Risk Management - Problems 5
(due date: June 15, 2025)

Jakub Gizbert-Studnicki
Maciej Trzetrzelewski
Maciej A. Nowak

Mark Kac Complex Systems Research Center
Jagiellonian University
Kraków, Poland
Summer semester (2024/25)
Jagiellonian University WFAIS.IF-Y491.0

Correlations and Random Matrices

1. Generate several (of order $M = 1000$) large matrices with dimension $N = 8$ and then 16 and 50, populated according to the GOE principle, i.e., first, fill all elements of matrix M from normal distribution $\mathcal{N}(\mu = 0, \sigma^2 = 2)$, add to such matrix its transpose, and divide the sum by the doubled dimension of the matrix, in brief $X = (M+M^\dagger)/(2\sqrt{N})$. Check if the spectrum of X fulfills the Wigner semicircle. What happens if you would neglect the factor N in denominator?

2. Calculate by the brute force the first 10 spectral moments of the above ensemble and compare to the analytic moments obtained from the expression for resolvent $G(z)$, treated as a series expanded around the infinity $z = \infty$. (*) Crosscheck the obtained first 5 non-zero moments with the Sloane encyclopedia of integer sequences, http://oeis.org

3. Check that the replacement of the Gaussian variables by any other iid distribution with finite variance does not change the result for Wigner's semicircle.

4. For sample set (e.g. for $N = 8$ or $N = 16$) of eigenvalues of the GOE ensemble, sort the eigenvalues for each matrix in the increasing order and find the difference between the neighbouring $\lambda_{n+1} - \lambda_n$, for $n \sim N/2$. Plot the histogram of these splittings divided by the mean splitting, with bean size small enough to see the fluctuations. Check the numerics with analytic formula given by the Wigner's surmise . (*) Why not use all the eigenvalues' splittings?

5. Check numerically the spectrum of the Wishart ensemble. Consider several large number of rows N and large number of columns T rectangular random matrices populated from $\mathcal{N}(0, 1)$ and check what happens to the spectrum of matrix $C = \frac{1}{T}XX^\dagger$ where † means transposition. Plot few cases when $r = N/T$ is smaller, equal ot larger than 1. Compare the numerics to Marchenko-Pastur formula.

6. * Fill two large (e.g. $N = 100$) matrices D_1, D_2 in such a way, that on the diagonal the elements are chosen from random binary distribution $\{-1, 1\}$ and all other elements are put to zero. Then rotate one of the matrices by the **random** orthogonal transformation (e.g. $OD_1O^\dagger$) and calculate numerically the spectrum of the ensemble $OD_1O^\dagger + D_2$.

Try to get an analytic result for such operation using the R-transform tricks.

7. The provided file (*SP500_406x1308.dat*) includes the daily data from Standard and Poor 500, in the form of the ratios of proces $x_{i,t}/x_{i,t-1}$. calculate the returns, then means and variances for each company during the observation period T, and finally construct the covariance matrix from the standardized data (i.e. for each company subtract the mean μ_i and divide by the standard deviation σ_i). Then calculate the spectrum of such matrix and compare results to Bouchaud et al seminal result, discussed during the lectures. Next, destroy all the correlations by multiple reshuffling of the columns in the dataset, and check, if in this case the spectrum agrees with the Marchenko-Pastur distribution.

   (*) Star means optional exercise (and more difficult...)