Please write ***Your name:*** ________________________________

Show all work. You should either write at a sentence explaining your reasoning, or annotate your math work with brief explanations. There is no need to simplify, and no calculators are needed.

. . . .

(1) Let X be the time that a car can run until a major repair. If $\mathbb{E}X = 2$, and X is exponentially distributed, what is $\mathbb{P}(2 < X < 5)$? ***Answer:*** $\mathbb{P}(2 < X < 5) = e^{-1} - e^{-2.5}$ because $\lambda = 1/2$.

(2) Given that this car has ran 2 years without a repair, what is the conditional probability that it will run 3 more years without a major repair? ***Answer:*** $\mathbb{P}(X > 5|X > 2) = \mathbb{P}(X > 3) = e^{-1.5}$ because an exponential random variable has the memoryless property.

(3) What is the probability density function of $Y = X^2$?
Answer: if $y = x^2 > 0$ then $\mathbb{P}(Y > y) = \mathbb{P}(X > x) = e^{-x/2} = e^{-\sqrt{y}/2}$. By differentiation we obtain that $f_Y(y) = \dfrac{1}{4\sqrt{y}} e^{-\sqrt{y}/2}$ if $y > 0$ and $f_Y(y) = 0$ if $y \leqslant 0$.

(4) Find $\mathrm{Var}(X)$ if X is uniformly distributed on the interval $[-1, 5]$. Show all steps.
Answer: $\mathbb{E}X = \frac{1}{6}\int_{-1}^{5} x\,dx = \frac{1}{6}x^2/2\Big|_{x=-1}^{3} = (25-1)/12 = 2$
$\mathbb{E}X^2 = \frac{1}{6}\int_{-1}^{5} x^2\,dx = \frac{1}{6}x^3/3\Big|_{x=-1}^{5} = (125+1)/18 = 7$ and so $\mathrm{Var}(X) = 7 - 2^2 = 3$
An easier answer: if Y is uniform on $[-1, 1]$, then $\mathbb{E}Y = 0$ and $\mathbb{E}Y^2 = \frac{1}{2}\int_{-1}^{1} x^2\,dx = \frac{1}{2}x^3/3\Big|_{x=-1}^{1} = 2/6 = 1/3$ and so $\mathrm{Var}(Y) = 1/3$. Then $X = 3Y + 2$ and so $\mathrm{Var}(X) = 3^2\mathrm{Var}(Y) = 3$.

[(*optional questions for extra credit*)]: Let a, b, c be positive numbers, $b > 1$, and the probability density function $f(x)$ of a random variable X be defined by $f(x) = ax^{-b}$ for $x > c$ and $f(x) = 0$ for $x \leqslant c$.

- What is the relation between a, b, c?
  Answer: if $b > 1$ then $\int_c^\infty ax^{-b}dx = \frac{a}{1-b}x^{1-b}\Big|_{x=c}^{\infty} = ac^{1-b}/(b-1) = 1$ or $ac^{1-b} = b - 1$.
- What is the necessary and sufficient condition for b so that $\mathbb{E}X < +\infty$?
  Answer: $b > 2$ if and only if $\int_c^\infty x^{-b+1}dx < \infty$
- What is the necessary and sufficient condition for b so that $\mathrm{Var}(X) < +\infty$?
  Answer: $b > 3$ if and only if $\int_c^\infty x^{-b+2}dx < \infty$