Temporal Logic

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Temporal Logic

Temporal Logic deals with statements true at certain times.
Formulas are of form

$$\phi \text{ at n, where } n \in \mathbb{N}, \text{ where } \phi \text{ is a logical formula}$$

Extension of classical logic ($\top, \bot, \vee, \wedge, \Rightarrow, \neg$) with additional logical connectives.

The Added Connectives

$X\phi$ at $n \Leftrightarrow \ \phi$ at $n+1$

$G\phi$ at $n \Leftrightarrow \ \forall m \geq n : \phi$ at m

$\phi U \psi$ at n
$\Leftrightarrow \ (\psi$ at $n)$ or $(\exists w$: ψ at w and $\forall n \leq n' < w : \phi$ at $n')$

All other connectives, for example $F\phi$ ("At some point in the future ϕ will be true") can be constructed from existing ones.

It is not necessary to use $\mathbb{N}$.

Interpretation of temporal logic

```
⟦_⟧ : Formula → ℙ
⟦ ` P ⟧ n = η P
⟦ ⊤ ⟧ n = ⊤ʰ
⟦ ⊥ ⟧ n = ⊥ʰ
⟦ P ∧ Q ⟧ n = ⟦ P ⟧ n ∧ʰ ⟦ Q ⟧ n
⟦ P ∨ Q ⟧ n = ⟦ P ⟧ n ∨ʰ ⟦ Q ⟧ n
⟦ P ⇒ Q ⟧ n = ⟦ P ⟧ n ⇒ʰ ⟦ Q ⟧ n
⟦ X P ⟧ n = ⟦ P ⟧ (suc n)
⟦ G P ⟧ n = ∀ʰ ℕ ((λ m → (n ≤ʰ m) ⇒ʰ ⟦ P ⟧ m))
⟦ P U Q ⟧ n = ∃ʰ ℕ  (λ n' → (
                (n ≤ʰ n') ∧ʰ ((⟦ Q ⟧) n' ∧ʰ (∀ʰ ℕ (λ m → (
                            ((n ≤ʰ m) ∧ʰ (m <ʰ n')) ⇒ʰ ⟦ P ⟧ m
                      ))))
      ))
```

Natural deduction

Ordinary logic rules +

$$\text{X-elim}\ \frac{\Delta \vdash X(\phi)\ \text{AT}\ n}{\Delta \vdash \phi\ \text{AT}\ n+1}$$

$$\text{X-intro}\ \frac{\Delta \vdash \phi\ \text{AT}\ n+1}{\Delta \vdash X(\phi)\ \text{AT}\ n}$$

$$\text{G-elim}\ \frac{n \leq m \qquad \Delta \vdash G(\phi)\ \text{AT}\ n}{\Delta \vdash \phi\ \text{AT}\ m}$$

$$\text{G-intro}\ \frac{(m, n \leq m) \mapsto \Delta \vdash \phi\ \text{AT}\ m}{\Delta \vdash G(\phi)\ \text{AT}\ n}$$

Soundness

$$\text{Soundness} : \Delta \vdash \delta \text{ AT } n \rightarrow \text{proof}(\llbracket \Delta \rrbracket) \rightarrow \text{proof}(\llbracket \delta \rrbracket \ n)$$

Proving Tautologies

$$\vdash G(\phi \Rightarrow \psi) \Rightarrow (G(\phi) \Rightarrow G(\psi)) \text{ AT } n$$

$$\vdash X(\neg\phi) \Leftrightarrow \neg X(\phi) \text{ AT } n$$

$$\vdash X(\phi \Rightarrow \psi) \Rightarrow (X(\phi) \Rightarrow X(\psi)) \text{ AT } n$$

$$\vdash G(\phi) \Rightarrow \phi \wedge X(G(\phi)) \text{ AT } n$$

$$\vdash G(\phi \Rightarrow X(\phi)) \Rightarrow (\phi \Rightarrow G(\phi)) \text{ AT } n$$

Problems

$$[\phi \text{ at } n \text{ , } \psi \text{ at } m] \vdash \vartheta \text{ AT } k$$

- Inference rules
- Mixing times
- Implementing the until operator U
- Proving the

$$\vdash G(\phi \Rightarrow X(\phi)) \Rightarrow (\phi \Rightarrow G(\phi)) \text{ AT } n$$

by splitting $n \leq suc\ m$ on $n \leq m$ and $n = suc\ m$

Future

Perhaps formalizing the operator U

Proving Axiom 6 using well-founded induction.