Feedback Exercise 9

Jelinek Jakub
2478625J

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1 FB1

Suppose that $f : A \to B$ is a surjective function. Define the following relation on A:

$$a_1 \sim a_2 \text{ } if \text{ } and \text{ } only \text{ } if \text{ } f(a_1) = f(a_2).$$

Show that this is an equivalence relation. Denote by $A/\sim$ the set of equivalence classes of $\sim$. Prove that

$$|A/\sim| = |B|.$$

To show that $\sim$ is an equivalence relation, we have to show that $\sim$ is (i) reflexive, (ii) symmetric and (iii) transitive. Let $a, b, c \in A$, then

(i) $\sim$ is reflexive because $a \sim a$ as $f(a) = f(a)$ by the definition of function.

(ii) $\sim$ is symmetric as if $a \sim b$ and so $f(a) = f(b)$, then $b \sim a$, because $f(b) = f(a)$.

(iii) $\sim$ is transitive, because if $f(a) = f(b)$ and $f(b) = f(c)$, then $f(a) = f(c)$.

As $a \sim b$ are in relation if and only if $f(a) = f(b)$, then the number of equivalence classes is the number of distinct $f(x)$, where $f(x) \in B$ and as the function $A \to B$ is surjective, $\forall f(x) \in B \; \exists x \in A$. Hence, the cardinality of the set of all equivalence classes and the cardinality of the target are equal.

$\frac{5}{5}$

rewrite as $\forall b \in B, \exists a \in A$
s.t. $f(a) = b$

2 FB2

Suppose that G is a group with identity element e. Let $\alpha, \beta\gamma \in G$ be arbitrary. Prove the following statements.

(i) $\alpha\beta\gamma = e$ implies $\beta\gamma\alpha = e$.

(ii) $\beta\alpha\gamma = \alpha^{-1}$ implies $\gamma\alpha\beta = \alpha^{-1}$.

As $e = xx^{-1}$, there are only two possibilities for (i):

a) $\alpha^{-1} = \beta\gamma$.

b) $\gamma^{-1} = \alpha\beta$.

I do not follow your method of proof. You must assume $\alpha\beta\gamma = e$ and show, from here, $\beta\gamma\alpha = e$.

Proof. If $\alpha\beta\gamma = e$

$\Rightarrow \alpha^{-1}\beta\gamma = \alpha^{-1}e$

$\beta\gamma = \alpha^{-1}$

$\beta\gamma\alpha = \alpha^{-1}\alpha = e$ ∎

Let's consider possibility a) first. If $\beta\gamma = \alpha^{-1}$, then $\alpha\beta\gamma = \alpha\alpha^{-1} = e$ and (i) is true.
Now, we have to prove that (i) is also true with possibility b). As $\alpha\beta = \gamma^{-1}$, we can prove it as

$$
\begin{aligned}
\beta\gamma\alpha &= e \\
\Rightarrow \beta\gamma\alpha\beta &= e\beta \\
\Rightarrow \beta\gamma\gamma^{-1} &= e\beta \\
\Rightarrow \beta &= e\beta \\
\Rightarrow \beta &= \beta.
\end{aligned}
$$

Therefore, (i) is true.
To prove that (ii) is true, we will use a similar approach as to prove (i):

$$
\begin{aligned}
\beta\alpha\gamma &= \alpha^{-1} \\
\Rightarrow \beta\alpha\gamma\gamma^{-1} &= \alpha^{-1}\gamma^{-1} \\
\Rightarrow \beta\alpha &= \alpha^{-1}\gamma^{-1} \\
\Rightarrow \alpha\beta\alpha &= \alpha\alpha^{-1}\gamma^{-1} \\
\Rightarrow \alpha\beta\alpha &= \gamma^{-1} \\
\Rightarrow \alpha\beta\alpha\alpha^{-1} &= \gamma^{-1}\alpha^{-1} \\
\Rightarrow \alpha\beta &= \gamma^{-1}\alpha^{-1} \\
\Rightarrow \gamma\alpha\beta &= \gamma\gamma^{-1}\alpha^{-1} \\
\Rightarrow \gamma\alpha\beta &= \alpha^{-1}.
\end{aligned}
$$

Hence, (ii) is true. nice.

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3 FB3

A parametric curve is described by the following equations

$$\frac{dx}{dt} = x, \;\; y = cos\ t, \;\; z = sin\ t,$$

and passes through $\langle 1, 1, 0\rangle$ when $t = 0$. By solving the ODE for $x(t)$, or otherwise, find an expression for x in terms of t and use this to write the space curve as a vector function. Hence, find the unit tangent to the curve $T(t)$ at the point $\langle 1, 1, 0\rangle$.

We start by solving ODE for $x(t)$. We can easily see that the equation is separable, so we can rearrange it to the following form

$$\frac{1}{x}\,dx = dt,$$

so now we can integrate both sides to get a solution.

$$\int \frac{1}{x}\,dx = \int dt$$

which is equivalent to

$$ln(x) + C_1 = t + C_2.$$

To get an expression for x, we take into account the initial condition of $x = 1$ when $t = 0$ and we use definition of logarithm to obtain

$$x = e^t.$$

Now, we can define $\mathbf{r(t)} = \langle e^t, cos\ t, sin\ t\rangle$ and to get unit tangent, we differentiate $\mathbf{r(t)}$ as

$$\mathbf{r(t)} = \langle e^t, -sin\ t, cos\ t\rangle,$$

and so

$$\begin{aligned}\mathbf{T(t)} &= \frac{\langle e^t, -sin\ t, cos\ t\rangle}{|\langle e^t, -sin\ t, cos\ t\rangle|}\\ &= \frac{\langle e^t, -sin\ t, cos\ t\rangle}{\sqrt{e^{2t} + (-sin(t))^2 + (cos(t))^2}}\\ &= \frac{\langle e^t, -sin\ t, cos\ t\rangle}{\sqrt{e^{2t} + (sin(t))^2 + (cos(t))^2}}\\ &= \frac{\langle e^t, -sin\ t, cos\ t\rangle}{\sqrt{e^{2t}+1}}.\end{aligned}$$

As was mentioned in the beginning, the curve passes through $\langle 1, 1, 0\rangle$ when $t = 0$.
Therefore, the wanted unit tangent is

$$\mathbf{T(0)} = \frac{\langle e^0, -sin(0), cos(0)\rangle}{\sqrt{e^{2\times 0}+1}} = \frac{\langle 1, 0, 1\rangle}{\sqrt{2}}.$$

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SOLVE
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