Exercises: Modular Arithmetic 1

Exercise 1

This exercise practices the basic skills you need to solve the subsequent exercises.

1.1 Divisors

Find all the divisors of the following numbers:

a. 10
b. 11
c. 12
d. 13
e. 99
f. 100

Determine whether each of the following are true:

a. $50 \mid 100$
b. $51 \mid 100$
c. $2 \nmid 4$
d. $2 \nmid 5$

1.2 Remainders

Find the remainders below:

a. $\text{rem}(17, 12)$
b. $\text{rem}(10, 5)$
c. $\text{rem}(10, 4)$
d. $\text{rem}(100, 20)$
e. $\text{rem}(20, 100)$
f. $\text{rem}(8, 10)$

1.3 Primes

Which of the following statements about primes are true?

a. A prime is a number that only has one divisor.
b. A prime is a number that has exactly two divisors.
c. All primes are odd.
d. There are 9 primes smaller than 20.

Which of the following numbers are primes?

a. 1
b. 2
c. 21
d. 31
e. 499
f. 512

Exercise 2

Find the prime factorization of

a. 120
b. 375
c. 47
d. 2145

Exercise 3

In exercise 1.1 you found all the divisors of a list of numbers. Use these results to find the greatest common divisors below:

a. $\gcd(10, 12)$
b. $\gcd(10, 11)$
c. $\gcd(11, 10)$
d. $\gcd(11, 13)$
e. $\gcd(10, 100)$
f. $\gcd(99, 100)$

Exercise 4

Use Euclid's algorithm to find the greatest common divisor in each of the following pairs of numbers:

a. $\gcd(9, 30)$
b. $\gcd(102, 38)$
c. $\gcd(62, 391)$

Exercise 5

Two numbers, a and b, are called relatively prime if $\gcd(a, b) = 1$. Answer the questions below.

a. Is 2 relatively prime to 5 ?
b. Is 7 relatively prime to 20 ?
c. Is 7 relatively prime to 14 ?
d. Are 6 and 9 relatively prime to each other?
e. Are 15 and 20 relatively prime to each other?
f. Is 1 relatively prime to 21 ?

Exercise 6

If n is some positive integer, we can calculate how many of the numbers between 1 and n that are relatively prime to n as $\varphi(n)$ - this function is called Euler's phi-function. Use Euler's phi-function to answer the following questions:

a. How many numbers between 1 and 5 are relatively prime to 5?
b. How many numbers between 1 and 17 are relatively prime to 17?
c. What is $\varphi(15)$?
d. What is $\varphi(14)$?
e. Let $1 \le n \le 25$. How many values of n fulfills $\gcd(n, 25) = 1$?
f. Let $1 \le n \le 8$. How many values of n fulfills $\gcd(n, 8) = 1$?

Exercise 7

Two numbers are said to be "congruent to each other modulo n" if they have the same remainder after division by n. For example, 1 is congruent to 11 modulo 5 , because both have the same remainder after division by 5: $\text{rem}(1, 5) = \text{rem}(11, 5)$. Which of the following numbers are congruent to each other modulo 4 ?

a. Is 1 congruent to 5 modulo 4?
b. Is 2 congruent to 5 modulo 4 ?
c. Is 0 congruent to 4 modulo 4 ?
d. Is 0 congruent to 8 modulo 4 ?
e. Is 3 congruent to 7 modulo 4 ?
f. Is 3 congruent to 10 modulo 4 ?

Exercise 8

If a is congruent to b modulo n, we can write this as $a \equiv b(\text{mod} n)$. For example, the statement that 1 is congruent to 11 modulo 5 can be written as $1 \equiv 11 \pmod 5$. Use this notation to write each of the statements from exercise 7 to which the answers was yes.

Exercise 9

Just as regular arithmetic revolves around the equality symbol, =, (e.g. solving equations like $5x-3 = 7$), modular arithmetic revolves around the congruence symbol, $\equiv (\text{mod} n)$. So in modular arithmetic, we solve congruences like $5x - 3 \equiv 7 \pmod 4$. Solve each of the congruences below (you are allowed to add or subtract on both sides just like in regular arithmetic):

a. $x - 1 \equiv 0(\text{mod} 4)$

b. $x + 2 \equiv 1(\text{mod} 4)$

c. $x - 10 \equiv 3(\text{mod} 4)$

d. $x + 2 \equiv 4(\text{mod} 7)$

e. $1 - x \equiv 2(\text{mod} 5)$

Exercise 10

In each of the exercises in exercise 9, find the smallest positive value for x which fulfills the congruence.