Graph

Graph

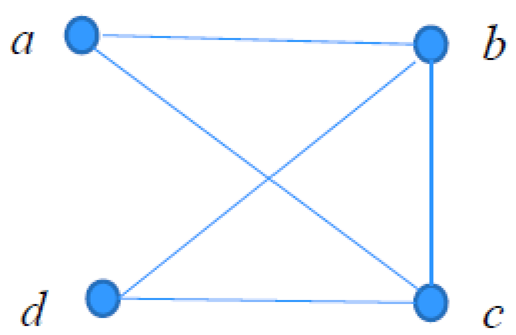
Graphs and graph theory can be used to model:

– Computer networks
– Social networks
– Communications networks
– Information networks
– Software design
– Transportation networks
– Biological networks

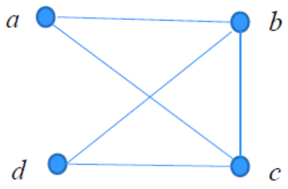
Graph

Definition: A *graph* *G = (V, E)* *consists of a nonempty set V of vertices* *(or nodes) and a set E of edges*.
Each edge has either one or two vertices associated with it, called its *endpoints.*
An edge is said to *connect its endpoints.*

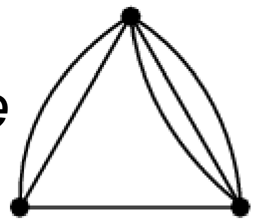
Example



Terminology



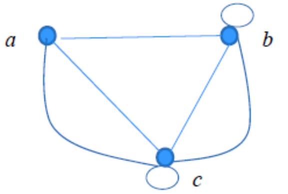
In a *simple graph each edge connects two different vertices and* no two edges connect the same pair of vertices.



Multigraphs may have multiple edges connecting the same two vertices. When *m different edges connect the vertices u and v, we* say that {*u,v*} *is an edge of multiplicity m.*

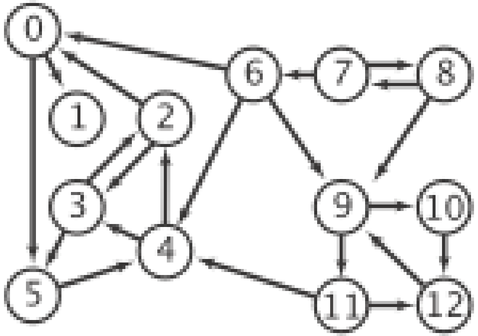
- An edge that connects a vertex to itself is called a *loop*.

- A *pseudograph may include loops, as well as multiple edges* connecting the same pair of vertices.



Directed graph

Definition: A directed graph (or digraph) G = (V, E) consists of a nonempty set V of vertices (or nodes) and a set E of directed edges (or arcs). Each edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u,v) is said to *start at u and end at v*.

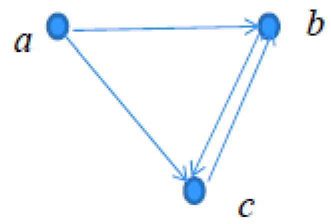


Remark:

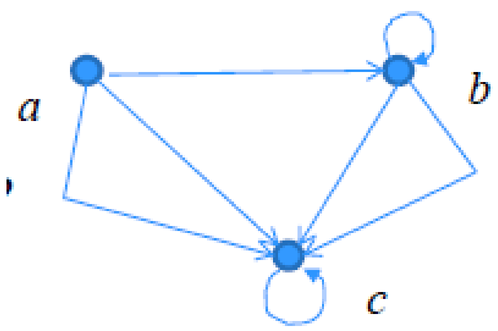
– Graphs where the end points of an edge are not ordered are said to be undirected graphs.

Directed graph

A simple directed graph has no loops and no multiple edges.
Example:



• A directed multigraph may have multiple directed edges. When there are m directed edges from the vertex u to the vertex v, we say that (u,v) is an edge of multiplicity m.



Example:

• multiplicity of (a,b) is ? 1
• and the multiplicity of (b,c) is ? 2

Undirected graph

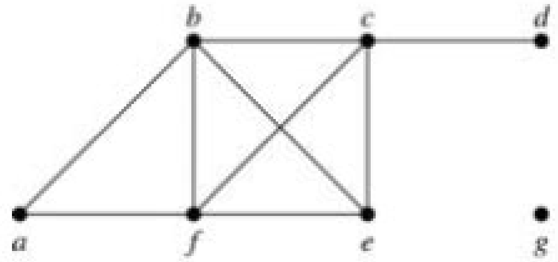
Definition 1. Two vertices u, v in an undirected graph G are called adjacent (or neighbors) in G if there is an edge e between u and v. Such an edge e is called incident with the vertices u and v and e is said to connect u and v.

Definition 2. The set of all neighbors of a vertex v of G = (V, E), denoted by N(v), is called the neighborhood of v. If A is a subset of V, we denote by N(A) the set of all vertices in G that are adjacent to at least one vertex in A. So,

Definition 3. The degree of a vertex in a undirected graph is the number of edges incident with it, except that a loop at a vertex contributes two to the degree of that vertex. The degree of the vertex v is denoted by deg(v).

Undirected graph

Example: What are the degrees and neighborhoods of the vertices in the graphs *G?*



Solution:

G: deg(a) = 2,

deg(b) = *deg(c)* = *deg(f)* = 4,

deg(d) = 1,

deg(*e*) = 3,

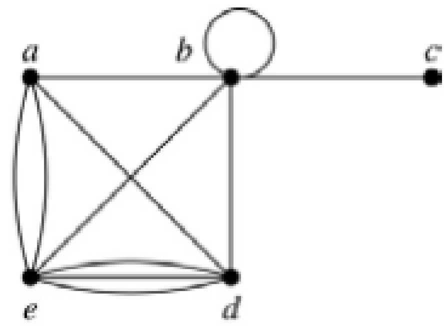
deg(g) = 0.

N(a) = {*b, f* }, *N(b)* = {*a, c, e, f* }, *N(c)* = {*b, d, e, f* },

N(d) = {*c*}, *N(e)* = {*b, c , f* }, *N(f)* = {*a, b, c, e*}, *N(g)* = ∅ .

Undirected graph

Example: What are the degrees and neighborhoods of the vertices in the graphs *H?*



Solution:

H: deg(a) = 4, deg(b) = deg(e) = 6, deg(c) = 1, deg(d) = 5.

N(a) = {b, d, e}, N(b) = {a, b, c, d, e}, N(c) = {b},

N(d) = {a, b, e}, N(e) = {a, b ,d}.

Undirected graph

Theorem 1 ***: If G = (V,E) is an*** undirected graph with *m edges, then*

$$2m = \sum_{v \in V} \deg(v)$$

Proof:

Each edge contributes twice to the degree count of all vertices. Hence, both the left-hand and right-hand sides of this equation equal twice the number of edges.

Undirected graph

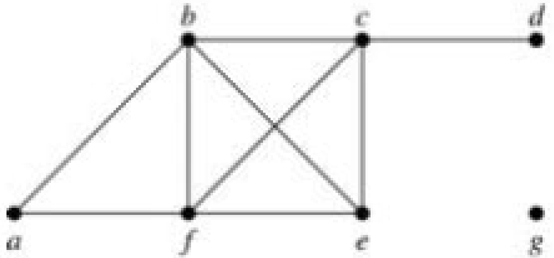
Theorem 2: An undirected graph has an even number of vertices of odd degree.

Proof: Let V_1 be the vertices of even degree and V_2 be the vertices of odd degree in an undirected graph *$G = (V, E)$ with m edges.*
Then

$$2m = \sum_{v \in V} \deg(v) = \sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v)$$

must be even since deg(v) *is* even for each $v \in V_1$

This sum must be even because $2m$ is even and the sum of the degrees of the vertices of even degrees is also even. Because this is the sum of the degrees of all vertices of odd degree in the graph, there must be an even number of such vertices.



Directed graph

Definition: An directed graph G = (V, E) consists of V, a nonempty set of vertices (or nodes), and E, a set of directed edges or arcs. Each edge is an ordered pair of vertices. The directed edge (u,v) is said to start at u and end at v.

Definition: Let (u,v) be an edge in G. Then u is the initial vertex of this edge and is adjacent to v and v is the terminal (or end) vertex of this edge and is adjacent from u. The initial and terminal vertices of a loop are the same.

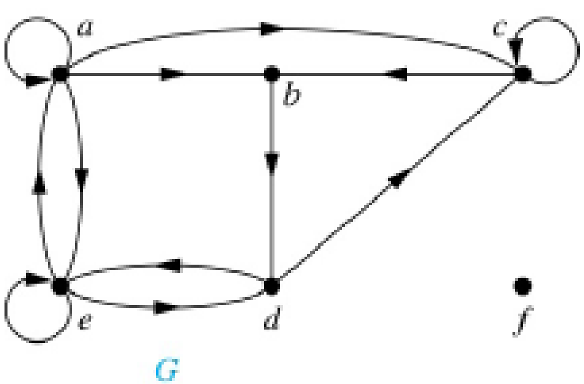
Directed graph

Definition: The *in-degree of a vertex v, denoted* $deg^{-}(v)$, *is* the number of edges which terminate at *v.*

The out-degree of v, denoted $deg^{+}(v)$, *is the number of edges with v as their initial* vertex.

Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of the vertex

Example: Assume graph G



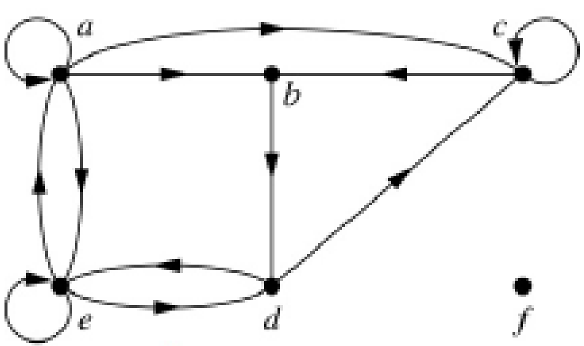
$deg^{-}(a)=2$, $deg^{-}(b)=2$, $deg^{-}(c)=3$,

$deg^{-}(d)=?$, $deg^{-}(e)=?$, $deg^{-}(f)=?$

Directed graph

Definition: The *in-degree of a vertex v, denoted* $deg^-(v)$*, is* the number of edges which terminate at *v. The out-degree of v,* denoted $deg^+(v)$*, is the number of edges with v as their initial* vertex. Note that a loop at a vertex contributes 1 to both the indegree and the out-degree of the vertex

Example: Assume graph G



G

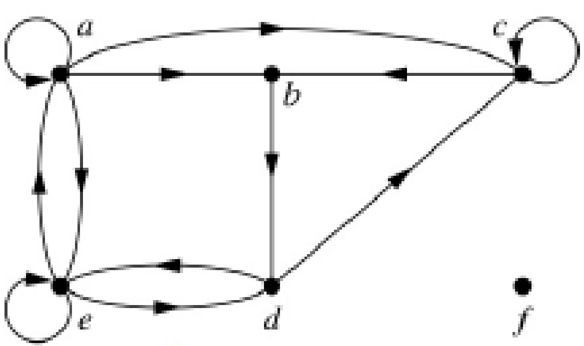
$deg^-(a)=2$, $deg^-(b)=2$, $deg^-(c)=2$,

$deg^-(d)=2$, $deg^-(e)=3$, $deg^-(f)=0$

Directed graph

Definition: The *in-degree of a vertex v, denoted* $deg^{-}(v)$*, is* the number of edges which terminate at *v. The out-degree of v,* denoted $deg^{+}(v)$*, is the number of edges with v as their initial* vertex. Note that a loop at a vertex contributes 1 to both the indegree and the out-degree of the vertex

Example: Assume graph G



G

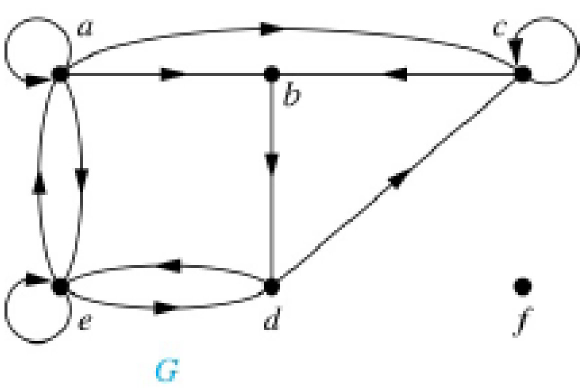
$Deg^{+}(a)=4$, $deg^{+}(b)=1$, $deg^{+}(c)=2$,

$deg^{+}(d)=?$, $deg^{+}(e)=?$, $deg^{+}(f)=?$

Directed graph

Definition: The *in-degree of a vertex v, denoted* $deg^-(v)$*, is* the number of edges which terminate at *v. The out-degree of v,* denoted $deg^+(v)$, *is the number of edges with v as their initial* vertex. Note that a loop at a vertex contributes 1 to both the indegree and the out-degree of the vertex

Example: Assume graph G



$Deg^+(a)=4$, $deg^+(b)=1$, $deg^+(c)=2$,

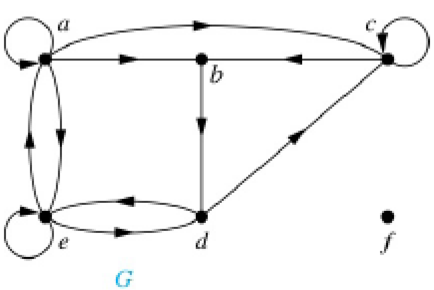
$deg^+(d)=2$, $deg^+(e)=3$, $deg^+(f)=0$

Directed graph

Theorem: Let *G = (V, E) be a graph with directed edges. Then:*

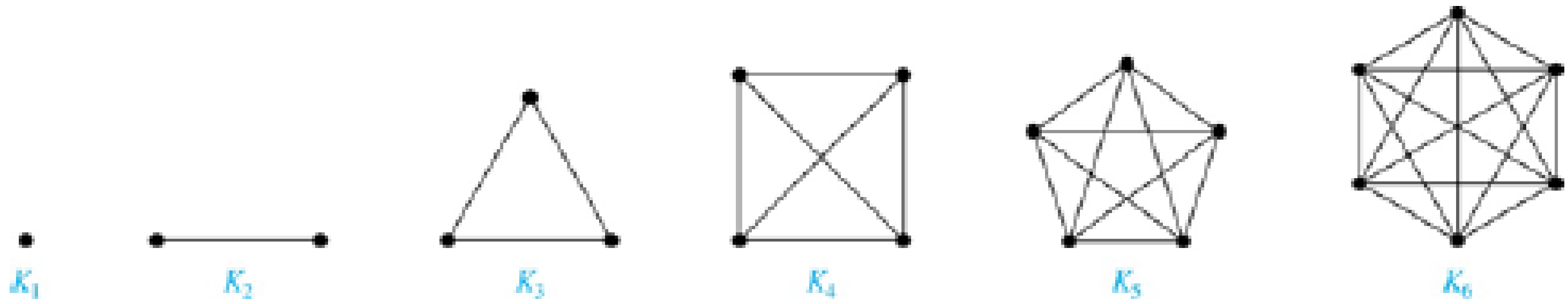
$$|E| = \sum_{v \in V} \deg^+(v) = \sum_{v \in V} \deg^-(v)$$

Proof:
The first sum counts the number of outgoing edges over all vertices and the second sum counts the number of incoming edges over all vertices. It follows that both sums equal the number of edges in the graph.



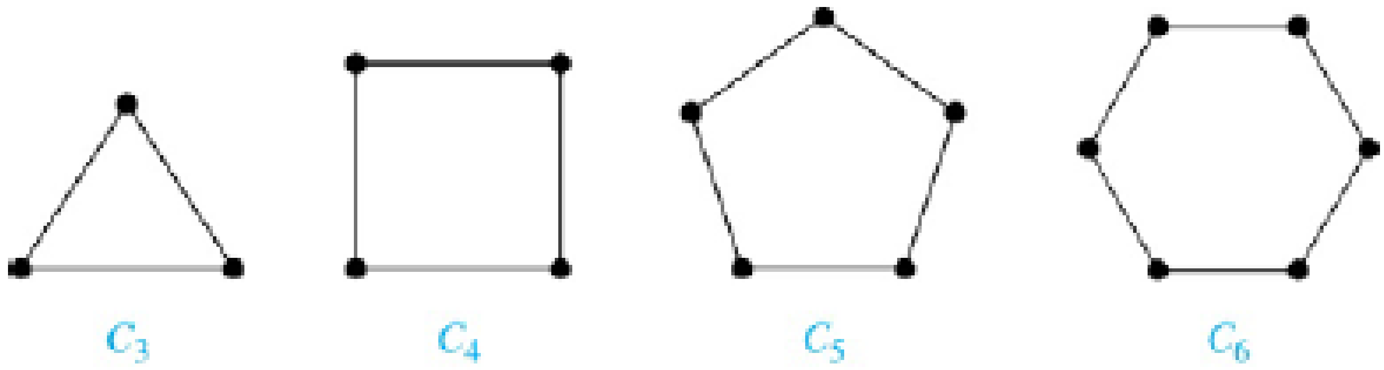
Complete graph

A *complete graph on n vertices, denoted by K_n, is the simple* graph that contains exactly one edge between each pair of distinct vertices.



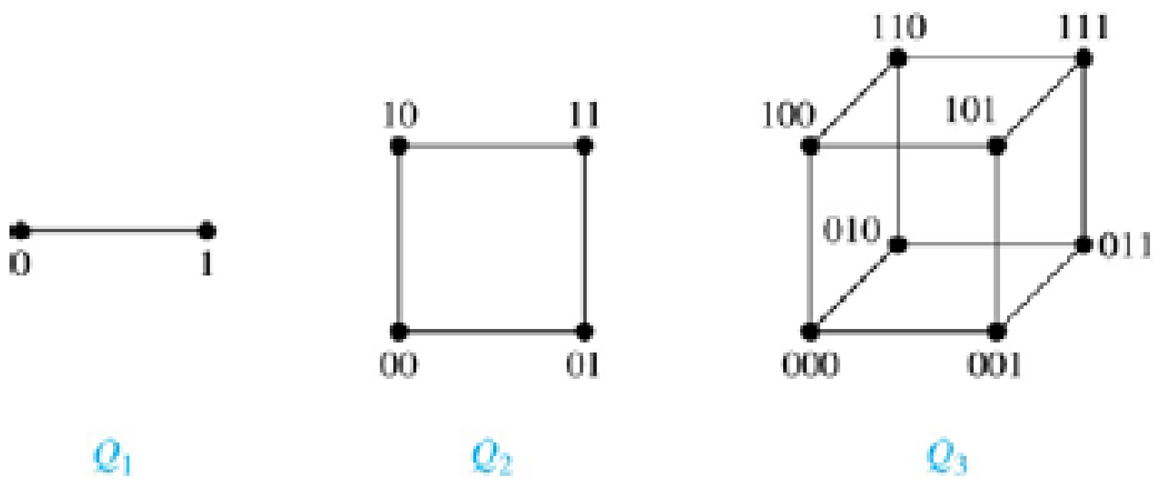
A Cycle

A *cycle* C_n for $n \geq 3$ consists of n *vertices* $v_1, v_2, \cdots, v_n$, *and edges* $\{v_1, v_2\}, \{v_2, v_3\}, \cdots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$.



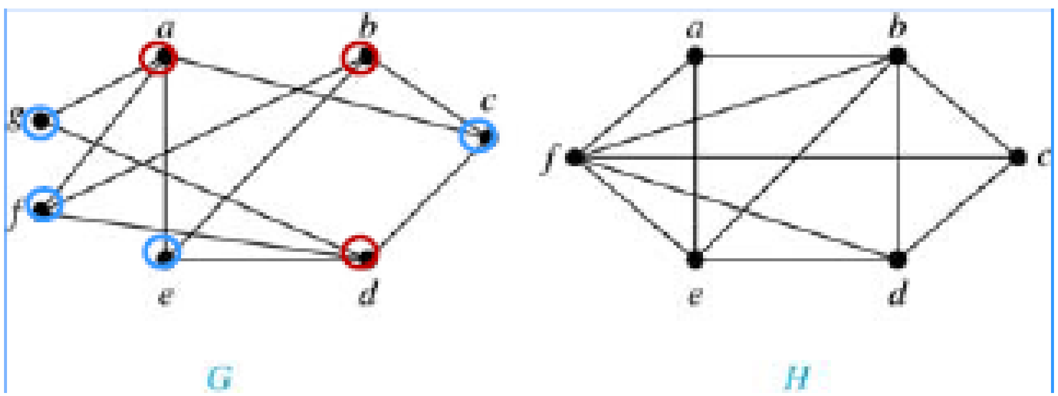
N-dimensional hypercube

An *n-dimensional hypercube, or n-cube*, Q_n, *is a graph with* 2^n vertices representing all bit strings of length n, *where there is an* edge between two vertices that differ in exactly one bit position.



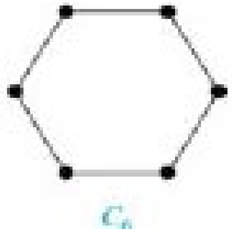
Bipartite Graph

Definition: A simple graph *G is bipartite if V can be partitioned* into two disjoint subsets V_1 *and* V_2 *such that every edge connects a vertex in* V_1 *and a vertex in* V_2. *In other words, there are no edges* which connect two vertices in V_1 *or in* V_2.



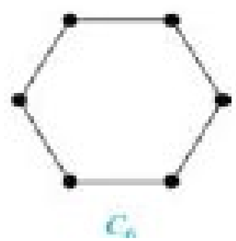
Bipartite Graph

Example: Show that C_6 *is bipartite.*

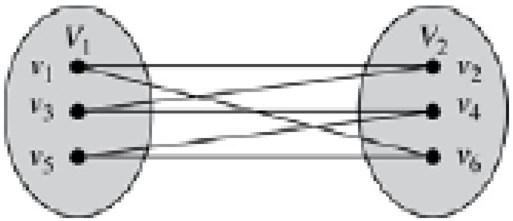


Bipartite Graph

Example: Show that C_6 *is bipartite.*

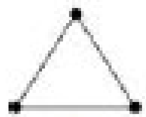


Solution: We can partition the vertex set into $V_1 = \{v_1, v_3, v_5\}$ *and* $V_2 = \{v_2, v_4, v_6\}$ *so that every edge of* C_6 *connects a vertex in* V_1 *and* V_2 *.*



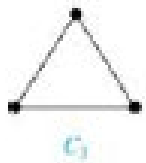
Bipartite Graph

Example: Show that C_3 *is not bipartite.*



Bipartite Graph

Example: Show that C_3 *is not bipartite.*

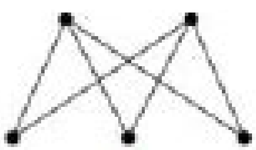


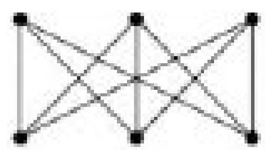
Solution: If we divide the vertex set of C_3 *into two nonempty sets, one of the* two must contain two vertices. But in C_3 *every vertex is connected* to every other vertex. Therefore, the two vertices in the same partition are connected. Hence, C_3 *is not bipartite.*

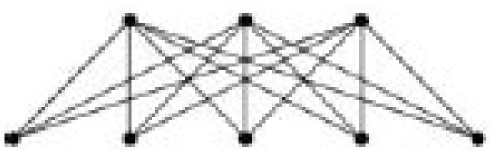
Complete Bipartite Graph

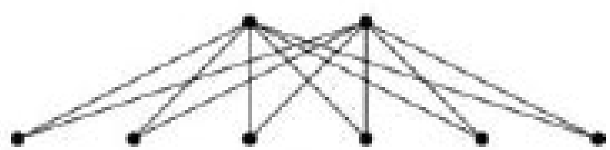
Definition: A *complete bipartite graph $K_{m,n}$ is a graph that has its* vertex set partitioned into two subsets *V_1 of size m and V_2 of size n such that there is an edge from every vertex in V_1 to every vertex* in V_2.

Example: We display four complete bipartite graphs here.





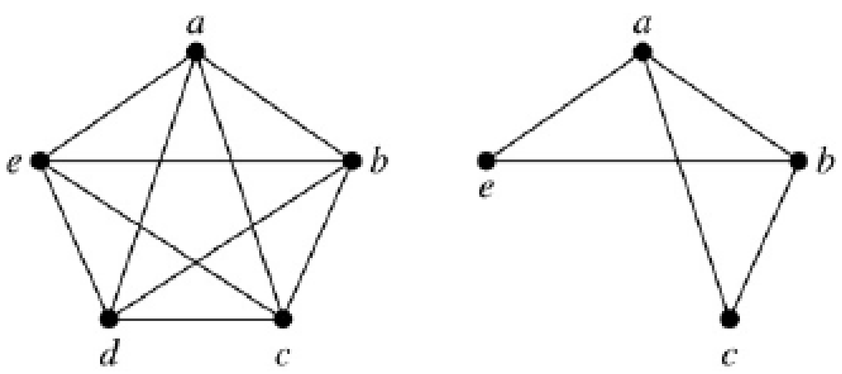




Subgraph

Definition: A *subgraph of a graph G = (V,E) is a graph (W,F),* where *$W \subset V$ and $F \subset E$.*
A subgraph H of G is a proper subgraph of *G if $H \neq G$.*

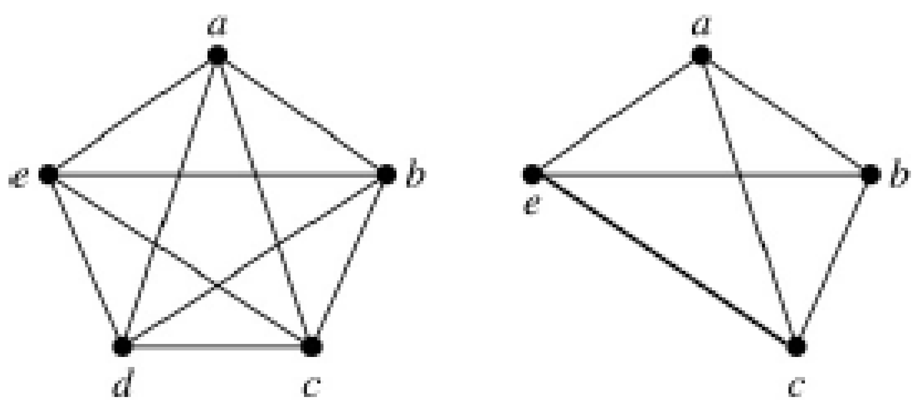
Example: *K_5 and one of its subgraphs.*



Subgraph

Definition: Let $G = (V, E)$ *be a simple graph.*
The subgraph induced by a subset W of the vertex set V is the graph (W,F), where the edge set F *contains an edge in* E *if and only if both* endpoints are in W.

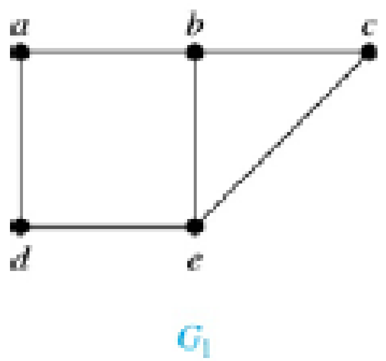
Example: K_5 *and the subgraph induced by* $W = \{a,b,c,e\}$.

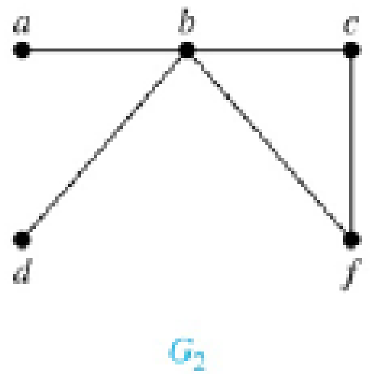


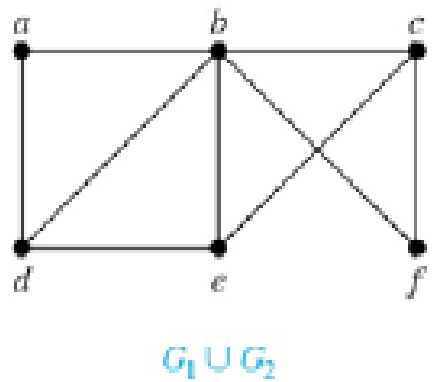
Union of Graph

Definition: The *union of two simple graphs* $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$ *is the simple graph with vertex set* $V_1 \cup V_2$ *and edge set* $E_1 \cup E_2$*. The union of* G_1 *and* G_2 *is denoted by* $G_1 \cup G_2$.

Example:



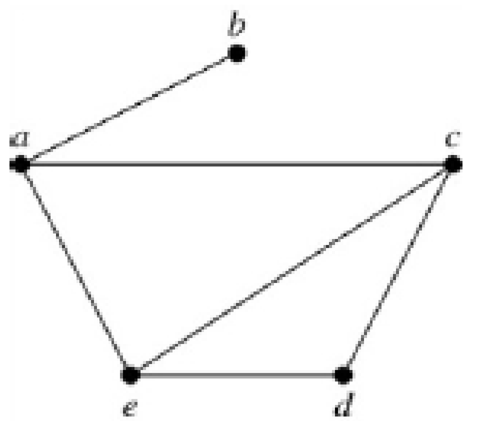




Representation: Adjacency List

Definition: An *adjacency list can be used to represent a graph with* no multiple edges by specifying the vertices that are adjacent to each vertex of the graph.

Example:

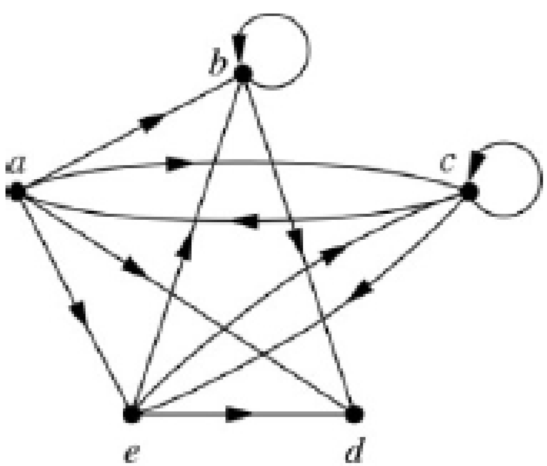


An adjacency list for a simple graph	
vertex	Adjacent vertex
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, d, c

Representation: Adjacency List

Definition: An *adjacency list can be used to represent a graph with* no multiple edges by specifying the vertices that are adjacent to each vertex of the graph.

Example:

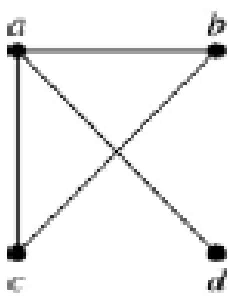


An adjacency list for a directed graph	
vertex	Adjacent vertex
a	b, c, d, e
b	b, d
c	a, c, e
d	
e	b, c, d

Adjacency Matrix

Definition: Suppose that $G = (V, E)$ is a simple graph where $|V| = n$. Arbitrarily list the vertices of G as $v_1, v_2, \ldots, v_n$. The adjacency matrix A_G of G, with respect to the listing of vertices, is the $n \times n$ zero-one matrix with 1 as its (i, j)th entry when v_i and v_j are adjacent, and 0 as its (i, j)th entry when they are not adjacent.

Example:



$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

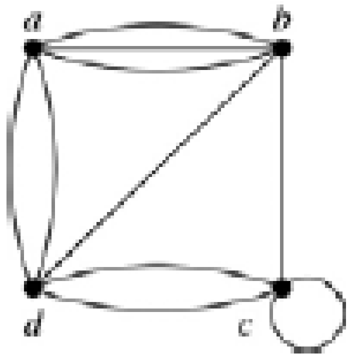
Adjacency Matrix

Adjacency matrices can also be used to represent graphs with loops and multiple edges.

• A loop at the vertex *vi is represented by a* 1 *at the (i, i)th position* of the matrix.

• When multiple edges connect the same pair of vertices *vi and vj, (or* if multiple loops are present at the same vertex), the (*i, j)th entry* equals the number of edges connecting the pair of vertices.

Example: The adjacency matrix of the pseudograph shown here using the ordering of vertices *a, b, c, d.*



$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$