

Lecture 1

Probability calculus in a nutshell

Risk Management

Summer semester 2024/25

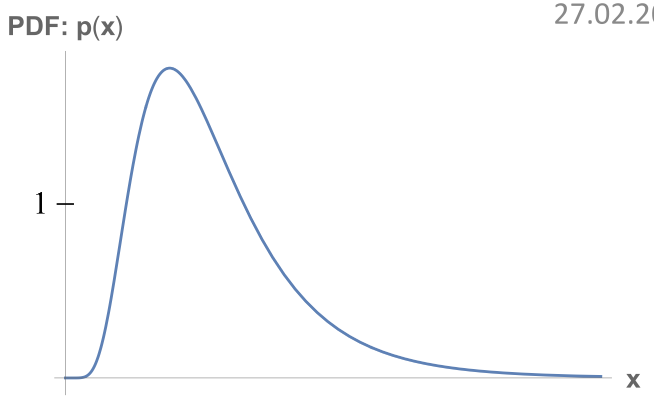
Probability calculus in a nutshell

- ❖Basics
- ❖Probability distributions
- ❖Central limit theorems
- ❖Extreme values

Basics: randomness

- Randomness
  - results from our ignorance (incomplete knowledge of reality, lack of information)
  - is related to complexity (many ”small agents” with non-linear interactions, small perturbations lead to large effects)
  - predictability is possible but on the basis of statistics (non-deterministic)
- Financial markets comply well with these ideas
  - incomplete knowledge about the economy/sector/company/behaviour of market participants/…
  - many small agents (investors) with different view of the market situation & behaviour
  - external “news” / “events” happen unexpectedly both in time & “nature”
- Prediction (generalization) is based on past market data (statistics)
  - but the “market” can change over time
  - predicting works as long as statistical properties don’t change much (~stationarity)
  - trade-off: larger data sets (better statistics) ⇔ ~stationarity (predictability)
  - in practice one uses “medium-term” data (usually from ~ 1-2 yrs) but it may not work well in case of rare unexpected events (“black-swans”), where in fact risk estimation is crucial ! ;-(

Basics: PDF



- ❖ We will focus on continuous random variables*, i.e. $X \in \mathbb{R}$
- ❖ Probability Density Function (PDF): $p(x)$
  - ❑ $p(x)dx$ is the probability of finding the random variable X in a small interval $[x\ ;\ x+dx]$
  - ❑ Note that $p(x)$ itself is a DENSITY ! $\Rightarrow p(x) \geq 0$, but it is NOT limited to [0,1] !, it has a unit $[X]^{-1}$
- ❖ PDF properties:
  - ❑ $\forall x\colon p(x) \geq 0$
  - ❑ $\int_{x_{min}}^{x_{max}} p(x)dx = 1$ (or in general: $\int_{-\infty}^{\infty} p(x)dx = 1$)
  - ❑ probability to find rand. var. X in the interval $[a\ ;\ b]$ is: $P(a < X < b) = \int_a^b p(x)dx$
  - ❑ $p(x)$ is invariant under: $x \rightarrow y(x)$ for any monotonic function $y(x)$ in a sense: $p(x)dx = p(y)dy$ (i.e., probability is conserved $\Rightarrow$ computing PDF $p(y)$ of the new rand. var. Y one must remember about jacobian of the transformation !)

*Prices of financial instruments change (almost, i.e., up to minimal „tics”) continuously

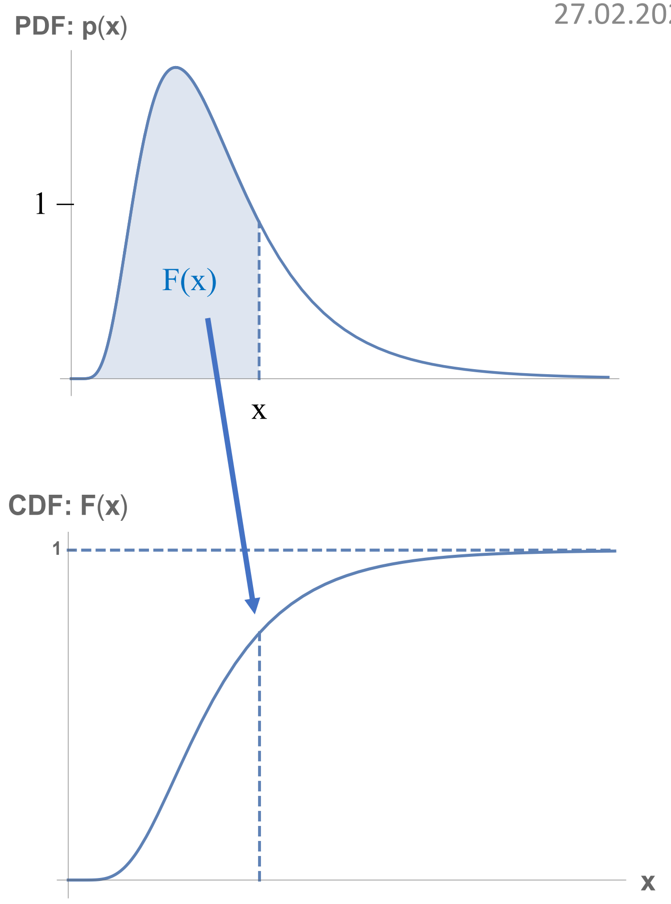
Basics: CDF

- Cumulative Distribution Function (CDF):

$$F(x) \equiv P_{\leq}(x) \equiv P(X \leq x) = \int_{-\infty}^{x} p(x')dx'$$

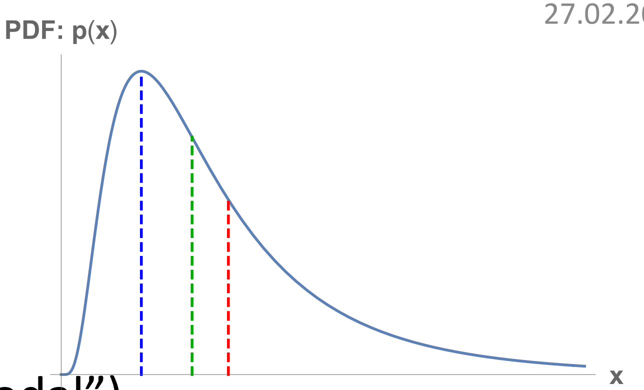
- CDF properties:
  - $\forall x: 0 \leq F(x) \leq 1$
  - $F(x)$ increases monotonically* with x
  - $F(-\infty) = 0$
  - $F(+\infty) = 1$
  - If $F(x)$ is differentiable $\Rightarrow$ $p(x) = \dfrac{d}{dx}F(x)$

- Sometimes one also defines: $P_{>}(x) \equiv 1 - P_{\leq}(x) \equiv 1 - F(x)$

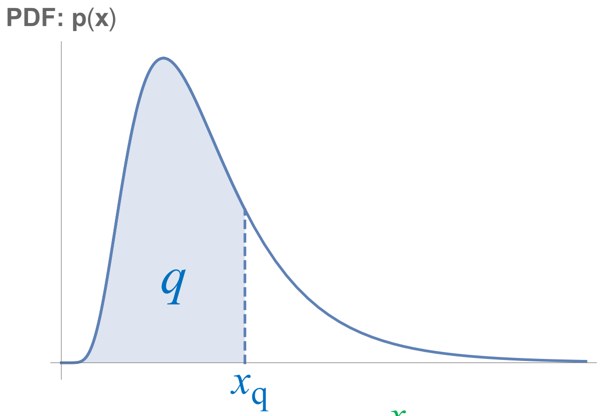


*In general: NOT strictly monotonic, as CDF can be piecewise constant, e.g. for discrete rand. var.

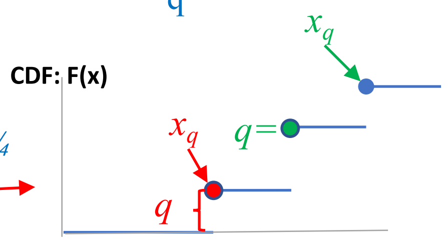
Basics: “typical” values



- **Mode** (most probable value): x_{max}
  - ❑ x_{max} is the maximum of the PDF $p(x)$ (need NOT be unique: “multimodal”)
- **Mean** (“expected” value): $E(X) \equiv \langle x \rangle \equiv \int_{-\infty}^{\infty} x p(x) dx$
  - ❑ exists only if PDF tails fall faster than $\sim x^{-2}$!
  - ❑ has ”good” properties (if exists): analytic, additive under convolution ($E(X_1+X_2)=E(X_1)+E(X_2)$), …
- **Median** (“middle” value): x_{med}
  - ❑ Definition: $P_<(x_{med}) = P_>(x_{med}) = ½$
  - ❑ if CDF $F(x)$ is strictly monotonic*, and thus invertible, then: $x_{med} = F^{-1}(½)$

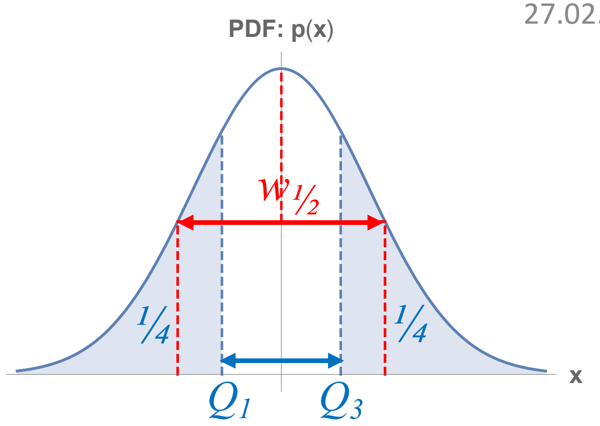


- **Quantile**: x_q
  - ❑ Definition: $P_<(x_q) = q$ and $P_>(x_q) = 1-q$
  - ❑ this is a generalization of the median ($x_{med} = x_{½}$)
  - ❑ if CDF $F(x)$ is strictly monotonic*, and thus invertible, then: $x_q = F^{-1}(q)$
  - ❑ “important” quantiles: (so called) quartiles: $Q_1 \equiv x_{¼}$, $Q_2 \equiv x_{½} \equiv x_{med}$, $Q_3 \equiv x_{¾}$



*If not, then one has to clarify the definition (conventions), e.g. $x_q=\inf\{x: F(x) \geq q\}$

Basics: "typical" deviations



- RMS (root mean square, standard deviation): $\boxed{\sigma \equiv \sqrt{\sigma^2}}$

Variance → $\boxed{\sigma^2 \equiv E\left((X - E(X))^2\right)} \equiv \int_{-\infty}^{+\infty} (x - \langle x \rangle)^2 \, p(x) dx$ empirical outliers get magnified !

  - exists only if PDF tails fall faster than $\sim x^{-3}$!
  - variance has "good" properties (if exists): analytic, additive under convolution, …

- MAE (mean absolute error, average absolute deviation): $\boxed{E_{abs}}$

$$\boxed{E_{abs} \equiv E\left(|X - m|\right)} \equiv \int_{-\infty}^{+\infty} |x - m| \, p(x) dx$$

not so prone to empirical outliers!

$m = m_1 \equiv E(x)$ or other "typical" value (e.g. $m = x_{med}$)

  - exists only if PDF tails fall faster than $\sim x^{-2}$!

- MAD (median absolute deviation): $MAD = \mathrm{Median}(|x - x_{med}|)$
- IQR (interquartile range): $IQR \equiv Q_3 - Q_1$
- FWHM (full width at half maximum): $w_{1/2}$ ⇒ def: $p\left(x_{max} \pm \frac{w_{1/2}}{2}\right) = \frac{p(x_{max})}{2}$
  - "good" only for symmetric PDFs

Basics: moments & characteristic function

- **Moment***: $\boxed{m_n \equiv E\left(X^n\right)} = \int_{-\infty}^{+\infty} x^n\, p(x)dx$ empirical outliers get VERY magnified ! (estimating from data becomes problematic for large n)
  - Exists only if PDF tails fall faster than $\sim x^{-(n+1)}$ ⇒ only up to some n for "heavy tail" PDFs !
  - Examples: $m_1 \equiv E(X)$ (mean), $m_2 = \sigma^2 + m_1^2$
  - In most cases (NOT always ! – see Exercises 1): knowledge of moments ⇔ knowledge of PDF
- **Characteristic function** (CF): $\boxed{\hat{p}(t) \equiv E(e^{itX}) = \int_{-\infty}^{+\infty} e^{itx} p(x)dx}$ ⟵ Fourier transform !
  - PDF normalization: $\hat{p}(0) \equiv \int_{-\infty}^{+\infty} p(x)dx = 1$
  - alternative way to define PDF: define CF and invert it (e.g. Levy α-stable distributions)

$$\boxed{p(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-itx}\hat{p}(t)dt}$$ ⟵ Inverse Fourier transform !

  - but one must be careful to get real and positive PDF (doesn't work for arbitrary function $\hat{p}(t)$!)

*Sometimes one also defines CENTRAL moments: $\mu_n \equiv E((X - m_1)^n)$ e.g., $\mu_1=0$, $\mu_2=\sigma^2$

Basics: moments & characteristic function

$$m_n \equiv E\left(X^n\right) = \int_{-\infty}^{+\infty} x^n \, p(x) dx$$

$$\hat{p}(t) \equiv E(e^{itX}) = \int_{-\infty}^{+\infty} e^{itx} p(x) dx$$

- CF is a "moment generating*" function
  - smart way to compute all moments "at once":

$$\boxed{m_n = (-i)^n \frac{d^n}{dt^n} \hat{p}(t) \Big|_{t=0}}$$

$$\hat{p}(t) = \int_{-\infty}^{+\infty} e^{itx} p(x) dx = \int_{-\infty}^{+\infty} \sum_n \frac{(itx)^n}{n!} p(x) dx = \sum_n \frac{(it)^n}{n!} \int_{-\infty}^{+\infty} x^n p(x) dx = \sum_n \frac{(it)^n}{n!} m_n$$

  - CF of a CONVOLUTION (sum of indep. rand. var. $X_1 \sim p_1(x_1)$ & $X_2 \sim p_2(x_2)$ $\Rightarrow$ $S = X_1 + X_2 \sim p_{1+2}(s)$)

Important for stable distributions (see later) !

$$\boxed{\hat{p}_{1+2}(t) = \hat{p}_1(t) \cdot \hat{p}_2(t)}$$

convolution

$$p_{1+2}(s) = \iint_{-\infty}^{+\infty} dx_1 dx_2 \; p_1(x_1) p_2(x_2) \delta(s - x_1 - x_2) = \int_{-\infty}^{+\infty} p_1(x) p_2(s - x) dx$$

$$\hat{p}_{1+2}(t) = \int_{-\infty}^{+\infty} ds \; e^{its} p(s) ds = \int_{-\infty}^{+\infty} ds \iint_{-\infty}^{+\infty} dx_1 dx_2 \; p_1(x_1) p_2(x_2) \delta(s - x_1 - x_2) e^{its} =$$

$$= \iint_{-\infty}^{+\infty} dx_1 dx_2 \; p_1(x_1) p_2(x_2) e^{it(x_1 + x_2)} = \int_{-\infty}^{+\infty} dx_1 p_1(x_1) e^{itx_1} \cdot \int_{-\infty}^{+\infty} dx_2 p_2(x_2) e^{itx_2} = \hat{p}_1(t) \cdot \hat{p}_2(t)$$

*Formally one distinguishes between CF (Fourier transform of PDF) and MGF (Laplace transform of PDF)

Basics: cumulants

$$\hat{p}(t) \equiv E(e^{itX}) = \int_{-\infty}^{+\infty} e^{itx} p(x) dx$$

$$\hat{p}_{1+2}(t) = \hat{p}_1(t) \cdot \hat{p}_2(t)$$

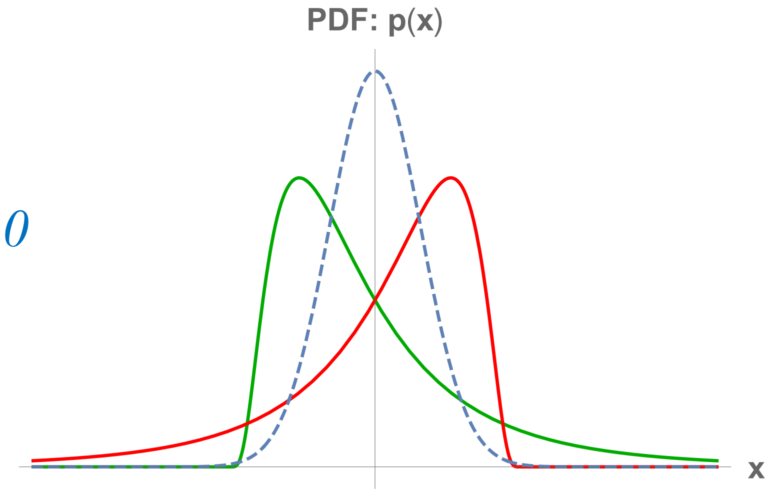
- Cumulant: C_n
  - Cumulants C_n are polynomials of moments m_k ($k=1, \ldots, n$), e.g.
    - $C_1 = m_1$
    - $C_2 = \sigma^2 = m_2 - m_1^2$
    - $C_3 = m_3 - 3m_2m_1 + 2m_1^3$
    - $C_4 = m_4 - 4m_3m_1 - 3m_2^2 + 12m_2m_1^2 - 6m_1^4$
    - …
  - They are generated by the log of CF: $C_n = (-i)^n \frac{d^n}{dt^n} \ln \hat{p}(t) \Big|_{t=0}$
  - Cumulants are ADDITIVE UNDER CONVOLUTION (i.e. sum of indep. rand. var.)

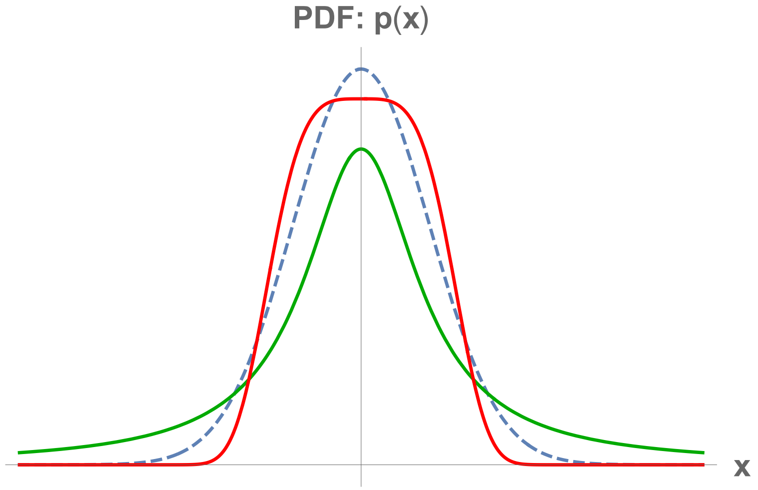
$$\ln \hat{p}_{1+2}(t) = \ln \hat{p}_1(t) + \ln \hat{p}_2(t) \quad \Rightarrow \quad \forall_n : C_n^{(1+2)} = C_n^{(1)} + C_n^{(2)}$$

  - Note: for a Gaussian distribution: $\forall n > 2: C_n = 0$

Basics: cumulants

- "Normalized" cumulants: $\boxed{\lambda_n \equiv \frac{C_n}{\sigma^n}}$
  - For a Gaussian distribution: $\forall n > 2: \lambda_n = 0$
  - Skewness: $\varsigma \equiv \lambda_3 = \frac{\langle (x - \langle x \rangle)^3 \rangle}{\sigma^3}$
    - λ=0 (symmetric)
    - λ>0 (positive skewness)
    - λ<0 (negative skewness)
  - (Excess) Kurtosis: $\kappa \equiv \lambda_4 = \frac{\langle (x - \langle x \rangle)^4 \rangle}{\sigma^4} - 3$
    - κ = 0 ("mesokurtic")
    - κ > 0 (" leptokurtic", "fat tail")
    - κ < 0 (" platykurtic", "thin tail")





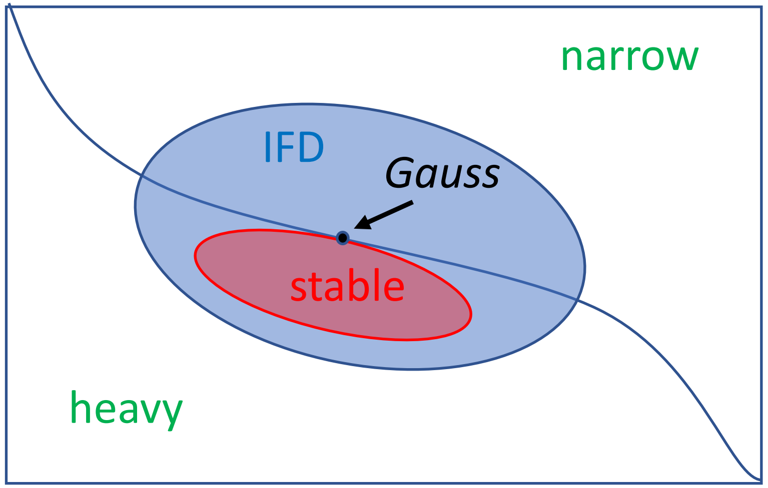
Probability calculus in a nutshell

- ❖Basics
- ❖Probability distributions
- ❖Central limit theorems
- ❖Extreme values

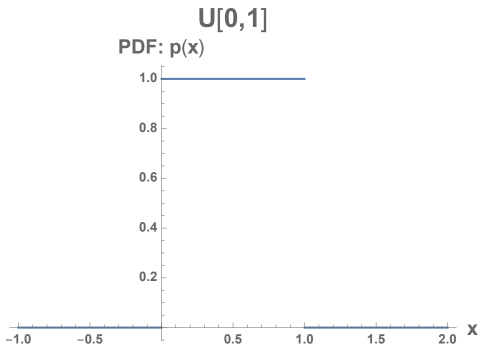
Distributions: general remarks

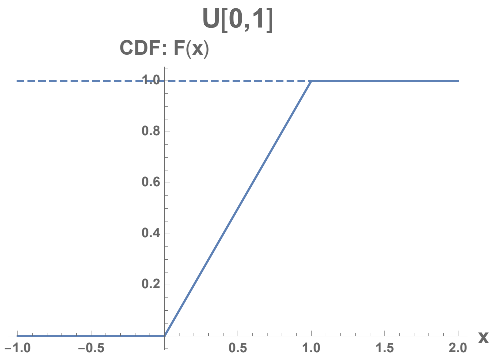
- Here we again focus on continuous random variables
- Important subclasses of distributions
  - All moments finite ("narrow"/exponential tails) vs. some or all moments ∞ ("heavy"/power tails)
  - "Stable" r.v.: functional form of distribution doesn't change under convolution (sum of indep. r. v. coming from a stable distribution has the same distribution up to a rescaling / shift of parameters)
  - "Infinitely divisible" r.v.: $\forall n \in \mathbb{N}: X = X_1 + \ldots + X_n$, where independent & identically distributed ("iid") rand. vars. $X_1, \ldots, X_n$ have some distribution (not necessarily the same PDF as X)
- Important distributions:
  - Uniform
  - Gaussian (Normal)
  - Log-Normal
  - (Levy) α-stable
  - (Distributions of extremes: Gumbel / Frechet / Weibull: see later!)
  - Examples of other distributions used in finance
  - …

ALL DISTRIBUTIONS



Distributions: Uniform





- ❖ **Uniform:** $U(a,b)$
  - ❑ Support: [a,b]
  - ❑ 2 parameters: a , $b \in \mathbb{R}$ $(a<b)$
  - ❑ PDF and CDF:

$$p(x) = \begin{cases} \frac{1}{b-a} & , x \in [a,b] \\ 0 & , \text{otherwise} \end{cases} \qquad F(x) = \begin{cases} 0 & , \quad x < a \\ \frac{x-a}{b-a} & , \quad x \in [a,b] \\ 1 & , \quad x > b \end{cases}$$

  - ❑ Typical values: $E(X) = x_{med} = \dfrac{a+b}{2}$
  - ❑ Typical deviations: $\sigma = \sqrt{\dfrac{1}{12}}(b-a)$ $\qquad E_{abs} = \dfrac{1}{4}(b-a)$
  - ❑ Skewness and kurtosis: $\lambda_3 = 0$ (symmetric) , $\lambda_4 = -6/5$ ("platykurtic", i.e. "thin tail": in fact no tail !)
  - ❑ CF: $$\hat{p}(t) = \begin{cases} \frac{\exp(itb)-\exp(ita)}{it(b-a)} & , \text{ for } t \neq 0 \\ 1 & , \text{ for } t = 0 \end{cases}$$

Distributions: Uniform

- "Standard" uniform distribution ($U(0,1)$) can be obtained from any other prob. distr. of X with a strictly monotonic CDF (i.e. F_X^{-1} exists), by a simple change of variables (so called: "probability integral transform")

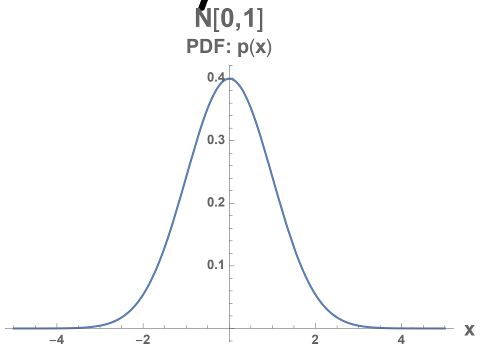
$$\boxed{X \sim p(x) \to Y = F_X(X) \sim \mathrm{U}[0,1]}$$

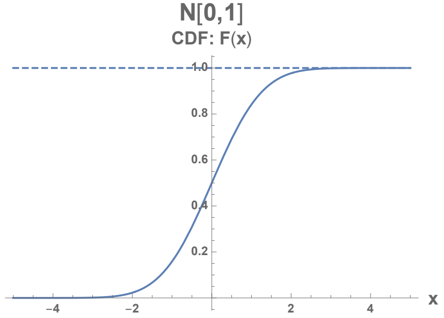
  - Proof: $Y \in [0,1]$ and: $F_Y(y) = P(Y \leq y) = P(F_X(X) \leq y) = P(X \leq F_X^{-1}(y)) = F_X(F_X^{-1}(y)) = y$

    CDF of $U(0,1)$

- This is useful:
  - Basis for GENERATING rand. var. from arbitrary distribution (if one can easily compute F_X^{-1})
    - Generate Y ~ Uniform[0,1]
    - X = $F_X^{-1}(Y)$ ~ requested distribution
  - TESTING distributions based on random samples
    - P-P plot (see Lecture 3)
    - Kolmogorov-Smirnov test (see Lecture 3)
  - Generating or testing correlated / coupled MULTIVARIATE rand. vars. ⇒ COPULAS (see Lecture 2)

Distributions: Gaussian / Normal





- ❖ Gaussian / Normal: $N(m,s)$
  - ❑ Support: $\mathbb{R}$
  - ❑ 2 parameters: $m \in \mathbb{R}$, $s > 0$
    (“standard”: m=0, s=1)
  - ❑ PDF and CDF:

$$\boxed{p(x) = \frac{1}{\sqrt{2\pi s^2}} \exp\left(-\frac{(x-m)^2}{2s^2}\right)} \qquad F(x) = \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{x-m}{\sqrt{2}s}\right)\right)$$

“error f-ction”

  - ❑ Typical values (symmetric): $E(X) = x_{max} = x_{med} = m$
  - ❑ Typical deviations: $\sigma = s \qquad E_{abs} = \sqrt{\frac{2}{\pi}}\, s$
  - ❑ Skewness, kurtosis (and all higher (“normalized”) cumulants): $\lambda_3 = \lambda_4 = 0$, ($\forall\ n > 2$: λ_n =0)
    (basis for comparison of other distributions to “normal”)

Distributions: Gaussian / Normal

$$p(x) = \frac{1}{\sqrt{2\pi s^2}} \exp\left(-\frac{(x-m)^2}{2s^2}\right)$$

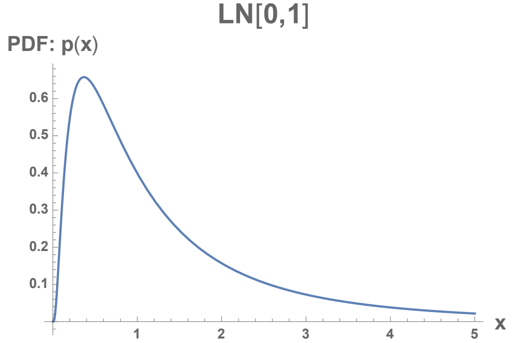
- **Gaussian / Normal: $N(m,s)$**
  - CF (also "Gaussian"): $\boxed{\hat{p}(t) = \exp\left(imt - \frac{s^2 t^2}{2}\right)}$ ⇒ $\ln \hat{p}(t) = imt - \frac{s^2 t^2}{2}$
  - Moments: $m_n = \begin{cases} 0 & , \; n - \text{odd} \\ (2n-1)(2n-3)\ldots s^{2n} & , \; n - \text{even} \end{cases}$
  - The only non-vanishing Cumulants: $C_1 = m_1 = m$ & $C_2 = s^2$
  - Quantiles: $x_q \equiv F^{-1}(q) = m + \sqrt{2}\, \sigma\, \mathrm{erf}^{-1}(2q-1)$

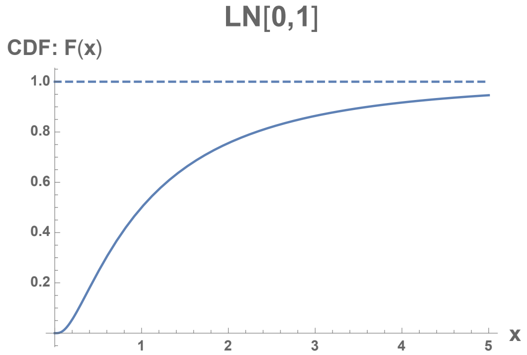
  - Gaussian distribution is called "Normal" as it is often observed in "Nature", BUT e.g. in financial data it is NOT the case (Gaus. has very low prob. of "extreme" events !, much lower than observed)

$P(X > c)$:

c	Standard Gauss. N[0,1]	Standard Cauchy	Standard Levy
1	$\simeq 0.16$	$\simeq 0.25$	$\simeq 0.62$
2	$\simeq 0.02$	$\simeq 0.15$	$\simeq 0.52$
3	$\simeq 0.0013$	$\simeq 0.10$	$\simeq 0.44$
4	$\simeq 3 \cdot 10^{-5}$	$\simeq 0.08$	$\simeq 0.38$
5	$\simeq 3 \cdot 10^{-7}$	$\simeq 0.06$	$\simeq 0.35$

Distributions: Log-Normal



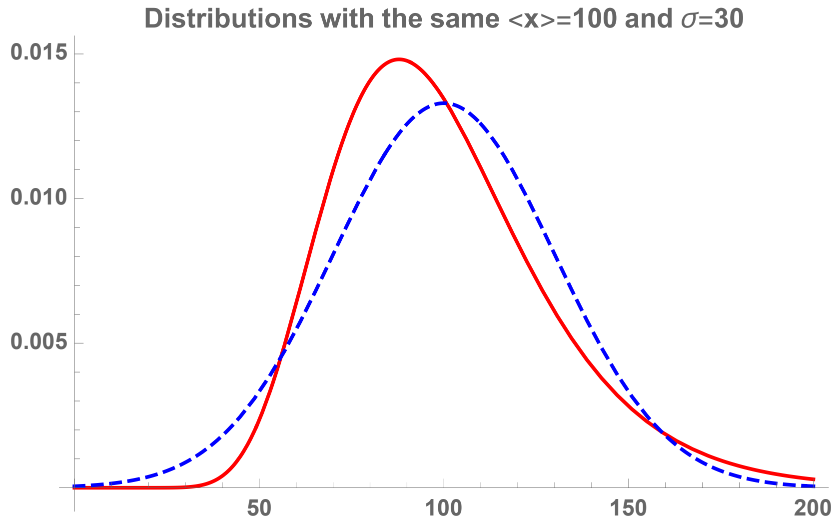


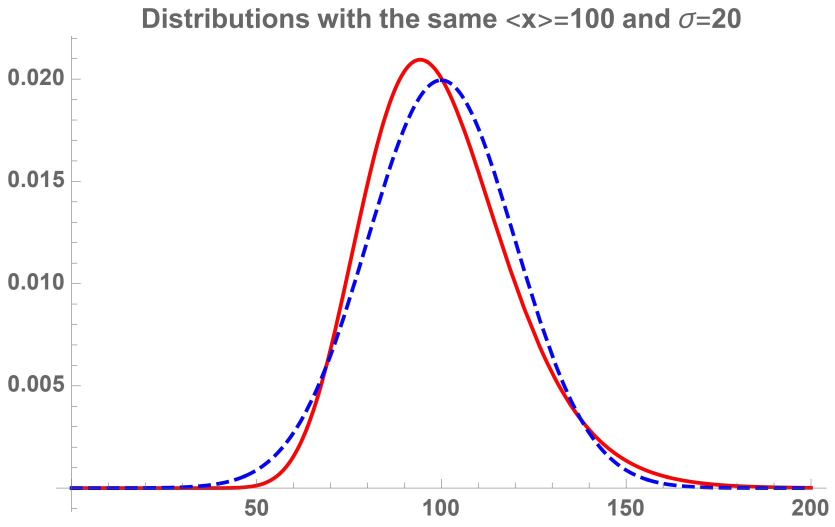
- Log-Normal: $LN(m,s)$
  - Support: $x > 0$!!!
  - 2 parameters: $m \in \mathbb{R}$, $s > 0$
    ("standard": m=0, s=1)
  - PDF and CDF: $$\boxed{p(x) = \frac{1}{\sqrt{2\pi s^2}\, x} \exp\left(-\frac{(\ln x - m)^2}{2s^2}\right)} \quad F(x) = \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{\ln x - m}{\sqrt{2}s}\right)\right)$$
  - if: $X \sim LN(m,s) \;\Rightarrow\; Y = \ln X \sim N(m,s)$ (Gaussian)
  - Typical values (assymetric !): $x_{max} = e^{m-s^2} \quad < \quad x_{med} = e^m \quad < \quad E(x) = e^{m+s^2/2}$
  - Typical deviations: $\sigma = \sqrt{e^{2m+s^2}\left(e^{s^2} - 1\right)}$
  - Skewness and kurtosis: $\lambda_3 > 0$ (positive skewness) & $\lambda_4 > 0$ (leptokurtic: tails "fatter" than Gauss.)
  - Moments: $m_n = \exp\left(m \cdot n + \frac{s^2}{2} n^2\right)$ This is a "patological" example of "indeterminate" distr.: one cannot determine PDF based on knowledge of all moments (see Exercises 1) !

Distributions: Log-Normal

$$p(x) = \frac{1}{\sqrt{2\pi s^2}\, x} \exp\left(-\frac{(\ln x - m)^2}{2s^2}\right)$$

- Log-Normal distribution is "popular" in financial modelling:
  - E.g.: Black-Scholes option pricing model
    or modelling currency exchange rates (virtue: $\ln$ €/\$ = $-\ln$ \$/ €, so probability is symmetric)
  - Good: no negative prices ($x > 0$!) ⇒ but in practice it usually doesn't matter much if we use Gaussian or Log-Normal: e.g. for the share price S=100\$ and yearly volatility 30% (i.e. we assume $E(S)=100$ and $\sigma=30$) one has $P_G(S<0) \simeq 4\cdot 10^{-4}$ which is negligible (of course $P_{LN}(S<0) = 0$).
  - The difference is more on a positive side ⇒ in the above example $P_G(S>200) \simeq 4\cdot 10^{-4}$ and $P_{LN}(S>200) \simeq 6\cdot 10^{-3}$, i.e. one order of magnitude bigger for a Log-Normal than for a Gaussian



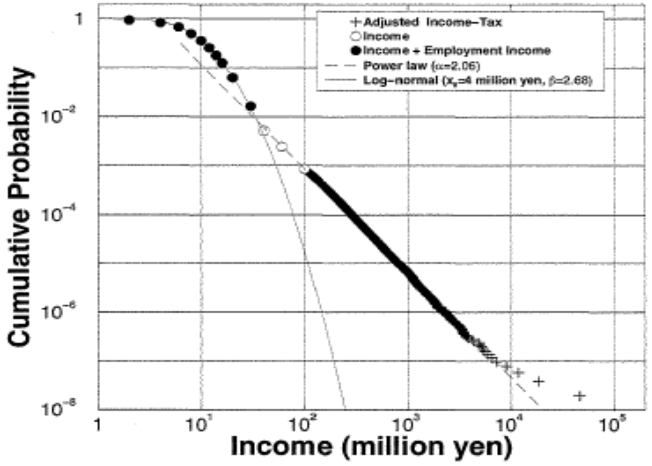


Distributions: “heavy tails”

❖Typical data observed on real markets are different ⇒ in finance large positive & negative jumps are much more frequent than in Gaussian or Log-Normal (“heavy tails”)

❖ Empirical distributions often behave as:

$$\boxed{p(x) \sim \frac{\alpha A^{\alpha}}{|x|^{\alpha+1}} \quad \text{for } |x| \gg x_0}$$



- ❑ So called “Pareto” tails (fall-off as power law)
- ❑ Moments of order $n \geq \alpha$ are divergent ! (for $\alpha \leq 1$: no moments !)
- ❑ Such distributions are “self-similar” (“scale-free”) for large $x \gg x_0$:

$$\boxed{\frac{p(\lambda x)}{p(x)} = \lambda^{-(\alpha+1)}}$$

  - ❑ E.g. for λ=10 and α=2 ($\lambda^{-(\alpha+1)}$=10^{-3}) one has: the number of people 10x richer is 1000x smaller (independent of initial value)
  - ❑ Rate of growth of the economy or a company is independent of its size
  - ❑ Size of cities
  - ❑ Size of people’s wealth, income, …

❖This leads to (a family of) the, so-called, (Levy) α-stable distributions

Distributions: (Levy) α-stable

Note: $\forall\alpha$: σ =„∞" ; for $\forall\alpha > 1$: E(x) < ∞

- (Levy) α-stable: $L_\alpha(\beta,c,m)$

$$p(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-itx}\hat{p}(t)dt \sim \frac{\alpha A_\pm^\alpha}{|x|^{\alpha+1}} \quad \text{for } x \to \pm\infty$$

  - Support: $\mathbb{R}$
  - 4 Parameters:

$$P_>(x) \sim \left(\frac{A_+}{x}\right)^\alpha \quad \text{for } x \to +\infty$$

    - $\alpha \in (0,2)$
    - $A_\pm > 0$ ("tail amplitudes") or alternatively:
      $c_\alpha > 0$ ("scale par.") & $\beta \in (-1,1)$ ("skewness par.")
    - m ∈ $\mathbb{R}$ ("location")

$$\beta = \frac{A_+^\alpha - A_-^\alpha}{A_+^\alpha + A_-^\alpha}$$

In the simplest case: β = m = 0
(symmetric, centered at zero)

$$A^\alpha = (c_\alpha)^\alpha \frac{\sin\frac{\pi\alpha}{2}}{\pi}\Gamma(\alpha)$$

  - PDF: in most cases NOT given by closed-end formula (in principle can be computed from CF)
  - CF is simple (for $\alpha \neq 1$):

$$\hat{p}(t) = \exp\left(itm - |c_\alpha t|^\alpha\left(1 - i\beta\tan\left(\frac{\pi\alpha}{2}\right)\text{sign}(t)\right)\right)$$

NOT analytic at t=0 !

$$\hat{p}(t) = \exp(-|c_\alpha\, t|^\alpha)$$

  - This distributions are manifestly stable (for given fixed α) as convolution (sum of indep. ran. vars.) leads to a simple rescaling of parameters

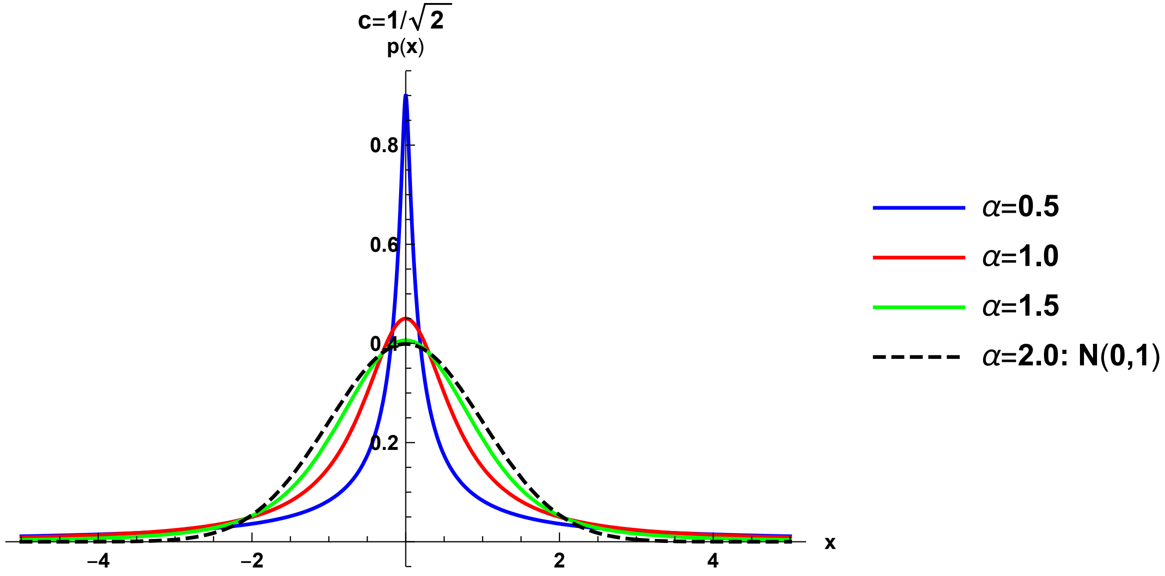
$$\hat{p}_{1+2}(t) = \hat{p}_1(t)\cdot\hat{p}_2(t) \Rightarrow$$

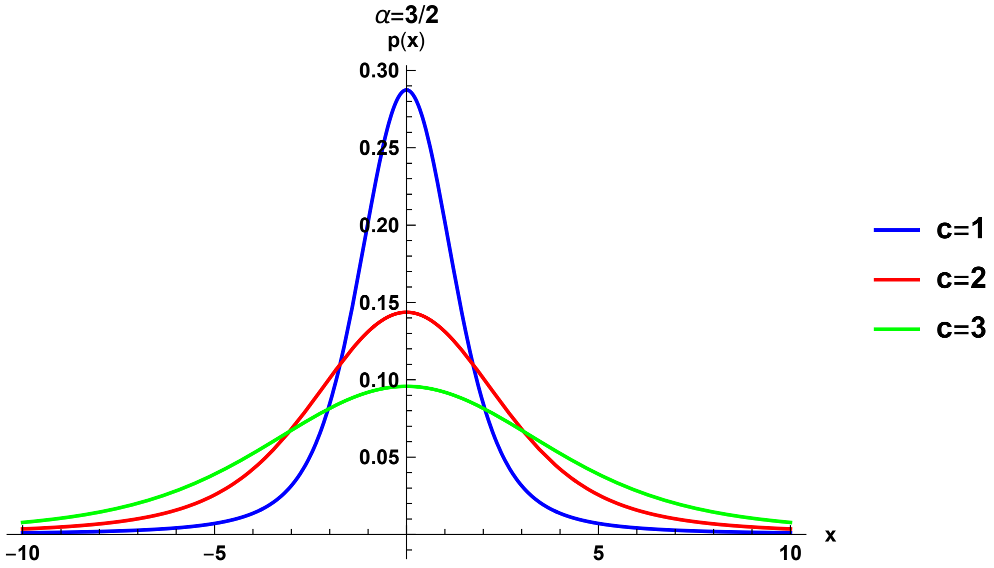
$$m_{1+2}=m_1+m_2$$

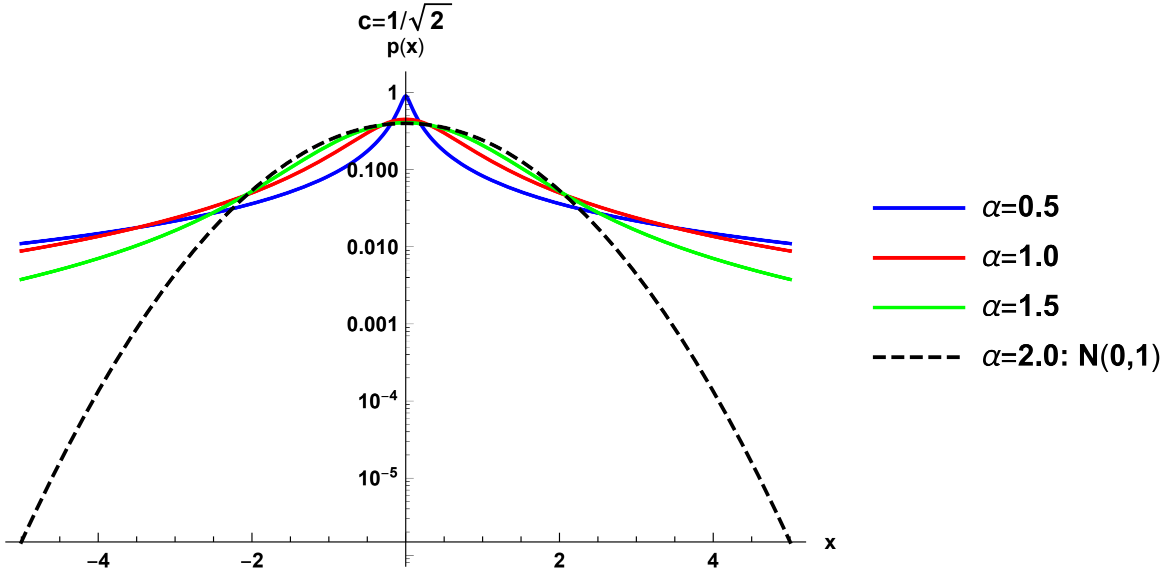
$$|c_{1+2}| = (|c_1|^\alpha + |c_2|^\alpha)^{1/\alpha}$$

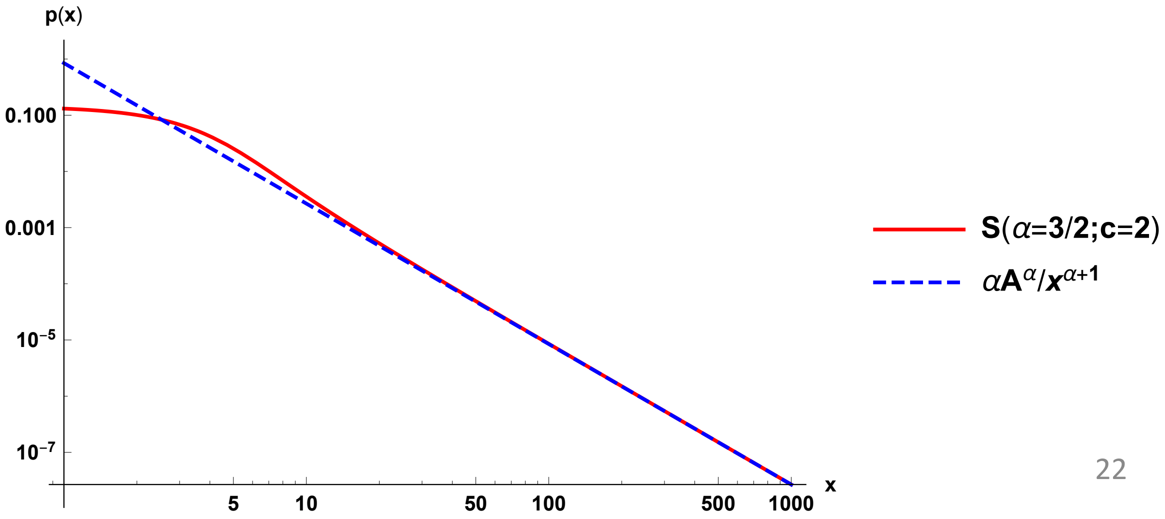
$$\beta_{1+2} = \frac{\beta_1|c_1|^\alpha + \beta_2|c_2|^\alpha}{|c_1|^\alpha + |c_2|^\alpha}$$

(Levy) α-stable distributions





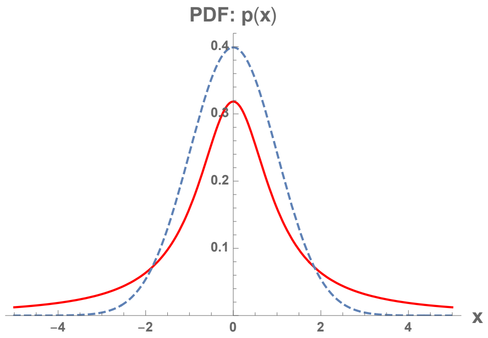


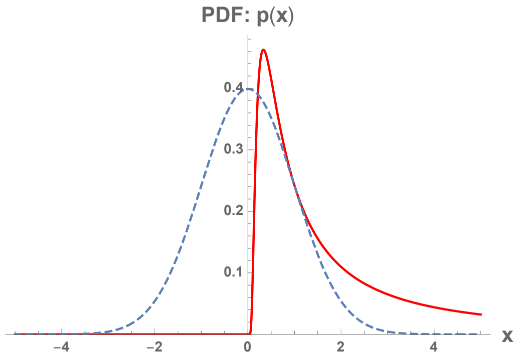


Distributions: (Levy) α-stable

$$p(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-itx}\hat{p}(t)dt$$

- (Levy) α-stable: $L_\alpha(\beta, c, m)$ - special cases
  - (Limiting case) for $\alpha=2$: Gaussian $N\left[m, \sqrt{2}c\right]$: $\hat{p}(t) = \exp\left(itm - c^2t^2\right)$
  - For $\alpha=1$ & $\beta=0$: "Cauchy" distribution: $$\hat{p}(t) = \exp\left(itm - c|t|\right)$$ $$p(x) = \frac{1}{\pi c\left(1 + \left(\frac{x-m}{c}\right)^2\right)}$$
  - For $\alpha=$ ½: "Levy" distribution: (here one assumes: $x \geq m$!) $$\hat{p}(t) = \exp\left(itm - \sqrt{-2ict}\right)$$ $$p(x) = \sqrt{\frac{c}{2\pi}}\frac{\exp\left(-\frac{c}{2(x-m)}\right)}{(x-m)^2}$$





- Stable distributions are important as they are attractors for a SUM: $S_N = X_1 + \ldots + X_N$ of N iid rand. vars. when $N\to\infty$ (see: CLTs)

Distributions: Other

- Other distributions used in financial modelling (see *handwritten notes* + *Bouchaud's book* + *wiki*)
  - Distributions of extremes (see later)
    - Gumbel
    - Frechet
    - Weibull
  - Poisson (discrete !)
  - Truncated Levy
  - Exponential
  - Exponential Power (Generalized Normal)
  - Pareto (Log-exponential)
  - Hyperbolic
  - Student's t
  - Inverse Gamma
  - …

Probability calculus in a nutshell

- ❖Basics
- ❖Probability distributions
- ❖Central limit theorems
- ❖Extreme values

CLTs

- We are interested in the probability distribution of the SUM of N independent & identically distributed (iid) random variables for $N \longrightarrow \infty$

$$S_N = X_1 + \ldots + X_N \quad \text{where: } \; X_i \sim iid$$

- This is a common situation in „Nature" & in finance ⇒ many "small" factors „add" into large effects

- Of course if X_i ~ *stable distribution* (i.e. a Gaussian: $N(m,s)$ or (Levy) α-stable: $L_\alpha(\beta,c,m)$) then (by definition) $\forall N$: S_N ~ <u>*the same (type of) stable distribution*</u> (but with <u>rescaled/shifted parameters</u>)
  - E.g. for $Xi \sim N(m,s)$ one has $\forall N$:

„Standardization" of a random variable

$$u_N = \frac{S_N - \overbrace{\boxed{Nm}}^{a_N}}{\underbrace{\boxed{s\sqrt{N}}}_{b_N}} \sim N(0,1)$$

In general: a_N & b_N are some (N-dependent) constants

- In general S_N (for a non-stable X_i) will have a different (type of) distribution than X_i

- But for $N \longrightarrow \infty$: S_N will converge to a stable distribution

CLTs: Gaussian CLT

$$\boxed{S_N = \sum_{i=1}^{N} X_i \ , \ X_i \sim iid}$$

❖ If X_i has two first moments finite (i.e. mean: $\langle x \rangle$ and variance: σ^2 exist) then for $N \to \infty$

$$u_N = \frac{S_N - \overbrace{\boxed{N\langle x\rangle}}^{a_N}}{\underbrace{\boxed{\sigma\sqrt{N}}}_{b_N}} \xrightarrow[N\to\infty]{} N(0,1)$$

i.e. S_N converges to a Gaussian distribution with mean: $a_N = N\langle x \rangle$ and variance: $b_N^{\,2} = N\sigma^2$

❖ More formally the convergence means that for any finite interval $[u1, u2]$:

$$\lim_{N\to\infty} P(u_1 \le u_N \le u_2) = \int_{u_1}^{u_2} du \boxed{\frac{1}{2\pi} e^{-u^2/2}} \longrightarrow \text{„Standard” Gaussian PDF}$$

CLTs: Gaussian CLT

$$\boxed{S_N = \sum_{i=1}^{N} X_i \;\; , \;\; X_i \sim iid}$$

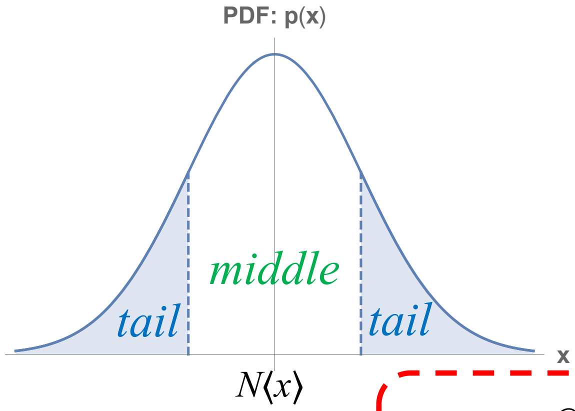
- Note that only two first moments must be finite (higher moments may not exist)

$$\boxed{u_N = \frac{S_N - N\langle x \rangle}{\sigma\sqrt{N}} \xrightarrow[N\to\infty]{} N(0,1)}$$

- But it is quite easy to "prove" CLT in the case of a "narrow" distribution (where all moments exist):
  - For all Cumulants of a convolution one has: $C_n^{(1+..+N)} = C_n^{(1)} + \ldots + C_n^{(N)}$
  - So for $X_i \sim$ iid: $C_n^{(S_N)} = N C_n^{(X_i)}$
  - And normalized cumulants: $\lambda_n^{(S_N)} \equiv \dfrac{C_N^{(S_N)}}{\left(\sigma^{(S_N)}\right)^n} = \dfrac{N C_n^{(X_i)}}{\left(\sqrt{N}\sigma\right)^n} = \dfrac{C_n^{(X_i)}}{\sigma^n} \cdot N^{1-n/2} = \lambda_n^{(X_i)} \boxed{N^{1-n/2}}$
  - As all normalized cumulants of order $n > 2$ vanish for $N\to\infty$ (only λ_1 & λ_2 survive) ⇒ one gets a Gaussian
- Gaussian CLT can be generalized:
  - X_i may be NOT independent, i.e. "slightly" correlated (correlator C_{ij} must fall fast enough with $|i\text{-}j|$
  - X_i may be NOT identical, but no one variance can "dominate"

ADDITIONAL CONDITIONS !

CLTs: Large deviations



$$S_N = \sum_{i=1}^{N} X_i \;\;,\;\; X_i \sim iid$$

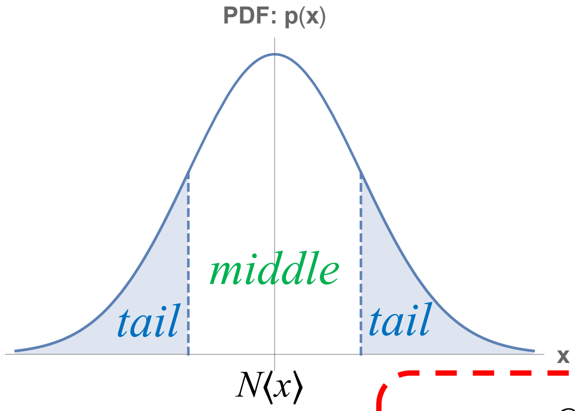
$$u_N = \frac{S_N - N\langle x\rangle}{\sigma\sqrt{N}} \xrightarrow[N\to\infty]{} N(0,1)$$

- ❖ Formally Gaussian CLT works for $N \to \infty$
- ❖ Question: how large N should be "in practice" and what is the "region" of CLT applicability ?
  - ❑ The problem is that the convergence: is NOT „uniform"

$$\lim_{N\to\infty} P(u_1 \le u_N \le u_2) = \int_{u_1}^{u_2} du\, \frac{1}{2\pi} e^{-u^2/2}$$

  - ❑ It is different in the "tails" where (for finite N) S_N can be very different from the Gaussian: tails of S_N "remember" the tails of original X_i distributions (important when these are "fat" tails !)
  - ❑ And different in the "middle", and what is the "middle" depends on the tails of X_i
    - ❑ If X_i has a "narrow" distribution (all moments exist): "middle" range $\propto N^{2/3}\,\sigma$ or $\propto N^{3/4}\,\sigma$
    - ❑ If X_i has "heavy" (power-like) tails: "middle" range $\propto \sqrt{N \ln N}\,\sigma$ (it grows very slowly with N !)

CLTs: Large deviations



$$S_N = \sum_{i=1}^{N} X_i \;\;,\;\; X_i \sim iid$$

$$u_N = \frac{S_N - N\langle x\rangle}{\sigma\sqrt{N}} \xrightarrow[N\to\infty]{} N(0,1)$$

- Formally Gaussian CLT works for $N\longrightarrow\infty$
- Question: how large N should be "in practice" and what is the "region" of CLT applicability ?
- "Proof" for a "narrow" distribution (all moments exist)
  - assume that u_N is well described by a Gaussian for $|u_N| \ll u^*(N)$
  - for a „narrow" distribution one has expansion in polynomials of u_n($\propto$ normalized cumulants):

$$\Delta P_>(u_N) \equiv P_>(u_N) - P_>^{(G)}(u_N) \approx \frac{e^{-u_N^2/2}}{\sqrt{2\pi}}\left(\frac{Q_1(u_N)}{N^{1/2}} + \frac{Q_2(u_N)}{N} + \dots\right)$$

$$P_>^{(G)}(u_N) \approx \frac{e^{-u_N^2/2}}{\sqrt{2\pi}} u_N^{-1} \qquad Q_1(u_N) = \frac{1}{6}\lambda_3(u_N^2 - 1) \qquad Q_2(u_N) = \frac{1}{8}\lambda_4\left(\frac{1}{3}u_N^3 - u_N\right) + \text{terms dependent on } \lambda_3$$

  - if skeness $\lambda_3 \neq 0$ then the leading term of the expansion is $\frac{Q_1(uN)}{N^{1/2}}$ and for small $u_N \sim 1$ (the "middle")

$$\frac{\lambda_3}{N^{1/2}} \ll 1 \quad \Rightarrow \quad N \gg \boxed{\lambda_3^2 \equiv N^*}$$

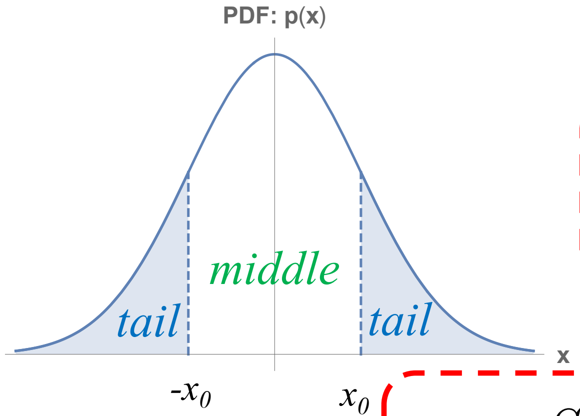
  - so for $N \gg N^* = \lambda_3{}^2$ and u_N in the „middle":

$$\frac{\Delta P_>(u_N)}{P_>^{(G)}(u_N)} \propto \frac{\lambda_3 u_N^2}{u_N^{-1} N^{1/2}} \ll 1 \quad \Rightarrow \quad |u_N| \ll \frac{N^{1/6}}{\lambda_3^{1/3}} = \left(\frac{N}{N^*}\right)^{1/6}$$

$$\boxed{|S_N - N\langle x\rangle| \ll} \sigma\sqrt{N}\left(\frac{N}{N^*}\right)^{1/6} \boxed{\propto N^{2/3}\sigma}$$

  - Similarly, if λ_3=0 & kurtosis $\lambda_4 \neq 0$: $\boxed{N^* = \lambda_4}$ & $\boxed{|S_N - N\langle x\rangle| \ll N^{3/4}\sigma}$

CLTs: Large deviations



$$S_N = \sum_{i=1}^{N} X_i \quad , \quad X_i \sim iid$$

$$u_N = \frac{S_N - N\langle x\rangle}{\sigma\sqrt{N}} \xrightarrow[N\to\infty]{} N(0,1)$$

- Formally Gaussian CLT works for $N\to\infty$
- Question: how large N should be "in practice" and what is the "region" of CLT applicability ?
- Example for a "heavy" tail: $p(x) \sim \dfrac{\alpha A^\alpha}{|x|^{\alpha+1}}$ for $|x| \gg x_0$ (but $\alpha > 2$, so $\langle x\rangle$ and σ^2 exist)
  - $X_i \sim$ „t-student" distribution ($\alpha = 3$)

$$p(x) = \frac{2a^3}{\pi\left(x^2+a^2\right)^2} \quad , \quad \langle x\rangle = 0 \quad , \quad \sigma^2 = a^2$$

  - assume again that u_N is well described by a Gaussian for $|u_N| \ll u^*(N)$
  - from Gaussian CLT: in the „middle": $p(S_N) \approx \dfrac{1}{\sqrt{2\pi N a^2}} \exp\left(-\dfrac{S_N^2}{2Na^2}\right)$ ⟵ Gaussian PDF
  - but the "heavy" tails „survive": $p(S_N) \approx \dfrac{2a^3 N}{\pi S_N^4}$ for $S_N \gg x_o \equiv u^*(N) a\sqrt{N}$
  - both regions „meet" for $S_N \simeq x_0$: $\dfrac{1}{\sqrt{2\pi N a^2}} \exp\left(-\dfrac{x_0^2}{2Na^2}\right) \approx \dfrac{2a^3 N}{\pi x_0^4}$ ⟹ $\boxed{x_0 \approx a\sqrt{N \ln N}}$

CLTs: Non-Gaussian CLT

$$S_N = \sum_{i=1}^{N} X_i \ , \ X_i \sim iid$$

- Gaussian CLT requires that at least two first moments exist
- This is NOT the case for "heavy" power-law tails distributions when $0 < \alpha < 2$

$$p(x) \sim \frac{\alpha A_\pm^\alpha}{|x|^{\alpha+1}} \quad \text{for } |x| \to \pm\infty$$

- In this case S_N converges to the (Levy) α-stable: $L_\alpha(\beta,c,m)$ with the same α as $X_i \sim p(x)$:
  - $\alpha^{(N)} = \alpha$ $\qquad A_\pm^{(N)} = N A_\pm$ $\qquad \beta^{(N)} = \frac{A_+^\alpha - A_-^\alpha}{A_+^\alpha + A_-^\alpha} = \beta$
  - If left & right tails have different exponents (i.e. $\alpha_+ \neq \alpha_-$) ➡ smaller α „wins" ! And then: S_N has totally assymetric (β = -1 or β = +1) (Levy) α-stable distribution with $\alpha = \min(\alpha_+ ; \alpha_-)$
  - All comments concerning Gaussian CLT apply, i.e. X_i do not have to be exactly iid, convergence is faster in the „middle" than in the „tails", …

Probability calculus in a nutshell

- ❖Basics
- ❖Probability distributions
- ❖Central limit theorems
- ❖Extreme values

Extreme values

❖ We are now interested in the probability distribution of the MAX. of N independent & identically distributed (iid) random variables for fixed N (and for $N \to \infty$)

$$X_{max} = max(X_1, \dots, X_N) \quad \text{where:} \quad X_i \sim iid$$

- This is important situation in risk menagement, e.g. if X_i represents daily losses from some portfolio (investment) then X_{max} is the maximum daily loss (in N days investment horizon)

- If $X_{max} \leq \Lambda$ then of course also $X_1 \leq \Lambda$ & .... & $X_N \leq \Lambda$, therefore:

$$\underbrace{P(X_{max} \leq \Lambda)}_{F_{max}(\Lambda)} \overset{\text{independent } X_i}{=} P(X_1 \leq \Lambda) \cdot \ldots \cdot P(X_N \leq \Lambda) \overset{\text{identical } X_i}{=} \underbrace{P(X_i \leq \Lambda)}_{F(\Lambda)}{}^N$$

CDF of X_{max} → $F_{max}(\Lambda)$ $\qquad$ $F(\Lambda)$ ← CDF of X_i

$$\boxed{F_{max}(\Lambda) = (F(\Lambda))^N}$$

- Approximately: (for $N \to \infty$) $\quad F_{max}(\Lambda) = (1 - P_>(\Lambda))^N \approx \exp(-NP_>(\Lambda))$ ← This works when $N \to \infty$

- PDF of X_{max} : $\quad p_{max}(\Lambda) = \dfrac{d}{d\Lambda} F_{max}(\Lambda)$

Extreme values: „narrow" X_i

$$\boxed{F_{max}(\Lambda) \approx \exp\left(-N P_>(\Lambda)\right)}$$

This works when $N \to \infty$

❖ Example 1: X_i has a "narrow" distribution (tails fall faster than any power-law), e.g. $X_i \sim$ *exponential distribution*:

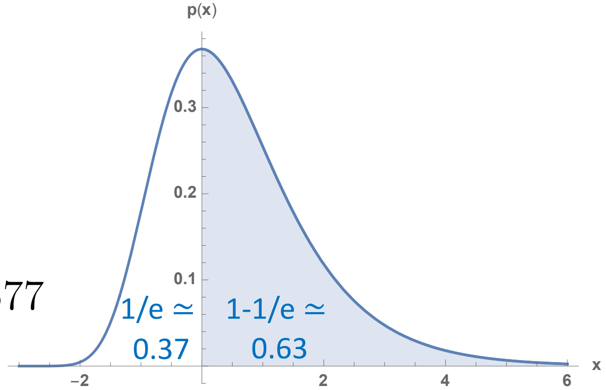
$$p(x_i) = \lambda e^{-\lambda x_i} \quad , \quad x_i \geq 0 \qquad F(x_i) = 1 - e^{-\lambda x_i} \qquad P_>(x_i) = e^{-\lambda x_i} \qquad F^{-1}(x_i) = -\frac{\ln(1 - x_i)}{\lambda}$$

$$\boxed{F_{max}(\Lambda)} \approx \exp\left(-N e^{-\lambda \Lambda}\right) = \boxed{\exp\left(-e^{-\underbrace{\lambda\Lambda - \ln N}_{u}}\right)}$$

$$U \equiv \frac{X_{max} - \boxed{a_N}}{\boxed{b_N}} \qquad \boxed{a_N} = \frac{\ln N}{\lambda} = \boxed{F^{-1}\left(1 - \frac{1}{N}\right)} \qquad \boxed{b_N} = \frac{1}{\lambda} = \boxed{F^{-1}\left(1 - \frac{1}{Ne}\right) - a_N}$$

❖ For $N \to \infty$: $U \sim$ *Gumbel distribution*:

- ❑ Support: u ∈ $\mathbb{R}$
- ❑ PDF and CDF: $\boxed{p(u) = e^{-(u + e^{-u})}}$ $\boxed{F(u) = e^{-e^{-u}}}$
- ❑ Typical values: $u_{max} = 0 \quad < \quad u_{med} = -\ln(\ln 2) \approx 0.367 \quad < \quad E(u) = \gamma \approx 0.577$
- ❑ Typical deviations: $\sigma = \pi/\sqrt{6}$
- ❑ Skewness and kurtosis: $\lambda_3 > 0$ (positive skewness) & $\lambda_4 > 0$ (leptokurtic: tails "fatter" than Gauss.)



Extreme values: „heavy” X_i

$$F_{max}(\Lambda) \approx \exp\left(-N P_>(\Lambda)\right)$$

This works when $N \to \infty$

- **Example 2**: X_i has a “heavy” tail distribution (power-law tails with exponent α), e.g. $X_i \sim$ *Pareto distribution (with α = ½)*:

$$p(x_i) = \frac{1}{2x_i^{3/2}} \quad , \quad x_i \geq 1 \qquad F(x_i) = 1 - \frac{1}{x_i^{1/2}} \qquad P_>(x_i) = \frac{1}{x_i^{1/2}} \qquad F^{-1}(x_i) = \frac{1}{(1-x_i)^2}$$

$$\boxed{F_{max}(\Lambda)} \approx \exp\left(-N\frac{1}{\Lambda^{1/2}}\right) = \boxed{\exp\left(-\left(\underbrace{\frac{\Lambda}{N^2}}_{u}\right)^{-1/2}\right)}$$

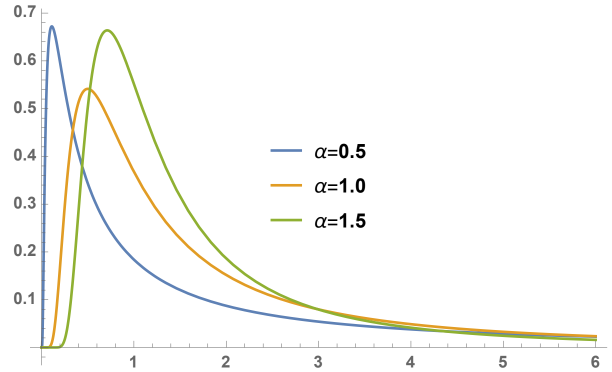
$$U \equiv \frac{X_{max} - \boxed{a_N}}{\boxed{b_N}} \qquad \boxed{a_N = 0} \qquad \boxed{b_N} = N^2 = \boxed{F^{-1}\left(1 - \frac{1}{N}\right)}$$

- For $N \to \infty$: $U \sim$ *Frechet distribution*:
  - Support: $u \geq 0$
  - PDF and CDF: $\boxed{p(u) = \alpha u^{-(\alpha+1)} e^{-u^{-\alpha}}}$ $\qquad \boxed{F(u) = e^{-u^{-\alpha}}}$
  - Typical values: $u_{max} = \left(\frac{\alpha}{\alpha+1}\right)^{1/\alpha} < u_{med} = (\ln 2)^{-1/\alpha} < E(u) = \Gamma\left(1 - \frac{1}{\alpha}\right)$

    $E(u)$ finite only for $\alpha > 1$
  - Typical deviations: $\sigma = \sqrt{\Gamma\left(1 - \frac{2}{\alpha}\right) - \left(\Gamma\left(1 - \frac{1}{\alpha}\right)\right)^2}$

    σ finite only for $\alpha > 2$
  - Skewness and kurtosis: $\lambda_3 > 0$ (positive skewness) & $\lambda_4 > 0$ (leptokurtic: tails “fatter” than Gauss.)

    λ_3 finite only for $\alpha > 3$ $\qquad$ λ_4 finite only for $\alpha > 4$



Extreme values: „limited” X_i

$$\boxed{F_{max}(\Lambda) \approx \exp\left(-N P_>(\Lambda)\right)}$$

This works when $N\to\infty$

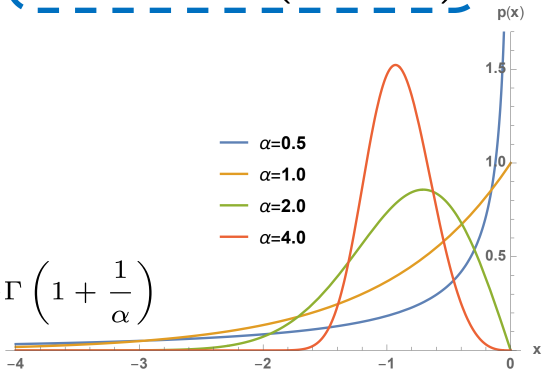
- Example 3: X_i has limited support ($X_i \leq x_+$), right tail of $1\text{-}F(x_i)$ grows with exponent α around x_+ when moving away form x_+, e.g. $X_i \sim$ *Uniform distribution* ($x_+ = 1$, $\alpha = 1$):

$$p(x_i) = 1 \quad , \quad 0 \leq x_i \leq 1 \qquad F(x_i) = x_i \qquad P_>(x_i) = 1 - x_i \qquad F^{-1}(x_i) = x_i$$

$$\boxed{F_{max}(\Lambda)} = \exp\left(-N(1-\Lambda)\right) = \boxed{\exp\left(-(\underbrace{-(\Lambda - 1)N}_{u})\right)}$$

$$U \equiv \frac{X_{max} - \boxed{a_N}}{\boxed{b_N}} \qquad \boxed{a_N} = 1 = \boxed{x_+} \qquad \boxed{b_N} = \frac{1}{N} = \boxed{a_N - F^{-1}\left(1 - \frac{1}{N}\right)}$$

- For $N\to\infty$: $-U \sim$ *Weibull distribution*:
  - Support: $u \leq 0$
  - PDF and CDF: $\boxed{p(u) = \alpha(-u)^{\alpha-1} e^{-(-u)^{\alpha}}}$ $\boxed{F(u) = e^{-(-u)^{\alpha}}}$
  - Typical values: $u_{max} = -\left(\frac{\alpha-1}{\alpha}\right)^{1/\alpha}$ (or 0 for $\alpha \leq 1$) , $u_{med} = -(\ln 2)^{1/\alpha}$, $E(u) = -\Gamma\left(1+\frac{1}{\alpha}\right)$
  - Typical deviations: $\sigma = \sqrt{\Gamma\left(1+\frac{2}{\alpha}\right) - \left(\Gamma\left(1+\frac{1}{\alpha}\right)\right)^2}$
  - Skewness and kurtosis: λ_3 (can be any sign, depend. on α) & λ_4 (can be any sign, depend. on α)



Extreme values: summary

$$F_{max}(\Lambda) \approx \exp\left(-N P_{>}(\Lambda)\right)$$

This works when $N \to \infty$

- Depending on the type of (tails of) distribution X_i the MAX. of N independent & identically distributed (iid) random variables: $X_{max} = max(X_1, \dots, X_N)$ will converge (for $N \to \infty$) to one of three universal (families of) distributions
- One can "standardize": $U \equiv \dfrac{X_{max} - a_N}{b_N}$

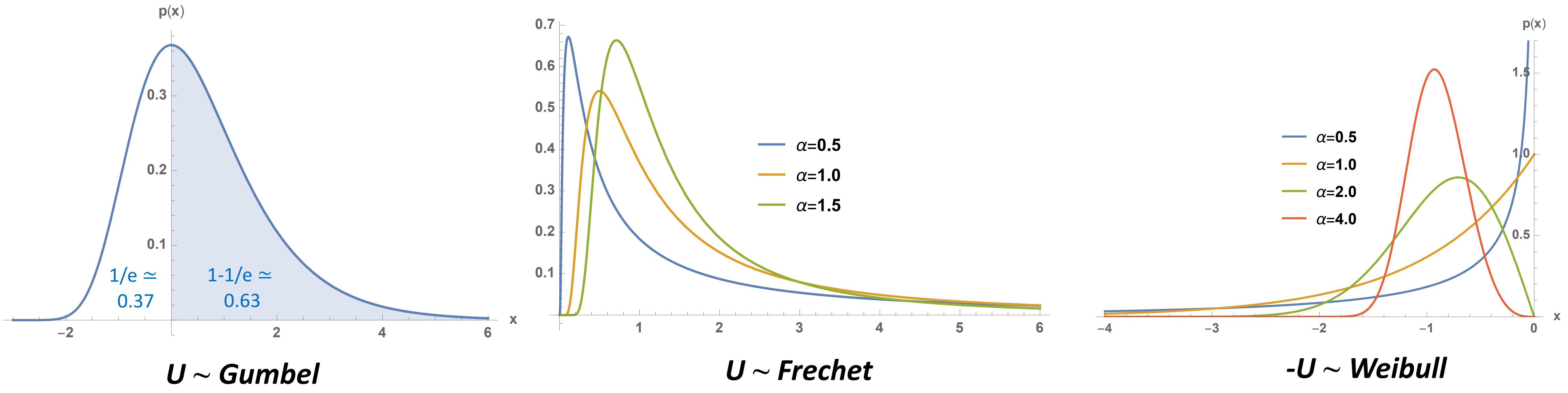
X_i distribution	Limiting distribution of X_{max}	PDF & CDF
Exponential („narrow") **tails** * (all moments exist) * Formally: more technical conditions apply	**$U \sim$ Gumbel** ($u \in \mathbb{R}$) $a_N = F^{-1}\left(1 - \frac{1}{N}\right)$ $b_N = F^{-1}\left(1 - \frac{1}{Ne}\right) - a_N$	$p(u) = e^{-(u+e^{-u})}$ $F(u) = e^{-e^{-u}}$
Power-law tails with exponent α $\frac{P_{>}(t\, x_i)}{P_{>}(t)} \xrightarrow[t\to\infty]{} x_i^{-\alpha}$	**$U \sim$ Frechet** ($u \geq 0$) $a_N = 0$ $b_N = F^{-1}\left(1 - \frac{1}{N}\right)$	$p(u) = \alpha u^{-(\alpha+1)} e^{-u^{-\alpha}}$ $F(u) = e^{-u^{-\alpha}}$
Limited suport ($p(x_i) = 0$ for $x_i > x_+$) (tail near x_+ with exponent α) $\frac{1 - F(x_+ + t\, x_i)}{1 - F(x_+ - t)} \xrightarrow[t\to 0]{} (-x_i)^{\alpha}$, $x_i < 0$	**$-U \sim$ Weibull** ($u \leq 0$) $a_N = x_+$ $b_N = a_N - F^{-1}\left(1 - \frac{1}{N}\right)$	$p(u) = \alpha(-u)^{\alpha-1} e^{-(-u)^{\alpha}}$ $F(u) = e^{-(-u)^{\alpha}}$

Extreme values: summary

$$F_{max}(\Lambda) \approx \exp\left(-N P_{>}(\Lambda)\right)$$

This works when $N \to \infty$

- ❖ Depending on the type of (tails of) distribution X_i the MAX. of N independent & identically distributed (iid) random variables: $X_{max} = max(X_1, \dots, X_N)$ will converge (for $N \to \infty$) to one of three universal (families of) distributions
- ❖ One can "standardize": $$U \equiv \frac{X_{max} - a_N}{b_N}$$



Summary

$$m_n \equiv E\left(X^n\right) = \int_{-\infty}^{+\infty} x^n \, p(x) dx$$

$$m_n = (-i)^n \frac{d^n}{dt^n} \hat{p}(t)\Big|_{t=0}$$

$$\hat{p}(t) \equiv E(e^{itX}) = \int_{-\infty}^{+\infty} e^{itx} p(x) dx$$

$$\hat{p}_{1+2}(t) = \hat{p}_1(t) \cdot \hat{p}_2(t)$$

convolution

$$C_n = (-i)^n \frac{d^n}{dt^n} \ln \hat{p}(t)\Big|_{t=0}$$

Basics: PDF, CDF, typical values / deviations

Moments, Characteristic Function

Cumulants, Skewness, Kurtosis

Basic Distributions: Uniform, Normal, α-stable

Central Limit Theorems (Gaussian & general)

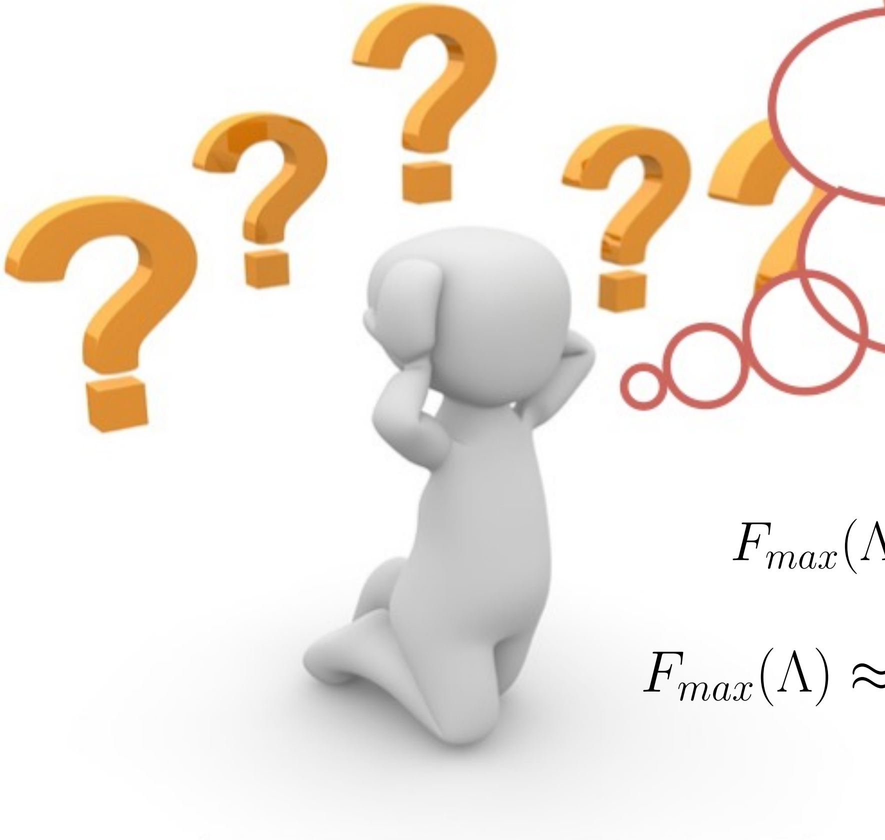
Extreme values – 3 classes

(Gumbel, Frechet, Weibull)

$$p(x) = \begin{cases} \frac{1}{b-a} & , x \in [a,b] \\ 0 & , \text{otherwise} \end{cases}$$

$$X \sim p(x) \to Y = F_X(X) \sim \mathrm{U}[0,1]$$

$$p(x) = \frac{1}{\sqrt{2\pi s^2}} \exp\left(-\frac{(x-m)^2}{2s^2}\right)$$



$$F_{max}(\Lambda) = \left(F(\Lambda)\right)^N$$

$$F_{max}(\Lambda) \approx \exp\left(-NP_>(\Lambda)\right)$$

$$\hat{p}(t) = \exp(-|c_\alpha \, t|^\alpha)$$

$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-itx} \hat{p}(t) dt \sim \frac{\alpha A_\pm^\alpha}{|x|^{\alpha+1}} \quad \text{for } x \to \pm\infty$$