Financial Instruments and Pricing
Winter semester 2024/25

Set 6 & 7
(due date: February 9th 2025)

Option pricing

1. **<u>N-step Binomial model</u>**
   In the N=3 step Binomial model example discussed in Lecture 6, p. 32, with parameters: $S=1$, $u=1.2$, $d=1/u$, $R=0.1$:
   a. Derive the Binomial tree with the B, Δ and the option price for the <u>American Call</u> option C with the exercise price $X=1$
   b. Derive the Binomial tree with the B, Δ and the option price for the <u>European put</u> option p with the exercise price $X=1$
   c. Derive the Binomial tree with the B, Δ and the option price for the <u>American Put</u> option P with the exercise price $X=1$
   d. Check (numerically) if the "call-put parity" (see Lecture 5, p. 36):

   $$c-p = S - X\,(1+R)^{-N}$$
   $$S - X \;\leq\; C - P \;\leq\; S - X\,(1+R)^{-N}$$

   is satisfied.

2. **<u>N-step Binomial model. Analytical pricing of European options.</u>**
   Assume that one wants to price the European option, which **<u>can be exercised only at expiration (i.e. after N time steps</u>**). Let (in general) the payoff function be: $f(S(N))$ (e.g. for the call option: $f(S(N)) = \max(S(N)–X , 0)$). Using the analytic formula from Lecture 6, p. 44:

   $$f = \frac{1}{(1+R)^N}\sum_{j=0}^{N}\binom{N}{j} g^j (1-g)^{N-j} f\left(Su^j d^{N-j}\right)$$

   a. Price (numerically) **<u>the European binary option: b</u>**. Make calculations for: $S = 1$, $X = 1$, $u = 1.05$, $d = 1/u$, $R = 0.025$, $N = 12$.
   b. Price (numerically) **<u>the European call option: c</u>**. Make calculations for: $S = 1$, $X = 1$, $u = 1.05$, $d = 1/u$, $R = 0.025$, $N = 12$.
   c. Price (numerically) **<u>the European put option: p</u>**. Make calculations for: $S = 1$, $X = 1$, $u = 1.05$, $d = 1/u$, $R = 0.025$, $N = 12$.
   d. Check (numerically) how the above option prices depend on: S, X, R, u ($d=1/u$) – draw plots similar to those in Lecture 6, pp. 16-17 (i.e. assume that one parameter is changed, and the rest is kept constant at values from point a).

3. Using the Binominal model show that the current value of a Forward contract with the forward/exercise price X expiring in N steps (payoff function $F(N) = S(N) – X$) is equal to $F(0) = S(0) – X\,(1+R)^{-N} = S(0) – PV(X)$. Note that this actually means that when the forward/exercise price is set at $X = S(0)\,(1+R)^N = FV(S(0))$ then $F(0) = 0$ and there is no any initial payment, i.e. $CF(0) = 0$.

   Hint: Use the arguments of risk-neutral/martingale pricing.

4. We have shown (see Lecture 7, p. 61) that the **Binomial model converges to the B-S model** for: $\Delta t \to 0$, $N \to \infty$ ($N\,\Delta t = T$). For finite N & Δt and for the following Binomial model parameters:

$$(1+R) = e^{r\,T/N} \qquad u = e^{\sigma\sqrt{\frac{T}{N}}} \qquad d = 1/u$$

one gets an approximation of the B-S model.
   a. Compare value of a **European call** option from the Binomial model with that from the B-S model. Make comparison in function of a number of the Binomial model steps N = 10, 20, 30,…. , 100 (show the results and also plot them in the function of N). Make calculations for the option with parameters: S=1, X=1, T=1, σ=0.2, r=0.1.
   b. Repeat point (a) for a **European put** option with the same parameters.
   c. * Use the **Binomial model** do compute the value of American Call and American Put options with the same parameters as in point (a).

5. *Using (numerical) **Monte-Carlo method**:
   a. Price the **European call option** with parameters S=1, X=1, T=1, σ =0.2, r=0.1 (for 1k, 10k and 100k trajectory realizations). Compute the option price (mean value of the discounted ! payoff) and its statistical error (standard deviation of the mean). Compare the results with the B-S price calculated analytically.
   b. For the same parameters price the **European Binary call** option, whose payoff is $b(T) = \Theta(S(T)-X)$ and compare it with the price computed analytically (see Lecture 7, p. 67)
   c. For the same parameters price a **“strange sinusoidal” European option**, whose payoff function is $V(T) = \sin S(T)$ for $S(T) \le \pi$ and $V(T) = 0$ for $S(T) > \pi$
   d. Price the **Asian call option** (in payoff function of the call option the share price is replaced by its arithmetic average during life of the option: $V(T) = \max(\langle S(t)\rangle_t - X\,, 0)$ where $0 \le t \le T$)

NOTE: Remember to use the risk neutral probability measure !
For the European options in **points (a) - (c)** there is no need to generate the whole trajectory but one can generate just S(T) at the option expiration T.
For point (d) one must generate trajectories (to compute the average price $\langle S(t)\rangle_t$) with some time step Δt. **Use Δt = 1/250** (i.e. approximately daily prices)

*NOTE: Exercises 4c and (whole) 5 ARE NOT OBLIGATORY ! But THEY GIVE A PREMIUM DURING FINAL EXAM.

Jakub Gizbert-Studnicki