CHAPTER 4

Conditional probability

4.1. Definition, Bayes' Rule and examples

Suppose there are 200 men, of which 100 are smokers, and 100 women, of which 20 are smokers. What is the probability that a person chosen at random will be a smoker? The answer is 120/300. Now, let us ask, what is the probability that a person chosen at random is a smoker given that the person is a women? One would expect the answer to be 20/100 and it is.

What we have computed is

$$\frac{\text{number of women smokers}}{\text{number of women}} = \frac{\text{number of women smokers } /300}{\text{number of women } /300},$$

which is the same as the probability that a person chosen at random is a woman and a smoker divided by the probability that a person chosen at random is a woman.

With this in mind, we give the following definition.

Definition 4.1 (Conditional probability)

If $\mathbb{P}(F) > 0$, we define *the probability of E given F* as

$$\mathbb{P}(E \mid F) := \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)}.$$

Note $\mathbb{P}(E \cap F) = \mathbb{P}(E \mid F)\mathbb{P}(F)$.

Example 4.1. Suppose you roll two dice. What is the probability the sum is 8?

Solution: there are five ways this can happen $\{(2,6),(3,5),(4,4),(5,3),(6,2)\}$, so the probability is $5/36$. Let us call this event A. What is the probability that the sum is 8 given that the first die shows a 3? Let B be the event that the first die shows a 3. Then $\mathbb{P}(A \cap B)$ is the probability that the first die shows a 3 and the sum is 8, or $1/36$. $\mathbb{P}(B) = 1/6$, so $\mathbb{P}(A \mid B) = \frac{1/36}{1/6} = 1/6$.

Example 4.2. Suppose a box has 3 red marbles and 2 black ones. We select 2 marbles. What is the probability that second marble is red given that the first one is red?

Solution: Let A be the event the second marble is red, and B the event that the first one is red. $\mathbb{P}(B) = 3/5$, while $\mathbb{P}(A \cap B)$ is the probability both are red, or is the probability that we chose 2 red out of 3 and 0 black out of 2. Then $\mathbb{P}(A \cap B) = \binom{3}{2}\binom{2}{0}/\binom{5}{2}$, and so $\mathbb{P}(A \mid B) = \frac{3/10}{3/5} = 1/2$.

Example 4.3. A family has 2 children. Given that one of the children is a boy, what is the probability that the other child is also a boy?

Solution: Let B be the event that one child is a boy, and A the event that both children are boys. The possibilities are bb, bg, gb, gg, each with probability $1/4$. $\mathbb{P}(A \cap B) = \mathbb{P}(bb) = 1/4$ and $\mathbb{P}(B) = \mathbb{P}(bb, bg, gb) = 3/4$. So the answer is $\frac{1/4}{3/4} = 1/3$.

Example 4.4. Suppose the test for HIV is 99% accurate in both directions and 0.3% of the population is HIV positive. If someone tests positive, what is the probability they actually are HIV positive?

Solution: Let D is the event that a person is HIV positive, and T is the event that the person tests positive.

$$\mathbb{P}(D \mid T) = \frac{\mathbb{P}(D \cap T)}{\mathbb{P}(T)} = \frac{(0.99)(0.003)}{(0.99)(0.003) + (0.01)(0.997)} \approx 23\%.$$

A short reason why this surprising result holds is that the error in the test is much greater than the percentage of people with HIV. A little longer answer is to suppose that we have 1000 people. On average, 3 of them will be HIV positive and 10 will test positive. So the chances that someone has HIV given that the person tests positive is approximately 3/10. The reason that it is not exactly 0.3 is that there is some chance someone who is positive will test negative.

Suppose you know $\mathbb{P}(E \mid F)$ and you want to find $\mathbb{P}(F \mid E)$. Recall that

$$\mathbb{P}(E \cap F) = \mathbb{P}(E \mid F)\mathbb{P}(F),$$

and so

$$\mathbb{P}(F \mid E) = \frac{\mathbb{P}(F \cap E)}{\mathbb{P}(E)} = \frac{\mathbb{P}(E \mid F)\mathbb{P}(F)}{\mathbb{P}(E)}$$

Example 4.5. Suppose 36% of families own a dog, 30% of families own a cat, and 22% of the families that have a dog also have a cat. A family is chosen at random and found to have a cat. What is the probability they also own a dog?

Solution: Let D be the families that own a dog, and C the families that own a cat. We are given $\mathbb{P}(D) = 0.36, \mathbb{P}(C) = 0.30, \mathbb{P}(C \mid D) = 0.22$ We want to know $\mathbb{P}(D \mid C)$. We know

$\mathbb{P}(D \mid C) = \mathbb{P}(D \cap C)/\mathbb{P}(C)$. To find the numerator, we use $\mathbb{P}(D \cap C) = \mathbb{P}(C \mid D)\mathbb{P}(D) = (0.22)(0.36) = 0.0792$. So $\mathbb{P}(D \mid C) = 0.0792/0.3 = 0.264 = 26.4\%$.

Example 4.6. Suppose 30% of the women in a class received an A on the test and 25% of the men received an A. The class is 60% women. Given that a person chosen at random received an A, what is the probability this person is a women?

Solution: Let A be the event of receiving an A, W be the event of being a woman, and M the event of being a man. We are given $\mathbb{P}(A \mid W) = 0.30, \mathbb{P}(A \mid M) = 0.25, \mathbb{P}(W) = 0.60$ and we want $\mathbb{P}(W \mid A)$. From the definition

$$\mathbb{P}(W \mid A) = \frac{\mathbb{P}(W \cap A)}{\mathbb{P}(A)}.$$

As in the previous example,

$$\mathbb{P}(W \cap A) = \mathbb{P}(A \mid W)\mathbb{P}(W) = (0.30)(0.60) = 0.18.$$

To find $\mathbb{P}(A)$, we write

$$\mathbb{P}(A) = \mathbb{P}(W \cap A) + \mathbb{P}(M \cap A).$$

Since the class is 40% men,

$$\mathbb{P}(M \cap A) = \mathbb{P}(A \mid M)\mathbb{P}(M) = (0.25)(0.40) = 0.10.$$

So

$$\mathbb{P}(A) = \mathbb{P}(W \cap A) + \mathbb{P}(M \cap A) = 0.18 + 0.10 = 0.28.$$

Finally,

$$\mathbb{P}(W \mid A) = \frac{\mathbb{P}(W \cap A)}{\mathbb{P}(A)} = \frac{0.18}{0.28}.$$

Proposition 4.1 (Bayes' rule)

If $\mathbb{P}(E) > 0$, then

$$\mathbb{P}(F \mid E) = \frac{\mathbb{P}(E \mid F)\mathbb{P}(F)}{\mathbb{P}(E \mid F)\mathbb{P}(F) + \mathbb{P}(E \mid F^c)\mathbb{P}(F^c)}.$$

Proof. We use the definition of conditional probability and the fact that

$$\mathbb{P}(E) = \mathbb{P}((E \cap F) \cup (E \cap F^c)) = \mathbb{P}(E \cap F) + \mathbb{P}(E \cap F^c).$$

This will be called the *law of total probability* and will be discussed again in Proposition 4.4 in more generality. Then

$$\begin{aligned}\mathbb{P}(F \mid E) &= \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(E)} = \frac{\mathbb{P}(E \mid F)\mathbb{P}(F)}{\mathbb{P}(E \cap F) + \mathbb{P}(E \cap F^c)} \\ &= \frac{\mathbb{P}(E \mid F)\mathbb{P}(F)}{\mathbb{P}(E \mid F)\mathbb{P}(F) + \mathbb{P}(E \mid F^c)\mathbb{P}(F^c)}.\end{aligned}$$

$\square$

Here is another example related to conditional probability, although this is not an example of Bayes' rule. This is known as the *Monty Hall problem* after the host of the TV show in the 60s called *Let's Make a Deal*.

Example 4.7. There are three doors, behind one a nice car, behind each of the other two a goat eating a bale of straw. You choose a door. Then Monty Hall opens one of the other doors, which shows a bale of straw. He gives you the opportunity of switching to the remaining door. Should you do it?

Solution: Let's suppose you choose door 1, since the same analysis applies whichever door you chose. Strategy one is to stick with door 1. With probability 1/3 you chose the car. Monty Hall shows you one of the other doors, but that doesn't change your probability of winning.

Strategy 2 is to change. Let's say the car is behind door 1, which happens with probability 1/3. Monty Hall shows you one of the other doors, say door 2. There will be a goat, so you switch to door 3, and lose. The same argument applies if he shows you door 3. Suppose the car is behind door 2. He will show you door 3, since he doesn't want to give away the car. You switch to door 2 and win. This happens with probability 1/3. The same argument applies if the car is behind door 3. So you win with probability 2/3 and lose with probability 1/3. Thus strategy 2 is much superior.

A problem that comes up in actuarial science frequently is *gambler's ruin*.

Example 4.8 (Gambler's ruin). Suppose you play the game by tossing a fair coin repeatedly and independently. If it comes up heads, you win a dollar, and if it comes up tails, you lose a dollar. Suppose you start with \$50. What's the probability you will get to \$200 without first getting ruined (running out of money)?

Solution: it is easier to solve a slightly harder problem. The game can be described as having probability 1/2 of winning 1 dollar and a probability 1/2 of losing 1 dollar. A player begins with a given number of dollars, and intends to play the game repeatedly until the player either goes broke or increases his holdings to N dollars.

For any given amount n of current holdings, the conditional probability of reaching N dollars before going broke is independent of how we acquired the n dollars, so there is a unique probability $\mathbb{P}(N \mid n)$ of reaching N on the condition that we currently hold n dollars. Of course, for any finite N we see that $\mathbb{P}(N \mid n) = 0$ and $\mathbb{P}(N \mid N) = 1$. The problem is to determine the values of $\mathbb{P}(N \mid n)$ for n between 0 and N.

We are considering this setting for $N = 200$, and we would like to find $\mathbb{P}(200 \mid 50)$. Denote $y(n) := \mathbb{P}(200 \mid n)$, which is the probability you get to 200 without first getting ruined if you start with n dollars. We saw that $y(0) = 0$ and $y(200) = 1$. Suppose the player has n dollars at the moment, the next round will leave the player with either $n + 1$ or $n - 1$ dollars, both with probability 1/2. Thus the current probability of winning is the same as a weighted average of the probabilities of winning in player's two possible *next* states. So we have

$$y(n) = \tfrac{1}{2}y(n+1) + \tfrac{1}{2}y(n-1).$$

Multiplying by 2, and subtracting $y(n) + y(n-1)$ from each side, we have

$$y(n+1) - y(n) = y(n) - y(n-1).$$

This says that slopes of the graph of $y(n)$ on the adjacent intervals are constant (remember that x must be an integer). In other words, the graph of $y(n)$ must be a line. Since $y(0) = 0$ and $y(200) = 1$, we have $y(n) = n/200$, and therefore $y(50) = 1/4$.

Another way to see what the function $y(n)$ is to use the telescoping sum as follows

$$\begin{aligned} y(n) &= y(n) - y(0) = (y(n) - y(n-1)) + ... + (y(1) - y(0)) \\ &= n(y(1) - y(0)) = ny(1). \end{aligned} \tag{4.1.1}$$

since the all these differences are the same, and $y(0) = 0$. To find $y(1)$ we can use the fact that $y(200) = 1$, so $y(1) = 1/200$, and therefore $y(n) = n/200$ and $y(50) = 1/4$.

Example 4.9. Suppose we are in the same situation, but you are allowed to go arbitrarily far in debt. Let $z(n)$ be the probability you ever get to \$200 if you start with n dollars. What is a formula for $z(n)$?

Solution: Just as above, we see that z satisfies the recursive equation

$$z(n) = \tfrac{1}{2}z(n+1) + \tfrac{1}{2}z(n-1).$$

What we need to determine now are boundary conditions. Now that the gambler can go to debt, the condition that if we start with 0 we never get to \$200, that is, probability of getting \$200 is 0, is **not** true. Following Equation 4.1.1 with $z(0) \neq 0$ we see that

$$\begin{aligned} z(n) - z(0) &= (z(n) - z(n-1)) + ... + (z(1) - z(0)) \\ &= n(z(1) - z(0)), \end{aligned}$$

therefore

$$z(n) = n(z(1) - z(0)) + z(0).$$

If we denote $a := z(1) - z(0)$ and $b := z(0)$ we see that as a function of n we have

$$z(n) = an + b.$$

We would like to find a and b now. Recall that this function is probability, so for any n we have $0 \leqslant z(n) \leqslant 1$. This is possible only if $a = 0$, that is,

$$z(1) = z(0),$$

so

$$z(n) = z(0)$$

for any n. We know that $z(200) = 1$, therefore

$$z(n) = 1 \text{ for all } n.$$

In other words, one is certain to get to \$200 eventually (provided, of course, that one is allowed to go into debt).

4.2. Further examples and applications

4.2.1. Examples, basic properties, multiplication rule, law of total probability.

Example 4.10. Landon is 80% sure he forgot his textbook either at the Union or in Monteith. He is 40% sure that the book is at the union, and 40% sure that it is in Monteith. Given that Landon already went to Monteith and noticed his textbook is not there, what is the probability that it is at the Union?

Solution: Calling $U =$ *textbook is at the Union*, and $U =$ *textbook is in Monteith*, notice that $U \subseteq M^c$ and hence $U \cap M^c = U$. Thus,

$$\mathbb{P}(U \mid M^c) = \frac{\mathbb{P}(U \cap M^c)}{\mathbb{P}(M^c)} = \frac{\mathbb{P}(U)}{1 - \mathbb{P}(M)} = \frac{4/10}{6/10} = \frac{2}{3}.$$

Example 4.11. Sarah and Bob draw 13 cards each from a standard deck of 52. Given that Sarah has exactly two aces, what is the probability that Bob has exactly one ace?

Solution: Let $A =$ *Sarah has two aces*, and let $B =$ *Bob has exactly one ace*. In order to compute $\mathbb{P}(B \mid A)$, we need to calculate $\mathbb{P}(A)$ and $\mathbb{P}(A \cap B)$. On the one hand, Sarah could have any of $\binom{52}{13}$ possible hands. Of these hands, $\binom{4}{2} \cdot \binom{48}{11}$ will have exactly two aces so that

$$\mathbb{P}(A) = \frac{\binom{4}{2} \cdot \binom{48}{11}}{\binom{52}{13}}.$$

On the other hand, the number of ways in which Sarah can pick a hand and Bob another (different) is $\binom{52}{13} \cdot \binom{39}{13}$. The the number of ways in which A and B can simultaneously occur is $\binom{4}{2} \cdot \binom{48}{11} \cdot \binom{2}{1} \cdot \binom{37}{12}$ and hence

$$\mathbb{P}(A \cap B) = \frac{\binom{4}{2} \cdot \binom{48}{11} \cdot \binom{2}{1} \cdot \binom{37}{12}}{\binom{52}{13} \cdot \binom{39}{13}}.$$

Applying the definition of conditional probability we finally get

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{\binom{4}{2} \cdot \binom{48}{11} \cdot \binom{2}{1} \cdot \binom{37}{12} \Big/ \binom{52}{13} \cdot \binom{39}{13}}{\binom{4}{2} \cdot \binom{48}{11} \Big/ \binom{52}{13}} = \frac{2 \cdot \binom{37}{12}}{\binom{39}{13}}$$

Example 4.12. A total of 500 married couples are polled about their salaries with the following results

	husband makes less than \$25K	husband makes more than \$25K
wife makes less than \$25K	212	198
wife makes more than \$25K	36	54

(a) Find the probability that a husband earns less than \$25K.
Solution:

$$\mathbb{P}(\text{husband makes } < \$25\text{K}) = \frac{212}{500} + \frac{36}{500} = \frac{248}{500} = 0.496.$$

(b) Find the probability that a wife earns more than \$25K, given that the husband earns as that much as well.
Solution:

$$\mathbb{P}\left(\text{wife makes } > \$25\text{K} \mid \text{husband makes } > \$25\text{K}\right) = \frac{54/500}{(198+54)/500} = \frac{54}{252} = 0.214$$

(c) Find the probability that a wife earns more than \$25K, given that the husband makes less than \$ 25K.
Solution:

$$\mathbb{P}\left(\text{wife makes } > \$25\text{K} \mid \text{husband makes } < \$25\text{K}\right) = \frac{36/500}{248/500} = 0.145.$$

From the definition of conditional probability we can deduce some useful relations.

Proposition 4.2

Let $E, F \in \mathcal{F}$ be events with $\mathbb{P}(E), \mathbb{P}(F) > 0$. Then
(i) $\mathbb{P}(E \cap F) = \mathbb{P}(E)\mathbb{P}(F \mid E)$,
(ii) $\mathbb{P}(E) = \mathbb{P}(E \mid F)\mathbb{P}(F) + \mathbb{P}(E \mid F^c)\mathbb{P}(F^c)$,
(iii) $\mathbb{P}(E^c \mid F) = 1 - \mathbb{P}(E \mid F)$.

Proof. We already saw (i) which is a rewriting of the definition of conditional probability $\mathbb{P}(F \mid E) = \frac{\mathbb{P}(E\cap F)}{\mathbb{P}(E)}$. Let us prove (ii): we can write E as the union of the pairwise disjoint sets $E \cap F$ and $E \cap F^c$. Using (i) we have

$$\begin{aligned}\mathbb{P}(E) &= \mathbb{P}(E \cap F) + \mathbb{P}(E \cap F^c) \\ &= \mathbb{P}(E \mid F)\mathbb{P}(F) + \mathbb{P}(E \mid F^c)\mathbb{P}(F^c).\end{aligned}$$

Finally, writing $F = E$ in the previous equation and since $\mathbb{P}(E \mid E^c) = 0$, we obtain (iii). $\square$

Example 4.13. Phan wants to take either a Biology course or a Chemistry course. His adviser estimates that the probability of scoring an A in Biology is $\frac{4}{5}$, while the probability of scoring an A in Chemistry is $\frac{1}{7}$. If Phan decides randomly, by a coin toss, which course to take, what is his probability of scoring an A in Chemistry?

Solution: denote by B the event that *Phan takes Biology*, and by C the event that *Phan takes Chemistry*, and by $A =$ the event that the *score is an* A. Then, since $\mathbb{P}(B) = \mathbb{P}(C) = \frac{1}{2}$ we have

$$\mathbb{P}(A \cap C) = \mathbb{P}(C)\mathbb{P}(A \mid C) = \frac{1}{2} \cdot \frac{1}{7} = \frac{1}{14}.$$

The identity $\mathbb{P}(E \cap F) = \mathbb{P}(E)\mathbb{P}(F \mid E)$ from Proposition 4.2(i) can be generalized to any number of events in what is sometimes called the *multiplication rule*.

Proposition 4.3 (Multiplication rule)

Let $E_1, E_2, \ldots, E_n \in \mathcal{F}$ be events. Then

$$\begin{aligned}&\mathbb{P}\left(E_1 \cap E_2 \cap \ldots \cap E_n\right)\\&= \mathbb{P}\left(E_1\right) \mathbb{P}\left(E_2 \mid E_1\right) \mathbb{P}\left(E_3 \mid E_1 \cap E_2\right) \cdots \mathbb{P}\left(E_n \mid E_1 \cap E_2 \cap \cdots \cap E_{n-1}\right).\end{aligned}$$

Example 4.14. An urn has 5 blue balls and 8 red balls. Each ball that is selected is returned to the urn along with an additional ball of the same color. Suppose that 3 balls are drawn in this way.

(a) What is the probability that the three balls are blue?
Solution: In this case, we can define the sequence of events $B_1, B_2, B_3, \ldots$, where B_i is the event that *the ith ball drawn is blue*. Applying the multiplication rule yields

$$\mathbb{P}(B_1 \cap B_2 \cap B_3) = \mathbb{P}(B_1)\mathbb{P}(B_2 \mid B_1)\mathbb{P}(B_3 \mid B_1 \cap B_2) = \frac{5}{13}\frac{6}{14}\frac{7}{15}.$$

(b) What is the probability that only 1 ball is blue?
Solution: denoting by $R_i =$ the even that *the ith ball drawn is red*, we have

$$\mathbb{P}\left(\text{only 1 blue ball}\right) = \mathbb{P}(B_1 \cap R_2 \cap R_3) + \mathbb{P}(R_1 \cap B_2 \cap R_3) + \mathbb{P}(R_1 \cap R_2 \cap B_3) = 3\frac{5 \cdot 8 \cdot 9}{13 \cdot 14 \cdot 15}.$$

Also the identity (ii) in Proposition 4.2 can be generalized by partitioning the sample space S into several pairwise disjoint sets $F_1, \ldots, F_n$ (instead of simply F and F^c).

Proposition 4.4 (Law of total probability)

Let $F_1, \ldots, F_n \subseteq S$ be mutually exclusive and exhaustive events, i.e. $S = \bigcup_{i=1}^n F_i$. Then, for any event $E \in \mathcal{F}$ it holds that

$$\mathbb{P}\left(E\right) = \sum_{i=1}^{n} \mathbb{P}\left(E \mid F_i\right) \mathbb{P}\left(F_i\right).$$

4.2.2. Generalized Bayes' rule. The following example describes the type of problems treated in this section.

Example 4.15. An insurance company classifies insured policyholders into *accident prone* or *non-accident prone*. Their current risk model works with the following probabilities.

The probability that an *accident prone* insured has an accident within a year is 0.4
The probability that a *non-accident prone* insured has an accident within a year is 0.2.

If 30% of the population is *accident prone*,

(a) what is the probability that a policy holder will have an accident within a year?
(b) Suppose now that the policy holder has had accident within one year. What is the probability that he or she is accident prone?

Solution:

(a) denote by A_1 = the event that *a policy holder will have an accident within a year*, and denote by A = the event that *a policy holder is accident prone*. Applying Proposition 4.2(ii) we have

$$\begin{aligned}\mathbb{P}(A_1) &= \mathbb{P}(A_1 \mid A)\,\mathbb{P}(A) + \mathbb{P}(A_1 \mid A^c)(1-\mathbb{P}(A)) \\ &= 0.4\cdot 0.3 + 0.2(1-0.3) = 0.26\end{aligned}$$

(b) Use Bayes' formula to see that

$$\mathbb{P}(A \mid A_1) = \frac{\mathbb{P}(A\cap A_1)}{\mathbb{P}(A_1)} = \frac{\mathbb{P}(A)\,\mathbb{P}(A_1 \mid A)}{0.26} = \frac{0.3\cdot 0.4}{0.26} = \frac{6}{14}.$$

Using the *law of total probability* from Proposition 4.4 one can generalize Bayes' rule, which appeared in Proposition 4.1.

Proposition 4.5 (Generalized Bayes' rule)

Let $F_1,\dots,F_n \subseteq S$ be mutually exclusive and exhaustive events, i.e. $S = \bigcup_{i=1}^n F_i$. Then, for any event $E \subseteq S$ and any $j = 1,\dots,n$ it holds that

$$\mathbb{P}(F_j \mid E) = \frac{\mathbb{P}(E \mid F_j)\,\mathbb{P}(F_j)}{\sum_{i=1}^n \mathbb{P}(E \mid F_i)\,\mathbb{P}(F_i)}$$

Example 4.16. Suppose a factory has machines I, II, and III that produce *iSung* phones. The factory's record shows that

Machines I, II and III produce, respectively, 2%, 1%, and 3% defective *iSung*s.
Out of the total production, machines I, II, and III produce, respectively, 35%, 25% and 40% of all *iSung*s.

An *iSung* is selected at random from the factory.

(a) What is probability that the *iSung* selected is defective?
Solution: By the law of total probability,

$$\begin{aligned}\mathbb{P}(D) &= \mathbb{P}(I)\,\mathbb{P}(D \mid I) + \mathbb{P}(II)\,\mathbb{P}(D \mid II) + P(III)\,\mathbb{P}(D \mid III) \\ &= 0.35\cdot 0.02 + 0.25\cdot 0.01 + 0.4\cdot 0.03 = \frac{215}{10,000}.\end{aligned}$$

(b) Given that the *iSung* is defective, what is the conditional probability that it was produced by machine III?
Solution: Applying Bayes' rule,

$$\mathbb{P}(III \mid D) = \frac{\mathbb{P}(III)\,\mathbb{P}(D \mid III)}{\mathbb{P}(D)} = \frac{0.4\cdot 0.03}{215/10,000} = \frac{120}{215}.$$

Example 4.17. In a multiple choice test, a student either knows the answer to a question or she/he will randomly guess it. If each question has m possible answers and the student knows the answer to a question with probability p, what is the probability that the student actually knows the answer to a question, given that he/she answers correctly?

Solution: denote by K the event that *a student knows the answer*, and by C the event that *a student answers correctly*. Applying Bayes' rule we have

$$\mathbb{P}(K \mid C) = \frac{\mathbb{P}(C \mid K)\,\mathbb{P}(K)}{\mathbb{P}(C \mid K)\,\mathbb{P}(K) + \mathbb{P}(C \mid K^c)\,\mathbb{P}(K^c)} = \frac{1 \cdot p}{1 \cdot p + \frac{1}{m}(1-p)} = \frac{mp}{1+(m-1)p}.$$

4.3. Exercises

Exercise 4.1. Two dice are rolled. Consider the events $A = \{\text{sum of two dice equals 3}\}$, $B = \{\text{sum of two dice equals 7 }\}$, and $C = \{\text{at least one of the dice shows a 1}\}$.

(a) What is $\mathbb{P}(A \mid C)$?
(b) What is $\mathbb{P}(B \mid C)$?
(c) Are A and C independent? What about B and C?

Exercise 4.2. Suppose you roll two standard, fair, 6-sided dice. What is the probability that the sum is at least 9 given that you rolled at least one 6?

Exercise 4.3. A box contains 1 green ball and 1 red ball, and a second box contains 2 green and 3 red balls. First a box is chosen and afterwards a ball withdrawn from the chosen box. Both boxes are equally likely to be chosen. Given that a green ball has been withdrawn, what is the probability that the first box was chosen?

Exercise 4.4. Suppose that 60% of UConn students will be at random exposed to the flu. If you are exposed and did not get a flu shot, then the probability that you will get the flu *(after being exposed)* is 80%. If you did get a flu shot, then the probability that you will get the flu *(after being exposed)* is only 15%.

(a) What is the probability that a person who got a flu shot will get the flu?
(b) What is the probability that a person who did not get a flu shot will get the flu?

Exercise 4.5. Color blindness is a sex-linked condition, and 5% of men and 0.25% of women are color blind. The population of the United States is 51% female. What is the probability that a color-blind American is a man?

Exercise 4.6. Two factories supply light bulbs to the market. Bulbs from factory X work for over 5000 hours in 99% of cases, whereas bulbs from factory Y work for over 5000 hours in 95% of cases. It is known that factory X supplies 60% of the total bulbs available in the market.

(a) What is the probability that a purchased bulb will work for longer than 5000 hours?
(b) Given that a light bulb works for more than 5000 hours, what is the probability that it was supplied by factory Y?
(c) Given that a light bulb work does not work for more than 5000 hours, what is the probability that it was supplied by factory X?

Exercise 4.7. A factory production line is manufacturing bolts using three machines, A, B and C. Of the total output, machine A is responsible for 25%, machine B for 35% and machine C for the rest. It is known from previous experience with the machines that 5% of the output from machine A is defective, 4% from machine B and 2% from machine C. A bolt is chosen at random from the production line and found to be defective. What is the probability that it came from Machine A?

Exercise 4.8. A multiple choice exam has 4 choices for each question. The student has studied enough so that the probability they will know the answer to a question is 0.5, the probability that the student will be able to eliminate one choice is 0.25, otherwise all 4 choices seem equally plausible. If they know the answer they will get the question correct. If not they have to guess from the 3 or 4 choices. As the teacher you would like the test to measure what the student knows, and not how well they can guess. If the student answers a question correctly what is the probability that they actually know the answer?

Exercise 4.9. A blood test indicates the presence of Amyotrophic lateral sclerosis (ALS) 95% of the time when ALS is actually present. The same test indicates the presence of ALS 0.5% of the time when the disease is not actually present. One percent of the population actually has ALS. Calculate the probability that a person actually has ALS given that the test indicates the presence of ALS.

Exercise 4.10. A survey conducted in a college found that 40% of the students watch show A and 17% of the students who follow show A, also watch show B. In addition, 20% of the students watch show B.

(1) What is the probability that a randomly chosen student follows both shows?
(2) What is the conditional probability that the student follows show A given that she/he follows show B?

Exercise 4.11. Use Bayes' formula to solve the following problem. An airport has problems with birds. If the weather is sunny, the probability that there are birds on the runway is 1/2; if it is cloudy, but dry, the probability is 1/3; and if it is raining, then the probability is 1/4. The probability of each type of the weather is 1/3. Given that the birds are on the runaway, what is the probability

(1) that the weather is sunny?
(2) that the weather is cloudy (dry or rainy)?

Exercise 4.12. Suppose you toss a fair coin repeatedly and independently. If it comes up heads, you win a dollar, and if it comes up tails, you lose a dollar. Suppose you start with \$20. What is the probability you will get to \$150 before you go broke? (See Example 4.8 for a solution).

Exercise* 4.1. Suppose we play gambler's ruin game in Example 4.8 not with a fair coin, but rather in such a way that you win a dollar with probability p, and you lose a dollar with probability $1-p$, $0<p<1$. Find the probability of reaching N dollars before going broke if we start with n dollars.

Exercise* 4.2. Suppose F is an event, and define $\mathbb{P}_F(E) := \mathbb{P}(E \mid F)$. Show that the conditional probability $\mathbb{P}_F$ is a probability function, that is, it satisfies the axioms of probability.

Exercise* 4.3. Show directly that Proposition 2.1 holds for the conditional probability $\mathbb{P}_F$. In particular, for any events E and F

$$\mathbb{E}\left(E^c \mid F\right) = 1 - \mathbb{E}\left(E \mid F\right).$$

4.4. Selected solutions

Solution to Exercise 4.1(A): Note that the sample space is $S = \{(i,j) \mid i,j = 1,2,3,4,5,6\}$ with each outcome equally likely. Then

$$\begin{aligned} A &= \{(1,2),(2,1)\} \\ B &= \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\} \\ C &= \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(3,1),(4,1),(5,1),(6,1)\} \end{aligned}$$

Then

$$\mathbb{P}(A \mid C) = \frac{\mathbb{P}(A \cap C)}{\mathbb{P}(C)} = \frac{2/36}{11/36} = \frac{2}{11}.$$

Solution to Exercise 4.1(B):

$$\mathbb{P}(B \mid C) = \frac{\mathbb{P}(B \cap C)}{\mathbb{P}(C)} = \frac{2/36}{11/36} = \frac{2}{11}.$$

Solution to Exercise 4.1(C): Note that $\mathbb{P}(A) = 2/36 \neq \mathbb{P}(A \mid C)$, so they are not independent. Similarly, $\mathbb{P}(B) = 6/36 \neq \mathbb{P}(B \mid C)$, so they are not independent.

Solution to Exercise 4.2: denote by E the event that *there is at least one 6* and by F the event that *the sum is at least 9*. We want to find $\mathbb{P}(F \mid E)$. Begin by noting that there are 36 possible rolls of these two dice and all of them are equally likely. We can see that 11 different rolls of these two dice will result in at least one 6, so $\mathbb{P}(E) = \frac{11}{36}$. There are 7 different rolls that will result in at least one 6 and a sum of at least 9. They are $\{(6,3),(6,4),(6,5),(6,6),(3,6),(4,6),(5,6)\}$, so $\mathbb{P}(E \cap F) = \frac{7}{36}$. This tells us that

$$\mathbb{P}(F \mid E) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(E)} = \frac{7/36}{11/36} = \frac{7}{11}.$$

Solution to Exercise 4.3: denote by B_i the event that *the ith box is chosen*. Since both are equally likely, $\mathbb{P}(B_1) = \mathbb{P}(B_2) = \frac{1}{2}$. In addition, we know that $\mathbb{P}(G \mid B_1) = \frac{1}{2}$ and $\mathbb{P}(G \mid B_2) = \frac{2}{5}$. Applying Bayes' rule yields

$$\mathbb{P}(B_1 \mid G) = \frac{\mathbb{P}(G \mid B_1)\mathbb{P}(B_1)}{\mathbb{P}(G \mid B_1)\mathbb{P}(B_1) + \mathbb{P}(G \mid B_2)\mathbb{P}(B_2)} = \frac{1/4}{1/4 + 1/5} = \frac{5}{9}.$$

Solution to Exercise 4.4(A): Suppose we look at students who have gotten the flu shot. Denote by E the event that *a student is exposed to the flu*, and by F the event that *a student gets the flu*. We know that $\mathbb{P}(E) = 0.6$ and $\mathbb{P}(F \mid E) = 0.15$. This means that $\mathbb{P}(E \cap F) = (0.6)(0.15) = 0.09$, and it is clear that $\mathbb{P}(E^c \cap F) = 0$. Since $\mathbb{P}(F) = \mathbb{P}(E \cap F) + \mathbb{P}(E^c \cap F)$, we see that $\mathbb{P}(F) = 0.09$.

Solution to Exercise 4.4(B): Suppose we look at students who have not gotten the flu shot. Let E be the event that *a student is exposed to the flu*, and let F be the event that *a student gets the flu*. We know that $\mathbb{P}(E) = 0.6$ and $\mathbb{P}(F \mid E) = 0.8$. This means that $\mathbb{P}(E \cap F) = (0.6)(0.8) = 0.48$, and it is clear that $\mathbb{P}(E^c \cap F) = 0$. Since $\mathbb{P}(F) = \mathbb{P}(E \cap F) + \mathbb{P}(E^c \cap F)$, we see that $\mathbb{P}(F) = 0.48$.

Solution to Exercise 4.5: denote by M the event *an American is a man*, by C the event *an American is color blind.* Then

$$\begin{aligned}\mathbb{P}(M \mid C) &= \frac{\mathbb{P}(C \mid M)\,\mathbb{P}(M)}{\mathbb{P}(C \mid M)\,\mathbb{P}(M) + \mathbb{P}(C \mid M^c)\,\mathbb{P}(M^c)} \\ &= \frac{(0.05)\,(0.49)}{(0.05)\,(0.49) + (0.0025)\,(0.51)} \approx 0.9505.\end{aligned}$$

Solution to Exercise 4.6(A): let H be the event *a bulb works over* 5000 *hours*, X be the event that *a bulb comes from factory* X, and Y be the event *a bulb comes from factory* Y. Then by the *law of total probability*

$$\begin{aligned}\mathbb{P}(H) &= \mathbb{P}(H \mid X)\,\mathbb{P}(X) + \mathbb{P}(H \mid Y)\,\mathbb{P}(Y) \\ &= (0.99)\,(0.6) + (0.95)\,(0.4) \\ &= 0.974.\end{aligned}$$

Solution to Exercise 4.6(B): By Part (a) we have

$$\begin{aligned}\mathbb{P}(Y \mid H) &= \frac{\mathbb{P}(H \mid Y)\,\mathbb{P}(Y)}{\mathbb{P}(H)} \\ &= \frac{(0.95)\,(0.4)}{0.974} \approx 0.39.\end{aligned}$$

Solution to Exercise 4.6(C): We again use the result from Part (a)

$$\begin{aligned}\mathbb{P}(X \mid H^c) &= \frac{\mathbb{P}(H^c \mid X)\,\mathbb{P}(X)}{\mathbb{P}(H^c)} = \frac{\mathbb{P}(H^c \mid X)\,\mathbb{P}(X)}{1 - \mathbb{P}(H)} \\ &= \frac{(1-0.99)\,(0.6)}{1-0.974} = \frac{(0.01)\,(0.6)}{0.026} \\ &\approx 0.23\end{aligned}$$

Solution to Exercise 4.7: denote by D the event that *a bolt is defective*, A the event that *a bolt is from machine* A, by B the event that *a bolt is from machine* C. Then by Bayes' theorem

$$\begin{aligned}\mathbb{P}(A \mid D) &= \frac{\mathbb{P}(D \mid A)\,\mathbb{P}(A)}{\mathbb{P}(D \mid A)\,\mathbb{P}(A) + \mathbb{P}(D \mid B)\,\mathbb{P}(B) + \mathbb{P}(D \mid C)\,\mathbb{P}(C)} \\ &= \frac{(0.05)\,(0.25)}{(0.05)\,(0.25) + (0.04)\,(0.35) + (0.02)\,(0.4)} = 0.362.\end{aligned}$$

Solution to Exercise 4.8: Let C be the event *a student gives the correct answer*, K be the event *a student knows the correct answer*, E be the event *a student can eliminate one incorrect answer*, and G be the event *a student have to guess an answer.* Using Bayes' theorem we have

$$\mathbb{P}(K \mid C) = \frac{\mathbb{P}(C \mid K)\mathbb{P}(K)}{\mathbb{P}(C)} =$$
$$\frac{\mathbb{P}(C \mid K)\mathbb{P}(K)}{\mathbb{P}(C \mid K)\mathbb{P}(K) + \mathbb{P}(C \mid E)\mathbb{P}(E) + \mathbb{P}(C \mid G)\mathbb{P}(G)} =$$
$$\frac{1 \cdot \frac{1}{2}}{1 \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4}} = \frac{24}{31} \approx .774,$$

that is, approximately 77.4% of the students know the answer if they give the correct answer.

Solution to Exercise 4.9: Let $+$ denote the event that *a test result is positive*, and by D the event that *the disease is present*. Then

$$\begin{aligned}\mathbb{P}(D \mid +) &= \frac{\mathbb{P}(+ \mid D)\, P(D)}{P(+ \mid D)\, P(D) + P(+ \mid D^c)P(D^c)} \\ &= \frac{(0.95)\,(0.01)}{(0.95)\,(0.01) + (0.005)\,(0.99)} = 0.657.\end{aligned}$$

Solution to Exercise 4.2: it is clear that $\mathbb{P}_F(E) = \mathbb{P}(E \mid F)$ is between 0 and 1 since the right-hand side of the identity defining $\mathbb{P}_F$ is. To see the second axiom, observe that

$$\mathbb{P}_F(S) = \mathbb{P}(S \mid F) = \frac{\mathbb{P}(S \cap F)}{\mathbb{P}(F)} = \frac{\mathbb{P}(F)}{\mathbb{P}(F)} = 1.$$

Now take $\{E_i\}_{i=1}^{\infty}, E_i \in \mathcal{F}$ to be pairwise disjoint, then

$$\begin{aligned}&\mathbb{P}_F\left(\bigcup_{i=1}^{\infty} E_i\right) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i \mid F\right) = \frac{\mathbb{P}\left(\left(\bigcup_{i=1}^{\infty} E_i\right) \cap F\right)}{\mathbb{P}(F)} \\ &= \frac{\mathbb{P}\left(\bigcup_{i=1}^{\infty} (E_i \cap F)\right)}{\mathbb{P}(F)} = \frac{\sum_{i=1}^{\infty} \mathbb{P}(E_i \cap F)}{\mathbb{P}(F)} \\ &= \sum_{i=1}^{\infty} \frac{\mathbb{P}(E_i \cap F)}{\mathbb{P}(F)} = \sum_{i=1}^{\infty} \mathbb{P}_F(E_i).\end{aligned}$$

In this we used the distribution law for sets $(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$ and the fact that $\{E_i \cap F\}_{i=1}^{\infty}$ are pairwise disjoint as well.