Homework 5 Exercises

5.1 ~~3, 8, 16, 18, 21~~

3. a) $P(1) = 1^2 = [1\times(1+1)\times(2\times1+1)] / 6$

   b) $P(1) = \underline{1^2} = \underline{[1\times(1+1)\times(2\times1+1)] / 6} = 6 / 6 = 1$

   Thus, $P(1)$ is true.

   c) Assume $P(k)$ is true for an arbitrary positive integer k, st.

   $1^2+2^2+ \dots + k^2 = [k(k+1)(2k+1)] / 6$

   d) Under the inductive hypothesis in c), we need to prove $P(k+1)$ is true, that is,

   $1^2+2^2+ \dots + (k+1)^2 = [(k+1)(k+2)(2(k+1)+1)] / 6$

   $=1^2+2^2+ \dots + (k+1)^2 = [(k+1)(k+2)(2k+3)] / 6$

   to prove $1^2+2^2+ \dots + (k+1)^2 = [(k+1)(k+2)(2k+3)] / 6$ is true.

   e) To prove $1^2+2^2+ \dots + (k+1)^2 = [(k+1)(k+2)(2k+3)] / 6$, we need the inductive hypothesis that $1^2+2^2+ \dots + k^2 = [k(k+1)(2k+1)] / 6$.

   $1^2+2^2+ \dots + k^2 = [k(k+1)(2k+1)] / 6$ by I.H.

   $1^2+2^2+ \dots + k^2 + (k+1)^2 = [k(k+1)(2k+1)] / 6 + (k+1)^2$ add $(k+1)^2$ at both sides

   $= [k(k+1)(2k+1)] / 6 + [6(k+1)^2] / 6$

   $= [k(k+1)(2k+1) + 6(k+1)^2] / 6$

   $= [(k+1)(k(2k+1) + 6(k+1))] / 6$

   $= [(k+1)(2k^2+k+6k+6)] / 6$

   $= [(k+1)(k+2)(2k+3)] / 6$

   $= [(k+1)(k+2)(2(k+1)+1)] / 6$

   This shows that $P(k+1)$ follows from $P(k)$, thus $P(k+1)$ is true.

   f) We have $1^2+2^2+ \dots + n^2 = [n(n+1)(2n+1)] / 6$ for $n = 1$ (base case)

   and $1^2+2^2+ \dots + k^2 = [k(k+1)(2k+1)] / 6$

   $\rightarrow$ $1^2+2^2+ \dots + (k+1)^2 = [(k+1)(k+2)(2(k+1)+1)] / 6$

   Therefore, by M.I., we have $1^2+2^2+ \dots + n^2 = [n(n+1)(2n+1)] / 6$ $\forall n\in\mathbb{Z}^+$.

8. $P(0) = \underline{2\times(-7)^0} = \underline{(1 - (-7)^1) / 4} = 8 / 4 = 2$

   Assume $P(k)$ is true for an arbitrary nonnegative integer k, st.

   $P(k) = 2 - 2\times7 + 2\times7^2 - \dots + 2\times(-7)^k = (1 - (-7)^{k+1}) / 4$

   Under the above inductive hypothesis, we need to prove that $P(k+1)$ is true, that is,

   to prove $P(k+1) = 2 - 2\times7 + 2\times7^2 - \dots + 2\times(-7)^{k+1} = (1 - (-7)^{k+2}) / 4$ is true.

   Assuming the I.H., it follows that

   $2 - 2\times7 + 2\times7^2 - \dots + 2\times(-7)^k + 2\times(-7)^{k+1} = (1 - (-7)^{k+1}) / 4 + 2\times(-7)^{k+1}$

   $= [(1 - (-7)^{k+1}) + 8\times(-7)^{k+1}] / 4$

   $= (1 + 7\times(-7)^{k+1}) / 4$

   $= (1 - (-7)\times(-7)^{k+1}) / 4$

   $= (1 - (-7)^{k+2}) / 4$

   This shows that $P(k+1)$ follows from $P(k)$, thus $P(k+1)$ is true.

   We have $2 - 2\times7 + 2\times7^2 - \dots + 2\times(-7)^n = (1 - (-7)^{n+1}) / 4$ for $n = 0$ (base case)

   and $2 - 2\times7 + 2\times7^2 - \dots + 2\times(-7)^k = (1 - (-7)^{k+1}) / 4$

   $\rightarrow$ $2 - 2\times7 + 2\times7^2 - \dots + 2\times(-7)^{k+1} = (1 - (-7)^{k+2}) / 4$

   Therefore, by M.I., we have $2 - 2\times7 + 2\times7^2 - \dots + 2\times(-7)^{n+1} = (1 - (-7)^{n+2}) / 4$

   $\forall n\in\{0\}\cup\mathbb{Z}^+$.

16. $P(1) = \underline{1 \times 2 \times 3} = 6 = \underline{(1 \times 2 \times 3 \times 4) / 4}$

Assume $P(k)$ is true for an arbitrary positive integer k, st.

$1 \times 2 \times 3 + 2 \times 3 \times 4 + ... + k(k+1)(k+2) = k(k+1)(k+2)(k+3) / 4$

Under the above inductive hypothesis, we need to prove that $P(k+1)$ is true, that is,

to prove $1 \times 2 \times 3 + 2 \times 3 \times 4 + ... + (k+1)(k+2)(k+3)= (k+1)(k+2)(k+3)(k+4) / 4$

Assuming the I.H., it follows that

$$\begin{aligned} 1 \times 2 \times 3+2 \times 3 \times 4+...+k(k+1)(k+2)+(k+1)(k+2)(k+3) &= k(k+1)(k+2)(k+3)/4+(k+1)(k+2)(k+3) \\ &= [k(k+1)(k+2)(k+3)+4(k+1)(k+2)(k+3)]/4 \\ &= [(k+1)(k+2)(k+3)(k+4)]/4 \end{aligned}$$

This shows that $P(k+1)$ follows from $P(k)$, thus $P(k+1)$ is true.

We have $1 \times 2 \times 3 + 2 \times 3 \times 4 + ... + n(n+1)(n+2) = n(n+1)(n+2)(n+3) / 4$ for $n = 1$,

and $1 \times 2 \times 3 + 2 \times 3 \times 4 + ... + k(k+1)(k+2) = k(k+1)(k+2)(k+3) / 4$

$\rightarrow$ $1 \times 2 \times 3 + 2 \times 3 \times 4 + ... + (k+1)(k+2)(k+3)= (k+1)(k+2)(k+3)(k+4) / 4$

Therefore, by M.I., we have $2 - 2 \times 7 + 2 \times 7^2 - ... + 2 \times (-7)^{n+1} = (1 - (-7)^{n+2}) / 4$

$\forall n \in \mathbb{Z}^+$.

18. a) $P(2) = \underline{2!} < \underline{2^2}$

b) $P(2) = \underline{2!} = 1 \times 2 = 2 < \underline{2^2} = 4$

BASIS STEP: $P(2)$ is true, because $2! < 2^2$.

c) Assume that $k! < k^k$ for the integer k greater than 1.

d) Under the inductive hypothesis in c), we need to prove $P(k+1)$ is true, that is,

to prove $(k+1)! < (k+1)^{k+1}$.

e) $(k+1)! = k!(k+1)$

$< k^k (k+1)$ by I.H

$< (k+1)^k (k+1)$ because k is an integer greater than 1, $k+1 > k$

$= (k+1)^{k+1}$

That is, $(k+1)! < (k+1)^{k+1}$.

This shows that $P(k+1)$ is true when $P(k)$ is true.

f) We have $n! < n^n$ for $n = 2$ (base case)

and $k! < k^k \rightarrow (k+1)! < (k+1)^{k+1}$

Therefore, by M.I., we have $n! < n^n \ \forall n \geqslant 1 \in \mathbb{Z}^+$.

21. $P(5) = 2^5 = 32 > 5^2$

BASIS STEP: $P(5)$ is true, because $2^5 > 5^2$.

INDUCTIVE STEP: Assume that $2^k > k^2$ for the integer k greater than 4.

Under this inductive hypothesis, we need to prove $P(k+1)$ is true, that is,

to prove $2^{k+1} > (k+1)^2$.

$2^{k+1} = 2^k \times 2$

$> k^2 \times 2$ by I.H.

$= k^2 + k^2$

$> k^2 + 2k+1$ because $k>4$, $k^2 > 2k+1$

$= (k+1)^2$

That is, $2^{k+1} > (k+1)^2$.

This shows that $P(k+1)$ is true when $P(k)$ is true.

We have $2^n > n^2$ for $n = 5$ (base case) and $2^k > k^2 \rightarrow 2^{k+1} > (k+1)^2$. Therefore, by M.I., we have $2^n > n^2 \ \forall n \geqslant 4 \in \mathbb{Z}^+$.

5.2 ~~3, 12, 17~~

3. a) BASIS STEP: $P(8)$ is true since 8 cents can be formed by one 3-cent stamp and one 5-cent stamp; $P(9)$ is true since 9 cents can be formed by three 3-cent stamps; $P(10)$ is true since 10 cents can be formed by two 5-cent stamps. This completes the basis step.

   b) Assume we can form postage of i cents, where $8 \leqslant i \leqslant k$ (k is an integer with $k \geqslant 10$).

   c) We need to prove that $P(k+1)$ is true, we can form postage of $k+1$ cents.

   d) By the I.H., we can assume that $P(k-2)$ is true since $k-2 \geqslant 8$, we can form postage of $k-2$ cents using 3-cent stamp and 5-cent stamp. Because $k+1 = k-2 + 3$, then we can form $k+1$ cents by adding another 3-cent stamp to stamps we use to form $k-2$ cents. Thus, we proved that if the inductive hypothesis is true, $P(k+1)$ then is true.

   e) We have shown the basis step in a) and inductive step of a strong induction proof in b) to d), thus we know $P(n)$ is true for all integers n greater than 8: n cents can be formed using 3-cent stamp and 5-cent stamp.

12. BASIS STEP: $P(1)$ is true since $1 = 2^0$.

    INDUCTIVE STEP: Inductive Hypothesis: Assume we can write every positive integer i as a sum of distinct powers of two, where $1 \leqslant i \leqslant k$ (k is a positive integer). We need to prove that $P(k+1)$ is true, that is, we can write $k+1$ as a sum of distinct powers of two. By I.H., $(k+1)/2$ is true. First, if $k+1$ is an even integer, we can write $k+1$ as the product of a 2^1 and the sum of distinct powers of two of $(k+1)/2$. This shows that $k+1$ is true when $k+1$ is even. Second, if $k+1$ is an odd number, then k is even. Since $\mathrm{P}(k)$ is true by I.H., then we can write $k+1$ as the product of a 2^0 and the sum of distinct powers of two of k. This shows that $k+1$ is true when $k+1$ is odd. Therefore, these two cases prove that $k+1$ is true.

    We have $P(1)$, as the base case, is true and have proved that $[P(1) \wedge ... \wedge P(k)] \rightarrow P(k+1)$, thus, $P(n)$ is true for $\forall n \geqslant 1 \in \mathbb{Z}^+$.

17. $P(n)$: In a triangulation of a n-sides simple polygon, at least two of the triangles in the triangulation have two sides that border the exterior of the polygon (n is an integer with $n \geqslant 4$).

    BASIS STEP: P(4) is true since two triangles in the triangulations in a rectangle border the exterior of the rectangle.

    INDUCTIVE STEP: Inductive Hypothesis: Assume that in a triangulation of a i-sides simple polygon, at least two of the triangles in the triangulation have two sides that border the exterior of the polygon ($4 \leqslant i \leqslant k$, $k \geqslant 4 \in \mathbb{Z}^+$). We need to prove that $P(k+1)$ is true. First, in a triangulation that divides $k+1$-sides polygon into one triangle and a smaller polygon, the number of sides of the smaller polygon is $k+1-2$(two sides of the triangle)+1(the diagonal), that is, k. By I.H., $P(k)$ is true. Since the $k+1$-sides polygon now has a triangle, which must border the exterior curve, and a k-sides polygon which is true for the statement (we can find at least two triangles bordering the exterior of the k-sides polygon which shares the curve with the $k+1$-sides polygon), $P(k+1)$ is true. Second, in a triangulation that divides $k+1$-sides into two smaller polygons with at least 4 sides, one has m sides ($4 \leqslant m \leqslant k-1$), and another one has $k+3$-m sides ($k+1-m+2$). By I.H., $P(m)$ and $P(k+3-m)$ are both true because $k-1 \geqslant m > 3$, $k+3-m < k$, we can find at least two triangles bordering the exterior in these two polygons, respectively; since these polygons share curve (except the diagonal divides the large polygon) with the k+1-sides polygon, we can know that there exists at least two triangles in each small polygons that border the exterior of the k+1-sides polygon. Thus, $P(k+1)$ is true in this case. Therefore, these two cases prove that k+1 is true.

We have $P(4)$, as the base case, is true and have proved that $[P(1)\wedge...\wedge P(k)]\rightarrow P(k+1)$, thus, $P(n)$ is true for $\forall n \geqslant 4 \in \mathbb{Z}^{+}$.

5.3 ~~1, 6, 9, 22, 34, 35, 57~~

1. a) $f(n+1) = f(n) + 2$

$f(1) = f(0+1) = f(0) + 2 = 3$

$f(2) = f(1+1) = f(1) + 2 = 5$

$f(3) = f(2+1) = f(2) + 2 = 7$

$f(4) = f(3+1) = f(3) + 2 = 9$

b) $f(n+1) = 3f(n)$

$f(1) = f(0+1) = 3f(0) = 3$

$f(2) = f(1+1) = 3f(1) = 9$

$f(3) = f(2+1) = 3f(2) = 27$

$f(4) = f(3+1) = 3f(3) = 81$

c) $f(n+1) = 2^{f(n)}$

$f(1) = f(0+1) = 2^{f(0)} = 2$

$f(2) = f(1+1) = 2^{f(1)} = 4$

$f(3) = f(2+1) = 2^{f(2)} = 16$

$f(4) = f(3+1) = 2^{f(3)} = 65536$

d) $f(n+1) = f(n)^2 + f(n) + 1$

$f(1) = f(0+1) = f(0)^2 + f(0) + 1 = 3$

$f(2) = f(1+1) = f(1)^2 + f(1) + 1 = 13$

$f(3) = f(2+1) = f(2)^2 + f(2) + 1 = 183$

$f(4) = f(3+1) = f(3)^2 + f(3) + 1 = 33673$

6. a) This is valid since for every positive integer, the value of the function at this integer is determined in an unambiguous way. $f(n)=(-1)^n$. Basis step: $f(0)=1=(-1)^1$. Inductive step: If $f(k)=(-1)^k$, then $f(k+1) = -f(k) = -(-1)^k = -1\times(-1)^k = (-1)^{k+1}$.

b) This is valid since since for every positive integer, the value of the function at this integer is determined in an unambiguous way. Basis step: $f(3) = 2 = 2^{3/3}$. $f(4) = 0$. $f(5) = 4 = 2^{\lfloor 5/3\rfloor+1}$. Inductive step: ***A***. $k = 3q$ (q is an positive integer). If $f(k)= 2^{k/3}$, then $f(k+3) = 2f(k) = 2\times2^{k/3} = 2^{k/3+1} = 2^{(k+3)/3}$. ***B***. $k = 3q + 1$ (q is an positive integer). If $f(k) = 0$, then $f(k+3) = 2f(k) = 2\times0 = 0$. ***C***. $k = 3q + 2$ (q is an positive integer). If $f(k) = 2^{\lfloor k/3\rfloor+1}$, then $f(k+3) = 2f(k) = 2\times2^{\lfloor k/3\rfloor+1} = 2^{\lfloor (k+3)/3\rfloor+1}$.

c) This is not valid since finding n needs the value of n+1.

d) This is valid since since for every positive integer, the value of the function at this integer is determined in an unambiguous way. Basis step: $f(2) = 2 = 2^{2-1}$. Inductive step: If $f(k)=2^{k-1}$, then $f(k+1) = 2f(k) = 2\times2^{k-1} = 2^k = 2^{(k+1)-1}$.

e) This is valid since since for every positive integer, the value of the function at this integer is determined in an unambiguous way. Basis step: $f(1) = 2 = 2^{2-1}$

9. $F(n) = F(n-1) + n$ for $n \geqslant 1$. $F(0) = 0$.

22. $P(k)$: $k \in S$ □$k \in \mathbb{Z}^+$. Base case: $P(1)$: $1 \in S$. Recursive step: $s+t \in S$ whenever $s \in S$ and $t \in S$. First, we need to prove that $\mathbb{Z}^+$□S. I.H.: Assume that $P(k)$ is true for an arbitrary positive integer k, that is, $k \in S$. Because $k \in S$ and $1 \in S$, it follows the second part of the recursive definition, $(k+1) \in S$, that is, $P(k+1)$ is true. Since $k \in \mathbb{Z}^+$ and $k \in S$, $\mathbb{Z}^+$□S. Second, we need to prove S□$\mathbb{Z}^+$. Suppose $m \in S$, since $s+t \in S$, $m = s+t$ or 1. Thus, m is an positive integer, which means $m \in \mathbb{Z}^+$. Therefore, S□$\mathbb{Z}^+$. From this two cases, $S=\mathbb{Z}^+$ as required.

34. a) w=0101, w^R=1010.
    b) w=1 1011, w^R=1101 1.
    c) w=1000 1001 0111, w^R=1110 1001 0001.

35. The reversal of an empty string, which has a 0 length, is itself, that is, $w^R(0)=w$. Then, for w that has $n+1$ length, denoted by $x\ y$ (x is the string w with n length and y is the last symbol in w that has $n+1$ length). Thus, the reversal of $w(n+1)$, $w(n+1)^R = y\ x^R$. Therefore, $w^R(0)=w$, $w(n+1)^R = y\ x^R$ (or $w(n+1)^R = y\ w(n)^R$).

57. $P(n)$: n is in well defined rule of F. BASIS STEP: $P(0)$ is true since $F(0)$ is specified. INDUCTIVE STEP: Inductive Hypothesis: Assume that k is in well defined rule of F, where $0 \leqslant k < n$. By I.H., $P(n-1)$ is true, thus, $P(n)$ is true since $P(0)$, $P(1)$,...$P(n-1)$ are all true, or well defined. We have $P(0)$, as the base case, is true and have proved that $[P(0) \wedge ... \wedge P(n-1)] \rightarrow P(n)$. Hence, $P(n)$ is true for $\forall\, n \geqslant 0 \in \mathbb{Z}^+$.