Risk Management - Problems 1
(due date: March 30, 2025)

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Summer semester (2024/25)
Jagiellonian University WFAIS.IF-Y491.0

Probability Theory

1. Calculate the characteristic function for the Gaussian distribution $X \sim N(\mu, \sigma)$. Then generalize the addition (stability) law for an arbitrary (let us say, n) number of independent Gaussian variables X_i with arbitrary mean μ_i and variance σ_i^2.

2. Calculate the characteristic function for the Cauchy distribution
$$P(x) = \frac{1}{\pi(1+x^2)}$$
Check if addition (stability) law holds.

3. Derive the formula for the cdf (cumulative distribution function): $P_{\leq}(x)$ of the Gaussian $N(\mu, \sigma)$ and the quantile (functional inverse of the cdf): $P_{\leq}^{-1}(x)$. Check numerically what percentage of values is within the range of $1, 2, ..., 10$ σ around the mean μ, respectively.

4. Suppose we have normal distribution $N(\mu, \sigma)$. Calculate the 0.9 quantile for $\mu = 2, \sigma = 0.3$ and the 0.15 quantile for $\mu = 100, \sigma = 6$.

5. Calculate kurtosis $\kappa = \lambda_4 = C_4/\sigma^4$
   (a) for the exponential distribution $P(x) = \lambda \exp(-\lambda x)\Theta(x)$, where $\Theta(x)$ is a step (Heaviside) function, and
   (b) for the continuous uniform distribution on interval $[-1/2; 1/2]$.

   Interpret the signs of the results.

6. Consider lognormal distribution
$$P_{LN}(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\ln^2(x/x_0)}{2\sigma^2}\right)$$
Calculate analytically the moments $m_n \equiv \langle x^n \rangle$. Write explicitly formulae for skewness and kurtosis.

7. Consider the family of distributions $P(x) = P_{LN}^*(x)[1 + \epsilon \sin(2\pi \ln x)]$, where $|\epsilon| \leq 1$ and $P_{LN}^*(x)$ is the lognormal distribution with $x_0 = \sigma = 1$. Show that the moments m_n of such distributions are identical to the moments of lognormal $P_{LN}^*(x)$, i.e. are independent on ϵ. Such case, where from the knowledge of all moments one cannot infer the corresponding distribution, is known as *indeterminate*.

8. Analyze how the maximum of N iid Gaussian variables $N(0,1)$ (with mean = 0 and variance = 1) converges to the Gumbel distribution.

   (a) Use the exact formula for the CDF of the maximum of N iid variables: $P_{\leq}^{max}(x) = (P_{\leq}(x))^N$ and draw the exact PDF of X_{max}: $p^{max}(x) = dP_{\leq}^{max}(x)/dx$ for $N = 10, 10^2, 10^3, 10^4$

   (b) On the same plot draw the PDF of the Gumbel distribution. Remember that the Gumbel PDF has to be appropriately rescaled with N, such that: $u = (X_{max} - a_N)/b_N$, where:

$$a_N = P_{\leq}^{-1}(1 - 1/N) \quad , \quad b_n = P_{\leq}^{-1}(1 - 1/(Ne)) - a_N \ ,$$

   and that in order to get the correct normalization the probability (not the PDF) is conserved, i.e. $p(u)du = p(x)dx$.

   NOTE: If you have problems with computing: $p^{max}(x) = dP_{\leq}^{max}(x)/dx$ you can instead plot $P_{\leq}^{max}(x)$ vs (rescaled) CDF of the Gumbel distribution.