Covering minimal separators in P_t-*free* graphs

Based on Covering minimal separators and potential maximal cliques in P_t-free graphs, 2021 by

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seminar 17.10.2024

H-*free* graph

A graph is H-*free* if it does not contain an induced subgraph isomorphic to H

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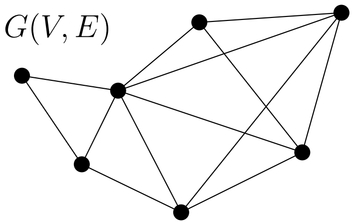

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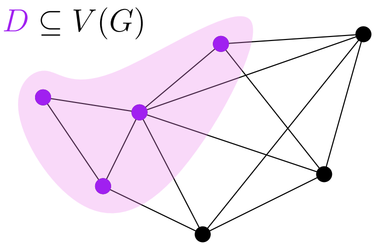

H-free graph

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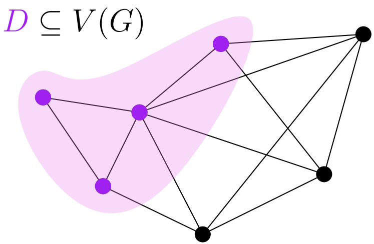

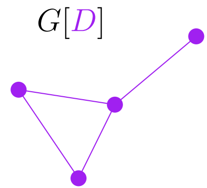

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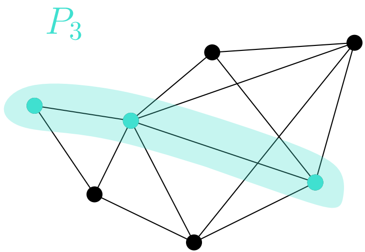
A graph is H-*free* if it does not contain an induced subgraph isomorphic to H

A graph is P_t-*free* if it does not contain an induced subgraph isomorphic to t-vertex path



P_3-*free* NO

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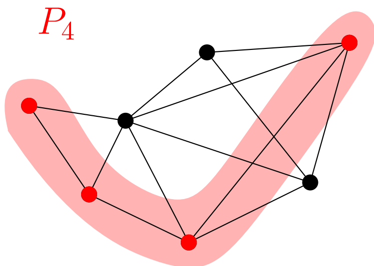

P_3-*free* NO

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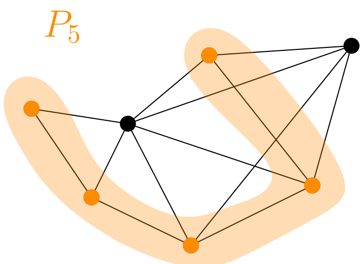

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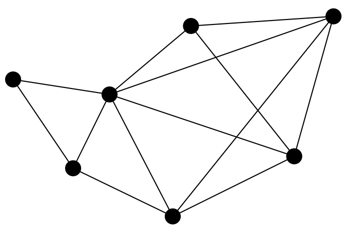

P_3-*free* NO

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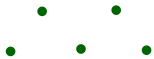
P_6-*free* YES

P_t-*free* graphs examples

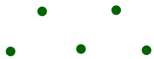
P_t-*free* graphs examples

P_2-*free*



P_t-*free* graphs examples

P_2-*free*



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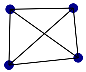

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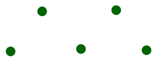

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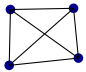
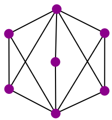
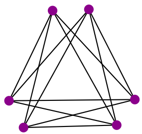


P_4-*free*

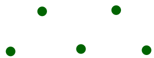




cographs!

P_t*-free* graphs examples

P_2*-free*



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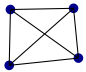

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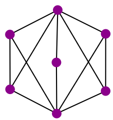

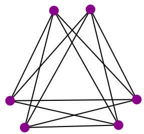

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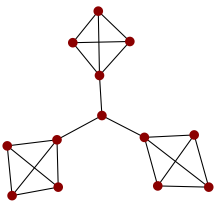
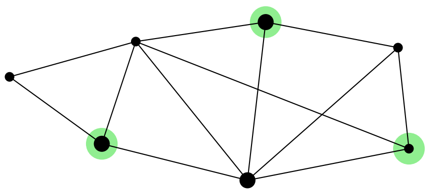
P_5*-free*



H-*free* graph problems

- Maximum Weight Indepented Set (MWIS)



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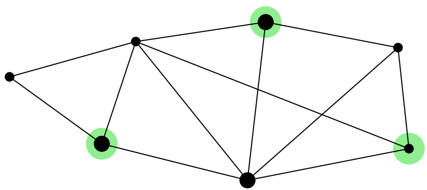

MWIS is NP-hard unless every connected component of H is a tree with at most three leaves

[Alekseev; 1982]

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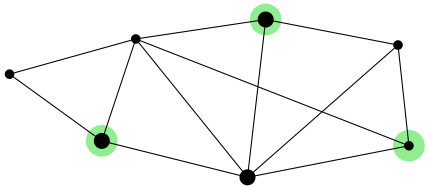

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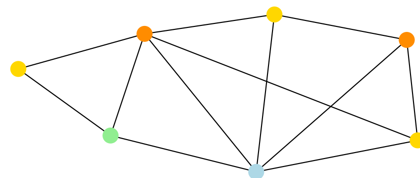
- k-Coloring



H-free graph problems

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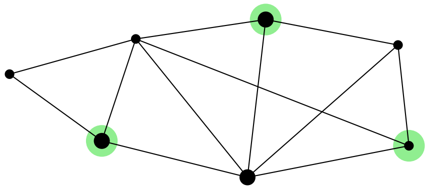

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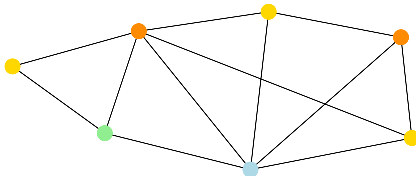

3-COLORING is NP-hard unless every connected component of H is a path

[Groenland, Okrasa, Rzażewski,... 2019]

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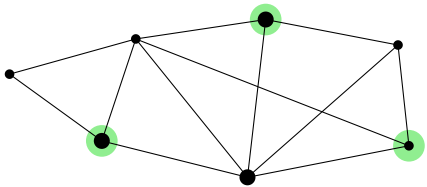

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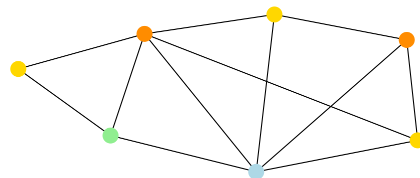

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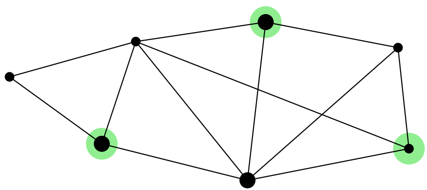
We expect MWIS and 3-Coloring to have polynomial solutions on P_t-*free* graphs for every fixed t

H-free graph problems

- Maximum Weight Indepented Set (MWIS)



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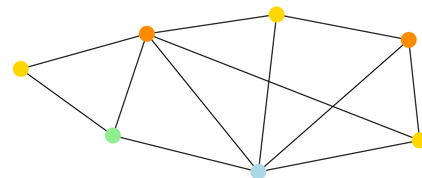

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We expect MWIS and 3-Coloring to have polynomial solutions on P_t-*free* graphs for every fixed t

Gyárfás' path argument gives subexponential-time algorithms for both MWIS and 3-Coloring in Pt -free graphs for any fixed t [2017, 2019]

P_t-*free* graph problems

P_4-*free* (*cographs*)

P_5-*free*

P_6-*free*

P_7-*free*

P_t-*free* graph problems

P_4-*free* (*cographs*) MWIS OK
3-Coloring OK

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P_t-*free* graph problems

P_4-*free* (*cographs*)

MWIS OK

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P_5-*free*

MWIS OK [Lokshtanov, Vatshelle, Villanger; 2014]

P_6-*free*

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P_t-*free* graph problems

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P_7-*free* 3-Coloring OK [Bonomo, Chudnovsky; 2018]

P_t-*free* graph problems

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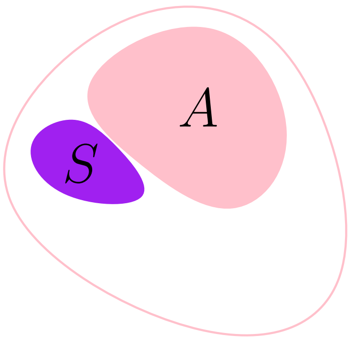
Minimal separator

G - graph, $S \subseteq V(G)$
A is a *full component* to S if:
A is conected componend of $G - S$ and
$N(A) = S$



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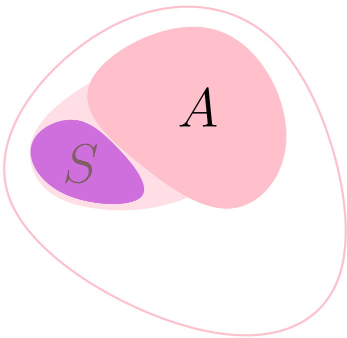

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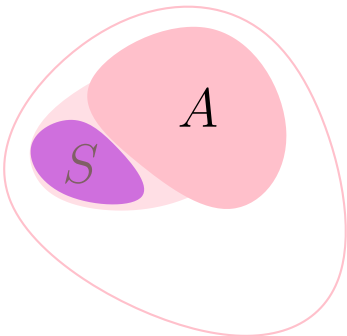
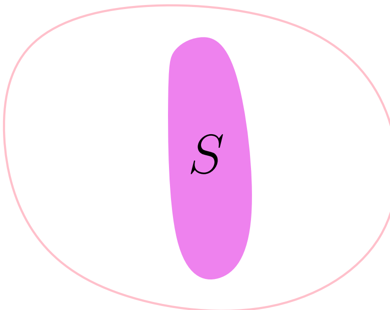


A set S is a minimal separator if it admits at least two *full components*



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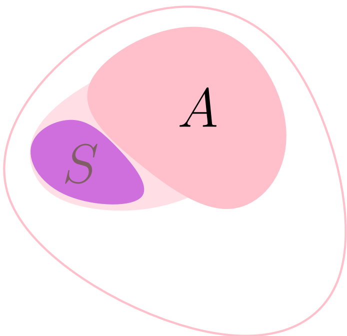

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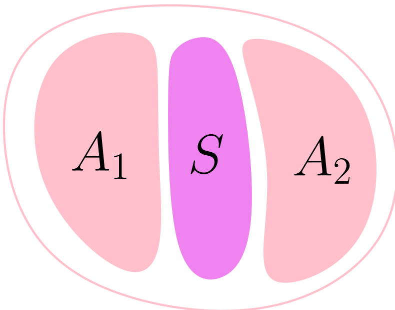


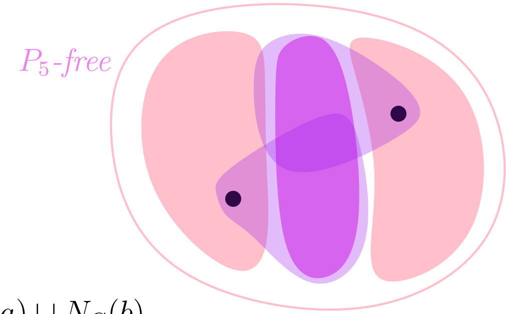
Covering minimal separator

Covering minimal separator

Lemma 1.
G - P_5-free graph
S - minimal separator in G
A, B - two full components of S
Then for every $a \in A$ and $b \in B$ it holds that $S \subseteq N_G(a) \cup N_G(b)$.

[Lokshtanov, Vatshelle, Villanger; 2014]

Covering minimal separator



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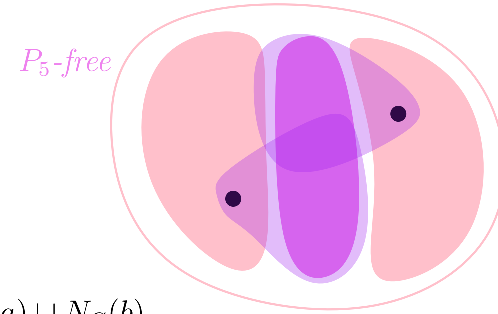

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Lemma 2.

G - P_6-free graph
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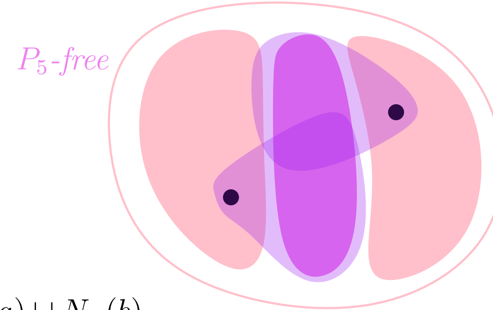
Then there exist nonempty sets $A_0 \subseteq A$ and $B_0 \subseteq B$ such that $|A_0| \leq 3, |B_0| \leq 3$, and $S \subseteq N_G(A) \sup N_G(B)$.

[Grzesik, Klimošová, Pilipczuk, Pilipczuk; 2019]

Covering minimal separator



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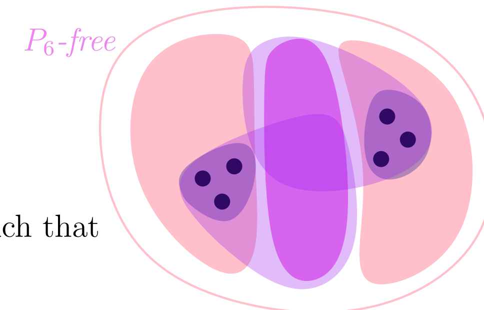

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Covering minimal separator

Theorem

G - P_7-free graph
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Then there exists a set $S_0 \subseteq V(G)$ of size at most 22 such that $S \subseteq N_G[S_0]$

[Grzesik, Klimošová, 2xPilipczuk; 2021]

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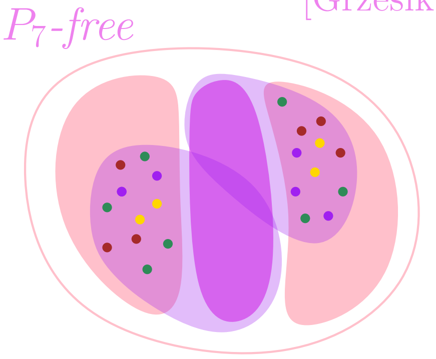

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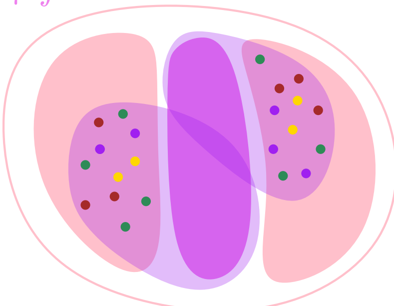
P_7-free



Proof?..

Modules

G - graph

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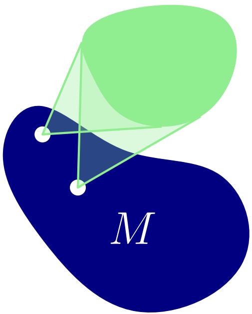

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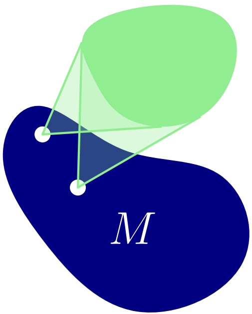

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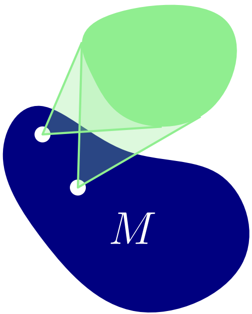

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All modules are *trivial* $\rightarrow$ graph is *prime*

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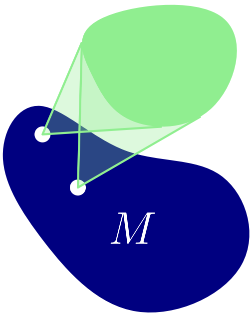

Trivial modules

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- $V(G)$
- $\{x\}, x \in V(G)$

All modules are *trivial* $\rightarrow$ graph is *prime*

Module M is *strong* if

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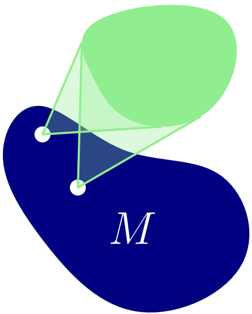

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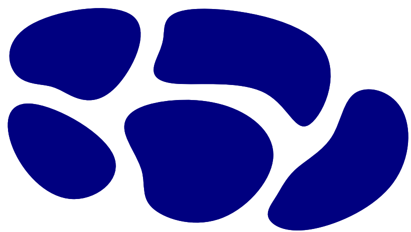
Modular partition $\mathrm{Mod}(G)$

family of maximal strong modules



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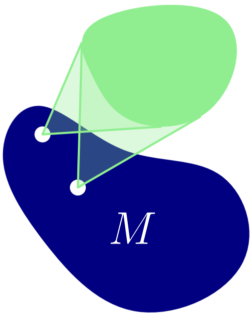

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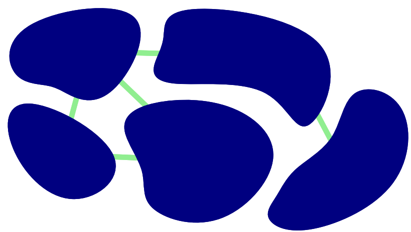

Quotient graph Quo(G)

- independent set
- or clicque
- or prime graph

Neighborhood Decomposition Lemma

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G - graph, $D \subseteq V(G)$, $|D| \geq 2$ and $G[D]$ is connected
$p, q \in D$ belong to different elements M^p, M^q of $\mathrm{Mod}(D)$ such that M^p and M^q are adjacent in Quo(D).
Then, for each $u \in N(D)$ at least one occures:

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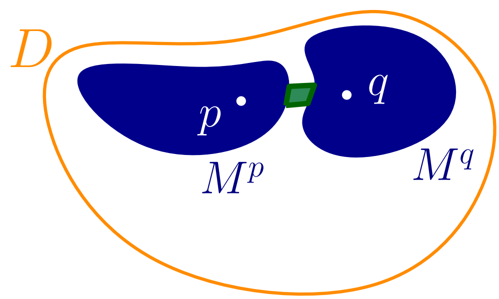

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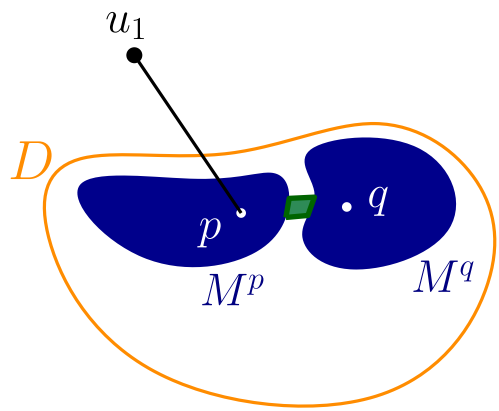

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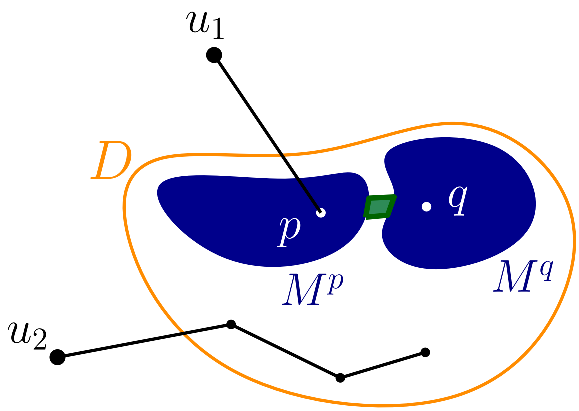
- there exists an induced P_4 in G such that u is one of its endpoints, while the other three vertices belong to D



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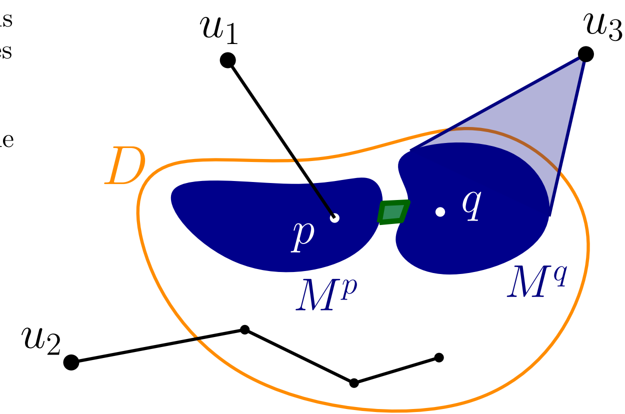
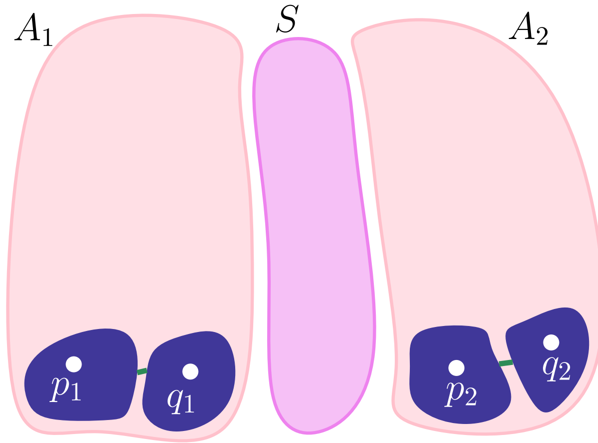
Then, for each $u \in N(D)$ at least one occures:

- $u \in N[p, q]$
- there exists an induced P_4 in G such that u is one of its endpoints, while the other three vertices belong to D
- $\text{Quo}(D)$ is a clique; neighborhood of u in D is the union of some collection of $M \in \text{Quo}(D)$



Proof



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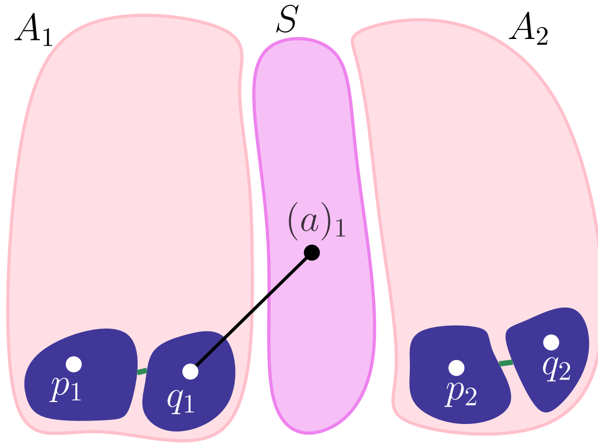

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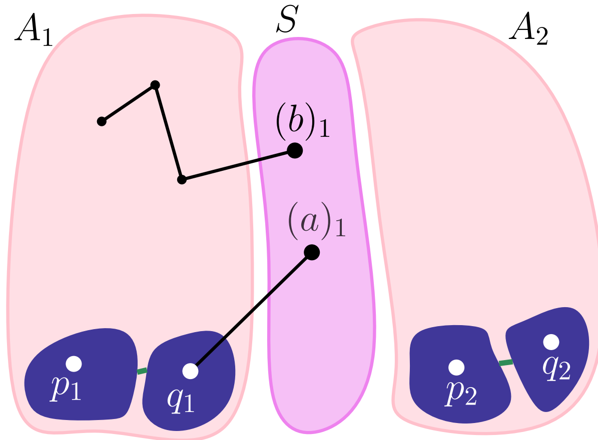

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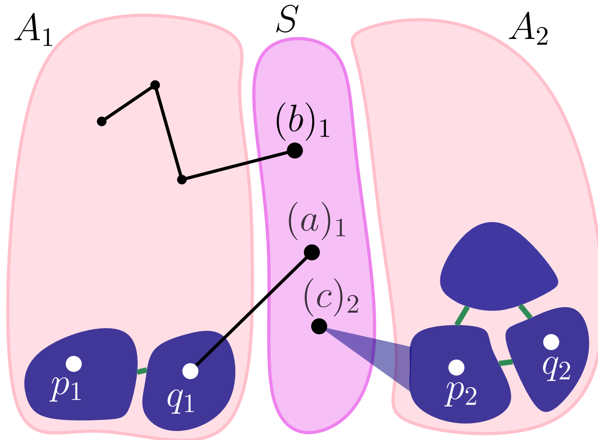
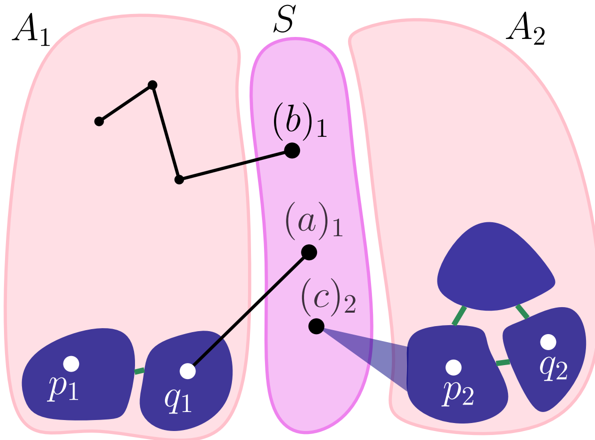
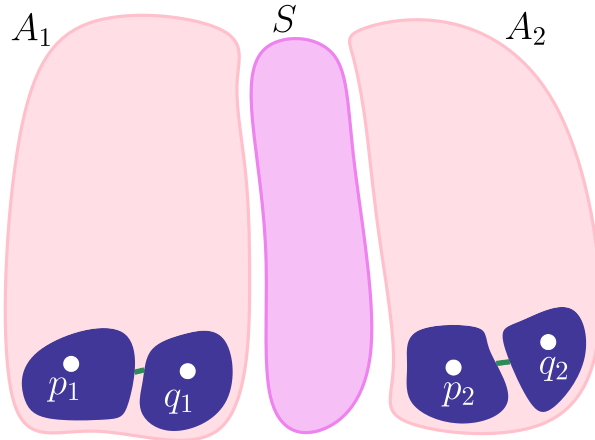


Proof



For $\alpha, \beta \in \{a, b, c\}$, let $S_{\alpha\beta}$ be the set of vertices $x \in S$ that are of type $(\alpha)_1$ and $(\beta)_2$

Proof



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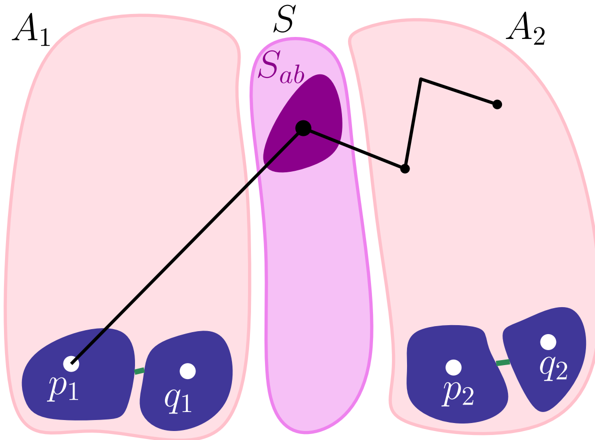

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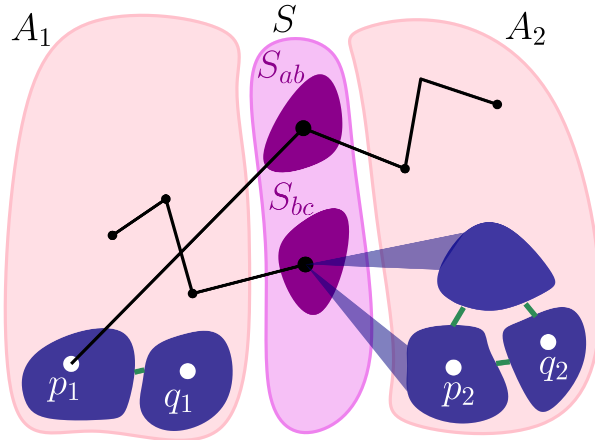

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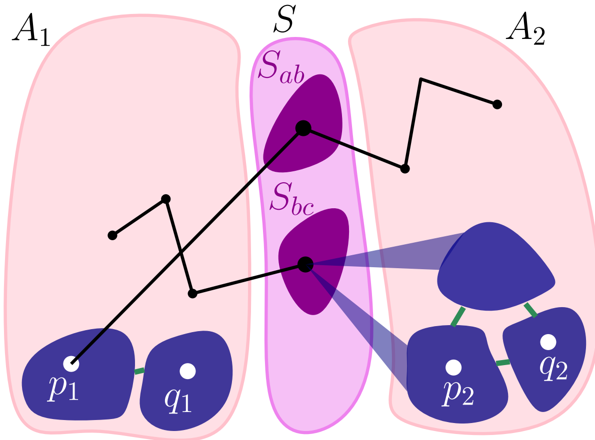
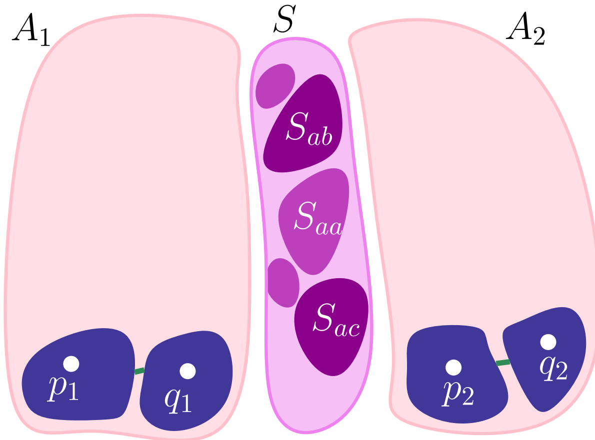


For $\alpha, \beta \in \{a, b, c\}$, let $S_{\alpha\beta}$ be the set of vertices $x \in S$ that are of type $(\alpha)_1$ and $(\beta)_2$

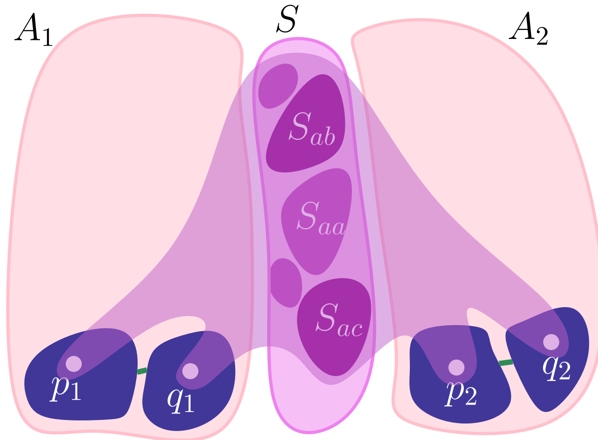
$$S_{bb} = \emptyset$$

Proof



$$R_a := \{p_1, q_1, p_2, q_2\}$$

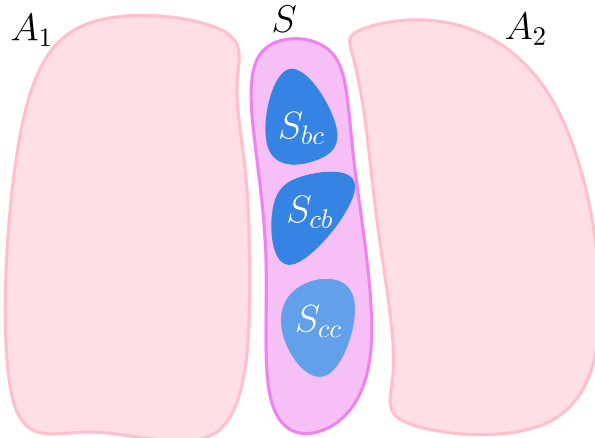
Proof



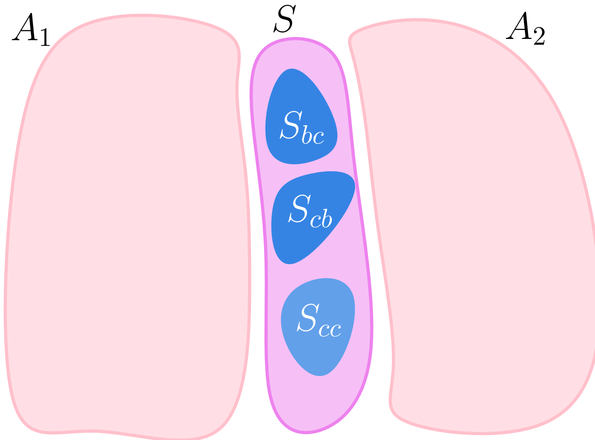
$R_a := \{p_1, q_1, p_2, q_2\}$

$S_{aa} \cup S_{ab} \cup S_{ba} \cup S_{ac} \cup S_{ca} \subseteq N(R_a)$

Proof

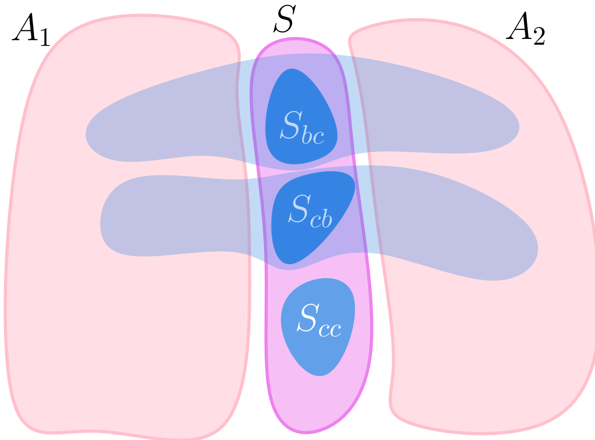


Proof



Constructing R_{bc}, R_{cb}, R_{cc} such that

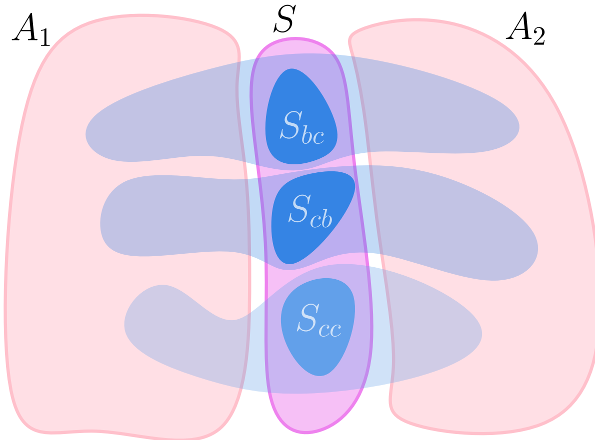
Proof



Constructing R_{bc}, R_{cb}, R_{cc} such that

- $S_{bc} \subseteq N[R_{bc}]$
- $S_{cb} \subseteq N[R_{cb}]$

Proof



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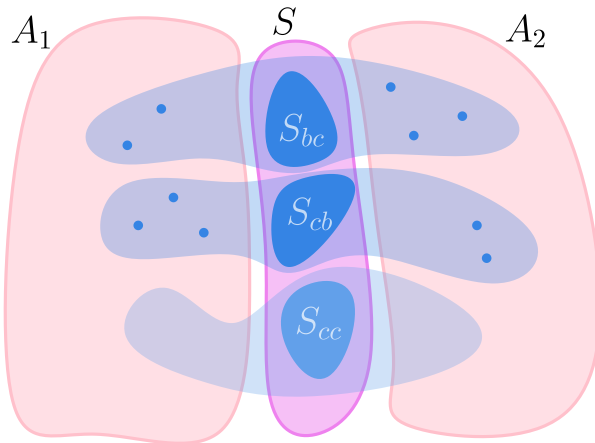

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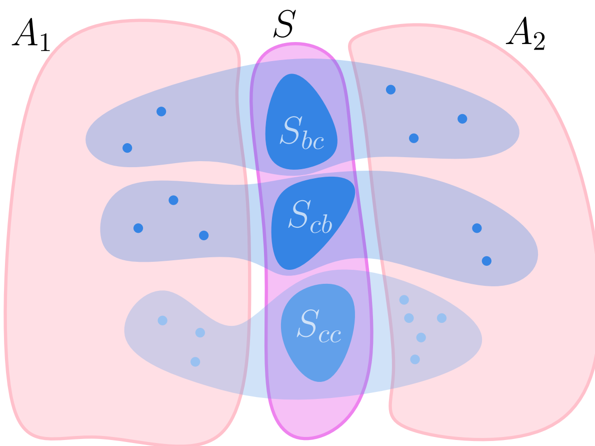
$|R_{bc}|, |R_{cb}| \leq 5$

Proof



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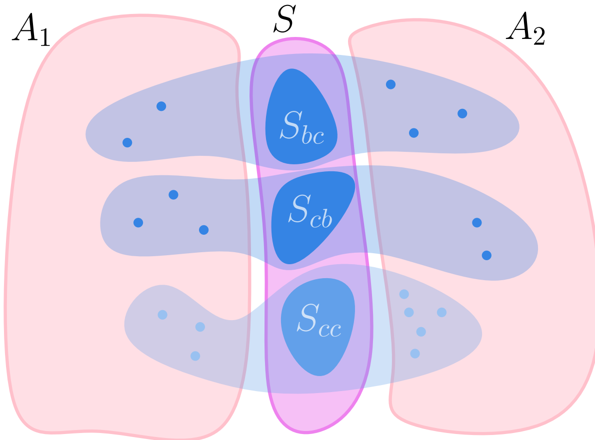
$|R_{cc}| \leq 8$

Proof



$|R_a| = 4$

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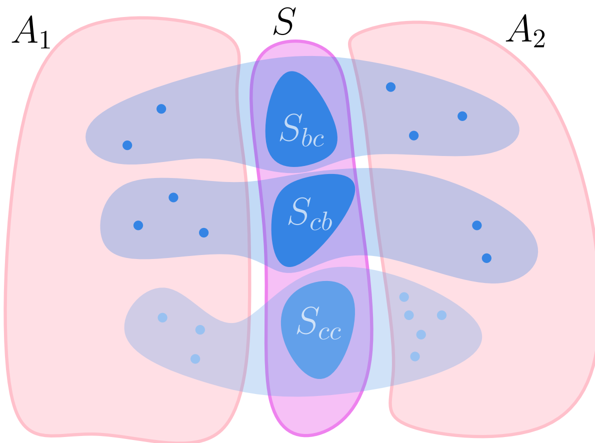

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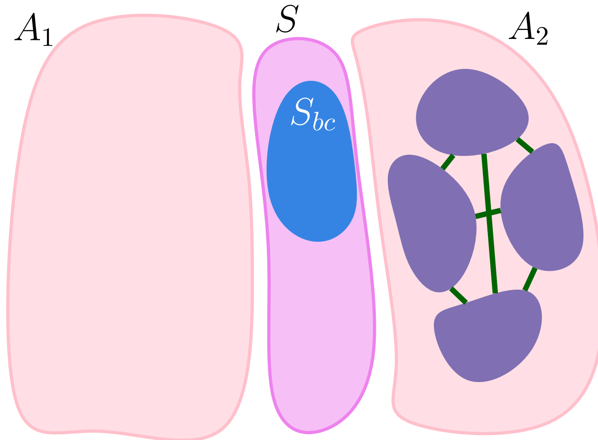
$|R_{bc}|, |R_{cb}| \leq 5$

$|R_{cc}| \leq 8$

$|R_a| = 4$

Then $S_0 := R_a \cup R_{bc} \cup R_{cb} \cup R_{cc}$ satisfies theorem statement

Proof



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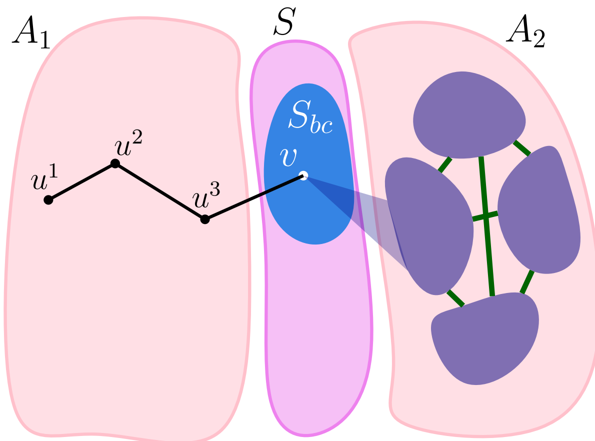
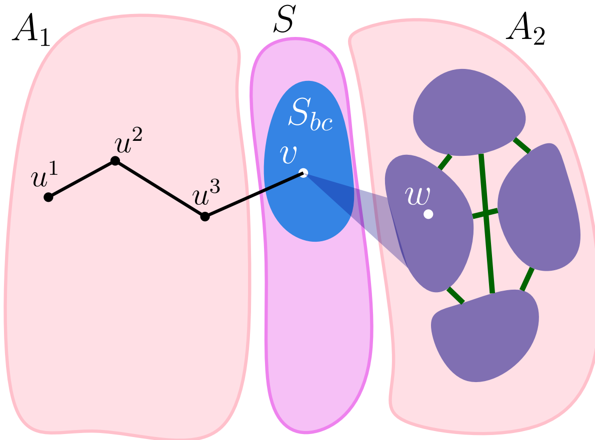


We choose $v \in S_{bc}$ to have minimal neighbourhood in A_2

Proof



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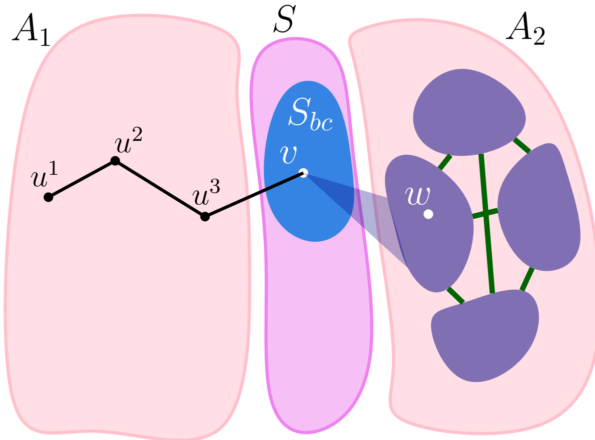

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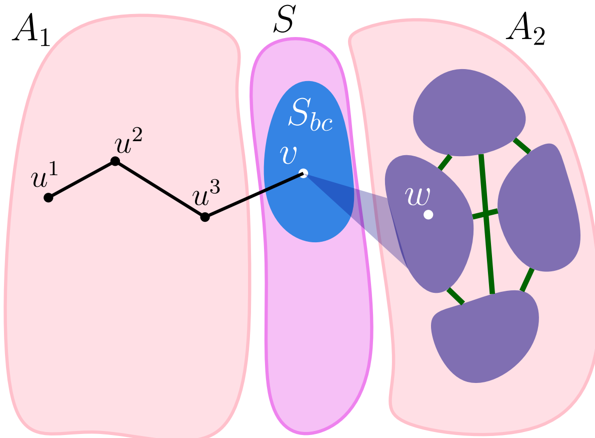

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Claim: $S_{bc} \subseteq N[R_{bc}]$

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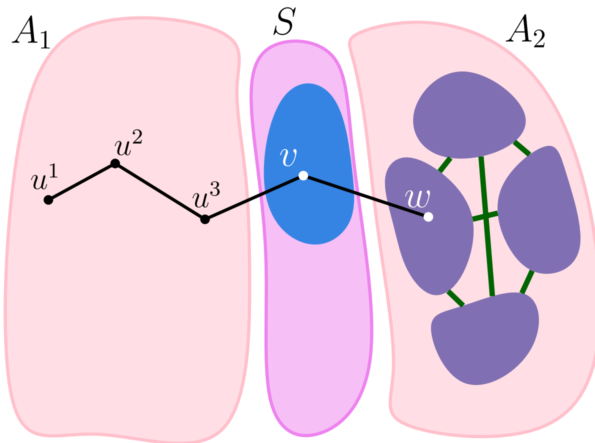

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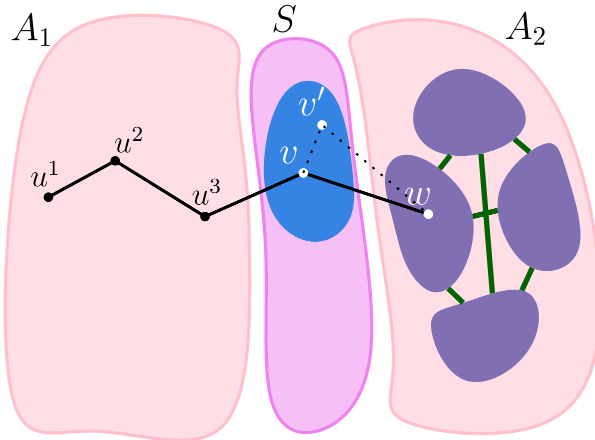

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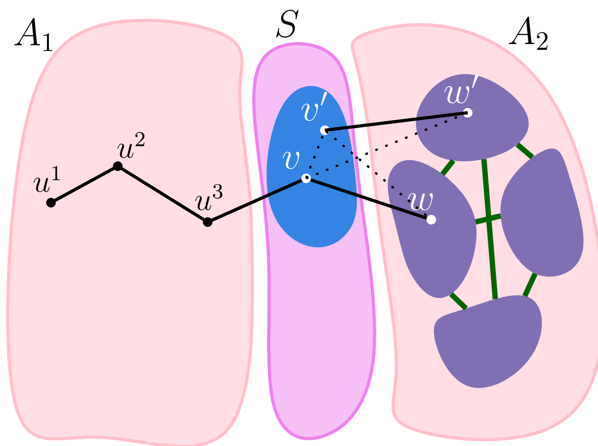

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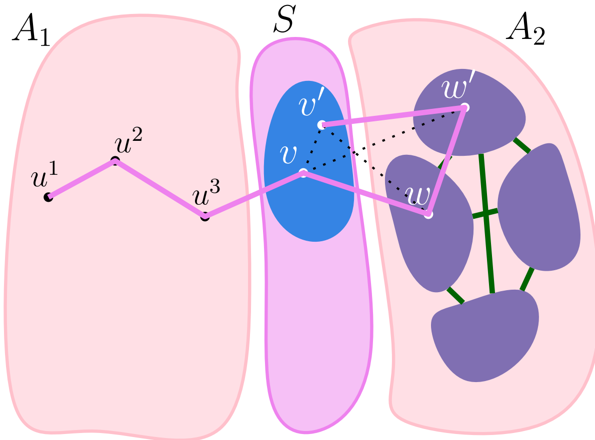

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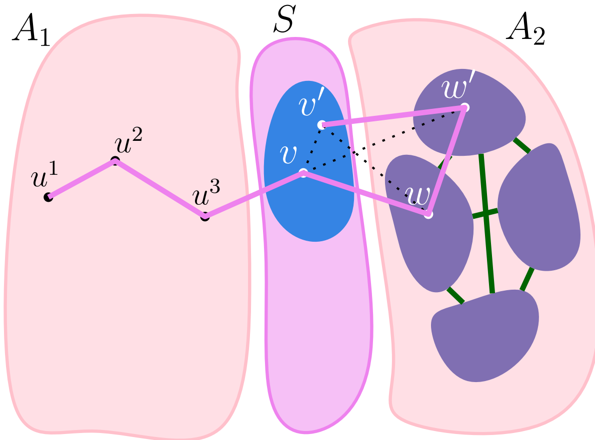
Path u^1-u^2-u^3-v-w-w'-v' gives contradiction

$\square$

Proof



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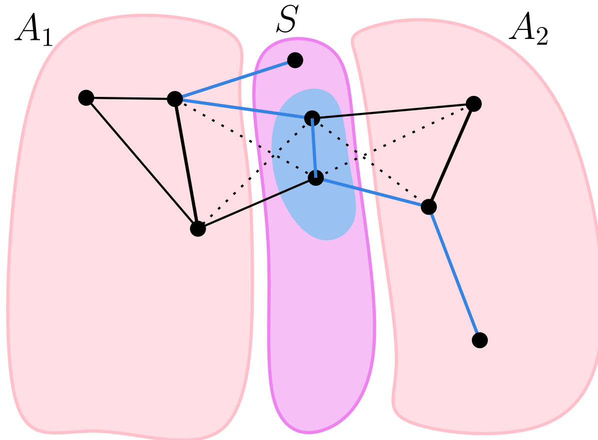
$\square$

proof of R_{cb} is analogous

Proof



R_{cc} case

Thank you