Risk Management - Problems 4
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Financial risk measures

1. Show that the Expected Shortfall of a continuous random variable X

$$ES_\alpha(X) \equiv E(-X|X \leq -VaR_\alpha(X)) = -\frac{1}{\alpha}\int_{-\infty}^{-VaR_\alpha(X)} xp(x)dx,$$

where $VaR_\alpha(X)$ is the Value at Risk defined as:

$$Pr(X \leq -VaR_\alpha(X)) = \int_{-\infty}^{-VaR_\alpha(X)} p(x)dx = \alpha \quad , \quad \alpha \in (0,1),$$

can be alternatively calculated as:

$$ES_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha VaR\gamma(X)d\gamma \tag{1}$$

Note: This formula is useful, e.g. when calculating ES_α from empirical data: one can easily compute VaR_α from the empirical CDF and then use (1) to compute ES_α - see Exercise 4 and 5.

2. Derive analytic formula for $VaR_\alpha(X)$ and $ES_\alpha(X)$ for

   (a) $X \sim$ exponential distribution, i.e. $p(x) = \lambda e^{-\lambda x} \quad , \quad x \geq 0$

   (b) $X \sim$ normal distribution with mean μ and standard deviation σ

   Make a plot of $VaR_\alpha(X)$ and $ES_\alpha(X)$ as a function of $\alpha \in (0,1)$ for a standard exponential distribution ($\lambda = 1$) and a standard normal distribution ($\mu = 0, \sigma = 1$).

   Note 1: For the normal distribution you can use, e.g the (inverse) error function $Erf^{-1}(x)$ or the standard Gaussian quantile $\Phi^{-1}(x)$

   Note 2: You can use some symbolic algebra software (e.g. Wolfram Mathematica) to solve the exercise, you do not have to calculate it "by hand"

3. Using results of Exercise 2 and assuming that the share price in time t is normally distributed, according to the (approximate) formula

$$S(t) = S(0) + S(0)\mu t + S(0)\sigma\sqrt{t}\ \xi\ , \tag{2}$$

   where ξ is a standard Gaussian random variable (mean: 0, variance: 1)

(a) Derive a functional relation between volatility σ and VaR_α and ES_α

(b) Compute daily VaR_α and ES_α for Gaussian share prices (2). Current share price is $S(0) = 100$ PLN, $\mu = 10\%/year$ and "annual" volatility $\sigma = 20\%/year$. Assume that a year has 250 business days, and assume a possibility of observing only one loss exceeding VaR_α in a one-year perspective: $\alpha = 1/250$.

(c) Compute daily VaR_α and ES_α if "annual volatility" increases to $30\%/year$

(d) Compute daily VaR_α and ES_α if one assumes observing 2, 3, ... losses $> VaR_\alpha$ in one year, choose the confidence level α accordingly

(e) Compute daily VaR_α and ES_α if one increases time-length of the investment to 2, 3, ... years (we consider 1 loss $> VaR_\alpha$ in that time), choose the confidence level α accordingly

(f) Compute weekly VaR_α and ES_α. Choose the confidence level α such that one can (statistically) expect 1 weekly loss exceeding VaR_α during one-year investment scope

4. Data file *dat_St.txt* contains a sample of 1000 daily share prices $S(t)$ generated for some geometric Brownian motion process (log rates of return: $R(t) = \ln(S(t)/S(0))$ are normally distributed).

(a) Based on this empirical data compute: (annualized) historical Volatility: $\sigma\sqrt{T} = \sqrt{250} \cdot sd$ (sd - standard deviation of daily log rates of return and we assume a year T has 250 business days) and (annualized) mean return $\mu T = 250 \cdot \langle . \rangle$ (where $\langle . \rangle$ is the mean daily log rate of return)

(b) Based on results of point (a) and Exercise 3 (i.e. using the Gaussian approximation (2)) compute daily $VaR_\alpha(\Delta S)$ and $ES_\alpha(\Delta S)$ for $\alpha = 0.01, 0.05, 0.10, 0.20$. Assume you invest PLN 10 mln in the shares.

(c) Based on results of point (a) and Exercise 3 (i.e. using the Gaussian approximation (2)) compute weekly $VaR_\alpha(\Delta S)$ and $ES_\alpha(\Delta S)$ for $\alpha = 0.01, 0.05, 0.10, 0.20$. Assume you invest PLN 10 mln in the shares.

(d) Based on the empirical data compute daily $VaR_\alpha(\Delta S)$ and $ES_\alpha(\Delta S)$ for $\alpha = 0.01, 0.05, 0.10, 0.20$. Assume you invest PLN 10 mln in the shares (remember to make appropriate rescaling of ΔS data, such that on each day the investment value is PLN 10 mln).
Note: compute empirical CDF: $\hat{F}(x) = \frac{\#\ sample\ elements\ \leq x}{n}$ (n - sample size) and then empirical $VaR_\alpha(X)$ using some convention, e.g. the "invese CDF" convention discussed in Lecture 3 p. 38, and then empirical $ES_\alpha(X)$ (use formula (1))

(e) Based on the empirical data compute weekly $VaR_\alpha(\Delta S)$ and $ES_\alpha(\Delta S)$ for $\alpha = 0.01, 0.05, 0.10, 0.20$. Assume you invest PLN 10 mln in the shares. In order to have weekly $S(t)$ data "decimate" the sample by taking every 5-th element (assume a week has 5 working days)

(f) On the same chart plot approximate and empirical daily $VaR_\alpha(\Delta S)$ and $ES_\alpha(\Delta S)$ (computed as in points (b) and (d), respectively) for $\alpha \in (0, 1)$. On another chart plot weekly $VaR_\alpha(\Delta S)$ and $ES_\alpha(\Delta S)$ (computed as in points (c) and (e), respectively)