Seminar

Hermitovi polinomi in normalna porazdelitev

Nejc Ševerkar & Tjaša Renko
mentor Janez Bernik

31. marec 2019

Povzetek

V tej nalogi bova predstavila povezavo med hermitovimi polinomi in standardno normalno porazdelitvijo.

Naloga

Hermitovi polinomi $(h_n)_{n\in\mathbb{N}}$ dveh realnih spremenljivk so definirani preko relacije

$e_a(x,t) := e^{ax-\frac{a^2t}{2}} = \sum_{n=0}^{\infty} \frac{a^n}{n!} h_n(x,t), \quad x,t \subset \mathbb{R}, a \in \mathbb{C}$

Naj bosta $X \sim N(0,1)$ in $Y \sim N(0,1)$ neodvisni standardni normalni spremenljivki.

(i) Izračunaj $E[e_a(X,1)e_b(X,1)](\{a,b\} \in \mathbb{C})$ in izpelji izraz za $E[h_n(X,1)h_m(X,1)]$ $(\{m,n\} \in \ \mathbb{N})$.

Pri prvem delu moramo izračunati $E\left(g\left(X\right)\right)$, kjer smo označili $g\left(x\right) = e_a\left(x,t\right) e_b\left(x,t\right)$. Za izračun upoštevamo izrek o matematičnem upanju transformacije slučajne spremenljivke, ki je razviden v računu:

$$
\begin{aligned}
E\left(e_a\left(X,t\right)e_b\left(X,t\right)\right) = E\left(g\left(X\right)\right) = \int_{-\infty}^{\infty} g\left(x\right) f_X\left(x\right) dx = \\
&= \int_{-\infty}^{\infty} e^{-\frac{t}{2}(a^2+b^2)} e^{x(a+b)} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= e^{-\frac{t}{2}\left(a^2+b^2\right)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}+x(a+b)} dx \\
&= e^{-\frac{t}{2}\left(a^2+b^2\right)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(x^2-2x(a+b)\right)} dx \\
&= e^{-\frac{t}{2}\left(a^2+b^2\right)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left((x-(a+b))^2-(a+b)^2\right)} dx \\
&= e^{-\frac{t}{2}\left(a^2+b^2\right)} e^{\frac{1}{2}(a+b)^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-(a+b))^2} dx \\
&= e^{\frac{1}{2}\left((a+b)^2-t\left(a^2+b^2\right)\right)} \int_{-\infty}^{\infty} f_Z(x) dx \quad (Z \sim N(a+b,1)) \\
&= e^{\frac{1}{2}\left((a+b)^2-t\left(a^2+b^2\right)\right)}
\end{aligned}
$$

Ko vstavimo $t = 1$ dobimo naslednji zvezi:

$$
E\left(e_a\left(X,1\right)e_b\left(X,1\right)\right) = e^{ab} = \sum_{i=0}^{\infty} \frac{(ab)^i}{i!}
$$

$$
E\left(e_a\left(X,1\right)e_b\left(X,1\right)\right) = E\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a^n b^m}{n!m!} h_n(x,1)h_m(x,1)\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a^n b^m}{n!m!} E\left(h_n(x,1)h_m(x,1)\right)
$$

Vidimo, da enakost med enačbama velja, ko $E\left(h_n\left(X,1\right)h_m\left(X,1\right)\right) = \begin{cases} n! & \text{za } m = n \\ 0 & \text{za } m \neq n \end{cases}$

(ii) Naj bo $a \in \mathbb{C}$ in $c \in \mathbb{R}$. Izrazi $E\left(e^{a(X+cY)}|X\right)$ z e_a, X in c ter $E\left((X+cY)^k|X\right)$ za $k \in \mathbb{N}_0$ z h_k, X, in c.

Najprej izpostavimo e^{aX} in uporabimo neodvisnost:

$$E\left(e^{a(X+cY)}|X\right) = e^{aX}E\left(e^{acY}\right)$$

Posebej izračunajmo $E\left(e^{acY}\right)$:

$$E\left(e^{acY}\right) = \int_{-\infty}^{\infty} e^{acy}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy =$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-\left(y^2-2acy\right)}{2}}dy =$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-\left((y-ac)^2-(ac)^2\right)}{2}}dy =$$

$$= e^{\frac{(ac)^2}{2}}$$

Torej dobimo:

$$E\left(e^{a(X+cY)}|X\right) = e^{aX}e^{\frac{(ac)^2}{2}} = e^{aX+\frac{a^2c^2}{2}} = e_a\left(X,-c^2\right)$$

Za izračun $E\left((X+cY)^k|X\right)$ uporabimo prejšnji račun in razvijemo $e^{a(X+cY)}$ v Taylorjevo vrsto:

$$E\left(e^{a(X+cY)}|X\right) = E\left(\sum_{n=0}^{\infty}\frac{a^n(X+cY)^n}{n!}|X\right) = \sum_{n=0}^{\infty}\frac{a^n}{n!}E((X+cY)^n|X)$$

Po definiciji hermitovih polinomov pa velja še $E\left(e^{a(X+cY)}|X\right) = \sum_{n=0}^{\infty}\frac{a^n}{n!}h_n\left(X,-c^2\right)$.
Dobimo torej $E\left((X+cY)^n|X\right) = h_n\left(X,-c^2\right)$.