CHAPTER 3

Independence

3.1. Independent events

Suppose we have a probability space of a sample space S, σ-field $\mathcal{F}$ and probability $\mathbb{P}$ defined on $\mathcal{F}$.

Definition (Independence)

We say that $E, F \in \mathcal{F}$ are *independent events* if

$$\mathbb{P}(E \cap F) = \mathbb{P}(E)\,\mathbb{P}(F)$$

Example 3.1. Suppose you flip two coins. The outcome of heads on the second is independent of the outcome of tails on the first. To be more precise, if A is tails for the first coin and B is heads for the second, and we assume we have fair coins (although this is not necessary), we have $\mathbb{P}(A \cap B) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = \mathbb{P}(A)\mathbb{P}(B)$.

Example 3.2. Suppose you draw a card from an ordinary deck. Let E be you drew an ace, F be that you drew a spade. Here $\frac{1}{52} = \mathbb{P}(E \cap F) = \frac{1}{13} \cdot \frac{1}{4} = \mathbb{P}(E) \cap \mathbb{P}(F)$.

Proposition 3.1

If E and F are independent, then E and F^c are independent.

PROOF.

$$\begin{aligned} \mathbb{P}(E \cap F^c) &= \mathbb{P}(E) - \mathbb{P}(E \cap F) = \mathbb{P}(E) - \mathbb{P}(E)\mathbb{P}(F) \\ &= \mathbb{P}(E)[1 - \mathbb{P}(F)] = \mathbb{P}(E)\mathbb{P}(F^c). \end{aligned}$$

$\square$

The concept of independence can be generalized to any number of events as follows.

Definition (Jointly independent events)

Let $A_1, \dots, A_n \subset S$ be a collection of n events. We say that they are *jointly (mutually) independent* if for all possible subcollections $i_1, \dots, i_r \in \{1, \dots, n\}$, $1 \leqslant r \leqslant n$, it holds that

$$\mathbb{P}\left(\bigcap_{k=1}^{r} A_{i_k}\right) = \prod_{k=1}^{r} \mathbb{P}(A_{i_k}).$$

For example, for three events, E, F, and G, they are independent if E and F are independent, E and G are independent, F and G are independent, and $\mathbb{P}(E \cap F \cap G) = \mathbb{P}(E)\mathbb{P}(F)\mathbb{P}(G)$.

Example 3.3 (Pairwise but not jointly independent events)**.** Throw two fair dice. Consider three events

$$\begin{aligned} E &:= \{\text{the sum of the points is } 7\}, \\ F &:= \{\text{the first die rolled } 3\}, \\ G &:= \{\text{the second die rolled } 4\}. \end{aligned}$$

The sample space consists of 36 elements $(i, j), i, j = 1, ..., 6$, each having the same probability. Then

$$\begin{aligned} E &:= \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}, \\ F &:= \{(3,1), (3,2), (3,3), (3,4), (3,5), (3,6), \}, \\ G &:= \{(1,4), (2,4), (3,4), (4,4), (5,4), (6,4)\}, \\ E \cap F &= E \cap G = F \cap G = E \cap F \cap G = \{(3,4)\}. \end{aligned}$$

Therefore

$$\begin{aligned} &\mathbb{P}(E) = \mathbb{P}(F) = \mathbb{P}(F) = \frac{1}{6}, \\ &\mathbb{P}(E \cap F) = \mathbb{P}(F \cap G) = \mathbb{P}(E \cap G) = \mathbb{P}(E \cap F \cap G) = \frac{1}{36}, \end{aligned}$$

so E, F, G are pairwise disjoint, but they are **not** jointly independent.

Example 3.4. What is the probability that exactly 3 threes will show if you roll 10 dice?

Solution: The probability that the 1st, 2nd, and 4th dice will show a three and the other 7 will not is $\frac{1}{6}^3 \frac{5}{6}^7$. Independence is used here: the probability is $\frac{1}{6}\frac{1}{6}\frac{5}{6}\frac{1}{6}\frac{5}{6} \cdots \frac{5}{6}$. The probability that the 4th, 5th, and 6th dice will show a three and the other 7 will not is the same thing. So to answer our original question, we take $\frac{1}{6}^3 \frac{5}{6}^7$ and multiply it by the number of ways of choosing 3 dice out of 10 to be the ones showing threes. There are $\binom{10}{3}$ ways of doing that.

This is a particular example of what are known as Bernoulli trials or the binomial distribution.

Proposition 3.2 (Bernoulli trials: binomial distribution)

If an experiment with probability of success p is repeated n times independently, the probability of having k successes for any $0 \leqslant k \leqslant n$ is given by

$$\mathbb{P}\{k \text{ successes in } n \text{ independent trials}\} = \binom{n}{k} p^k (1-p)^{n-k}$$

Proof. The probability there are k successes is the number of ways of putting k objects in n slots (which is $\binom{n}{k}$) times the probability that there will be k successes and $n-k$ failures in exactly a given order. So the probability is $\binom{n}{k} p^k (1-p)^{n-k}$. □

The name *binomial* for this distribution comes from the simple observation that if we denote by

$$E_k := \{k \text{ successes in } n \text{ independent trials}\}.$$

Then $S = \bigcup_{k=0}^{n} E_k$ is the disjoint decomposition of the sample space, so

$$\mathbb{P}(S) = \mathbb{P}\left(\bigcup_{k=0}^{n} E_k\right) = \sum_{k=0}^{n} \mathbb{P}(E_k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p + (1-p))^n = 1$$

by the binomial formula. This shows we indeed have the second axiom of probability for the binomial distribution. We denote by $\operatorname{Binom}(n, p)$ the binomial distribution with parameters n and p.

3.2. Further examples and explanations

3.2.1. Examples.

Example 3.5. A card is drawn from an ordinary deck of cards (52 cards). Consider the events $F :=$ *a face is drawn*, $R :=$ *a red color is drawn*.

These are independent events because, for one card, being a face does not affect it being red: there are 12 faces, 26 red cards, and 6 cards that are red and faces. Thus,

$$\mathbb{P}(F)\,\mathbb{P}(R) = \frac{12}{52}\cdot\frac{26}{52} = \frac{3}{26},$$
$$\mathbb{P}(F\cap R) = \frac{6}{52} = \frac{3}{26}.$$

Example 3.6. Suppose that two unfair coins are flipped: the first coin has the heads probability 0.5001 and the second has heads probability 0.5002. The events $A_T =:=$ *the first coin lands tails*, $B_H :=$ *the second coin lands heads* are independent. Why? The sample space $S = \{\text{HH, HT, TH, TT}\}$ has 4 elements, all of them of different probabilities, given as products. The events correspond to $A_T = \{\text{TH,TT}\}$ and $B_H = \{\text{HH,TH}\}$ respectively, and the computation of the probabilities is given by

$$\mathbb{P}(A_T\cap B_H) = 0.4999\cdot 0.5002 = \mathbb{P}(A_T)\,\mathbb{P}(B_H).$$

Example 3.7. An urn contains 10 balls, 4 red and 6 blue. A second urn contains 16 red balls and an unknown number of blue balls. A single ball is drawn from each urn and the probability that both balls are the same color is 0.44. How many blue balls are there in the second urn?

Solution: define the events $R_i :=$ *a red ball is drawn from urn* i, $B_i :=$ *a blue ball is drawn from urn* i, and let x denote the (unknown) number of blue balls in urn 2, so that the second urn has $16+x$ balls in total. Using the fact that the events $R_1\cap R_2$ and $B_1\cap B_2$ are independent (check this!), we have

$$\begin{aligned}
0.44 &= \mathbb{P}\left((R_1\cap R_2)\bigcup(B_1\cap B_2)\right) = \mathbb{P}(R_1\cap R_2)+\mathbb{P}(B_1\cap B_2)\\
&= \mathbb{P}(R_1)\,\mathbb{P}(R_2)+\mathbb{P}(B_1)\,\mathbb{P}(B_2)\\
&= \frac{4}{10}\frac{16}{x+16}+\frac{6}{10}\frac{x}{x+16}.
\end{aligned}$$

Solving this equation for x we get $x = 4$.

3.2.2. Bernoulli trials. Recall that successive independent repetitions of an experiment that results in a success with some probability p and a failure with probability $1-p$ are called *Bernoulli trials*, and the distribution is given in Proposition 3.2. Sometimes we can view an experiment as the successive repetition of a *simpler* one. For instance, rolling 10 dice can be seen as rolling one single die ten times, each time independently of the other.

Example 3.8. Suppose that we roll 10 dice. What is the probability that at most 4 of them land a two?

Solution: We can regard this experiment as consequently rolling one single die. One possibility is that the first, second, third, and tenth trial land a two, while the rest land something else. Since each trial is independent, the probability of this event will be

$$\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6}=\left(\frac{1}{6}\right)^4\cdot\left(\frac{5}{6}\right)^6.$$

Note that the probability that the 10th, 9th, 8th, and 7th dice land a two and the other 6 do not is the same as the previous one. To answer our original question, we thus need to consider the number of ways of choosing 0, 1, 2, 3 or 4 trials out of 10 to be the ones showing a two. This means

$$\mathbb{P}(\text{exactly 0 dice land a two})=\binom{10}{0}\cdot\left(\frac{1}{6}\right)^0\cdot\left(\frac{5}{6}\right)^{10}=\left(\frac{5}{6}\right)^{10}.$$

$$\mathbb{P}(\text{exactly 1 dice lands a two})=\binom{10}{1}\cdot\left(\frac{1}{6}\right)\cdot\left(\frac{5}{6}\right)^9.$$

$$\mathbb{P}(\text{exactly 2 dice land a two})=\binom{10}{2}\cdot\left(\frac{1}{6}\right)^2\cdot\left(\frac{5}{6}\right)^8.$$

$$\mathbb{P}(\text{exactly 3 dice land a two})=\binom{10}{3}\cdot\left(\frac{1}{6}\right)^3\cdot\left(\frac{5}{6}\right)^7.$$

$$\mathbb{P}(\text{exactly 4 dice land a two})=\binom{10}{4}\cdot\left(\frac{1}{6}\right)^4\cdot\left(\frac{5}{6}\right)^6.$$

The answer to the question is the sum of these five numbers.

3.3. Exercises

Exercise 3.1. Let A and B be two independent events such that $\mathbb{P}(A \cup B) = 0.64$ and $\mathbb{P}(A) = 0.4$. What is $\mathbb{P}(B)$?

Exercise 3.2. In a class, there are 4 male math majors, 6 female math majors, and 6 male actuarial science majors. How many actuarial science females must be present in the class if sex and major are independent when choosing a student selected at random?

Exercise 3.3. Following Proposition on 3.1, prove that E and F are independent if and only if E and F^c are independent.

Exercise 3.4. Suppose we toss a fair coin twice, and let E be the event that both tosses give the same outcome, F that the first toss is a heads, and G is that the second toss is heads. Show that E, F and G are pairwise independent, but not jointly independent.

Exercise 3.5. Two dice are simultaneously rolled. For each pair of events defined below, compute if they are independent or not.

(a) $A_1 = \{\text{the sum is } 7\}$, $B_1 = \{\text{the first die lands a } 3\}$.
(b) $A_2 = \{\text{the sum is } 9\}$, $B_2 = \{\text{the second die lands a } 3\}$.
(c) $A_3 = \{\text{the sum is } 9\}$, $B_3 = \{\text{the first die lands even}\}$.
(d) $A_4 = \{\text{the sum is } 9\}$, $B_4 = \{\text{the first die is less than the second}\}$.
(e) $A_5 = \{\text{two dice are equal}\}$, $B_5 = \{\text{the sum is } 8\}$.
(f) $A_6 = \{\text{two dice are equal}\}$, $B_6 = \{\text{the first die lands even}\}$.
(g) $A_7 = \{\text{two dice are not equal}\}$, $B_7 = \{\text{the first die is less than the second}\}$.

Exercise 3.6. Are the events A_1, B_1 and B_3 from Exercise 3.5 independent?

Exercise 3.7. A hockey team has 0.45 chances of losing a game. Assuming that each game is independent from the other, what is the probability that the team loses 3 of the next upcoming 5 games?

Exercise 3.8. You make successive independent flips of a coin that lands on heads with probability p. What is the probability that the 3rd heads appears on the 7th flip?

Hint: express your answers in terms of p; do not assume $p = 1/2$.

Exercise 3.9. Suppose you toss a fair coin repeatedly and independently. If it comes up heads, you win a dollar, and if it comes up tails, you lose a dollar. Suppose you start with $\$M$. What is the probability you will get up to $\$N$ before you go broke? Give the answer in terms of M and N, assuming $0 < M < N$.

Exercise 3.10. Suppose that we roll n dice. What is the probability that at most k of them land a two?

3.4. Selected solutions

Solution to Exercise 3.1: Using independence we have $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A)\mathbb{P}(B)$ and substituting we have

$$0.64 = 0.4 + \mathbb{P}(B) - 0.4\mathbb{P}(B).$$

Solving for $\mathbb{P}(B)$ we have $\mathbb{P}(B) = 0.4$.

Solution to Exercise 3.2: Let x denote the number of actuarial sciences females. Then

$$\begin{aligned} \mathbb{P}(\text{male} \cap \text{math}) &= \frac{4}{16+x}, \\ \mathbb{P}(\text{male}) &= \frac{10}{16+x} \\ \mathbb{P}(\text{math}) &= \frac{10}{16+x}. \end{aligned}$$

Then using independence $\mathbb{P}(\text{male} \cap \text{math}) = \mathbb{P}(\text{male})\,\mathbb{P}(\text{math})$ so that

$$\frac{4}{16+x} = \frac{10^2}{(16+x)^2} \implies 4 = \frac{100}{16+x}$$

and solving for x we have $x = 9$.

Solution to Exercise 3.3: Proposition 3.1 tells us that if E and F are independent, then E and F^c are independent. Let us now assume that E and F^c are independent. We can apply Proposition 3.1 and say that E and $(F^c)^c$ are independent. Since $(F^c)^c = F$ (draw a Venn diagram), the assertion is proved.

Solution to Exercise 3.4:

Solution to Exercise 3.7: These are Bernoulli trials. Each game is a trial and the probability of loosing is $p = 0.45$. Using Proposition 3.2 with $k = 3$ and $n = 5$ we have

$$\mathbb{P}(3 \text{ loses in } 5 \text{ trials}) = \binom{5}{3} 0.45^3 \cdot 0.55^2.$$

Solution to Exercise 3.8: The 3rd head appearing on the 7th flip means that exactly two heads during the previous 6 flips appear and the 7th is heads. Since the flips are independent we have that the probability we search is

$$\mathbb{P}(\textit{2 heads in 6 trials } \textbf{AND} \textit{ heads in the 7th flip}) = \mathbb{P}(\textit{2 heads in 6 trials})\mathbb{P}(H).$$

Using Bernoulli trials, $\mathbb{P}(\textit{2 heads in 6 trials}) = \binom{6}{2} p^2(1-p)^4$ and therefore the total probability is

$$\binom{6}{2} p^2(1-p)^4 \cdot p = \binom{6}{2} p^3(1-p)^4.$$

Solution to Exercise 3.10:

$$\sum_{r=0}^{k} \binom{n}{r} \cdot \left(\frac{1}{6}\right)^r \cdot \left(\frac{5}{6}\right)^{n-r}.$$