Graded homework — Asymptotics 2

TEAM-Berger-Nussbaum

1) <u>Little-Oh (o)</u> = $f(n) < c \cdot g(n)$

is not reflexive $\Rightarrow f(n) < c \cdot g(n)$ // $f(n) = 0$ & $g(n) = 0$ einsetzen

$0 < c \cdot 0$

$\Rightarrow$ There is no c that matches this inequality!

is transitive $\Rightarrow f(n) = o(g(n)) \Rightarrow f(n) < \boxed{c_1 \cdot g(n)}$

<u>$a<b$ && $b<c \Rightarrow a<c$</u> $\Rightarrow g(n) = o(h(n)) \Rightarrow \boxed{g(n)} < c_2 \cdot h(n)$ // $\cdot c_1$

$c_1 \cdot g(n) < c_1 \cdot c_2 \cdot h(n)$

$\Rightarrow f(n) < \underbrace{c_1 \cdot c_2}_{} \cdot h(n)$

$\Downarrow$

$f(n) = o(h(n))$ $\quad \underline{\underline{f(n) < c_3 \cdot h(n)}}$

<u>Little-Omega (ω)</u> = $f(n) > c \cdot g(n)$

is not reflexive $\Rightarrow f(n) > c \cdot g(n)$ // $f(n) = 0$ & $g(n) = 0$ einsetzen

$0 > c \cdot 0$

$\Rightarrow$ There is no c that matches the inequality

is transitive $\Rightarrow f(n) = \omega(g(n)) \Rightarrow f(n) > \boxed{c_1 \cdot g(n)}$

$\Rightarrow g(n) = \omega(h(n)) \Rightarrow \boxed{g(n)} > c_2 \cdot h(n)$ // $\cdot c_1$

$c_1 \cdot g(n) > c_1 \cdot c_2 \cdot h(n)$

$\Rightarrow f(n) > \underbrace{c_1 \cdot c_2}_{} \cdot h(n)$

$\Downarrow$

$f(n) = \omega(h(n)) \Rightarrow \underline{\underline{f(n) > c_3 \cdot h(n)}}$

2) Big-Oh (O) $\quad f(n) \le c \cdot g(n)$

is reflexive $\Rightarrow$ $f(n) \le c \cdot f(n)$

if $(c = 1)$ $\Rightarrow$ $f(n) \le 1 \cdot f(n)$ // ✓

if $(c > 1)$ $\Rightarrow$ $f(n) \le c \cdot f(n)$ // ✓ because right part will only grow...

if $(c < 1)$ $\Rightarrow$ $f(n) \le c \cdot f(n)$ // ✗

$\Rightarrow$ Big-Oh (O) is reflexive under the condition, that $c \ge 1$.

is transitive $\Rightarrow$ $f(n) = O(g(n)) \Rightarrow f(n) \le \boxed{c_1 \cdot g(n)}$

$g(n) = O(h(n)) \Rightarrow \boxed{g(n)} \le c_2 \cdot h(n)$ $\quad // \cdot c_1$

$\Downarrow$

$c_1 \cdot g(n) \le c_1 \cdot c_2 \cdot h(n)$

$\Rightarrow f(n) \le \underbrace{c_1 \cdot c_2}\cdot h(n)$

$f(n) = O(h(n)) \Rightarrow \underline{\underline{f(n) \le c_3 \cdot h(n)}}$

Big Omega (Ω) $\quad f(n) \ge c \cdot g(n)$

is reflexive $\Rightarrow$ $f(n) \ge c \cdot f(n)$

if $(c = 1)$ $\Rightarrow$ $f(n) \ge 1 \cdot f(n)$ // ✓

if $(c < 1)$ $\Rightarrow$ $f(n) \ge c \cdot f(n)$ // ✓ because right part will only shrink...

if $(c > 1)$ $\Rightarrow$ $f(n) \ge c \cdot f(n)$ // ✗

$\Rightarrow$ Big Omega (Ω) is reflexive under the condition, that $c \le 1$

is transitive $\Rightarrow$ $f(n) = \Omega(g(n)) \Rightarrow f(n) \ge \boxed{c_1 \cdot g(n)}$

$g(n) = \Omega(h(n)) \Rightarrow \boxed{g(n)} \ge c_2 \cdot h(n)$ $\quad // \cdot c_1$

$\Downarrow$

$c_1 \cdot g(n) \ge c_1 \cdot c_2 \cdot h(n)$

$\Rightarrow f(n) \ge \underbrace{c_1 \cdot c_2}\cdot h(n)$

$f(n) = \Omega(h(n)) \Rightarrow \underline{\underline{f(n) \ge c_3 \cdot h(n)}}$

Big Theta (Θ)

$$\underbrace{f(n) \le c_1 \cdot g(n)}_{\text{Big-Oh}} \cap \underbrace{f(n) \ge c_2 \cdot g(n)}_{\text{Big-Omega}}$$

is reflexive $\Rightarrow$ $f(n) \le c_1 \cdot f(n) \cap f(n) \ge c_2 \cdot f(n)$

if ($c_1 \ge 1$) AND if ($c_2 \le 1$)

$\Rightarrow$ ✓ because is only growing AND
because is only shrinking

else

$\Rightarrow$ ×

$\Rightarrow$ Big Θ is reflexive under the condition, that
c of $O(g(n))$ is ≥ 1 &&
c of $\Omega(g(n))$ is ≤ 1

is transitive $\Rightarrow$

$$f(n) = O(g(n)) \cap \Omega(g(n)) \Rightarrow f(n) \le c_1 \cdot g(n) \cap f(n) \ge c_2 \cdot g(n)$$

$$g(n) = O(h(n)) \cap \Omega(h(n)) \Rightarrow g(n) \le c_3 \cdot h(n) \cap g(n) \ge c_4 \cdot h(n) \quad // \cdot c_1$$

$$\underbrace{c_1 \cdot g(n)} \le c_1 \cdot c_3 \cdot h(n) \cap c_1 \cdot g(n) \ge c_1 \cdot c_4 \cdot h(n)$$

$$\Rightarrow f(n) \le c_1 \cdot c_3 \cdot h(n) \cap c_1 \cdot g(n) \ge c_1 \cdot c_4 \cdot h(n) \quad // \cdot c_2$$

$$\cancel{c_2} \cdot f(n) \le \cancel{c_2} \cdot c_1 \cdot c_3 \cdot h(n) \cap c_1 \cdot \underbrace{c_2 \cdot g(n)} \ge c_1 \cdot c_2 \cdot c_4 \cdot h(n)$$

$$\Rightarrow f(n) \le c_1 \cdot c_3 \cdot h(n) \cap \cancel{c_1} \, f(n) \ge \cancel{c_1} \cdot c_2 \cdot c_4 \cdot h(n)$$

$$\Rightarrow f(n) \le \underbrace{c_1 \cdot c_3} \cdot h(n) \cap f(n) \ge \underbrace{c_2 \cdot c_4} \cdot h(n)$$

$$f(n) \le c_5 \cdot h(n) \cap f(n) \ge c_6 \cdot h(n)$$

4) Big Theta (Θ)

is symmetric $\Rightarrow$ $f(n) = \Theta(g(n))$ // Declaration

$$\Downarrow \ldots \text{(work)}$$

$g(n) = \Theta(f(n))$ // Target

$$f(n) = O(g(n)) \cap \Omega(g(n)) \Rightarrow f(n) \leq c_1 \cdot g(n) \cap \boxed{f(n) \geq c_2 \cdot g(n)}$$

$$g(n) = O(f(n)) \cap \Omega(f(n)) \Rightarrow \boxed{g(n) \leq} c_3 \cdot f(n) \cap g(n) \geq c_4 \cdot f(n)$$

① // : c2

$$f(n) \leq c_1 \cdot g(n) \cap \frac{1}{c_2} \cdot f(n) > \frac{\cancel{c_2}}{\cancel{c_2}} \cdot g(n)$$

$$g(n) \leq c_3 \cdot f(n) \cap g(n) \geq c_4 \cdot f(n)$$

$$\Rightarrow c_3 = \frac{1}{c_2}$$

② // : c1

$$\frac{1}{c_1} f(n) \leq \frac{\cancel{c_1}}{\cancel{c_1}} \cdot g(n) \cap \frac{1}{c_2} \cdot f(n) > g(n)$$

$$g(n) \leq c_3 \cdot f(n) \cap g(n) \geq c_4 \cdot f(n)$$

$$\Rightarrow c_4 = \frac{1}{c_1}$$

In summary

$$c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n) \Rightarrow \frac{1}{c_2} \cdot f(n) \leq g(n) \leq \frac{1}{c_1} \cdot f(n)$$