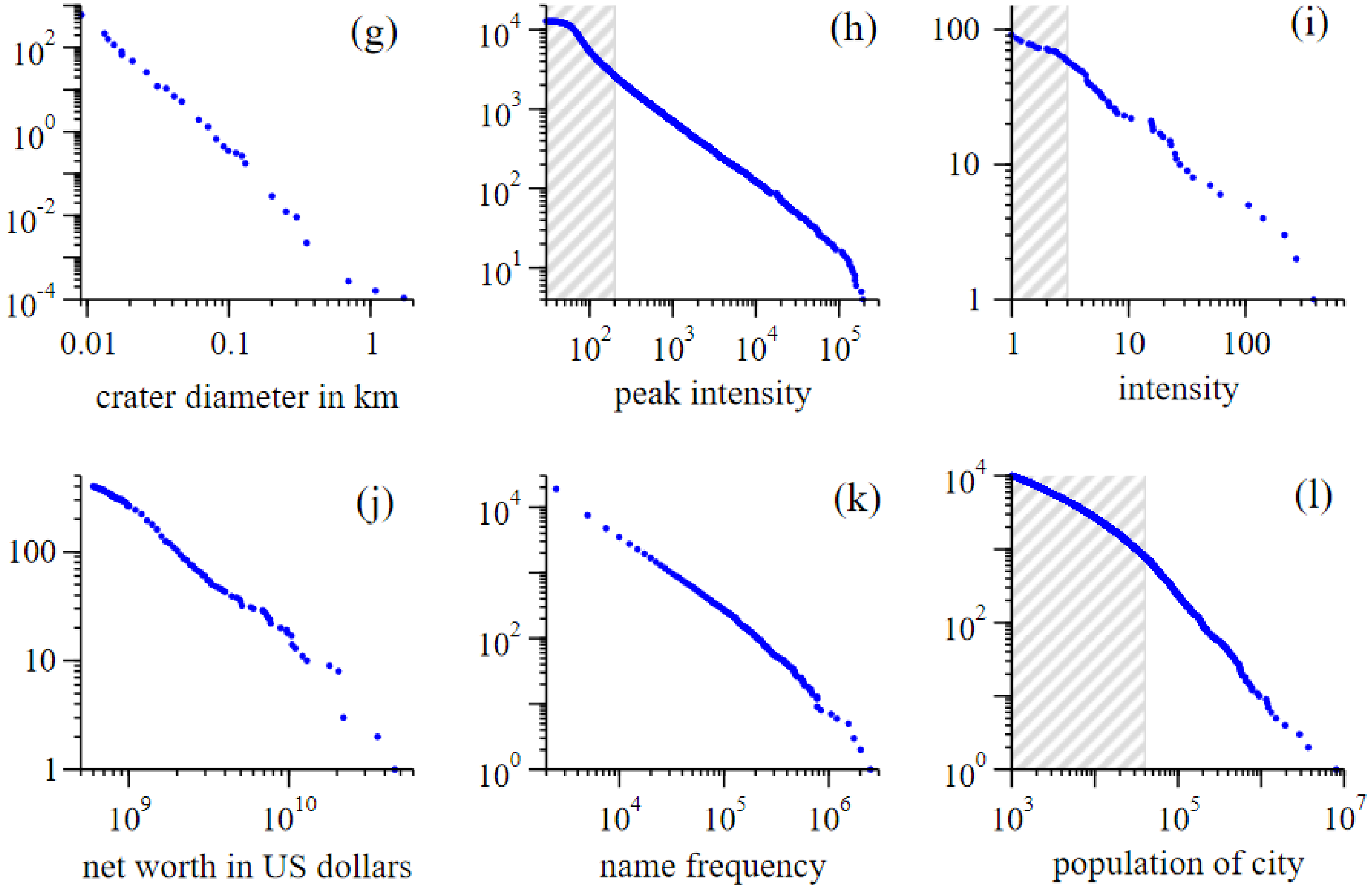
Scale-free networks are degree dependent



Examples: Power-law distribution



More examples: Power-law distribution



Sources

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- Mateos, G. Degrees, Power Laws and Popularity, University of Rochester, 2018.
- Zafarani, R., Abbasi, M. A. and Liu, H. *Social Media Mining: An Introduction*, Cambridge University Press, 2014.
- Barabási, A. Network Science, http://networksciencebook.com