Ex. 1

$$S_{N-1}^2 = \frac{1}{N-1}\sum_{i=1}^{N}(X_i - \bar{X}_N)^2$$

$$\bar{X}_N = \frac{1}{N}\sum_{i=1}^{N} X_i$$

$$\sum_{i=1}^{N}(X_i - \bar{X}_N)^2 = \sum_{i=1}^{N}(X_i^2 - 2X_i\bar{X}_N + \bar{X}_N^2) = \sum_{i=1}^{N} X_i^2 - 2\bar{X}_N\sum_{i=1}^{N} X_i + \sum_{i=1}^{N}\bar{X}_N^2$$

$$\bar{X}_N = \frac{1}{N}\cdot\sum_{i=1}^{N} X_i \rightarrow \sum_{i=1}^{N} X_i = N\cdot\bar{X}_N \rightarrow \sum_{i=1}^{N}\bar{X}_N^2 = N\cdot\bar{X}_N^2$$

Kontynuując:

$$\sum_{i=1}^{N}(X_i - \bar{X}_N)^2 = \sum_{i=1}^{N} X_i^2 - 2\cdot\bar{X}_N\cdot N\cdot\bar{X}_N + N\cdot\bar{X}_N^2 =$$

$$= \sum_{i=1}^{N} X_i^2 - 2N\bar{X}_N^2 + N\cdot\bar{X}_N^2 = \sum_{i=1}^{N} X_i^2 - N\bar{X}_N^2$$

$$S_{N-1}^2 = \frac{1}{N-1}\left(\sum_{i=1}^{N} X_i^2 - N\bar{X}_N^2\right)$$

$$Var(X_i) = \sigma^2$$

$$E[X_i] = \mu$$

$$Var(X) = E[X^2] - (E[X])^2 \rightarrow E[X^2] = Var(X) + (E[X])^2 =$$

$$= \sigma^2 + \mu^2$$

dla $\bar{X}_N = \frac{1}{N}\sum_{i=1}^{N} X_i \rightarrow E[\bar{X}_N^2] = \frac{1}{N}\cdot\sigma^2 + \mu^2$

$$E[S_{N-1}^2] = \frac{1}{N-1}\left(E\left[\sum_{i=1}^{N} X_i^2\right] - NE[\bar{X}_N^2]\right) =$$

$$= \cancel{\frac{1}{N-1}}$$

$$E\left[\sum_{i=1}^{N} X_i^2\right] = \sum_{i=1}^{N} E[X_i^2] = \sum_{i=1}^{N}(\sigma^2 + \mu^2) = N(\sigma^2 + \mu^2)$$

$$N\cdot E[\bar{X}_N^2] = N\left(\frac{1}{N}\sigma^2 + \mu^2\right) = \sigma^2 + N\mu^2$$

$$E[S_{N-1}^2] = \frac{1}{N-1}\left(N(\sigma^2 + \mu^2) - (\sigma^2 + N\mu^2)\right) =$$

$$= \frac{1}{N-1}(N\sigma^2 + N\mu^2 - \sigma^2 - N\mu^2) = \frac{1}{N-1}(N\sigma^2 - \sigma^2) =$$

$$= \frac{1}{N-1}(N-1)\sigma^2 = \sigma^2$$

Wniosek: $E[S^2_{N-1}] = \sigma^2$