6. Sequences, series and recurrenc

Consider:

$$\sum_{k=0}^{\infty} \frac{4(-1)^k}{2k+1}$$

Let's try $k = [5, 10, 100, 1000]$

Consider:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = s$$

$$\frac{1}{2}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\right) = \frac{1}{2} s$$

$$\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = \frac{1}{2} s$$

$$s - \frac{1}{2} = \frac{1}{2} s$$

$$s - \frac{1}{2} s = \frac{1}{2}$$

$$\frac{1}{2} s = \frac{1}{2}$$

$$\underline{s = 1}$$

As a sum:

$$s = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + = \sum_{k=1}^{\infty} \frac{1}{2^i}$$

Let's see for which i we get $s = 1$ in Java

Sequence:

A progression of numbers:

$\{a_1, a_2, a_3, \ldots\}$ or $\{a_0, a_1, a_2, \ldots\}$

e.g. $\{3, 7, 11, 15, 19, 23, 27 \ldots\}$

Series:

The sum of a sequence is called a Series

Sum notation:

This is when it stops.

$$\sum_{k=0}^{n} 2^k$$

This is the element that increments

This increments starting at 0

↳ may start on where as long as start $\leq n$.

ex:

$$\sum_{k=0}^{4} 2^k = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 = 1+2+4+8+16$$
$$= \underline{\underline{31}}$$

$$\sum_{k=1}^{5} 2^k - 1 = (2^1-1) + (2^2-1) + (2^3-1) + (2^4-1) + (2^5-1)$$
$$= 1 + 3 + 7 + 15 + 31 = \underline{\underline{57}}$$

$$\sum_{i=0}^{3} (i+1)! = (0+1)! + (1+1)! + (2+1)! + (3+1)!$$
$$= 1! + 2! + 3! + 4! = 1 + 2\cdot1 + 3\cdot2 + 4\cdot3\cdot2$$
$$= 1+2+6+24 = \underline{\underline{33}}$$

Explicit

a_n is expressed in terms of n:

$a_n = 3\cdot 2^n$

$a_n = 3(n-1)+7$

Recursive

a_n is expressed in terms of a_{n-1}:

$a_n = a_{n-1} - 6$

↳ Recurrence Relation

→ $\boxed{\text{Do factorial and problem in Java}}$

Note:

- Indexing can start anywhere usually 0 or 1. Be cautious!
- You always need some initial condition. This is called The Base Case!
- Going from recursive to explicit is sometimes called "solving the recurrence"

Exercise 1

Given the sequence $\{a_n\}_{n=1}^{5} = \{1,3,5,7,9\}$

a. What is the value of a_3?

b. Find the value of $\sum_{n=1}^{5} a_n$.

a) $a_3 = 5$

b) $= 1+3+5+7+9 = 25$

Exercise 2

Expand the following series and find the sum

$$\sum_{n=0}^{4} 2n$$

$= 2\cdot 0 + 2\cdot 1 + 2\cdot 2 + 2\cdot 3 + 2\cdot 4 = 20$

Exercise 3

List the first four terms of the following sequence, beginning with $n = 0$

$$\{a_n\}_{n=0} = \frac{(-1)^n}{(n+1)!}$$

$\left\{1, -\frac{1}{2}, \frac{1}{6}, -\frac{1}{24}\right\}$

Exercise 4

Find the sum of the first six terms of $\{a_n\}_{n=1}$ where $a_n = 2a_{n-1} + a_{n-2}$, $a_1 = 1$ and $a_2 = 1$.

$$\left.\begin{aligned}
a_1 &= 1 \\
a_2 &= 1 \\
a_3 &= 2\cdot 1 + 1 = 3 \\
a_4 &= 2\cdot 3 + 1 = 7 \\
a_5 &= 2\cdot 7 + 3 = 17 \\
a_6 &= 2\cdot 17 + 7 = 41
\end{aligned}\right\}$$

$$\text{sum} = 1+1+3+7+17+41 = \underline{\underline{70}}$$

Exercise 5

Write the following series using summation notation, beginning with $n = 1$:

$$2 \ -4 \ +6 \ -8 \ +10$$

$$\sum_{n=1}^{5} (-1)^{n+1} \cdot 2n \quad \text{or} \quad \sum_{n=0}^{4} (-1)^{n} \cdot 2(n+1)$$

Exercise 6

Write the following using summation notation

$$\frac{5}{6+3} + \frac{5}{7+3} + \frac{5}{8+3} + \cdots + \frac{5}{31+3}$$

$$\sum_{n=6}^{31} \frac{5}{n+3} \quad \text{or} \quad \sum_{n=1}^{26} \frac{5}{(n+5)+3}$$

$$\text{or} \quad \sum_{n=0}^{25} \frac{5}{(n+6)+3}$$

<u>Arithmetic Sequence:</u>

$$\left.\begin{matrix} 3, \; 7, \; 11, \; 15, \; 19 \\ +4 \;\; +4 \;\; +4 \;\; +4 \end{matrix}\right\} \begin{matrix} \text{Keep adding } 4 \\ d = \text{difference} = 4 \end{matrix}$$

Explicit:

$a_n = a_1 + (n-1)\cdot d$ or $a_n = a_0 + n\cdot d$

Recursive:

$a_1 = ?$ $a_0 = ?$

$a_{n+1} = a_n + d$ $a_n = a_{n-1} + d$

ex:

9, 5, 1, -3:

Explicit

$a_n = 9 + (n-1)\cdot(-4)$

$a_n = 9 + n\cdot(-4)$

Recursive

$a_1 = 9$

$a_{n+1} = a_n - 4$

ex:

Find the 12th term of 2, 6, 10

Recursive: $a_1 = 2$

$a_{n+1} = ?$

Explicit: $a_{12} = 2 + (12-1)\cdot 4 = 46$

$a_{11} = 2 + 11\cdot 4 = 46$

Geometric Sequence:

a_1	a_2	a_3	a_4	...		
3	6	12	24	48	92	184

×2 ×2 ×2 ×2 ×2

Explicit:

$a_n = a_1 \cdot r^{n-1}$ or $a_n = a_0 \cdot r^n$

Recursive:

$a_1 =$

$a_{n+1} = a_n \cdot r$

or

$a_0 =$

$a_n = a_{n-1} \cdot r$

Ex: $18, -6, 2, -\frac{2}{3}$

$\times(-1/3) \quad \times(-1/3) \quad \times(-1/3)$

explicit:

$a_n = 18 \cdot (-1/3)^{n-1}$

Recursive:

$a_1 = 18$

$a_{n+1} = a_n \cdot (-1/3)$

Series:

Aritmetic

Partial sum:

$$S_n = \left(\frac{a_1 + a_n}{2}\right) \cdot n$$

ex: $3, 7, 11, 15, 19, \ldots$

1: $S_4 = \left(\frac{3+15}{2}\right) \cdot 4$

$= \underline{\underline{36}}$

$$S_n = \left(\frac{a_0 + a_{n+1}}{2}\right)(n+1)$$

S_∞ undefined

Geometric:

Partial sum:

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

ex: $3, 6, 12, 24, 48, 96 \ldots$

$$S_6 = \frac{3(1-2^6)}{1-2}$$

$$= \frac{3 \cdot (-63)}{-1} = 189$$

$$S_n = \frac{a_0(1-r^{n+1})}{1-r}$$

$$S_5 = \frac{3(1-2^{5+1})}{1-2} = 189$$

$$S_\infty = \frac{a_1}{1-r}, \ |r| < 1$$

Ex: $a_n = 10 + 2(n+1)$

Find recurrence and explicit:

$$\left.\begin{aligned} a_0 &= 12 \\ a_1 &= 14 \\ a_2 &= 16 \end{aligned}\right\} d = 2 \rightarrow$$

$a_n = a_{n-1} + 2$ | $a_{n+1} = a_n + 2$

$a_n = 12 + 2 \cdot n$ | $a_n = 12 + 2(n-1)$

Ex:

$a_n = 5 + 12(n-1)$ ← Explicit

a_1 (5), d (12)

$a_1 = 5$

$a_n = a_{n-1} + 12$

Sometimes we write:

$$f(n) = f(n-1) + 12 \quad , \quad \text{i.e } a_n = f(n)$$

ex: $a_n = a_{n-1} + 3$, $a_1 = 2$

Find explicit:

$a_n = 2 + 3(n-1)$

Using forward subst.

Ex1:

$a_{n+1} = a_n + 3 \quad , a_1 = 2$

$a_1 = 2$

$a_2 = 2 + 3$

$a_3 = (2+3) + 3 = 2 + 2 \cdot 3$

$a_4 = (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3$

$a_5 = (2 + 3 \cdot 3) + 3 = 2 + 4 \cdot 3$

$\vdots$

$a_n = 2 + (n-1) \cdot 3$

Ex2:

$a_n = a_{n-1} + 3 , a_0 = 3$

$a_0 = 3$

$a_1 = 3 + 3 = 2 \cdot 3$

$a_2 = (2 \cdot 3) + 3 = 3 \cdot 3$

$a_3 = (3 \cdot 3) + 3 = 4 \cdot 3$

$\vdots$

$a_n = (n+1) \cdot 3 = 3n + 3$

Ex: $a_n = 2 \cdot a_{n-1} - 1 \quad , a_1 = 3$

$a_1 = 3$

$a_2 = 2 \cdot 3 - 1$

$a_3 = 2 \cdot (2 \cdot 3 - 1) - 1 = 2^2 \cdot 3 - 3$

$a_4 = 2(2^2 \cdot 3 - 3) - 1 = 2^3 \cdot 3 - 7$

$a_5 = 2(2^3 \cdot 3 - 7) - 1 = 2^4 \cdot 3 - 15$

$\vdots$

?

$a_1 = 2 + 1$

$a_2 = 2(2+1) - 1 = 2^2 + 1$

$a_3 = 2(2^2 + 1) - 1 = 2^3 + 1$

$a_4 = 2(2^3 + 1) - 1 = 2^4 + 1$

$\vdots$

$a_n = 2(2^{n-1} + 1) - 1 = 2^n + 1$

Changing Limits

$$\sum_{j=1}^{n} a_j = \sum_{i=0}^{n-1} a_{i+1}$$

TABLE 2 Some Useful Summation Formulae.	
Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r-1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x\rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x\rvert < 1$	$\frac{1}{(1-x)^2}$

Ex: Visitors to a webpage is currently 293 and increases by 2.6%.

Find recurrence:

$a_0 = 293$

$a_n = a_{n-1} \cdot 1.026$

Explicit:

$a_n = a_0 \cdot r^n$

$= 293 \cdot 1.026^n$

Recall, $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+ \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} (1/2)^n = 1$

We have a similar closed form:

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad , |x|<1$$

$$x=\frac{1}{2}: \quad \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = \frac{1}{1-1/2} = 2$$

$$\left(\frac{1}{2}\right)^0 + \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k = 2$$

$$\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k = 1$$

<u>Ex:</u>

$$\sum_{k=0}^{\infty} \frac{3^k-2^k}{4^k} = \sum_{k=0}^{\infty} \left(\frac{3^k}{4^k} - \frac{2^k}{4^k}\right) = \sum_{k=0}^{\infty} \left(\left(\frac{3}{4}\right)^k - \left(\frac{1}{2}\right)^k\right)$$

$$= \sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k - \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$$

$$= \frac{1}{1-3/4} - \frac{1}{1-1/2} = 4-2 = \underline{\underline{2}}$$