A NONLOCAL OPERATOR MODEL FOR MORPHOLOGICAL IMAGE PROCESSING

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ABSTRACT

A nonlocal operator model for morphological image processing is proposed in this paper. Compared with the previous nonlocal morphological operators, the derivations of the proposed model are easier to inherit useful properties and physical interpretations from the traditional morphology. Two important properties of the model have been clarified in theory. And for instances, based on the model, two basic morphological operators (i.e., erosion and dilatation) are developed with convincing experimental results. In addition, the generality and usability of the model are also discussed.

Index Terms—Mathematical morphology, nonlocal operator model, image processing

1. INTRODUCTION

Mathematical morphology (MM) is a well-known nonlinear methodology, which provides a wide range of operators to address various image processing problems [1][2]. MM has originally been developed for binary images and extended to grey-levels based on the complete lattice theory [3]. The original image (denoted by *I*) is probed with another small subsets called structuring element (*SE*). In traditional approaches, *SE* is utilized to design the two basic operators, i.e., erosion and dilation, which are denoted by ε and δ respectively. And other morphological operators, such as opening and closing, are obtained as the combinations of these two ones. The designation of *SE* is usually according to some priori knowledges about the specific tasks, which leads to clear physical interpretations. Meanwhile, as the basic operators, ε and δ both have several useful properties, which are crucial to the actual applications. For examples, their ordering relations property is necessary to the definition of morphological gradient which is generally used to detect boundaries and/or edges in an image [1][2][7]; and erosion and dilatation should form an adjunction (i.e., adjunction property), which is the essential property to guarantee $\delta\varepsilon$ and $\varepsilon\delta$ satisfy the algebraic properties of opening and closing [3]. It is the useful properties and the clear physical interpretations that promote the applications of traditional MM [1][2].

In traditional MM, all the definitions of *SE*, ε , δ and other operators only utilize the local information of the image processed. To break through the local limit of traditional MM, graph-based morphology has been studied and more general operators have been obtained successfully [4][5][6]. Starting from nonlocal means filter (NLM) [8], nonlocal methods have drawn much more attentions in image processing community [9][10]. Unlike whose local counterparts, the nonlocal methods can effectively take advantage of the periodic (nonlocal) information in the input image. Inspired by the success of NLM, nonlocal strategies have been borrowed to design mathematical morphological operators and achieved better results [11][12][13]. However, as pointed out in [14], several useful properties owned by the traditional operators have been missed in these extended works. And the absence of the properties will inevitably obstruct further applications of the corresponding operators. Among the properties, adjunction of erosion and dilatation is fundamental. To hold the property in nonlocal fashion, reference [14] suggests that *SE* should be fixed once they have been derived from an initial input *I*. Followed this effective rule, the nonlocal morphological erosion and dilation operators are developed in [15]. As expected, the adjunction property there has been kept, thus ε and δ can be effectively used to construct other nonlocal morphology operators. However, there are still several limitations in the proposed methodology in [15]: i) the ordering relations property of erosion and dilation cannot be kept; ii) the physical interpretation of *SE* is not so as clear as it in the traditional MM. And these limitations will be further discussed in the next station.

In this paper, the emphasis is put on the proposed nonlocal operator model, whose derivations can more easily inherit useful properties and clear physical interpretations from the traditional ones. The paper is organized into five sections. Section 2 reviews the traditional MM and analyzes the limitations of its nonlocal extensions. Section 3 presents the nonlocal operator model with two theories to clarify its advantage. To show the effectiveness of the proposed model, Section 4 derives two basic nonlocal operators (i.e., erosion and dilation) from it with convincing experiments. To end, Section 5 gives the conclusion of the whole paper.

2. A NONLOCAL OPERATOR MODEL FOR MORPHOLOGY

2.1 The Traditional Mathematical Morphology

The traditional MM stems from set theory, which considers a digital image I as a numerical function from a set Ω (definition domain) to a set F (value field). In the case of n-dimensional grey-level images, Ω is a subset of Z^n, and F is the set of grey values of the image. Thus, a grey-level image can naturally be represented by a function $x \mapsto I(x)$ with $x \in \Omega$ and $I(x) \in F$. That is, I maps each pixel x into a grey-level value t, i.e., $I(x)=t$. Denote the functions set from Ω to F with $Fun(\Omega, F)$, it can be induced that $I \in Fun(\Omega, F)$ and $Fun(\Omega, F)$ is a complete lattice [1]. A structuring element (SE) in mathematical morphology (MM) is nothing but a small set used to probe the image under study, whose shape and size must be adapted to the specific tasks. Therefore, SE generally requires clear physical interpretations in practical applications [1][2]. SEs in MM can be divided into two types. On the one hand, a n-dimensional SE, i.e., $SE \subseteq \Omega$, is named flat SE. On the other hand, an n+1 dimensional SE is called non-flat SE. In this paper, $SE(x)$ is abused to denote the SE whose origin coincides with x. Based on the theoretical backgrounds above and with the similar symbol-system adopted in [15], the two basic operators, i.e., erosion (ε) and dilation (δ), can be defined as follows. For the case of flat SE,

$$\varepsilon_{SE}(I)(x) = \bigwedge_{y \in SE(x)} I(y) \qquad x \in \Omega, \tag{1}$$

$$\delta_{SE}(I)(x) = \bigvee_{y \in \widehat{SE}(x)} I(y) \qquad x \in \Omega. \tag{2}$$

Where $\widehat{SE}$ is the transposed SE (i.e., reflection w.r.t. the origin). The two operators above are called flat erosion and flat dilation respectively. For the case of non-flat SE,

$$\varepsilon_{SE}(I)(x) = \bigwedge_{y \in SE(x)} \{I(y) - SE(x)(y)\} \quad x \in \Omega, \tag{3}$$

$$\delta_{\mathrm{SE}}(I)(x) = \bigvee_{y \in \widehat{SE}(x)} \{I(y) + \widehat{SE}(x)(y)\} \quad x \in \Omega. \tag{4}$$

Where $SE(x)(y)$ denotes the grey value (i.e., the (n+1)-th dimension [2]) of $SE(x)$ at the pixel y. Accordingly, the operators defined in Eqs. (3-4) are called non-flat erosion and non-flat dilation respectively.

As the basic operators, erosion and dilation in traditional MM have some important properties. And two of them are introduced here. One property is ordering relation, which can be formulated as

$$\varepsilon(f) \le f \le \delta(f), \tag{5}$$

where f denotes the image under studied. With this property, a morphological gradient, denoted by $\rho(f)$, can be defined as

$$\rho(f) = \delta(f) - \varepsilon(f). \tag{6}$$

As pointed out in [1][2][7], the morphological gradient is an effective operator generally used to locate boundaries and/or edges in a digital image. The other property focused here is the adjunction. To form an adjunction, for any two given images f and g, the operators ε and δ should satisfy the following equivalence:

$$\delta(f) \le g \Leftrightarrow f \le \varepsilon(g). \tag{7}$$

It is the adjunction property that guarantees $\delta\varepsilon$ and $\varepsilon\delta$ possessing the algebraic properties of opening and closing [3].

2.2 Limitations of the Previous Work

The nonlocal morphology has been considered in previous work [11][12][13][14][15]. In order to formulate the nonlocal MMs, this subsection starts from nonlocal means filter (NLM) [8]. A main strategy of NLM is using a weighted average of the whole (nonlocal) input image as the output of a pixel's grey value:

$$\mathrm{NLM}(I)(x) = \sum_{y \in \Omega} W_I(x, y) I(y), \qquad x \in \Omega \tag{8}$$

Where the weight $W_I(x, y)$ is defined by computing the similarity between a patch P_I centered around the pixel x and another patch centered around pixel y, specifically,

$$W_I(x, y) = \frac{\widehat{W}_I(x, y)}{\sum_{z \in \Omega} \widehat{W}_I(x, z)} \tag{9}$$

and

$$\begin{aligned} \widehat{W}_I(x, y) &= \widehat{W}_I(P_I(x), P_I(y)) \\ &= \exp\left(-\frac{\|P_I(x)\text{-}P_I(y)\|_{2,\alpha}^2}{h^2} \right). \end{aligned} \tag{10}$$

Here, $\|\bullet\|_{2,\alpha}$ is the Gaussian weighted L_2 norm and α is the standard deviation of Gaussian kernel; and h is a smoothing parameter. For the weights defined in Eq. (9), there are two constrains as follows

$$0 \le W_I(x, y) \le 1, \tag{11}$$

and

$$\sum_{y \in \Omega} W_I(x, y) = 1. \tag{12}$$

More specifics about NLM can be referred to [8].

Inspired by the nonlocal strategy introduced above, a generalization of traditional morphological operators from local to nonlocal is proposed in [11]. Where, the two basic nonlocal operators are generalized as

$$\varepsilon_{NLSE}(I)(x) = \bigwedge_{y \in SE(x)} \{I(y) - \beta W_I(x, y)\} \quad x \in \Omega, \tag{13}$$

$$\delta_{NLSE}(I)(x) = \bigvee_{y \in SE(\mathrm{x})} \{I(y) + \beta W_I(x, y)\} \quad x \in \Omega. \tag{14}$$

Where β is an operator parameter. The mainly difference between the traditional morphological operators defined in Eqs. (3-4) and the nonlocal generation above mainly lies in two aspects: one is the structuring element $SE(x)$, which is nonlocal in the latter beyond its local counterpart in the former; the other is $SE(x)(y)$ in Eqs. (3-4), which is replaced by $W_I(x,y)$ in the nonlocal generation.

Another nonlocal extension of MM is addressed in [12], which undergoes a complex process including dictionary learning, manifold learning and out of sample extension. Thus its physical interpretation is no longer as clear as it in traditional MM.

To design nonlocal morphological operators with adjunction property, an essential conclusion is given in [14]: "*one has to fix the adaptive neighborhood once they have been derived from an initial input image I.*" Inspired by the conclusion, reference [15] provides a definition of structuring elements system, which is fixed by image *I* and donated by $\{SE_I(x)\}_{x\in\Omega}$). Where, for any $x, y\in \Omega$,

$$x\in SE_I(x), \tag{15}$$

$$y\in SE_I(x) \Rightarrow x\in SE_I(y), \tag{16}$$

should be both satisfied. Then, a nonlocal morphology is proposed in [15], whose basic operators are defined as follows,

$$\varepsilon_{SE_I,\mathbb{W}_I}(f)(x) = \bigwedge_{y\in SE_I(x)} \{f(y)-\mathbb{W}_I(x,y)\} \quad x\in\Omega, \tag{17}$$

$$\delta_{SE_I,\mathbb{W}_I}(f)(x) = \bigvee_{y\in SE_I(x)} \{f(y)+\mathbb{W}_I(x,y)\} \quad x\in\Omega. \tag{18}$$

Where *f* is the image under studied; $\{SE_I(x)\}_{x\in\Omega}$ is a structuring elements system; and

$$\mathbb{W}_I(x,y) = \log\left(W_I(x,y)\right). \tag{19}$$

In the same reference, it has been proved that, for any two images *f* and *g*,

$$\delta_{SE_I,\mathbb{W}_I}(f)\le g \Leftrightarrow f\le \varepsilon_{SE_I,\mathbb{W}_I}(g). \tag{20}$$

That is, adjunction property between $\delta_{SE_I,\mathbb{W}_I}$ and $\varepsilon_{SE_I,\mathbb{W}_I}$ is satisfied, which guarantees that $\delta_{SE_I,\mathbb{W}_I}\varepsilon_{SE_I,\mathbb{W}_I}$ and $\varepsilon_{SE_I,\mathbb{W}_I}\delta_{SE_I,\mathbb{W}_I}$ are opening and closing in the algebraic sense [3].

However, there are still two deficiencies in the nonlocal extensions. On the one hand, $\mathbb{W}_I\le 0$, which can be deduced from Eq. (11) and (19). Thus, the ordering relation among $\delta_{SE_I,\mathbb{W}_I}(f)$, *f*, and $\varepsilon_{SE_I,\mathbb{W}_I}(f)$ (shown in Eq. (5)) is no longer guaranteed. On the other hand, different from the multiplication in NLM (Eq. (8)), the addition/subtraction between two different dimensions (i.e., grey value *I*(*y*) and weight W_I (or its logarithm $\mathbb{W}_I$)) is involved in the nonlocal extensions. As noted in [2], "*when nonflat SEs are used, their grey scale values should have the same units and scalings as those of the input image.*" The dimension discrepancy will discount the clarity of the operators' physical interpretation, thus further impede their applications.

3. A NONLOCAL OPERATOR MODEL FOR MORPHOLOGY

3.1 The Model Proposed

Inspired by the definition of dissimilarity measure in [16], a similarity weight system is defined as

Definition 1. *A similarity weight system* $W_I:\Omega\times\Omega\mapsto R^+$ *on I is a weight function such for all* $x,y\in\Omega$

$$0\le W_I(x,y)=W_I(y,x)\le W_I(x,x)=1. \tag{21}$$

Note that, the $\widehat{W}_I$ in NLM, proposed in Eq. (10), is actually a similarity system.

Meanwhile, inspired by the strategies in [14][15], the structuring elements need also to be fixed in this work. And to overcome the limitation of dimension discrepancy, a naive idea is manipulating the weights and the grey values with multiplication. Based on the principles, a nonlocal morphology model is proposed, whose basic operators are given as,

$$E_{SE_I,\mathcal{W}_I}(f)(x) = \bigwedge_{y\in SE_I(x)} \left\{f(y)-\lambda\log\left(\mathcal{W}_I(x,y)\right)D_I(x,y)\right\} \quad x\in\Omega, \tag{22}$$

$$D_{SE_I,\mathcal{W}_I}(f)(x) = \bigvee_{y\in SE_I(x)} \left\{f(y)+\lambda\log\left(\mathcal{W}_I(x,y)\right)D_I(x,y)\right\} \quad x\in\Omega. \tag{23}$$

Where *f* is the image under studied; $\{SE_I\}$ is a nonlocal structuring elements system fixed by initial image *I*; λ is the operator parameter; $\mathcal{W}_I$ is a similarity weight system describing the similarity between two pixels (*x* and *y*) in *I*; $D_I(x,y)$ is the difference grey value between *x* and *y*, which is symmetrical, i.e.,

$$D_I(x,y)=D_I(y,x), \tag{24}$$

and can be formulated as $|I(x)-I(y)|$, $\|P_I(x)\text{-}P_I(y)\|_{1,\alpha}$ (Gaussian weighted L_1 norm) and so on. It is should also be noted that, besides $\widehat{W}_I$ in NLM, many other forms, including multi-kernel-induced weights [10], robust weights based on L_1 norm [17] etc., can be assigned as the similarity weight system $\mathcal{W}_I$ in the model. In these means, the model is general.

It is obviously that the limitations of dimension discrepancy in the previous nonlocal extensions [14][15] are overcome in the model with the multiplication between the logarithm of the weight and grey value.

The two theorems below indicate that the two important properties of traditional MM have been inherited by the nonlocal model.

Theorem 1 (*ordering property*)
$E_{SE_I,\mathcal{W}_I}(f)\le f\le D_{SE_I,\mathcal{W}_I}(f)$, *for all I*, $f\in Fun(\Omega,F)$.

Proof.

$\mathcal{W}_I(x,x)=1,\ \forall x\in\Omega$ *by Eq. (21)*

$\Rightarrow \log\left(\mathcal{W}_I(x,x)\right)=0,\ \forall x\in\Omega$

$\Rightarrow f(x)=f(x)-\lambda\log\left(\mathcal{W}_I(x,x)\right)D_I(x,x),\ \forall x\in\Omega$

$\Rightarrow f(x)\ge \bigwedge_{y\in SE_I(x)}\left\{f(y)-\lambda\log\left(\mathcal{W}_I(x,y)\right)D_I(x,y)\right\},\ \forall x\in\Omega$ *by Eq. (15)*

$\Rightarrow f \ge E_{SE_I,\mathcal{W}_I}(f)$, *by Eq. (22)*

$f \le D_{SE_I,\mathcal{W}_I}(f)$ *can be deduced in a similar way. Thus,*

$E_{SE_I,\mathcal{W}_I}(f) \le f \le D_{SE_I,\mathcal{W}_I}(f)$.

Theorem 2 (*adjunction property*)

$D_{SE_I,\mathcal{W}_I}(f) \le g \Leftrightarrow f \le E_{SE_I,\mathcal{W}_I}(g)$, *for all* $I, f, g \in Fun(\Omega, F)$.

Proof. $D_{SE_I,\mathcal{W}_I}(f) \le g \quad \forall x \in \Omega$

$\Leftrightarrow \bigvee_{y \in SE_I(x)} \{ f(y) + \lambda \log(\mathcal{W}_I(x,y)) D_I(x,y) \} \le g(x), \forall x \in \Omega$ *by Eq.(23)*

$\Leftrightarrow f(y) + \lambda \log(\mathcal{W}_I(x,y)) D_I(x,y) \le g(x), \quad \forall x \in \Omega, \forall y \in SE_I(x)$

$\Leftrightarrow f(y) \le g(x) - \lambda \log(\mathcal{W}_I(x,y)) D_I(x,y), \quad \forall y \in \Omega, \forall x \in SE_I(y)$ *by Eq. (16)*

$\Leftrightarrow f(y) \le \bigwedge_{x \in SE_I(y)} \{ g(x) - \lambda \log(\mathcal{W}_I(x,y)) D_I(x,y) \}, \forall y \in \Omega$

$\Leftrightarrow f(y) \le \bigwedge_{x \in SE_I(y)} \{ g(x) - \lambda \log(\mathcal{W}_I(y,x)) D_I(y,x) \}, \forall y \in \Omega$

by eqs. (21) and (24).

$\Leftrightarrow f \le E_{SE_I,\mathcal{W}}(g)$. *by Eq.(22)*

4. A DERIVATION OF THE MODEL

To get a derivation of the model proposed in Eqs. (22-23), a naïve method is borrowing $\widehat{W}_I$ from NLM as the assignment of $\mathcal{W}_I$, and letting $D_I(x,y) = |I(x) - I(y)|$. Then, the model is specified as

$$E_{SE_I,\widehat{W}_I}(f)(x) = \bigwedge_{y \in SE_I(x)} \left\{ f(y) - \lambda \log\left(\widehat{W}_I(x,y)\right) |I(x) - I(y)| \right\} \quad x \in \Omega , \tag{25}$$

$$D_{SE_I,\widehat{W}_I}(f)(x) = \bigvee_{y \in SE_I(x)} \left\{ f(y) + \lambda \log\left(\widehat{W}_I(x,y)\right) |I(x) - I(y)| \right\} \quad x \in \Omega . \tag{26}$$

To illustrate the properties of the basic operators above, an artificial image ("Chessboard") and a natural image ("Lenna") are selected in the experiments. For the former, $\lambda = 0.001$; and for the latter, $\lambda = 0.002$. The other parameters in the model are respectively set to the recommendations in the original document [8]. Similarly as in [15], the morphological gradient, i.e., $D_{SE_I,\widehat{W}_I} - E_{SE_I,\widehat{W}_I}$, is used to test the ordering property; and the idempotent of opening (i.e., $D_{SE_I,\widehat{W}_I} E_{SE_I,\widehat{W}_I}$) and closing (i.e., $E_{SE_I,\widehat{W}_I} D_{SE_I,\widehat{W}_I}$) is used to test the adjunction property. The visual results are provided in Fig. 1 and Fig. 2.

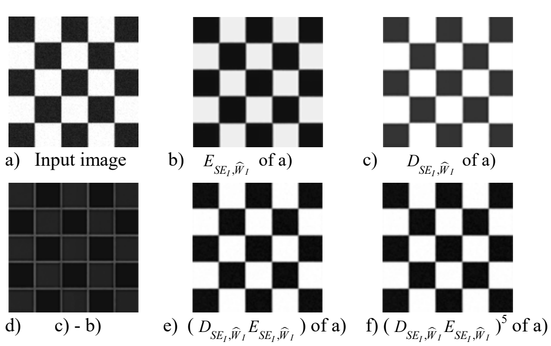

Fig. 1. Experiments on image "Chessboard"

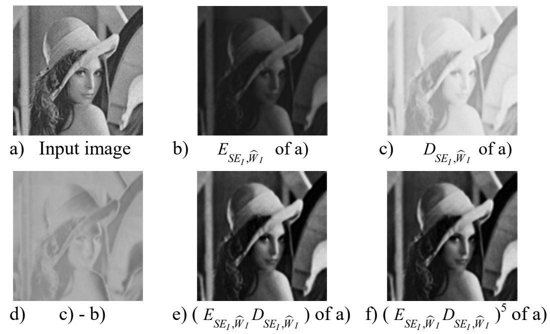

Fig. 2. Experiments on image "Lenna"

From the experimental results, it can be indicated that the two important properties have been inherited in the nonlocal extensions.

5. CONCLUSIONS

Focusing on limitations of the previous works, this paper proposed a nonlocal operator model for morphological image processing. The derivation of this model can easily inherit the advantages from traditional MM, such as clear physical interpretation and important properties. The conclusion has also been supported both in theories and experiments. As a general model, more derivations and properties of which are worth being explored in future work.

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