機率與統計 HW7 何寬羿 資訊 114 C34104032

Textbook exercise:

(7.12)

$\because X_1, X_2$ independent

$\therefore$ joint prob. dist. $f(x_1, x_2) = e^{-x_1} \cdot e^{-x_2} = e^{-(x_1+x_2)}$, $\begin{matrix} x_1 > 0 \\ x_2 > 0 \end{matrix}$

且 $\begin{cases} Y_1 = X_1 + X_2 \\ Y_2 = \dfrac{X_1}{X_1+X_2} \end{cases} \longrightarrow \begin{cases} x_1 = w_1(y_1, y_2) = y_1 y_2 \\ x_2 = w_2(y_1, y_2) = y_1 - y_1 y_2 \end{cases}$, $\begin{matrix} y_1 > 0 \\ 1 > y_2 > 0 \end{matrix}$

$$\Rightarrow g(y_1, y_2) = f[w_1(y_1,y_2), w_2(y_1,y_2)]\,|J|$$

$$= f(y_1y_2, y_1 - y_1y_2) \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = e^{-(y_1y_2 + y_1 - y_1y_2)} \cdot \left| \begin{vmatrix} y_2 & y_1 \\ 1-y_2 & -y_1 \end{vmatrix} \right|$$

$$= e^{-y_1} \cdot \left| \underbrace{-y_1y_2 - y_1(1-y_2)}_{-y_1} \right| = y_1 e^{-y_1}, \quad \begin{matrix} y_1 > 0 \\ 1 > y_2 > 0 \end{matrix}$$

$$\Rightarrow g(y_1) = \int_0^1 g(y_1, y_2)\, dy_2 = \int_0^1 y_1 e^{-y_1} dy_2 = y_1 e^{-y_1} y_2 \Big|_{y_2=0}^{y_2=1} = y_1 e^{-y_1} \quad (y_1 > 0)$$

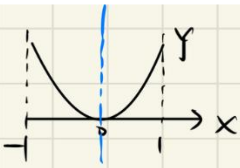
$$g(y_2) = \int_0^\infty g(y_1, y_2)\, dy_1 = \int_0^\infty y_1 e^{-y_1} dy_1 \xrightarrow{\text{Gamma Function}} = \Gamma(2) = 1! = 1 \quad (1 > y_2 > 0)$$

$\Rightarrow$ 由於 $g(y_1) \cdot g(y_2) = y_1 e^{-y} \cdot 1 = y_1 e^{-y} = g(y_1, y_2)$, $\begin{matrix} y_1 > 0 \\ 1 > y_2 > 0 \end{matrix}$

故, Y_1 and Y_2 are independent #

(7.14)

依題意, $Y = X^2$ is not "one to one"



$(1 > x > -1)$

$\Rightarrow \begin{cases} x_1 = w_1(y) = \sqrt{y} \\ x_2 = w_2(y) = -\sqrt{y} \end{cases}$, $1 > y > 0$

所求 $g(y) = f(\sqrt{y})\,|\overset{w_1'(y)}{J_1}| + f(-\sqrt{y})\,|\overset{w_2'(y)}{J_2}| = \dfrac{1+\sqrt{y}}{2} \cdot \dfrac{1}{2}\dfrac{1}{\sqrt{y}} + \dfrac{1-\sqrt{y}}{2} \cdot \dfrac{1}{2}\dfrac{1}{\sqrt{y}}$

$$= \frac{1+\sqrt{y}+1-\sqrt{y}}{4\sqrt{y}} = \frac{1}{2\sqrt{y}} = \frac{\sqrt{y}}{2y}, \quad 1 > y > 0 = g(y) \ \#$$

g(y) = 0, elsewhere.

(7.18)

* $g(x;p) = pq^{x-1}$, $x = 1,2,\cdots$ (discrete) ; $p+q=1$

$M_x(t) = E(e^{tx}) = \sum_x e^{tx}\cdot g(x;p) = \sum_x e^{tx}\cdot pq^{x-1} = \frac{p}{q}\sum_{x=1}^{\infty}(qe^t)^x$

當$|qe^t|<1$才會收斂

$qe^t < 1$ (∵ $q, e^t > 0$)

$\Rightarrow e^t < \frac{1}{q}$

$\Rightarrow t < \ln 1/q = -\ln q$

$= \frac{pe^t}{1-qe^t}$, $t < -\ln q$ # proved

* $\mu = \mu_1' = M_x'(0)$

$$= \frac{(pe^t)(1-qe^t)-(-qe^t)(pe^t)}{(1-qe^t)^2}\Big|_{t=0} = \boxed{\frac{pe^t}{(1-qe^t)^2}}\Big|_{t=0} = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

$\Rightarrow$ mean $= \frac{1}{p}$ # mean

↓再微分一次

* $\mu_2' = M_x''(0) = \frac{(pe^t)(1-qe^t)^2 - 2\cdot(1-qe^t)(-qe^t)\cdot(pe^t)}{(1-qe^t)^4}\Big|_{t=0} = \frac{\overset{1-q}{p}\cdot(1-q)^2 - 2(1-q)\cdot(-q)\cdot \overset{1-q}{p}}{(1-q)^4}$

上下同除$(1-q)^2$

$$= \frac{(1-q)+2q}{(1-q)^2} \quad \frac{1-q=p}{q=1-p} \quad \frac{p+2(1-p)}{p^2} = \frac{2-p}{p^2}$$

$\Rightarrow$ variance $= \sigma^2 = \mu_2' - (\mu_1')^2 = \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2 = \frac{1-p}{p^2}$ # variance

(7.22)

$M_x(t) = (1-2t)^{-\frac{\nu}{2}}$

所求：

* mean $= \mu_1' = M_x'(0) = \nu(1-2t)^{-\frac{\nu}{2}-1}\Big|_{t=0} = \nu$ # proved

* $\mu_2' = M_x''(0) = -2\nu\left(-\frac{\nu}{2}-1\right)\cdot(1-2t)^{-\frac{\nu}{2}-2}\Big|_{t=0} = \nu(\nu+2)$

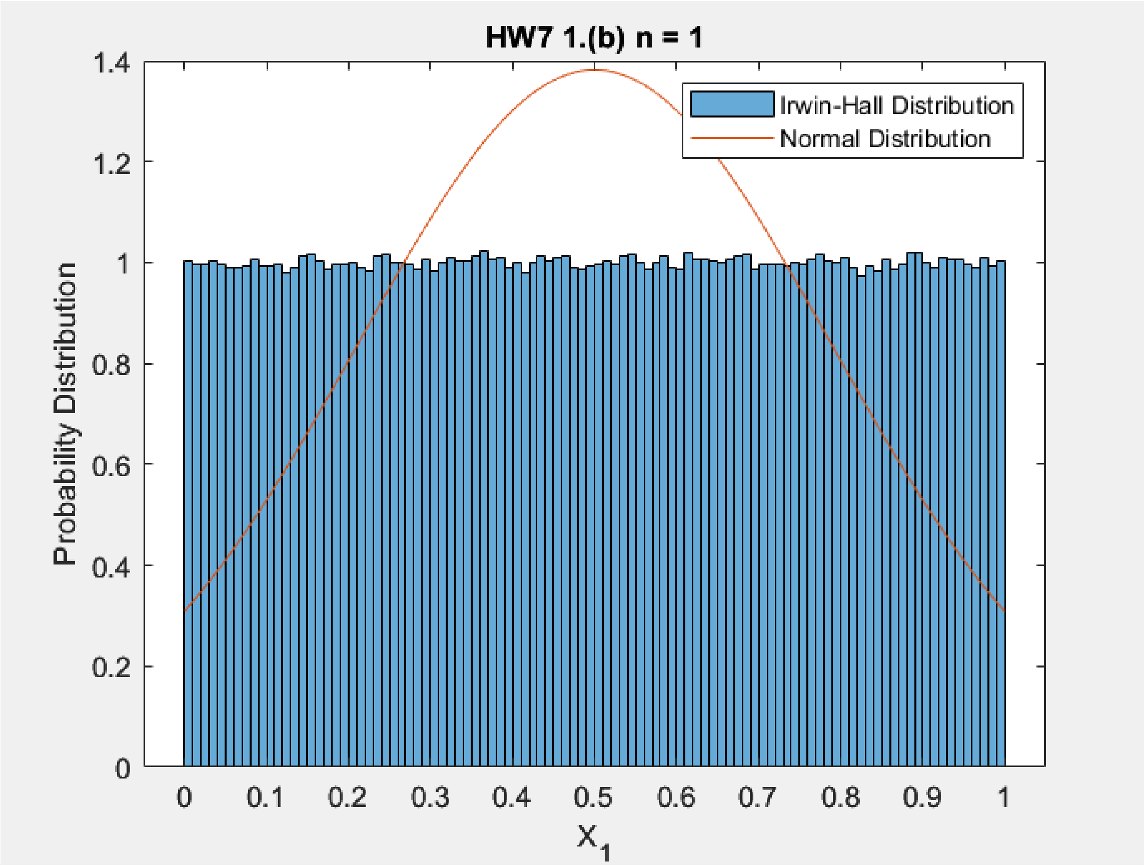
$\Rightarrow$ variance $= \mu_2' - (\mu_1')^2 = \nu(\nu+2) - \nu^2 = 2\nu$ # proved

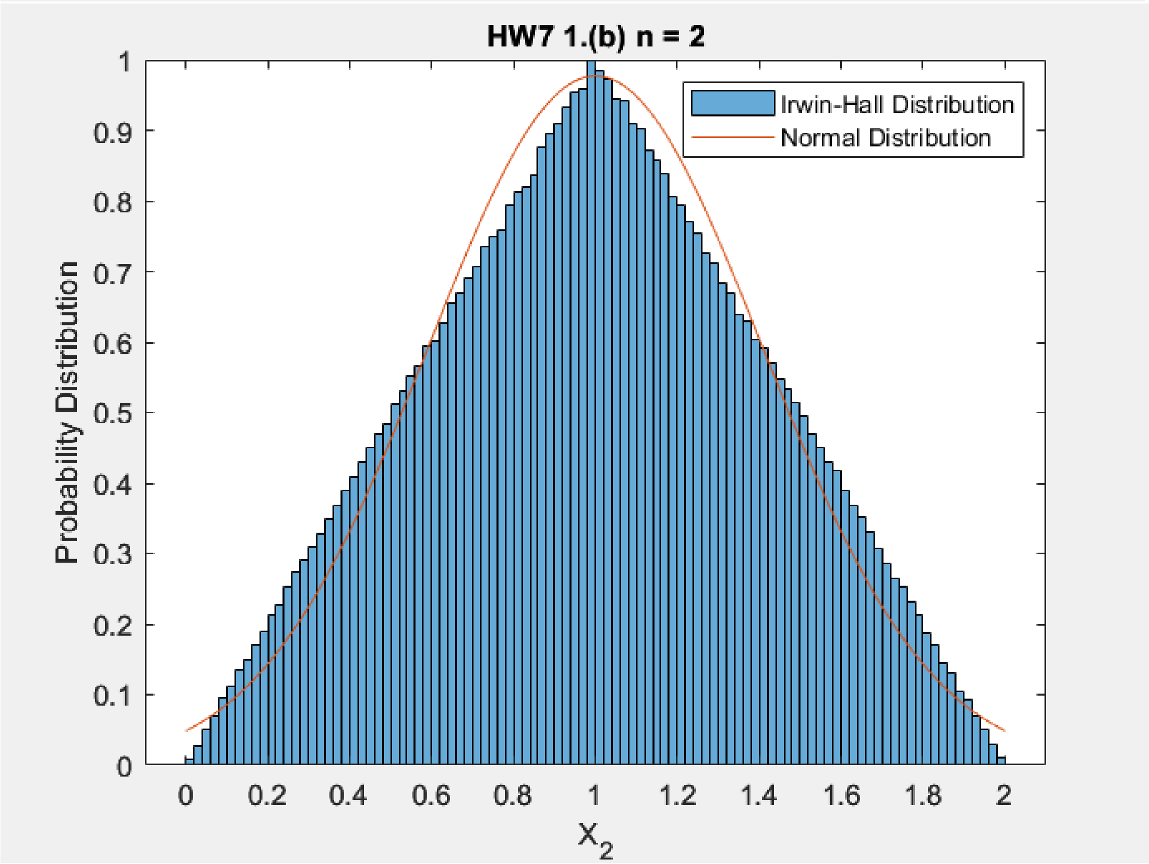
Matlab

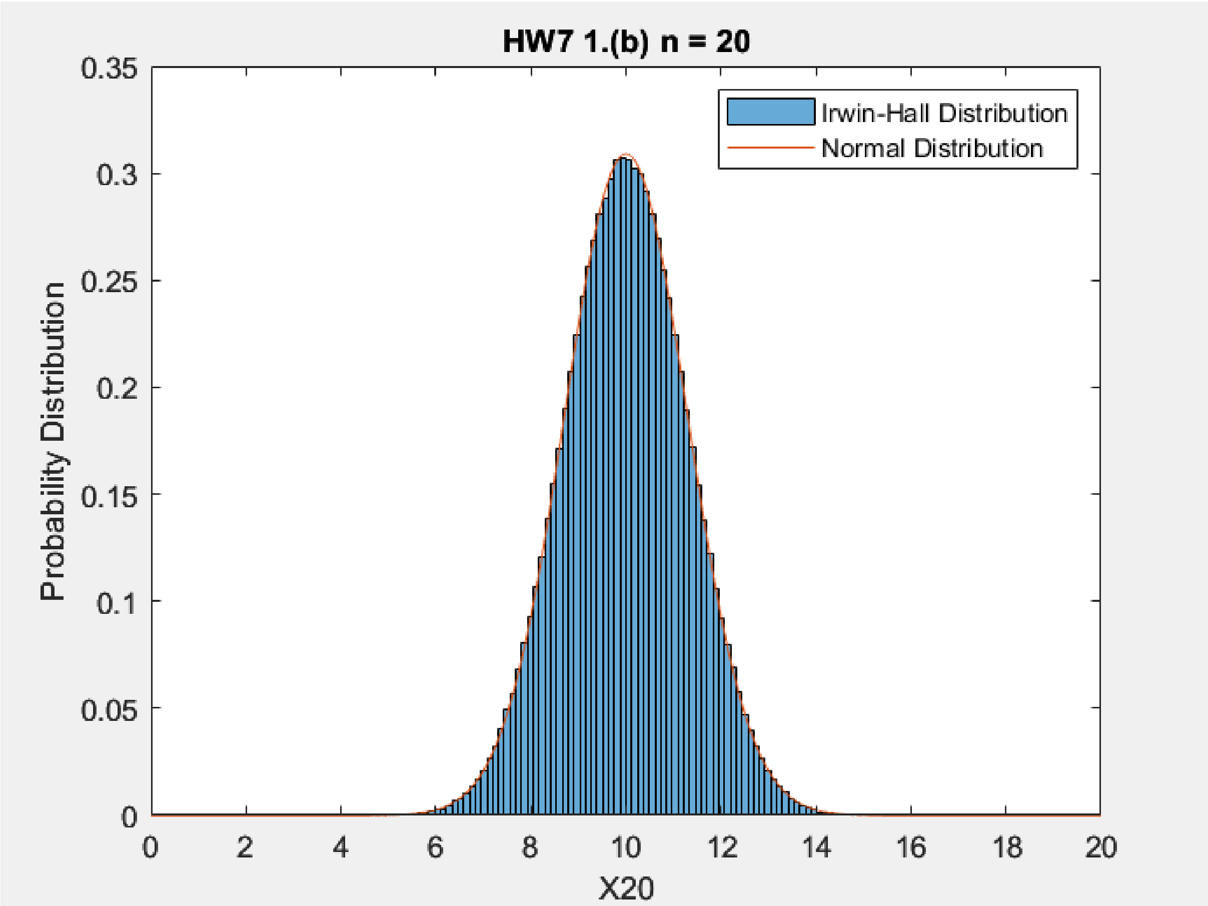
1.(a) Function 已實作在 HW7_1a.m

1.(b)

(histograms are normalized to values of probability density function for ease of comparison with a normal distribution.)





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% When n is too small, the errors of using a Irwin-Hall distribution
% to approximate a normal distribution will be very large. That is, the
% approximation is bad when n is small.(When n = 1, two distributions are
% completely different; when n = 2, the approximation is slightly better,
% but still do not look like normal distribution.).
%
% However, when n is huge enough(like the case n = 20 above),
% the approximation will be great, the errors are small.
% Also, this is a good example to demonstrate the central limit theorem.
% when n is huge enough, Irwin-Hall Dist.(Originally not normal dist.)
% will be like Normal Dist.
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