第六章 · 多元函数

第五节 · 复合函数和隐函数求导法则

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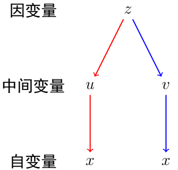
5.1 复合函数的求导法则

全导数

若 $z = f(u,v)$, $u = g(x), v = h(x)$, 则我们有复合函数 $z = f(g(x),h(x))$. 此时我们有全导数

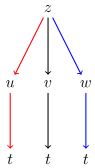


$$\frac{dz}{dx} = \frac{\partial z}{\partial u}\cdot\frac{du}{dx} + \frac{\partial z}{\partial v}\cdot\frac{dv}{dx}$$

口诀：分段用乘, 分叉用加, 单路全导, 叉路偏导.

情形 1: 中间变量多于两个

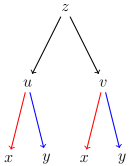
$z = f(u,v,w)$, 其中 $u = g(t), v = h(t)$, $w = \omega(t)$



$$\frac{dz}{dt} = \frac{\partial z}{\partial u}\cdot\frac{du}{dt} + \frac{\partial z}{\partial v}\cdot\frac{dv}{dt} + \frac{\partial z}{\partial w}\cdot\frac{dw}{dt}$$

情形 2: 中间变量是多元函数

例如 $z=f(u,v)$, 其中 $u=g(x,y)$, $v=h(x,y)$



$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

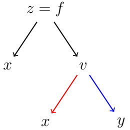
$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

. . . .

问题 若 $v=h(y)$, 求 $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$

情形 3: 中间变量是自变量

$z=f(x,v)$, 其中 $v=h(x,y)$



$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}=f_1'+f_2'h_1'$$

$$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}=f_2'h_2'$$

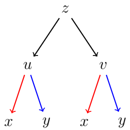
注 这里 $\frac{\partial z}{\partial x}$ 与 $\frac{\partial f}{\partial x}$ 不同.

$\frac{\partial z}{\partial x}$ 是把 $z=f(x,h(x,y))$ 中的 y 看做常量对 x 求偏导.

$\frac{\partial f}{\partial x}$ 是把 $f(x,v)$ 中的 v 看做常量对 x 求偏导.

例 1 设 $z=e^u\sin v, u=xy, v=x+y$, 求 $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$

解

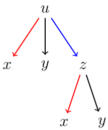


$$\begin{aligned}\frac{\partial z}{\partial x}&=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\\&=e^u\sin v\cdot y+e^u\cos v\cdot 1\\&=e^{xy}[y\sin(x+y)+\cos(x+y)]\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial y}&=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}=e^u\sin v\cdot x+e^u\cos v\cdot 1\\&=e^{xy}[x\sin(x+y)+\cos(x+y)]\end{aligned}$$

例 2 $u=f(x,y,z)=\mathrm{e}^{x^2+y^2+z^2}, z=x^2\sin y$. 求 $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$.

解

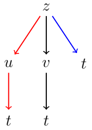


$$\begin{aligned}\frac{\partial u}{\partial x}&=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x}\\&=2x\mathrm{e}^{x^2+y^2+z^2}+2z\mathrm{e}^{x^2+y^2+z^2}\cdot 2x\sin y\\&=2x\left(1+2x^2\sin^2 y\right)\mathrm{e}^{x^2+y^2+x^4\sin^2 y}\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial y}&=\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y}=2y\mathrm{e}^{x^2+y^2+z^2}+2z\mathrm{e}^{x^2+y^2+z^2}\cdot x^2\cos y\\&=2\left(y+x^4\sin y\cos y\right)\mathrm{e}^{x^2+y^2+x^4\sin^2 y}\end{aligned}$$

例 3 $z = uv + \sin t, u = \mathrm{e}^t, v = \cos t$. 求全导数 $\frac{dz}{dt}$.

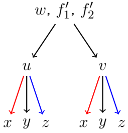
解



$$\frac{dz}{dt} = \frac{\partial z}{\partial u}\cdot\frac{du}{dt} + \frac{\partial z}{\partial v}\cdot\frac{dv}{dt} + \frac{\partial z}{\partial t}$$
$$= v\mathrm{e}^t - u\sin t + \cos t$$
$$= \mathrm{e}^t(\cos t - \sin t) + \cos t$$

例 4 设 $w = f(x+y+z, xyz)$, 其中 f 具有二阶连续偏导数, 求 $\frac{\partial w}{\partial x}$, $\frac{\partial^2 w}{\partial x \partial z}$.

解 设 $u = x+y+z$, $v = xyz$, 则 $w = f(u,v)$



$$\frac{\partial w}{\partial x} = f_1'\cdot 1 + f_2'\cdot yz$$
$$= f_1'(x+y+z, xyz)$$
$$+ yz f_2'(x+y+z, xyz)$$

这里 $f_1' = \frac{\partial f}{\partial u}, f_{12}'' = \frac{\partial^2 f}{\partial u \partial v}, \cdots$

$$\frac{\partial^2 w}{\partial x \partial z} = f_{11}''\cdot 1 + f_{12}''\cdot xy + (yz)' f_2' + yz\left[f_{21}''\cdot 1 + f_{22}''\cdot xy\right]$$
$$= f_{11}'' + y(x+z)f_{12}'' + xy^2 z f_{22}'' + y f_2'$$

5.2 多元复合函数的全微分

全微分的形式不变性

设有 $z = f(u,v)$，u, v 为自变量，则全微分为

$$\mathrm{d}z = \frac{\partial z}{\partial u}\mathrm{d}u + \frac{\partial z}{\partial v}\mathrm{d}v$$

全微分的形式不变性

若又有 $u=g(x,y)$, $v=h(x,y)$，u, v 为中间变量，则全微分仍为

$$
\begin{aligned}
dz &= \frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy \\
&= \left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\right)dx+\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\right)dy \\
&= \frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy\right)+\frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy\right) \\
&= \frac{\partial z}{\partial u}du+\frac{\partial z}{\partial v}dv
\end{aligned}
$$

可见无论 u, v 是自变量还是中间变量，其全微分表达形式都一样，这性质叫做全微分形式不变性.

利用全微分形式不变性再解例1

$$
\begin{aligned}
dz &= d\left(\mathrm{e}^{u}\sin v\right)=e^{u}\sin v du+\mathrm{e}^{u}\cos v dv \\
&= \mathrm{e}^{xy}[\sin(x+y)d(xy)+\cos(x+y)d(x+y)] \\
&= \mathrm{e}^{xy}[\sin(x+y)(ydx+xdy)+\cos(x+y)(dx+dy)] \\
&= \mathrm{e}^{xy}[y\sin(x+y)+\cos(x+y)]dx \\
&\qquad +\mathrm{e}^{xy}[x\sin(x+y)+\cos(x+y)]dy
\end{aligned}
$$

$$
\therefore\quad \frac{\partial z}{\partial x}=\mathrm{e}^{xy}[y\cdot\sin(x+y)+\cos(x+y)]
$$

$$
\frac{\partial z}{\partial y}=\mathrm{e}^{xy}[x\cdot\sin(x+y)+\cos(x+y)]
$$

5.3 隐函数的求导公式

5.4 微积分 1: 隐函数的求导公式

显函数与隐函数

定义 1　若方程 $F(x, y) = 0$ 能确定 y 是 x 的函数，或将一个显函数 $y = f(x)$ 隐藏在方程 $F(x, y)$ 中使得方程恒等 $F(x, y) = 0$. 那么称这种方式表示的函数是隐函数.

例如　当 $y = x - 1$ 时，得恒等式方程

$$y - x + 1 = 0$$

所以称 $y - x + 1 = 0$ 为隐函数.

注　y 在隐函数 $F(x, y) = 0$ 中不是变量，是关于 x 的函数，所以 $y = y(x)$.

隐函数的求导方法

由隐函数转换成显函数，称为隐函数显化，例如：隐函数 $y-x+1 = 0$ 可以显化为 $y = x - 1$ 从而可以对可显化的隐函数求导. 然而并非所有隐函数都可以显化，例如：

$$y^5 + 2y - x - 3x^7 = 0$$

问题 1　如何对无法显化的隐函数求导？

解法　由于在 $F(x,y) = 0$ 中 $y = y(x)$，所以对方程两边同时对 x 求导.

$$\frac{d}{dx}F(x,y) = 0$$

例 5　求由方程 $y^5 + 2y - x - 3x^7 = 0$ 确定的隐函数 $y = y(x)$ 的导数 $\dfrac{dy}{dx}$.

解　方程两边对 x 求导

$$5y^4\frac{dy}{dx} + 2\frac{dy}{dx} - 1 - 21x^6 = 0$$

$$\therefore \frac{dy}{dx} = \frac{1 + 21x^6}{5y^4 + 2}$$

例 6　求椭圆 $\frac{x^2}{16} + \frac{y^2}{9} = 1$ 在点 $\left(2, \frac{3}{2}\sqrt{3}\right)$ 处的切线方程.

解　椭圆方程两边对 x 求导

$$\frac{x}{8} + \frac{2}{9}y \cdot y' = 0$$

$$y' = -\frac{9}{16}\frac{x}{y}$$

所以在点 $\left(2, \frac{3}{2}\sqrt{3}\right)$ 处，$y' = -\frac{\sqrt{3}}{4}$.

故切线方程为

$$y - \frac{3}{2}\sqrt{3} = -\frac{\sqrt{3}}{4}(x - 2)$$

5.5 微积分 2: 隐函数的求导公式

隐函数的导数 1

定理 1 设 $F(x,y)$ 在 (x_0,y_0) 的某一邻域内满足

(1) 具有连续偏导;

(2) $F(x_0,y_0)=0$;

(3) $F_y(x_0,y_0)\neq 0$.

则方程 $F(x,y)=0$ 在点 x_0 的某邻域内可唯一确定一个单值连续函数 $y=f(x)$, 满足条件 $y_0=f(x_0)$ 并有连续导数

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{F'_x}{F'_y}$$

定理证明从略，仅就求导公式推导如下：

若函数 $y=f(x)$ 可以写为方程 $F(x,\,y)=0$ 的形式, 则称 $F(x,\,y)=0$ 为隐函数.

设 $z=F(x,y)$, 其中 $y=f(x)$

$$\begin{tikzcd}
 & z=F \arrow[dl] \arrow[dr] & \\
x & & y \arrow[d] \\
 & & x
\end{tikzcd}$$

$$F(x,y)\equiv 0$$

$\Downarrow$ 两边对 x 求导

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{dy}{dx}=0$$

$\Downarrow$ 在 $(x_0,\,y_0)$ 的某邻域内 $F'_y\neq 0$

$$\frac{dy}{dx}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}=-\frac{F'_x}{F'_y}$$

例 7 验证方程 $y-xe^y+x=0$ 在点 $(0,0)$ 的某个邻域内能唯一确定一个有连续导数的函数 $y=f(x)$, 使当 $x=0$ 时, $y=0$, 并求 $\left.\frac{dy}{dx}\right|_{x=0}$.

解法 1

1. 设 $F(x,y)=y-xe^y+x$, 则 $F'_x=-e^y+1$, $F'_y=1-xe^y$, 且在点 $(0,\,0)$ 的邻域内连续;
2. 又因 $F(0,0)=0$;
3. $F'_y(0,0)=1\neq 0$.

由定理1, 得

$$\frac{dy}{dx}=-\frac{F'_x}{F'_y}=-\frac{1-e^y}{1-xe^y}\Rightarrow\left.\frac{dy}{dx}\right|_{x=0}=0.$$

解法 2

$$y - xe^y + x = 0$$

方程两边对 x 求导, 得

$$\frac{dy}{dx} - \left(e^y + xe^y\frac{dy}{dx}\right) + 1 = 0.$$

解得

$$\frac{dy}{dx} = -\frac{1-e^y}{1-xe^y} \Rightarrow \left.\frac{dy}{dx}\right|_{x=0} = 0.$$

注　本章所学方法是把隐函数方程 $F(x, y) = 0$ 中的 x, y 当做两个自变量, 对 F 求各边变量的偏导数. 然而微积分 1 中所学的方法是把 $F(x, y) = 0$ 中的 x 当做自变量, y 当做关于 x 的函数, $y = f(x)$. 对 F 求导 x.

例 8　方程 $xy^2 - \ln y = a$ 确定了 y 是 x 的函数, 求 $\frac{dy}{dx}$

解　利用隐函数求导公式:

令 $F(x, y) = xy^2 - \ln y - a$, 则 $F'_x = y^2$, $F'_y = 2xy - \frac{1}{y}$

$$\therefore \frac{dy}{dx} = -\frac{F'_x}{F'_y} = -\frac{y^2}{2xy - \frac{1}{y}} = \frac{y^3}{1-2xy^2}.$$

方程两边对 x 求导:

$$y^2 + x \cdot 2yy' - \frac{1}{y} \cdot y' = 0.$$

$$\therefore \frac{dy}{dx} = \frac{y^3}{1-2xy^2}$$

例 9　设方程 $x^4 + y^4 = 1$ 确定了隐函数 $y = f(x)$，求 $\frac{\mathrm{d}y}{\mathrm{d}x}$ 和 $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

解　隐函数的一阶导数 $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x^3}{y^3}$. 因此二阶导数

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{x^3}{y^3}\right)\\ &= -\frac{\frac{\mathrm{d}}{\mathrm{d}x}(x^3)\cdot y^3 - x^3\cdot\frac{\mathrm{d}}{\mathrm{d}x}(y^3)}{y^6}\\ &= -\frac{3x^2\cdot y^3 - x^3\cdot 3y^2\frac{\mathrm{d}y}{\mathrm{d}x}}{y^6} = -\frac{3x^2}{y^7}\end{aligned}$$

注　注意不要错误认为 $\frac{\mathrm{d}}{\mathrm{d}x}(y^3) = 0$.

隐函数的导数 2

定理 2　设 $F(x, y, z)$ 在 (x_0, y_0, z_0) 的某一邻域内满足

(1) 具有连续偏导;

(2) $F(x_0, y_0, z_0) = 0$;

(3) $F_y(x_0, y_0, z_0) \neq 0$.

则方程 $F(x, y, z) = 0$ 在点 (x_0, y_0) 的某邻域内可唯一确定一个单值连续函数 $z = f(x, y)$, 满足条件 $z_0 = f(x_0, y_0)$ 并有连续导数

$$\frac{\partial z}{\partial x} = -\frac{F'_x}{F'_z} \qquad \frac{\partial z}{\partial y} = -\frac{F'_y}{F'_z}$$

定理证明从略，仅就求导公式推导如下：

若函数 $z = f(x, y)$ 可以写为方程 $F(x, y, z) = 0$ 的形式, 则称 $F(x, y, z) = 0$ 为隐函数.

设 $w = F(x,y,z)$, 其中 $z = f(x,y)$

$$
\begin{tikzcd}
 & w=F \arrow[dl] \arrow[d] \arrow[dr] & \\
x & y & z \arrow[dl] \arrow[d] \\
 & x & y
\end{tikzcd}
$$

$$F(x,y,f(x,y)) \equiv 0$$

$\Downarrow$ 两边对 x, y 求偏导

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} = 0, \quad \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial y} = 0$$

$\Downarrow F'_z \neq 0$

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{F'_x}{F'_z}, \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{F'_y}{F'_z}$$

例 10 求由方程 $e^{-xy} - 2z + e^z = 0$ 所确定的隐函数的偏导数 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

解 设 $F(x,y,z) = e^{-xy} - 2z + e^z$, 则

$$\frac{\partial F}{\partial x} = -ye^{-xy}, \quad \frac{\partial F}{\partial y} = -xe^{-xy}, \quad \frac{\partial F}{\partial z} = -2 + e^z$$

由定理2, 得

$$\frac{\partial z}{\partial x} = -\frac{F'_x}{F'_z} = \frac{ye^{-xy}}{e^z - 2};$$

$$\frac{\partial z}{\partial y} = -\frac{F'_y}{F'_z} = \frac{xe^{-xy}}{e^z - 2}.$$

例 11 设 $x^2 + y^2 + z^2 - 4z = 0$, 求 $\frac{\partial^2 z}{\partial x^2}$

解 利用隐函数求导:

$$2x + 2z\frac{\partial z}{\partial x} - 4\frac{\partial z}{\partial x} = 0 \quad \rightarrow \frac{\partial z}{\partial x} = \frac{x}{2-z}$$

$\Downarrow$ 两边对 x 求偏导

$$2 + 2\left(\frac{\partial z}{\partial x}\right)^2 + 2z\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x^2} = 0$$

$$\therefore \frac{\partial^2 z}{\partial x^2} = \frac{1 + \left(\frac{\partial z}{\partial x}\right)^2}{2-z} = \frac{(2-z)^2 + x^2}{(2-z)^3}$$

解 利用公式:

设 $F(x,y,z) = x^2 + y^2 + z^2 - 4z$,则 $F'_x = 2x$, $F'_z = 2z - 4$

$$\therefore \quad \frac{\partial z}{\partial x} = -\frac{F'_x}{F'_z} = -\frac{x}{z-2} = \frac{x}{2-z}$$

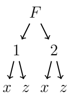
两边对 x 求偏导, 得

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{x}{2-z}\right) = \frac{(2-z) + x\frac{\partial z}{\partial x}}{(2-z)^2} = \frac{(2-z)^2 + x^2}{(2-z)^3}$$

例 12 设 $F(x,y)$ 具有连续偏导数, 已知方程 $F\left(\frac{x}{z},\frac{y}{z}\right)=0$, 求 dz

解 利用偏导数公式:

设 $z=f(x,y)$ 是由方程确定的隐函数 $F\left(\frac{x}{z},\frac{y}{z}\right)=0$, 则



$$\frac{\partial z}{\partial x}=-\frac{F_1'\cdot\frac{1}{z}}{F_1'\cdot\left(-\frac{x}{z^2}\right)+F_2'\cdot\left(-\frac{y}{z^2}\right)}=\frac{zF_1'}{xF_1'+yF_2'}$$

$$\frac{\partial z}{\partial y}=-\frac{F_2'\cdot\frac{1}{z}}{F_1'\cdot\left(-\frac{x}{z^2}\right)+F_2'\cdot\left(-\frac{y}{z^2}\right)}=\frac{zF_2'}{xF_1'+yF_2'}$$

$$\therefore\ dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=\frac{z}{xF_1'+yF_2'}\left(F_1'dx+F_2'dy\right)$$

解 微分法:

由定理 6.4.2, 全微分公式得

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy.$$

对方程两边求微分:

$$F\left(\frac{x}{z},\frac{y}{z}\right)=0$$

$$F_1'\cdot d\left(\frac{x}{z}\right)+F_2'\cdot d\left(\frac{y}{z}\right)=0$$

$$F_1'\cdot\left(\frac{zdx-xdz}{z^2}\right)+F_2'\cdot\left(\frac{zdy-ydz}{z^2}\right)=0$$

$$\frac{xF_1'+yF_2'}{z^2}dz=\frac{F_1'dx+F_2'dy}{z}$$

$$dz=\frac{z}{xF_1'+yF_2'}\left(F_1'dx+F_2'dy\right)$$

方程组确定的隐函数

隐函数存在定理可以推广到方程组情形. 比如

$$\begin{cases}F(x,y,u,v)=0\\G(x,y,u,v)=0\end{cases}\overset{?}{\Rightarrow}\begin{cases}u=u(x,y)\\v=v(x,y)\end{cases}$$

由 F、G 的偏导数组成的行列式

$$J=\frac{\partial(F,G)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial F}{\partial u}&\frac{\partial F}{\partial v}\\\frac{\partial G}{\partial u}&\frac{\partial G}{\partial v}\end{vmatrix}$$

称为 F、G 的雅可比行列式.

定理 设 F, G 在 (x_0,y_0,u_0,v_0) 邻域有连续偏导数

$$\begin{cases}F(x,y,u,v)=0\\G(x,y,u,v)=0\\F(x_0,y_0,u_0,v_0)=0\\G(x_0,y_0,u_0,v_0)=0\end{cases}\overset{J\neq 0}{\Rightarrow}\begin{cases}u=u(x,y)\\v=v(x,y)\\u_0=u(x_0,y_0)\\v_0=v(x_0,y_0)\end{cases}$$

而且隐函数也有连续偏导数

$$\frac{\partial u}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)}\qquad\frac{\partial v}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,x)}$$

$$\frac{\partial u}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(y,v)}\qquad\frac{\partial v}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,y)}$$

方程组确定的隐函数

例 13　设 $xu - yv = 0$，$yu + xv = 1$，求偏导数 $\dfrac{\partial u}{\partial x}, \dfrac{\partial v}{\partial x}$.

解　方程组两边对 x 求导，并移项得 $\begin{cases} x\frac{\partial u}{\partial x} - y\frac{\partial v}{\partial x} = -u \\ y\frac{\partial u}{\partial x} + x\frac{\partial v}{\partial x} = -v \end{cases}$

由题设 $J = \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix} = \begin{vmatrix} x & -y \\ y & x \end{vmatrix} = x^2 + y^2 \neq 0$

故有 $\begin{cases} \frac{\partial u}{\partial x} = \frac{1}{J}\begin{vmatrix} -u & -y \\ -v & x \end{vmatrix} = -\dfrac{xu + yv}{x^2 + y^2} \\ \frac{\partial v}{\partial x} = \frac{1}{J}\begin{vmatrix} x & -u \\ y & -v \end{vmatrix} = -\dfrac{xv - yu}{x^2 + y^2} \end{cases}$

5.6　内容小结

内容小结

1. 复合函数求导的链式法则: 分段用乘, 分叉用加, 单路全导, 叉路偏导
2. 全微分形式不变性: 对 $z = f(u, v)$ 不论 u, v 是自变量还是中间变量,
$$dz = f_u(u,v)du + f_v(u,v)dv$$
3. 隐函数 (组) 存在定理
4. 隐函数 (组) 求导方法
   - 方法 1: 利用复合函数求导法则直接计算;
   - 方法 2: 利用微分形式不变性;
   - 方法 3. 代公式.

本节完!