Relation

Cartesian product (review)

Let
$A=\{a_1, a_2, ..a_k\}$ and
$B=\{b_1,b_2,..b_m\}$.

The Cartesian product A x B is defined by a set of ordered pairs
$\{(a_1 , b_1), (a_1, b_2), \ldots (a_1, b_m), \ldots, (a_k,b_m)\}$.

Cartesian product defines a product set, or a set of all ordered arrangements of elements in sets in the Cartesian product.

Binary relation

Definition: Let A and B be two sets. A binary relation from A to B is a subset of a Cartesian product A x B.

- Let $R \subseteq A \times B$ means R is a set of ordered pairs of the form (a,b) where a $\in$ A and b $\in$ B.

- We use the notation **a R b to denote (a,b) $\in$ R and a $\not R$ b to denote (a,b) $\notin$ R. If a R b, we say a is related to b by R.**

Example: Let A={a,b,c} and B={1,2,3}.
- Is R={(a,1),(b,2),(c,2)} a relation from A to B? **Yes.**
- Is Q={(1,a),(2,b)} a relation from A to B? **No.**
- Is P={(a,a),(b,c),(b,a)} a relation from A to A? **Yes.**

Representing Binary relation

We can graphically represent a binary relation R as follows:

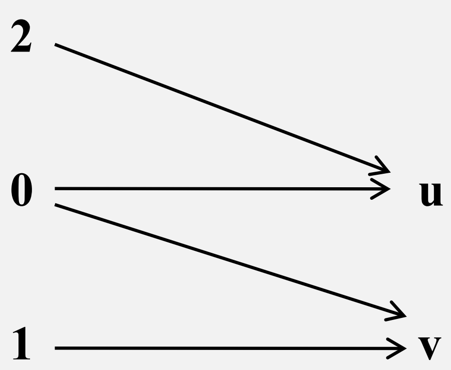
- if **a R b then draw an arrow from a to b.**

a $\rightarrow$ b

Example:

- Let A = {0, 1, 2}, B = {u,v} and

  R = { (0,u), (0,v), (1,v), (2,u) }
- Note: $R \subseteq A \times B$
- **Graph:**



Representing Binary relation

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Example:

- Let A = {0, 1, 2}, B = {u,v} and

  R = { (0,u), (0,v), (1,v), (2,u) }
- Note: $R \subseteq A \times B$
- **Table:**

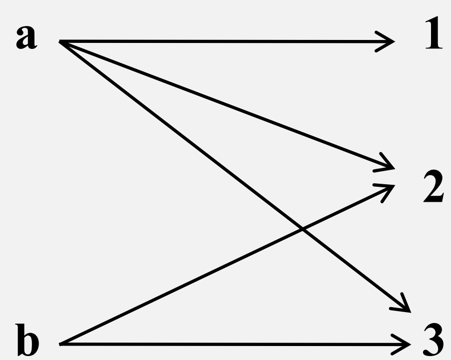
R	u	v
0	x	x
1		x
2	x	

R	u	v
0	1	1
1	0	1
2	1	0

Relation and Function

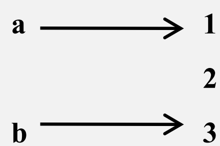
Relations represent **one to many relationships between** elements in A and B.

- **Example:**



- What is the difference between a **relation and a function from A to B?**

A function defined on sets A,B. A → B assigns to each element in the domain set A, exactly one element from B. So it is **a** special relation**.**



Relation on the set

Definition: A relation on the set A is a relation from A to itself.

Example 1:

- Let A = {1,2,3,4} and R_{div} = {(a,b)| a divides b}

- What does R_{div} consist of?

- R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}

R	1	2	3	4
1	x	x	x	x
2		x		x
3			x	
4				x

Relation on the set

Definition: A relation on the set A is a relation from A to itself.

Example 2:

- Let A = {1,2,3,4}.
- Define a $R_{\neq}$ b if and only if a $\neq$ b.

$R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}

R	1	2	3	4
1		x	x	x
2	x		x	x
3	x	x		x
4	x	x	x	

Binary Relation

Theorem: The number of binary relations on a set A, where | A | = n is:

$$2^{n^2}$$

Proof:

- If | A | = n then the cardinality of the Cartesian product | A x A | = n^2.
- R is a binary relation on A if $R \subseteq A \times A$ (that is, R is a subset of A x A).
- The number of subsets of a set with k elements : 2^k
- The number of subsets of A x A is : $2^{A \times A} = 2^{n^2}$

Binary Relation

Example: Let A = {1,2}

- What is A x A = **{(1,1),(1,2),(2,1),(2,2)}**

- **List of possible relations (subsets of A x A):**

Ø	1
{(1,1)} {(1,2)} {(2,1)} {(2,2)}	4
{(1,1), (1,2)} {(1,1),(2,1)} {(1,1),(2,2)} {(1,2),(2,1)} {(1,2),(2,2)} {(2,1),(2,2)}	6
{(1,1),(1,2),(2,1)} {(1,1),(1,2),(2,2)} {(1,1),(2,1),(2,2)} {(1,2),(2,1),(2,2)}	4
{(1,1),(1,2),(2,1),(2,2)}	1
Use formula: $\mathbf{2^4 = 16}$	16

Reflexive Relation

Definition (reflexive relation) : A relation R on a set A is called reflexive if (a,a) ϵ R for every element a ϵ A.

Example 1:

- Assume relation R_{div} ={(a b), if a |b} on A = {1,2,3,4}

R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}

- **Is R_{div} reflexive?**

- **Answer: Yes. (1,1), (2,2), (3,3), and (4,4) ϵ A.**

Reflexive Relation

Reflexive relation

- R_{div} ={(a b), if a |b} on A = {1,2,3,4}

- R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}

$$\begin{matrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

A relation R is reflexive if and only if R has 1 in every position on its main diagonal.

Reflexive Relation

Definition (reflexive relation) : A relation R on a set A is called reflexive if (a,a) $\in$ R for every element a $\in$ A.

Example 2:

- Relation R_{fun} on A = {1,2,3,4} defined as:
- R_{fun} = {(1,2),(2,2),(3,3)}.

- **Is R_{fun} reflexive?**

- **No. It is not reflexive since (1,1) ,(4,4) $\notin$ R_{fun}.**

Irreflexive Relation

Definition (irreflexive relation): A relation R on a set A is called irreflexive if $(a,a) \notin R$ for every $a \in A$.

Example 1:

• Assume relation R≠ on A={1,2,3,4}, such that a $R_{\neq}$ b if and only if $a \neq b$.

$R_{\neq}$ = {(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}

• Is $R_{\neq}$ irreflexive?

Answer: Yes. Because (1,1),(2,2),(3,3) and (4,4) $\notin R_{\neq}$

Irreflexive Relation

Irreflexive relation

- R≠ on A={1,2,3,4}, such that **a $R_{\neq}$ b if and only if a ≠ b.**
- **$R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}**

$$\begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}$$

A relation R is irreflexive if and only if R has 0 in every position on its main diagonal

Irreflexive Relation

Definition (irreflexive relation): A relation R on a set A is called irreflexive if $(a,a) \notin R$ for every $a \in A$.

Example 2:

- R_{fun} on A = {1,2,3,4} defined as:
- $R_{fun} = \{(1,2),(2,2),(3,3)\}$.
- **Is R_{fun} irreflexive?**
- **Answer: No. Because (2,2) and (3,3) $\in$ R_{fun}**

Symmetric Relation

Definition (symmetric relation): A relation R on a set A is called symmetric if

. $$\forall a,b \in A (a,b) \in R \rightarrow (b,a) \in R$$

Example 1:

- R_{div} ={(a b), if a |b} on A = {1,2,3,4}
- R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}

- **Is R_{div} symmetric?**

- **Answer: No. It is not symmetric since (1,2) ϵ R but (2,1) $\notin$ R.**

Symmetric Relation

Definition (symmetric relation): A relation R on a set A is called symmetric if

$$\forall a,b \in A (a,b) \in R \rightarrow (b,a) \in R$$

Example 2:

- $R_{\neq}$ on A={1,2,3,4}, such that a $R_{\neq}$ b if and only if a ≠ b.

$R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}

- Is $R_{\neq}$ symmetric ?

Answer: Yes. $\forall a,b \in A (a,b) \in R \rightarrow (b,a) \in R$

Symmetric Relation

Symmetric relation:

- $R_{\neq}$ on A={1,2,3,4}, such that **a $R_{\neq}$ b if and only if a ≠ b.**
- **$R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}**

.

0	1	1	1
1	0	1	1
1	1	0	1
1	1	1	0

A relation R is symmetric if and only if $m_{ij} = m_{ji}$ for all i,j.

Symmetric Relation

Definition (symmetric relation): A relation R on a set A is called symmetric if

$$\forall a,b \in A (a,b) \in R \rightarrow (b,a) \in R$$

Example 3:

- Relation R_{fun} on A = {1,2,3,4} defined as:
- R_{fun} = {(1,2),(2,2),(3,3)}.

- Is R_{fun} symmetric?

- **Answer: No. For (1,2) ϵ R_{fun} there is no (2,1) ϵ R_{fun}**

Anti-symmetric Relation

Definition (anti-symmetric relation): A relation on a set A is called **anti-symmetric if**

- [(a,b) ϵ R and (b,a) ϵ R] → a = b where a, b ϵ A.

Example 3:

- Relation R_{fun} on A = {1,2,3,4} defined as:
- R_{fun} = {(1,2),(2,2),(3,3)}.
- Is R_{fun} anti-symmetric?
- Answer: Yes. It is anti-symmetric

Anti-symmetric Relation

Antisymmetric relation

- relation $R_{fun} = \{(1,2),(2,2),(3,3)\}$

$$\begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

A relation is antisymmetric if and only if $m_{ij} = 1 \rightarrow m_{ji} = 0$ for $i \neq j$.

Transitive Relation

Definition (transitive relation): A relation R on a set A is called transitive if

- [(a,b) ϵ R and (b,c) ϵ R] → (a,c) ϵ R for all a, b, c ϵ A.

- **Example 1:**

- R_{div} ={(a b), if a |b} on A = {1,2,3,4}

- R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}

- **Is R_{div} transitive?**

- **Answer: Yes.**

Transitive Relation

Definition (transitive relation): A relation R on a set A is calledtransitive if

- [(a,b) ϵ R and (b,c) ϵ R] → (a,c) ϵ R for all a, b, c ϵ A.

- Example 2:

- $R_{\neq}$ on A={1,2,3,4}, such that a R≠ b if and only if a ≠ b.
- $R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}

- Is $R_{\neq}$ transitive ?

- Answer: No. It is not transitive since (1,2) ϵ R and (2,1) ϵ R but (1,1) is not an element of R.

Transitive Relation

Definition (transitive relation): A relation R on a set A is called transitive if

- $[(a,b) \in R$ and $(b,c) \in R] \rightarrow (a,c) \in R$ for all a, b, $c \in A$.

Example 3:

- Relation R_{fun} on A = {1,2,3,4} defined as:

- $R_{fun} = \{(1,2),(2,2),(3,3)\}$.

- Is R_{fun} transitive?

- Answer: Yes. It is transitive.

Equivalence Relation

Definition: A relation R on a set A is called an equivalence relation if it is reflexive, symmetric and transitive.

Example: Let A = {0,1,2,3,4,5,6} and

- R= {(a,b)| a,b ϵ A, a ≡ b mod 3} (a is congruent to b modulo 3)

Congruencies:

- 0 mod 3 = 0 1 mod 3 = 1 2 mod 3 = 2 3 mod 3 = 0
- 4 mod 3 = 1 5 mod 3 = 2 6 mod 3 = 0

Relation R has the following pairs:

- (0,0) (0,3), (3,0), (0,6), (6,0)
- (3,3), (3,6) (6,3), (6,6) (1,1),(1,4), (4,1), (4,4)
- (2,2), (2,5), (5,2), (5,5)

Equivalence Relation

Example: Let A = {0,1,2,3,4,5,6} and

- R= {(a,b)| a,b ϵ A, a ≡ b mod 3} (a is congruent to b modulo 3)

Relation R has the following pairs:

- (0,0)
- (3,3), (3,6) (6,3), (6,6)
- (2,2), (2,5), (5,2), (5,5)

(0,3), (3,0), (0,6), (6,0)

(1,1),(1,4), (4,1), (4,4)

Is R reflexive? Yes

Is R symmetric? Yes

Is R transitive? Yes