Optimization-Based Fluid-Structure Interaction Decoupling Approach

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The Finite Element Circus
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FSI Domain

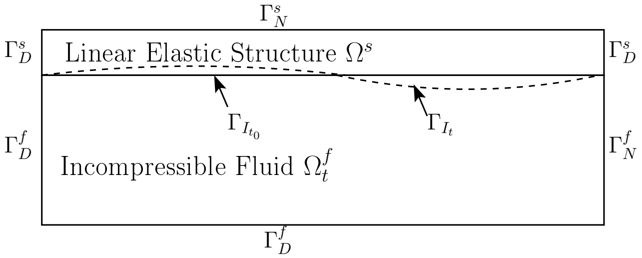


Figure: Fluid-Structure Interaction Domain

FSI Equations

Fluid: Navier-Stokes, Structure: Linear Elastic

$$
\begin{aligned}
\rho_f \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) - 2\nu_f \, \nabla \cdot D(\mathbf{u}) + \nabla p &= \mathbf{f}_f \quad \text{in } \Omega_t^f \,, \\
\nabla \cdot \, \mathbf{u} &= 0 \quad \text{in } \Omega_t^f \,, \\
\rho_s \frac{\partial^2 \boldsymbol{\eta}}{\partial t^2} - 2\nu_s \, \nabla \cdot D(\boldsymbol{\eta}) - \lambda \nabla (\nabla \cdot \boldsymbol{\eta}) &= \mathbf{f}_s \quad \text{in } \Omega^s \,,
\end{aligned}
$$

where $\mathbf{u}$ denotes the velocity vector of fluid, p the pressure of fluid, ρ_f the density of the fluid, ν_f the fluid viscosity, $\boldsymbol{\eta}$ the displacement of the structure, ρ_s the structure density; $D(\mathbf{v}) := (\nabla \mathbf{v} + \nabla \mathbf{v}^T)/2$

Initial Conditions and Boundary Conditions

$$
\begin{aligned}
2\nu_f D(\mathbf{u})\mathbf{n}_f - p\mathbf{n}_f &= \mathbf{u}_N \quad \text{on } \Gamma_N^f\,, \\
\mathbf{u} &= \mathbf{0} \quad \text{on } \Gamma_D^f\,, \\
2\nu_s D(\boldsymbol{\eta})\mathbf{n}_s + \lambda(\nabla\cdot\boldsymbol{\eta})\mathbf{n}_s &= \boldsymbol{\eta}_N \quad \text{on } \Gamma_N^s\,, \\
\boldsymbol{\eta} &= \mathbf{0} \quad \text{on } \Gamma_D^s\,, \\
\mathbf{u}|_{t=0} &= \mathbf{u}^0 \quad \text{in } \Omega_0^f\,, \\
\boldsymbol{\eta}|_{t=0} &= \boldsymbol{\eta}^0 \quad \text{in } \Omega^s\,, \\
\boldsymbol{\eta}_t|_{t=0} &= \dot{\boldsymbol{\eta}}^0 \quad \text{in } \Omega^s\,,
\end{aligned}
$$

where $\dot{\boldsymbol{\eta}}^0|_{\Gamma_{I_{t_n}}} = \mathbf{u}_0|_{\Gamma_{I_{t_n}}}$.

Interface Conditions

The moving boundary Γ_{I_t} is determined by the displacement $\boldsymbol{\eta}$ at time t. The interface conditions between the fluid and the structure are obtained by enforcing continuity of the velocity and the stress force:

Continuity of Velocity

$$\frac{\partial \boldsymbol{\eta}}{\partial t} = \mathbf{u} \quad \text{on } \Gamma_{I_t},$$

Continuity of Stress

$$2\nu_f D(\mathbf{u})\mathbf{n}_f - p\mathbf{n}_f = -(2\nu_s D(\boldsymbol{\eta})\mathbf{n}_s + \lambda(\nabla \cdot \boldsymbol{\eta})\mathbf{n}_s) \quad \text{on } \Gamma_{I_t}.$$

Variational formulation of the fluid governing equations

Using the Reynold's Transport theorem,
$\frac{d}{dt}\int_{\Omega_t} \phi \mathbf{v}\, d\Omega = \int_{\Omega_t} \left(\frac{\partial \phi}{\partial t}\mid_{\mathbf{y}} + \phi \nabla_{\mathbf{x}} \cdot \mathbf{z}\right) \mathbf{v}\, d\Omega$, with $\phi = \mathbf{u}$,
the chain rule, and integration by parts, the variational formulation of the flow equations becomes:

$$
\begin{aligned}
\rho_f \frac{d}{dt}(\mathbf{u}, \mathbf{v})_{\Omega_t^f} + \rho_f\left((\mathbf{u}-\mathbf{z})\cdot\nabla\mathbf{u}, \mathbf{v}\right)_{\Omega_t^f} - \rho_f(\mathbf{u}(\nabla\cdot\mathbf{z}), \mathbf{v})_{\Omega_t^f} + 2\nu_f(D(\mathbf{u}), D(\mathbf{v}))_{\Omega_t^f} \\
-(p, \nabla\cdot\mathbf{v})_{\Omega_t^f} - (2\nu_f D(\mathbf{u})\cdot\mathbf{n}_f - p\mathbf{n}_f, \mathbf{v})_{\Gamma_{I_t}} \\
= (\mathbf{f}_f, \mathbf{v})_{\Omega_t^f} + (\mathbf{u}_N, \mathbf{v})_{\Gamma_N^f} \;\forall \mathbf{v} \in \boldsymbol{H}_D^1(\Omega_t^f),
\end{aligned}
$$

$$
(q, \nabla\cdot\mathbf{u})_{\Omega_t^f} = 0 \quad \forall q \in L^2(\Omega_t^f).
$$

Time discretization of the flow equations

Time discretization by implicit Euler yields

$$\begin{aligned}
\rho_f \Big[(\mathbf{u}^n, \mathbf{v})_{\Omega^f_{t_n}} &- (\mathbf{u}^{n-1}, \mathcal{V}(\mathbf{v}))_{\Omega^f_{t_{n-1}}}\Big] + \Delta t\, \rho_f \Big[((\mathbf{u}^n - \mathbf{z}^n)\cdot\nabla\mathbf{u}^n, \mathbf{v})_{\Omega^f_{t_n}} - (\mathbf{u}^n(\nabla\cdot\mathbf{z}^n), \mathbf{v})_{\Omega^f_{t_n}}\Big] \\
&+ \Delta t \Big[2\nu_f (D(\mathbf{u}^n), D(\mathbf{v}))_{\Omega^f_{t_n}} - (p^n, \nabla\cdot\mathbf{v})_{\Omega^f_{t_n}}\Big] \\
&- \Delta t\, (2\nu_f D(\mathbf{u}^n)\cdot\mathbf{n}_f - p^n\mathbf{n}_f, \mathbf{v})_{\Gamma_{I_t}} \\
&= \Delta t \Big[(\mathbf{f}^n_f, \mathbf{v})_{\Omega^f_{t_n}} + (\mathbf{u}^n_N, \mathbf{v})_{\Gamma^f_N}\Big] \ \forall \mathbf{v} \in \boldsymbol{H}^1_D(\Omega^f_{t_n}),
\end{aligned}$$

$$(q, \nabla\cdot\mathbf{u}^n)_{\Omega^f_{t_n}} = 0 \quad \forall q \in L^2(\Omega^f_{t_n}).$$

- It is expected that the overall order of the time discretization (fluid & structure) will be only first order.
- Results given later are easily extended to Crank-Nicolson time discretization for the flow.
- Second order time scheme of the structure will be used for analysis because of extra accuracy needed.

Second order time discretization of the structure

A second order time discretization of the structure problem is given by

$$
\begin{aligned}
&\rho_s \left(\dot{\boldsymbol{\eta}}^n - \dot{\boldsymbol{\eta}}^{n-1}, \boldsymbol{\xi}\right)_{\Omega^s} \\
&\quad + \Delta t \left[2\,\nu_s \left(\frac{D(\boldsymbol{\eta}^n) + D(\boldsymbol{\eta}^{n-1})}{2}, D(\boldsymbol{\xi}) \right)_{\Omega^s} + \lambda \left(\nabla \cdot \left(\frac{\boldsymbol{\eta}^n + \boldsymbol{\eta}^{n-1}}{2} \right), \nabla \cdot \boldsymbol{\xi} \right)_{\Omega^s} \right] \\
&\quad - \Delta t \left(2\,\nu_s \left(\frac{(D(\boldsymbol{\eta}^n) + D(\boldsymbol{\eta}^{n-1})) \cdot \mathbf{n}_s}{2} \right) + \lambda \left(\nabla \cdot \left(\frac{\boldsymbol{\eta}^n + \boldsymbol{\eta}^{n-1}}{2} \right) \right) \mathbf{n}_s, \boldsymbol{\xi} \right)_{\Gamma_{I_0}} \\
&\quad = \Delta t \left[\left(\frac{\mathbf{f}_s^n + \mathbf{f}_s^{n-1}}{2}, \boldsymbol{\xi} \right)_{\Omega^s} + \left(\frac{\boldsymbol{\eta}_N^n + \boldsymbol{\eta}_N^{n-1}}{2}, \boldsymbol{\xi} \right)_{\Gamma_N^s} \right] \forall \boldsymbol{\xi} \in \boldsymbol{H}_D^1(\Omega^s)\,,
\end{aligned}
$$

$$
\Delta t \left(\frac{\dot{\boldsymbol{\eta}}^n + \dot{\boldsymbol{\eta}}^{n-1}}{2}, \boldsymbol{\gamma} \right)_{\Omega^s} - \left(\boldsymbol{\eta}^n - \boldsymbol{\eta}^{n-1}, \boldsymbol{\gamma} \right)_{\Omega^s} = 0 \;\forall \boldsymbol{\gamma} \in \mathbf{H}^1(\Omega^s)\,.
$$

Optimization Constraints

- Set $\mathbf{g}^n := \left(2\nu_f D(\mathbf{u}^n)\cdot\mathbf{n}_f - p\mathbf{n}_f - \frac{1}{2}((\mathbf{u}^n - \mathbf{z}^n)\cdot\mathbf{n}_f)\mathbf{u}^n\right)|_{\Gamma_{I_{t_n}}}$ as our control
- $\frac{1}{2}((\mathbf{u}^n - \mathbf{z})\cdot\mathbf{n}_f)\mathbf{u}^n$ will be approximately zero since at an optimal solution $-\mathbf{g}^n$ can be used as the stress for the structure
- $-(\mathbf{g}^n \circ \Psi_n^{-1})J_{t_n}$ representing $(2\nu_s D(\boldsymbol{\eta}^n)\cdot\mathbf{n}_s + \lambda(\nabla\cdot\boldsymbol{\eta}^n)\mathbf{n}_s)|_{I_{t_\mathbf{0}}}$

Making this substitution and introducing
$c(\mathbf{u},\mathbf{v},\mathbf{w}) = \frac{1}{2}((\mathbf{u}\nabla\mathbf{v},\mathbf{w})_{\Omega^f_{t_n}} - (\mathbf{u}\nabla\mathbf{w},\mathbf{v})_{\Omega^f_{t_n}})$,
the flow constraints become

$$
\begin{aligned}
\rho^f[(\mathbf{u}^n,\mathbf{v})_{\Omega^f_{t_n}} &- (\mathbf{u}^{n-1},\mathcal{V}(\mathbf{v}))_{\Omega^f_{t_{n-\mathbf{1}}}}] + \Delta t\,\rho^f[c(\mathbf{u}^n,\mathbf{u}^n,\mathbf{v})_{\Omega^f_{t_n}} + \frac{1}{2}((\mathbf{u}^n\cdot\mathbf{n}_f)\mathbf{u}^n,\mathbf{v})_{\Gamma^f_N} \\
&- \frac{1}{2}((\nabla\cdot\mathbf{z}^n)\mathbf{u}^n,\mathbf{v})_{\Omega^f_{t_n}} - c(\mathbf{z}^n,\mathbf{u}^n,\mathbf{v})_{\Omega^f_{t_n}}] \\
&+ \Delta t\,2\nu_f(D(\mathbf{u}^n),D(\mathbf{v}))_{\Omega^f_{t_n}} + \Delta t\,(p^n,\nabla\cdot\mathbf{v})_{\Omega^f_{t_n}} \\
&= \Delta t\,(\mathbf{f}^n_f,\mathbf{v})_{\Omega^f_{t_n}} + \Delta t\,(\mathbf{u}^n_N,\mathbf{v})_{\Gamma^f_N} + \Delta t\,(\mathbf{g}^n,\mathbf{v})_{\Gamma_{I_{t_n}}} \quad \forall\mathbf{v}\in\boldsymbol{H}^1_D(\Omega^f_{t_n}), \\
(q,\nabla\cdot\mathbf{u}^n)_{\Omega^f_{t_n}} &= 0 \quad q\in L^2(\Omega^f_{t_n}).
\end{aligned}
$$

Optimization Constraints

Also, the structure equations constraint can be rewritten as

$$
\begin{aligned}
\rho_s \left(\dot{\boldsymbol{\eta}}^n - \dot{\boldsymbol{\eta}}^{n-1}, \boldsymbol{\xi}\right)_{\Omega^s}
&+ \Delta t \left[2\,\nu_s \left(\frac{D(\boldsymbol{\eta}^n) + D(\boldsymbol{\eta}^{n-1})}{2}, D(\boldsymbol{\xi}) \right)_{\Omega^s} + \lambda \left(\nabla \cdot \left(\frac{\boldsymbol{\eta}^n + \boldsymbol{\eta}^{n-1}}{2} \right), \nabla \cdot \boldsymbol{\xi} \right)_{\Omega^s} \right] \\
&= \Delta t \left[\left(\frac{\mathbf{f}_s^n + \mathbf{f}_s^{n-1}}{2}, \boldsymbol{\xi} \right)_{\Omega^s} + \left(\frac{\boldsymbol{\eta}_N^n + \boldsymbol{\eta}_N^{n-1}}{2}, \boldsymbol{\xi} \right)_{\Gamma_N^s} \right] \\
&\quad - \frac{\Delta t}{2} \left(\mathcal{V}(\mathbf{g}^n) J_{t_n} + \mathcal{V}(\mathbf{g}^{n-1}) J_{t_{n-1}}, \boldsymbol{\xi} \right)_{\Gamma_{I_{t_0}}} \quad \forall \boldsymbol{\xi} \in \boldsymbol{H}_D^1(\Omega^s),
\end{aligned}
$$

$$
\Delta t \left(\frac{\dot{\boldsymbol{\eta}}^n + \dot{\boldsymbol{\eta}}^{n-1}}{2}, \boldsymbol{\gamma} \right)_{\Omega^s} - \left(\boldsymbol{\eta}^n - \boldsymbol{\eta}^{n-1}, \boldsymbol{\gamma} \right)_{\Omega^s} = 0 \; \forall \boldsymbol{\gamma} \in \mathbf{H}^1(\Omega^s) \, .
$$

Description of the optimization problem

An arbitrary $\mathbf{g}^n$ will not satisfy the semi-discrete weak formulation with matching boundary conditions (i.e. the monolithic problem).

Therefore, we will use the 'stress' function $\mathbf{g}^n$ as a control in each time step to attempt to enforce the continuity of velocity and continuity of stress, i.e., we wish to minimize the functional

$$\mathcal{J}_n^{\delta}(\mathbf{u}^n, p^n, \boldsymbol{\eta}^n, \dot{\boldsymbol{\eta}}^n, \mathbf{g}^n) = \frac{1}{2} \int_{\Gamma_{I_{t_n}}} |\mathbf{u}^n - \mathcal{V}(\dot{\boldsymbol{\eta}}^n)|^2 \, d\Gamma_{I_{t_n}} + \frac{\delta}{2} \int_{\Gamma_{I_{t_n}}} |\mathbf{g}^n|^2 \, d\Gamma_{I_{t_n}} \,,$$

subject to the flow and structure constraint equations, where δ is a positive constant penalty parameter.

A new functional

Expected difficulties:

- Not possible to get a stability estimate for $\dot{\boldsymbol{\eta}}^n$ in $\mathbf{H}^1(\Omega^s)$
- An optimal $\hat{\dot{\boldsymbol{\eta}}}^n$ can be shown only in $\mathbf{L}^2(\Omega^s)$
- The previous functionals are not well-defined (trace of optimal $\dot{\boldsymbol{\eta}}^n$ is not well-defined)

We introduce a first order finite difference approximation of $\dot{\boldsymbol{\eta}}^n$, and define the new optimization problem as

$$\mathcal{J}_n^\delta(\mathbf{u}^n, p^n, \boldsymbol{\eta}^n, \dot{\boldsymbol{\eta}}^n, \bar{\dot{\boldsymbol{\eta}}}^n, \mathbf{g}^n) = \frac{1}{2}\int_{\Gamma_{I_{t_n}}} \left| \mathbf{u}^n - \frac{\mathcal{V}(\boldsymbol{\eta}^n) - \mathcal{V}(\boldsymbol{\eta}^{n-1})}{\Delta t} \right|^2 \, d\Gamma_{I_{t_n}} + \frac{\delta}{2}\int_{\Gamma_{I_{t_n}}} |\mathbf{g}^n|^2 \, d\Gamma_{I_{t_n}} \,,$$

subject to the flow and structure constraints.

Recent Analytical Results

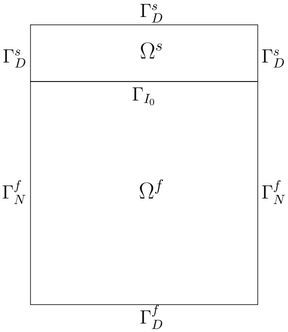
P. K., H. Lee, Analysis of a fluid-structure interaction problem recast in an optimal control setting, submitted 2014.

- Existence of an optimal solution
- Existence of Lagrange multipliers
- Derivation of the optimality system
  - State Equations
  - Adjoint Equations
  - Necessary Condition

P. K., H. Lee, Convergence of a fluid-structure interaction problem decoupled by a Neumann control over a single time-step, submitted 2014.

- Error estimates over a single time step
  - Standard Stokes and Linear Elasticity error rates
- Convergence of the gradient method from any initial guess, using a sufficiently small timestep

Numerical Experiment



Notable features:

- Manufactured problem, known solution
- Domain does not move
- All computations made using deal.II [1] with
  - $\mathbb{Q}^2$, $\mathbb{Q}^1$ for fluid velocity, pressure
  - $\mathbb{Q}^2$, $\mathbb{Q}^2$ for structure displacement, velocity

On $\Omega_t^f = \Omega^f = [0,1] \times [0,1]$ and $\Omega^s = [0,1] \times [1, 1.25]$,

$$\begin{aligned}
\mathbf{u}_1 &= \cos(x+t)\sin(y+t) + \sin(x+t)\cos(y+t),\\
\mathbf{u}_2 &= -\sin(x+t)\cos(y+t) - \cos(x+t)\sin(y+t),\\
p &= 2\nu_f(\sin(x+t)\sin(y+t) - \cos(x+t)\cos(y+t)) + 2\nu_s\cos(x+t)\sin(y+t),\\
\boldsymbol{\eta}_1 &= \sin(x+t)\sin(y+t),\\
\boldsymbol{\eta}_2 &= \cos(x+t)\cos(y+t).
\end{aligned}$$

Convergence over a single time step

h	$\|\mathbf{u}^n - \mathbf{u}^{true}\|_{L^2}$	Rate	$\|\mathbf{u}^n - \mathbf{u}^{true}\|_{H^1}$	Rate	$\left\|p^{n-\frac{1}{2}} - p^{true}\right\|_{L^2}$	Rate
1/9	1.1474e-05	-	1.0856e-03	-	1.4352e-01	-
1/14	3.0366e-06	3.01	4.4687e-04	2.01	2.4530e-02	4.00
1/21	8.9572e-07	3.01	1.9800e-04	2.01	4.8644e-03	3.99
1/31	2.7770e-07	3.01	9.0689e-05	2.00	1.0397e-03	3.96
1/46	8.4869e-08	3.00	4.1143e-05	2.00	2.2699e-04	3.86
1/69	2.5150e-08	3.00	1.8283e-05	2.00	5.4895e-05	3.50

h_x	h_y	$\|\boldsymbol{\eta}^n - \boldsymbol{\eta}^{true}\|_{L^2}$	Rate	$\|\boldsymbol{\eta}^n - \boldsymbol{\eta}^{true}\|_{H^1}$	Rate	$\|\dot{\boldsymbol{\eta}}^n - \dot{\boldsymbol{\eta}}^{true}\|_{L^2}$	Rate
1/9	1/36	1.7471e-06	-	1.6544e-04	-	4.8645e-06	-
1/14	1/56	4.5645e-07	3.04	6.7128e-05	2.04	1.2725e-06	3.04
1/21	1/84	1.3370e-07	3.03	2.9516e-05	2.03	3.7381e-07	3.02
1/31	1/124	4.1234e-08	3.02	1.3453e-05	2.02	1.1561e-07	3.01
1/46	1/184	1.2548e-08	3.01	6.0823e-06	2.01	3.5289e-08	3.01
1/69	1/276	3.7029e-09	3.01	2.6949e-06	2.01	1.0464e-08	3.00

- Computation over a single timestep, t=0.8 s, Δt=1e-6
- Using steepest descent as optimization method, ϵ_{tol}=1e-10
- Finite difference in objective

Convergence over all time steps

h	Δt	$\|\mathbf{u}^n - \mathbf{u}^{true}\|_{L^\infty(\mathbf{L^2})}$	Rate	$\|\mathbf{u}^n - \mathbf{u}^{true}\|_{L^2(\mathbf{H^1})}$	Rate	$\left\|p^{n-\frac{1}{2}} - p^{true}\right\|_{\mathbf{L^2}}$	Rate
1/6	1/15	1.1955e-02	-	1.0426e-01	-	8.9887e-03	-
1/9	1/27	2.6513e-03	3.71	3.9199e-02	2.41	3.5714e-03	2.28
1/14	1/53	5.0970e-04	4.07	1.2511e-02	2.82	1.5589e-03	2.04
1/21	1/97	8.6752e-05	4.01	3.5995e-03	2.82	6.4282e-04	2.00
1/31	1/173	2.3133e-05	3.26	1.2054e-03	2.70	2.8616e-04	2.00
1/46	1/312	6.5748e-06	3.23	4.3259e-04	2.63	1.3150e-04	2.00

h_x	h_y	Δt	$\|\boldsymbol{\eta}^n - \boldsymbol{\eta}^{true}\|_{L^2(\mathbf{L^2})}$	Rate	$\|\boldsymbol{\eta}^n - \boldsymbol{\eta}^{true}\|_{L^2(\mathbf{H^1})}$	Rate	$\|\dot{\boldsymbol{\eta}}^n - \dot{\boldsymbol{\eta}}^{true}\|_{L^\infty(\mathbf{L^2})}$	Rate
1/4	1/16	1/16	1.0642e-04	-	1.3550e-03	-	2.5272e-03	-
1/6	1/24	1/28	2.0519e-05	4.06	3.5795e-04	3.28	6.4563e-04	3.37
1/9	1/36	1/54	3.7975e-06	4.16	1.1056e-04	2.90	1.6216e-04	3.41
1/14	1/54	1/106	7.5280e-07	3.68	3.8761e-05	2.38	4.8158e-05	2.76
1/21	1/81	1/194	1.9953e-07	3.27	1.6277e-05	2.14	1.6575e-05	2.63
1/31	1/122	1/346	5.9897e-08	3.08	7.2861e-06	2.06	5.8173e-06	2.68

- Computation over timesteps from t=0.5 to t=1.0 s
- Using Gauss-Newton as outer optimization routine, ϵ_{tol}=1e-10
- Using CG as interior optimization routine, ϵ_{tol}=1e-13
- **NO** finite difference in objective

Significance of these results

- No deterioration in convergence in time or space over many timesteps
- Very few outer optimization iterations are performed per timestep (2-4 generally)
- The assembled matrix does not change between inner optimization iterations
  - We can **reuse the matrix factorization** over all CG based optimization iterations!
- We can solve the coupled FSI problem in a **constant multiple** of the computational effort needed to solve the forward problems, had the correct boundary condition been known
- We use partitioned solvers, so the forward solves are cheap in comparison to a monolithic approach

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