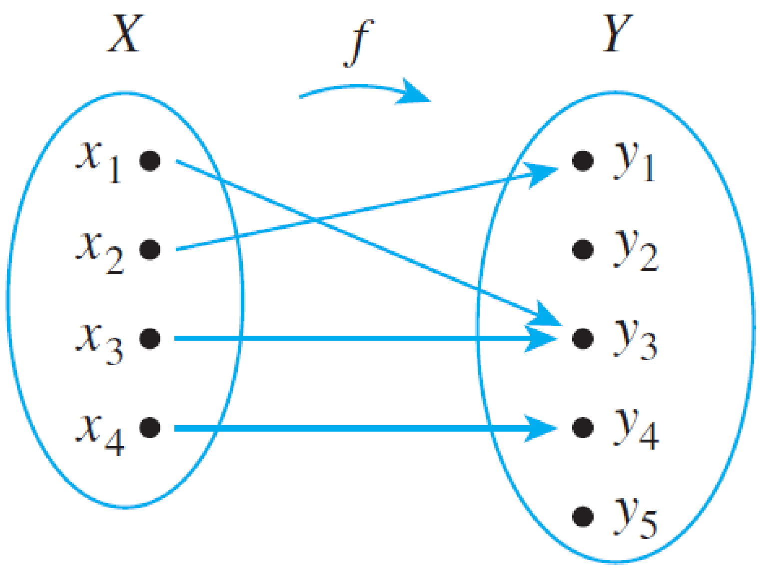
Gør tanke til handling

VIA University College

Functions

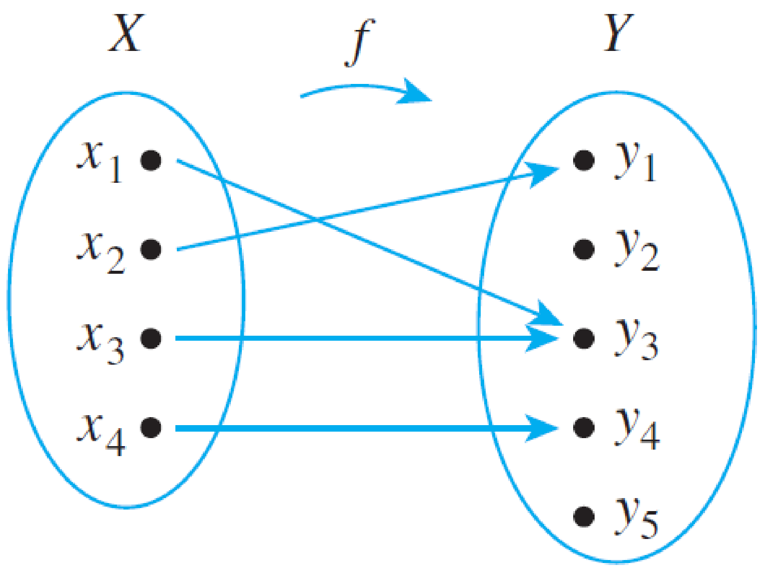
Definition of a function

A function from a set X to a set Y is an assignment of **exactly one** element of Y to each element of X.



We write $f(x_1) = y_3$ if y_3 is the element of Y assigned to the element x_1 of A (which is the case in function above).

Domain, codomain and range



X is called the **domain** of the function. This is the set the function is *from*.

Y is called the **codomain** of the function. This is the set the function is *to.*

The *subset* of the codomain which is "hit" by the function is called the **range** of the function – in this case, the range is $\{y_1, y_3, y_4\}$.

Real functions

A **real function** is a function from $\mathbb{R}$ (or some subset of $\mathbb{R}$) to $\mathbb{R}$.

Domain Codomain

Examples

- $f(x) = 3x - 5$ is a real function from $\mathbb{R}$ to $\mathbb{R}$

Real numbers larger than or equal to 0.

- $g(x) = \sqrt{x}$ is a real function from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}$

- $\mathrm{h}(x) = \frac{1}{x} + 2$ is a real function from $\mathbb{R}\backslash\{0\}$ to $\mathbb{R}$. Real numbers except 0.

The domain of a real function

The domain of a real function can only include values for which the function is actually defined.

- That's why the domain of $g(x) = \sqrt{x}$ from the last slide was $\mathbb{R}_{\geq 0}$: you cannot take the square root of a negative number, so these have to be excluded from the domain.
- Likewise, you cannot divide by 0, so the domain of $h(x) = \frac{1}{x} + 2$ is $\mathbb{R} \backslash \{0\}$.

The range of a real function

The **range** of a real function is the set of all numbers that the function can output. A couple of examples:

- The function $f(x) = 3x - 5$ can output all real numbers, so the range of f is $\mathbb{R}$.
- The function $g(x) = \sqrt{x}$ can only output real numbers equal to a larger than 0, so the range of g is $\mathbb{R}_{\geq 0}$.
- The function $h(x) = \frac{1}{x} + 2$ can output all real numbers except 2, so the range of h is $\mathbb{R} \backslash \{2\}$.

Real numbers except 2.

Examples

1) What is the domain and range of the function $f(x) = \frac{1}{x-5}$?

Answer: Domain: <u>All real numbers except 5</u>: $\mathbb{R}\backslash\{5\}$

Range: <u>All real numbers except 0</u>: $\mathbb{R}\backslash\{0\}$

2) What is the domain and range of the function $g(x) = \sqrt{x+3}$?

Answer: Domain: <u>All real numbers greater than or equal to -3</u>: $[-3, \infty)$

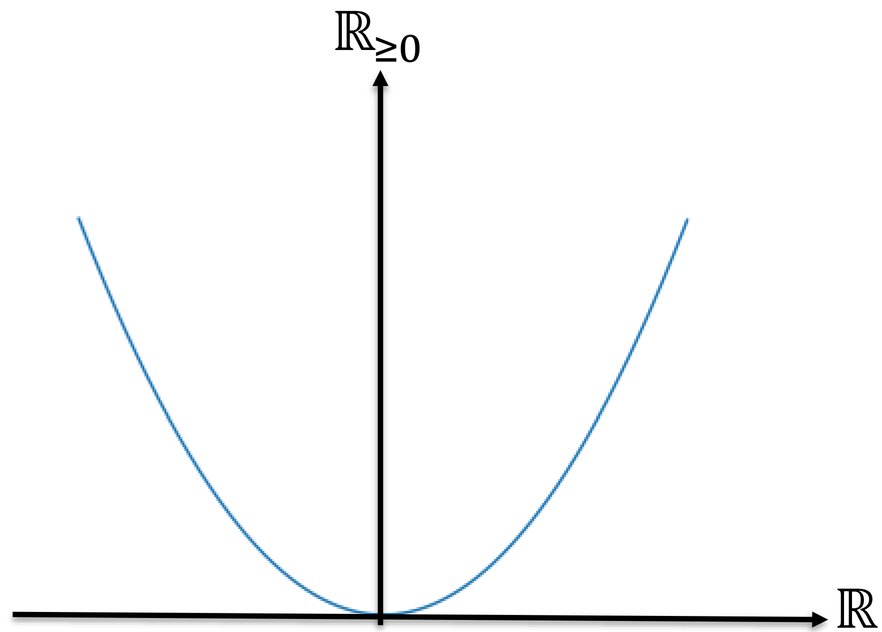
Range: <u>All positive real number</u>: $\mathbb{R}_{\geq 0}$

The graph of a real function

A great way to visualize a real function is by drawing it's graph.

<u>Example:</u>

To the left is shown the graph of the function $f(x) = x^2$ from $\mathbb{R}$ to $\mathbb{R}_{\geq 0}$



The graph of $f(x) = x^2$

Important real functions

In the following slides, a number of especially important functions will be introduced.

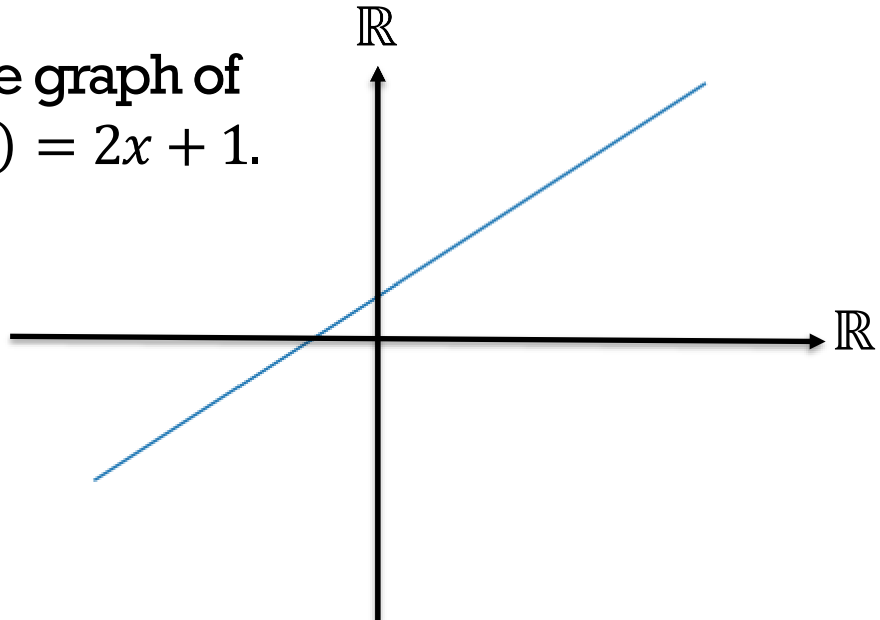
Linear functions

A linear function is a function of the form

$$f(x) = ax + b$$

<u>Example:</u>

The graph of
$f(x) = 2x + 1.$



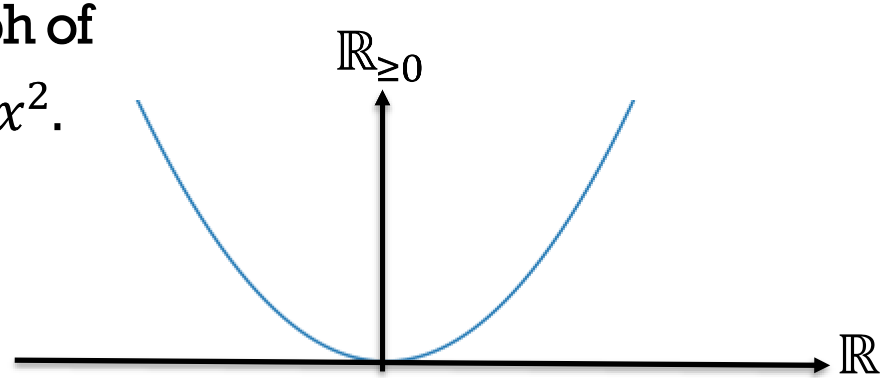
Polynomials

A linear function, $y = ax + b$, is actually a special case of the class of functions called **polynomials.** These are functions of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

<u>Example</u>

The graph of
$f(x) = x^2$.



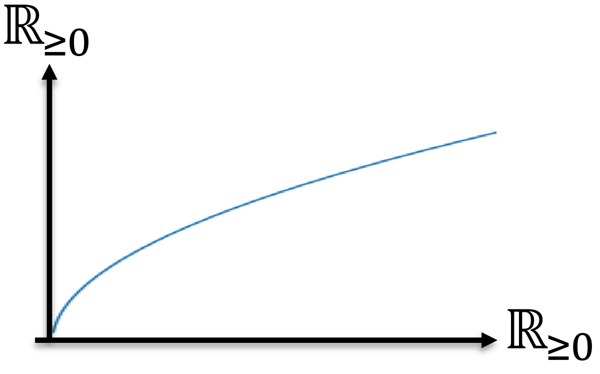
Power functions

Power-functions are functions of the form

$$f(x) = a \cdot x^b$$

Example

The graph of
$f(x) = x^{0.5}$.

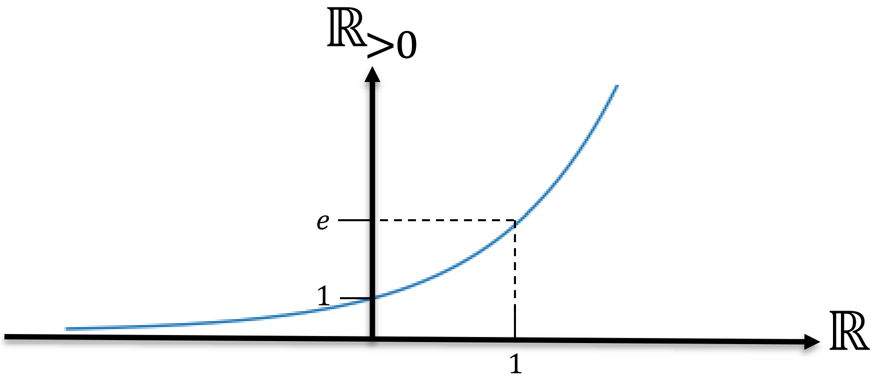


Exponential functions

Exponential functions are functions of the form

$$f(x) = a \cdot b^x$$

<u>Example</u>



The graph of $f(x) = e^x$.

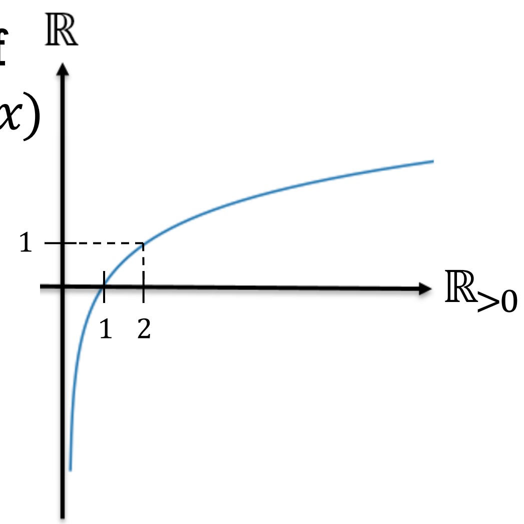
Logarithms

Logarithmic functions are functions of the form:

$$f(x) = \log_b(x).$$

In this expression, $\log_b$ is called the **logarithm of base b**.

The graph of
$f(x) = \log_2(x)$



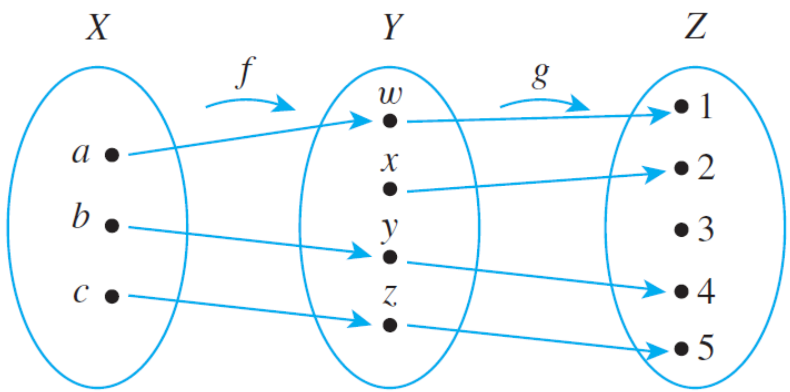
This logarithm of base 2
is often written as $\lg(x)$

Note that the logarithm is only defined for positive values of x!

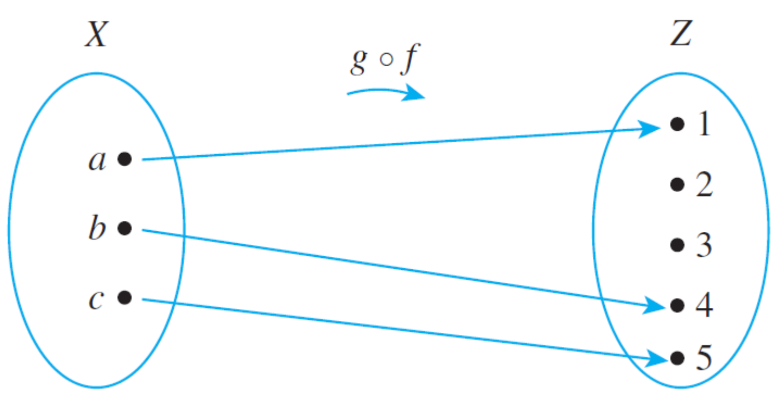
Composite functions

Composite functions

Consider the functions f and g below:



The range of the function f is part of the domain of the function g. Therefore, it is possible to take any output of f and use that as an input to the function g. The function obtained by doing this is called **the composition between f and g**, and is denoted $g \circ f$:



Composite functions

If the range of one function (f) is a subset of the domain of another function (g), we can take the output of f and put that into g. For example:

$$f(x) = 8x^2 \text{ from } \mathbb{R} \text{ to } \mathbb{R}^+$$
$$g(x) = x + 1 \text{ from } \mathbb{R} \text{ to } \mathbb{R}.$$

$$g\big(f(x)\big) = g(8x^2) = 8x^2 + 1.$$

We denote the composite function $g \circ f$, i.e. $(g \circ f)(x) = g(f(x))$.

Example

Consider the real functions f and g given by

$$f(x) = x^2 \quad \text{and} \quad g(x) = 2x + 4.$$

Since the range of g is equal to the domain of f, we can construct the composite function $f \circ g$:

1) Construct the composite function $f \circ g$.

   Answer: $(f \circ g)(x) = f\big(g(x)\big) = f(2x + 4) = (2x + 4)^2 = 4x^2 + 16x + 16.$

2) What is the domain and range of $f \circ g$?

   Answer: The domain is the domain of g, $\mathbb{R}$, and the range is the range of f, $\mathbb{R}^+$.

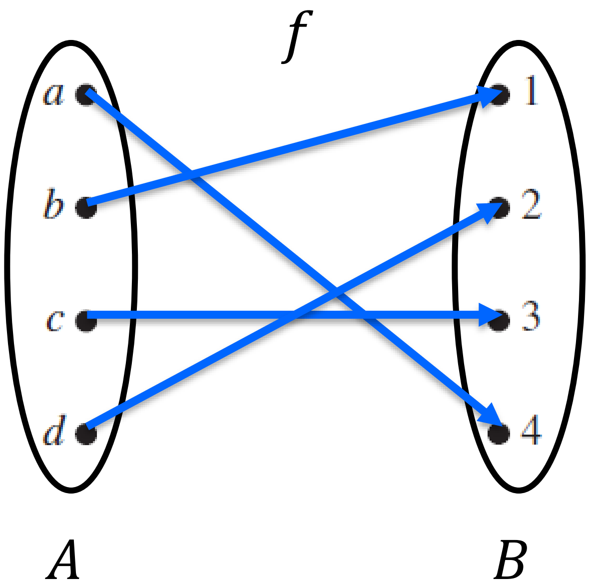
3) Find the value of $(f \circ g)(3)$.

   Answer: $(f \circ g)(3) = (2 \cdot 3 + 4)^2 = 10^2 = 100.$

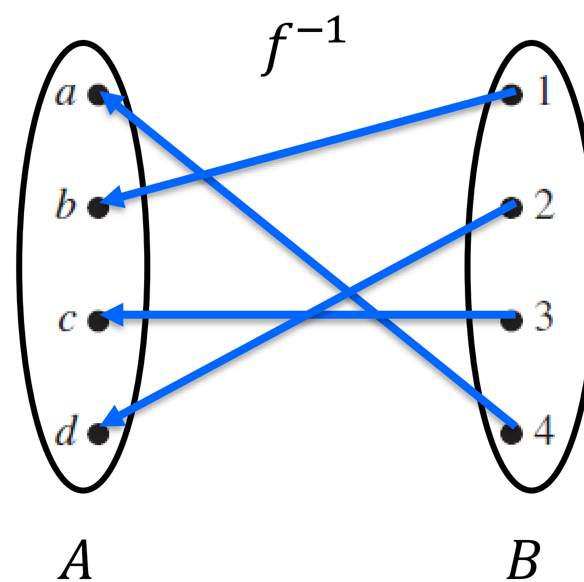
The inverse of a function

The inverse of a function

The inverse of a function f is the function, f^{-1}, whose domain is equal to the range of f and whose range is equal to the domain of f, and goes the opposite way of f.



Find the inverse function by turning each arrow around:



NOTE: It is only possible to find an inverse of f if

- **$f's$ range is equal to its codomain (if this is the case, we say that f is "onto")**
- **f maps all inputs to different outputs (if this is the case, we say that f is "one-to-one")**

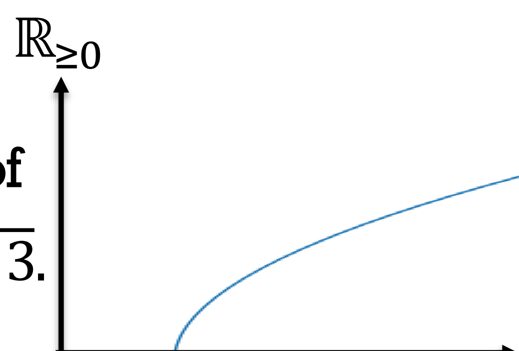
The graph of the inverse function

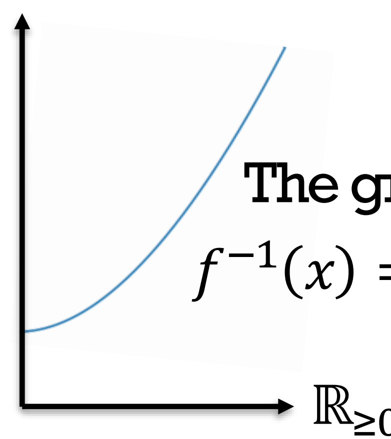
Consider the function $f\colon [3, \infty[\to \mathbb{R}_{\geq 0}$ given by $f(x) = \sqrt{x-3}$.

The function f^{-1} has the domain $\mathbb{R}_{\geq 0}$ and the range $[3, \infty[$, and for each output y of f, f^{-1} outputs the value x that produces that output.

Graphically, this is obtained by interchaning the x- and y-axis in the graph of f, and reflecting the function in the line $x = y$:

The graph of $f(x) = \sqrt{x-3}$.





The graph of $f^{-1}(x) = x^2 + 3$

The inverse of a real function

Consider the function $f(x) = 3x + 1$. We can determine the inverse function by isolating x in the expression $y = 3x + 1$:

$$x = \frac{y - 1}{3}.$$

This expression tells us how to obtain x if we know y.

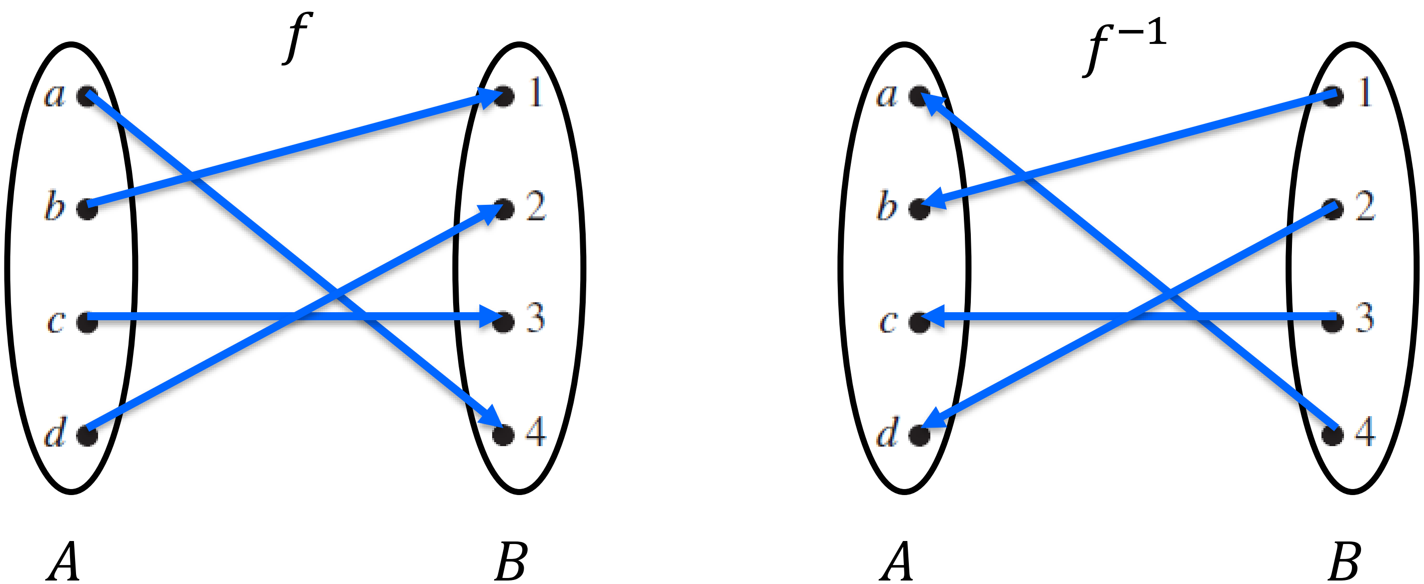
At last, we write it as a function:

$$f^{-1}(x) = \frac{x - 1}{3}.$$

Composition of f and f^{-1}

What do you get if you take $(f \circ f^{-1})(4)$? How about $(f^{-1} \circ f)(a)$?

Answer: $(f \circ f^{-1})(4) = 4$ and $(f^{-1} \circ f)(a) = a$.



<u>General rule</u>: For any (onto and one-to-one) function f,

$(f \circ f^{-1})(x) = f\big(f^{-1}(x)\big) = x$ and $(f^{-1} \circ f)(x) = f^{-1}\big(f(x)\big) = x$.

Checking the inverse

We can use the general rule from the last slide to check if the inverse function is correct.

<u>Example</u>

Consider the function $f(x) = 5x + 7$. To check if $f^{-1}(x) = \frac{1}{5}x - \frac{7}{5}$ is the inverse function, try to use the composition between f and f^{-1}:

$$(f \circ f^{-1})(x) = f\big(f^{-1}(x)\big) = f\left(\frac{1}{5}x - \frac{7}{5}\right) = 5 \cdot \left(\frac{1}{5}x - \frac{7}{5}\right) + 7 = x.$$

This shows that f^{-1} is indeed the inverse function to f.

More about logarithms and exponential functions

Exponents and logarithms

<u>Remember:</u>

The logarithm was first introduced in order to be able to isolate x in expressions like

$$a = b^x$$

We do this by taking the logarithm of base b on both sides:

$$\log_b(a) = x$$

This implies that logarithms are the inverse of exponential functions.

Conventions

The most used logarithms are the logarithm of base 2, the logarithm of base $e = 2.718\ldots$ and the logarithm of base 10. We will use the following conventions:

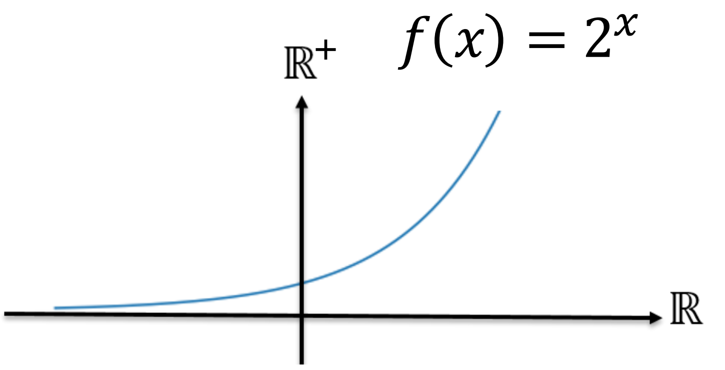
$\log_2$ will be written as $\lg$

$\log_e$ will be written as $\ln$

$\log_{10}$ will be written as $\log$

The logarithm of base e is also known as "the natural logarithm".

Example:
The inverse of an exponential functions



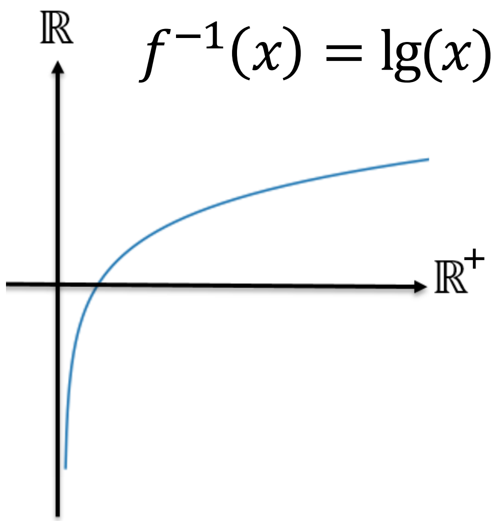
Find the inverse of the exponential function $f(x) = 2^x$

<u>Solution</u>

We isolate x:

$$y = 2^x \Leftrightarrow \lg(y) = x$$

So $f^{-1}(x) = \lg(x)$.



Calculation rules for exponents and logarithms

Below is a collection of useful calculation rules for exponents and logarithms.

$$b^m \cdot b^n = b^{m+n}$$

$$\frac{b^m}{b^n} = b^{m-n}$$

$$(b^m)^r = b^{m \cdot r}$$

$$b^0 = 1$$

$$b^1 = b$$

Radicals (roots):

$$a^{\frac{1}{y}} = \sqrt[y]{a}$$

$$\log_b(mn) = \log_b m + \log_b n$$

$$\log_b\left(\frac{m}{n}\right) = \log_b m - \log_b n$$

$$\log_b(m^r) = r \cdot \log_b m$$

$$\log_b 1 = 0$$

$$\log_b b = 1$$

Base-conversion:

$$\log_b m = \frac{\log_q m}{\log_q b}$$

Exponents and logarithms in practice

In the last slides we will see a few examples of where exponential functions and logarithms are used in practice.

Example – exponential growth

Imagine that you have a balance of 93 cents on your bank account on January 1st year 2000. The account has an interest of 2.25% which is compounded annually,

a) After how many years will the account balance exceed 2$?

b) What will the account balance be on January 1st year 3000?

<u>Solution</u>

We first find a functional expression for the account balance after x years:

$$f(x) = 0.93 \cdot 1.0225^x$$

a) We solve the equation $2 = 0.93 \cdot 1.0225^x$:

$$a)\ \ 2 = 0.93 \cdot 1.0225^x \Leftrightarrow \frac{2}{0.93} = 1.0225^x \Leftrightarrow x = \log_{1.0225}\left(\frac{2}{0.93}\right)$$

We can evaluate the last expression by using the base-conversion formula $\log_b(m) = \frac{\log_q(m)}{\log_q(b)}$

with $b = 1.0225, m = \frac{2}{0.93}$ and (e.g.) $q = 10$: $\quad x = \log_{1.0225}\left(\frac{2}{0.93}\right) = \frac{\log(2/0.93)}{\log(1.0225)} = 34.4$

So the balance will exceed 2$ after 35 years.

b) We insert $x = 3000 - 2000 = 1000$: $f(1000) = 0.93 \cdot 1.0225^{1000} = 4.28 \cdot 10^9$

So the balance on the 1st of January year 3000 will be 4.28 billion dollars.

Example - time complexity of an algorithm

The *time complexity* of an algorithm describes the amount of time it takes to run an algorithm expressed as a function of the size of the input. Typically, one compute the *worst case* time it will take to run the algorithm.

Example

Say you have a sorted list of $\mathrm{n} = 13$ grades given in a specific class, and you want to search through these to see whether any student got the grade 10.

- The most straight-forward way of doing this is simply to scan through the numbers one by one, until you find the number you want (or hit the end). This is called **linear search**, and the time it takes as the function of input size is linear; it could e.g. be $f(n) = 2n$ (measured in ms, i.e. milli-seconds).
- Alternatively, one could use a **binary search**. The time it takes in this case as a function of input size is logarithmic and could e.g. be $f(n) = \lg(n)$

a) How long time does it take to search through the list if you use a linear search? How long time does it take using the bineary search?

b) What if the input size increases to $n = 10{,}000$?

Answers:

a) Linear search: 26 ms; Binary search: 3.7 ms;

b) Linear search: 20,000 ms = 20 seconds; Binary search: 13 ms

Reference: https://www.geeksforgeeks.org/linear-search-vs-binary-search/