Homework 7 Exercises:

7.1 ~~1, 3, 4, 8, 9, 12, 19, 25, 35, 37~~

1. The probability is **1/13** (4/52=1/13).
3. The probability is **1/2** (50/100=1/2).
4. The probability is **0.082** (30/366=0.082).
8. The probability is **0.096** (C(51,4)/C(52,5)=0.096).
9. The probability is **0.904** (C(51,5)/C(52,5)=0.904).
12. The probability is **0.299** ([C(48,4)C(4,1)]/C(52,5)=0.299).
19. $[C(13,5)[C(4,1)]^5 - C(13,5)\times 4 - C(10,1)\times[C(4,1)]^5 - C(10,1)\times 4] / C(52,5)$ = **0.501**
   (Possible selections of five cards with different kinds - Number of flushes -
   Number of straights (from A to 10, A-2-3-4-5 to 10-J-Q-K-A) - Number of both flush and straight) / All possible selections of five cards
25. a) $C(6,6)/C(50, 6)$ = **6.293e-8** b) $C(6,6)/C(52, 6)$ = **4.912e-8**
   c) $C(6,6)/C(56, 6)$ = **3.080e-8** d) $C(6,6)/C(60, 6)$ = **1.997e-8**
35. a) 18/38 = **0.474** b) 18/38×18/38 = **0.224**
   c) 1/38 + 1/38 = **0.052** d) $(36/38)^5$ = **0.763**
   e) $[C(6,1)C(32,1)]/[C(38,1)]^2$ = **0.133**
37. Two dice: 4/36 (6,3; 3,6; 4,5; 5,4).
   Three dice: 25/216 (1,6,2; 1,2,6; 2,6,1;2,1,6; 6,1,2; 6,2,1; 1,3,5; 1,5,3; 3,1,5; 3,5,1; 5,3,1; 5,1,3; 4,4,1; 1,4,4; 4,1,4; 2,2,5; 2,5,2; 5,2,2; 2,3,4; 2,4,3; 3,4,2; 3,2,4; 4,2,3; 4,3,2; 3,3,3).
   25/216 = 0.116 > 4/36. Thus, rolling a total of 9 when **three dice** are rolled is more likely.

7.2 ~~3, 5, 7, 11, 15, 19, 28~~

3. p(2 or 4) = 3(1- p(2 or 4)) → p(2 or 4) = 3/4. Since rolling a 2 or a 4 is equally likely, $p(2)$ = **3/8** = $p(4)$. Then, $p(1) = p(3) = p(5) = p(6)$ = (1-3/4)/4 = (1/4)/4 = **1/16**.
5. There are 6 ways to get a 7 with rolling two dice: (1,6), (2,5), (3,4), (4,3), (5,2), and (6,1). The probabilities of each are: (1/7×1/7)=1/49, (1/7×1/7)=1/49, (1/7×1/7)=1/49, (2/7×2/7)=4/49, (1/7×1/7)=1/49, and (1/7×1/7)=1/49, which have a sum up to **9/49**.
7. a) p(1 precedes 4) = **1/2** (P(4,2)/P(4,4)=1/2).
   b) p(4 precedes 1) = **1/2** (P(4,2)/P(4,4)=1/2).
   c) p(4 precedes 1 and 2) = **3/8** ([P(2,2)+P(3,3)]/P(4,4))=1/2).
   d) p(4 precedes 1, 2, and 3) = **1/4** (P(3,3)/P(4,4)=1/4).
   e) p(4 precedes 3 and 2 precedes 1) = **1/4** (there are 6 combinations out of 24 possible ways: 4321, 4231, 4213, 2431, 2413, and 2143).
11. $p(E\cup F)\geqslant p(E)=0.7$, thus $p(E\cup F)\geqslant 0.7$. Then, $1\geqslant p(E\cap F)\geqslant 0 \rightarrow 1\geqslant p(E)+p(F)-p(E\cup F)\geqslant 0$ $\rightarrow 1\geqslant 0.7+0.5-p(E\cup F)\geqslant 0 \rightarrow 1\geqslant 1.2-p(E\cup F)\geqslant 0$, since the sum of two events must be greater than their union, we can get, $1\geqslant 1.2-p(E\cup F)\rightarrow 0\geqslant 0.2-p(E\cup F)\rightarrow p(E\cup F)\geqslant 0.2$, as required.
15. BASIS STEP: The base case is when n=2, that is, $p(E_1\cup E_2)\leqslant p(E_1)+p(E_2)$ is true, because $p(E_1\cup E_2)=p(E_1)+p(E_2)-p(E_1\cap E_2)$ and $p(E_1\cap E_2)\geqslant 0$, $p(E_1\cup E_2)\leqslant p(E_1)+p(E_2)$.
   INDUCTIVE STEP: Assume that $p(E_1\cup E_2\cup\ldots\cup E_k)\leqslant p(E_1)+p(E_2)+\ldots+p(E_k)$ for the integer $k\geqslant 2$. Under this inductive hypothesis, we need to prove $P(k+1)$ is true, that is,
   to prove $p(E_1\cup E_2\cup\ldots\cup E_k\cup E_{k+1})\leqslant p(E_1)+p(E_2)+\ldots+p(E_k)+p(E_{k+1})$.
   $p(E_1\cup E_2\cup\ldots\cup E_k\cup E_{k+1})= p((E_1\cup E_2\cup\ldots\cup E_k)\cup(E_{k+1}))\leqslant p(E_1\cup E_2\cup\ldots\cup E_k)+p(E_{k+1})\leqslant$

$p(E_1)+p(E_2)+...+p(E_k)+p(E_{k+1})$ (by I.H.)

This shows that $P(k+1)$ is true when $P(k)$ is true.

We have $p(E_1 \cup E_2 \cup ... \cup E_n) \leqslant p(E_1)+p(E_2)+...+p(E_n)$ for $n = 2$ (base case) and $p(E_1 \cup E_2 \cup ... \cup E_k) \leqslant p(E_1)+p(E_2)+...+p(E_k) \rightarrow p(E_1 \cup E_2 \cup ... \cup E_k \cup E_{k+1}) \leqslant p(E_1)+p(E_2)+...+p(E_k)+p(E_{k+1})$. Therefore, by M.I., we have $p(E_1 \cup E_2 \cup ... \cup E_n) \leqslant p(E_1)+p(E_2)+...+p(E_n)$ $\forall n \geqslant 2 \in \mathbb{Z}^+$.

19. a) The probability is **1/12** ($1\times 1/12=1/12$).

   b) The probability is **1-(12/12×11/12×10/12×...×(13-*n*)/12)**(the probability of at least two born in the same month of the year equals to 1-the probability of no people have the same born month, thus p(at least two people)=1-p(no people)=1-(12/12×11/12× 10/12×...×(13-n)/12)).

   c) The number of people is **five**. When $n = 2$, the probability is 1/12; when $n = 3$, the probability is 17/72; when $n = 4$, the probability is 41/96 (0.427<0..5); when $n = 5$, the probability is 89/144 (0.618>0.5).

28. a) $C(5,3)0.51^3 0.49^2 =$ **0.318**

   b) $1-0.49^5 =$ **0.972**

   c) $1-0.51^5 =$ **0.965**

   d) $0.49^5+0.51^5 =$ **0.0628**

7.3 ~~1, 4, 5, 9, 15, 19~~

1. $p(E \mid F) = p(E\cap F)/p(F) \rightarrow 2/5 = p(E\cap F)/1/2 \rightarrow p(E\cap F) = 0.2$

   $\therefore$ $p(F \mid E) = p(F\cap E)/p(E) = 0.2/1/3 =$ **0.6**

4. Let S denote the event of selecting the second box, and O denote the event of selecting an orange ball. By applying *Bayer's Theorem*, we can obtain that,

$$p(S \mid O) = \frac{p(O \mid S)\cdot p(S)}{p(O \mid S)\cdot p(S) + p(O \mid \bar{S})\cdot p(\bar{S})} = \frac{5/11\times 1/2}{5/11\times 1/2+3/7\times 1/2} \approx 0.5147$$

   Thus, the probability that Ann picked a ball from the second box if she has selected an orange ball is **0.5147**.

5. Let S denote the event of using steroids, and P denote the event of positive result to steroids. By applying *Bayer's Theorem*, we can obtain that,

$$p(S \mid P) = \frac{p(P \mid S)p(S)}{p(P \mid S)p(S)p(P \mid \bar{S})p(\bar{S})} = \frac{0.96\times 0.08}{0.96\times 0.08+0.09\times 0.92} \approx 0.4812$$

   Thus, the probability that a randomly selected bicyclist who tests positive for steroids actually uses steroids is **0.4812**.

9. a) Let P denote the event of positive result to HIV, and I denote the event of being infected with HIV. By applying *Bayer's Theorem*, we can obtain that,

$$p(I \mid P) = \frac{p(P \mid I)p(I)}{p(P \mid I)p(I)p(P \mid \bar{I})p(\bar{I})} = \frac{0.98\times 0.08}{0.98\times 0.08+0.03\times 0.92} \approx 0.7396$$

   Thus, the probability is **0.7396**.

   b) $p(\bar{I} \mid P) = 1 - p(I \mid P) = 0.2604$

   Thus, the probability is **0.2604**.

c) $$p(I \mid \overline{P}) = \frac{p(\overline{P} \mid I)p(I)}{p(\overline{P} \mid I)p(I)p(\overline{P} \mid \bar{I})p(\bar{I})} = \frac{0.02 \times 0.08}{0.02 \times 0.08 + 0.97 \times 0.92} \approx 0.0018$$

Thus, the probability is **0.0018**.

d) $$p(\bar{I} \mid \overline{P}) = 1 - p(I \mid \overline{P}) \approx 0.9982$$

Thus, the probability is **0.9982**.

15. a) The probability is **1/3**.

b) If j and k are the same number, which means the same door, then $p(M=j \mid W=k)=0$. And if number of winning door is not the same as j, there are two cases: first, if one selects the winning door, then Monty has two doors and selects randomly (p=1/2); or, if one does not select the winning door, then Monty must open the only left losing door (p=1). Thus, we can conclude that: **A**. $p(M=j \mid W=k)=0$ if j and k are the same number; **B**. $p(M=j \mid W=k)=1/2$ if j and k are different numbers and $i = k$; **C**. $p(M=j \mid W=k)=1$ if j and k are different numbers, $i \neq k$ and $i \neq j$. Hence, we can know the probability when j=1, 2, or 3 and k=1, 2, or 3 based on these three cases.

c) $$p(W=j \mid M=k) = \frac{p(M=k \mid W=j)p(W=j)}{p(M=k \mid W=j)p(W=j) + p(M=k \mid W \neq j)p(W \neq j)} = \frac{1 \times 1/3}{1 \times 1/3 + 1/2 \times 1/3 + 0 \times 1/3} \approx 0.6667$$

Thus, the probability is **0.6667**.

d) Part (c) tells us that if one selects a door that is different with the door Monty opens, one should change the door since the probability that the only closed door is the winning door is 0.6667, which is as twice as the probability of selecting the winning door randomly (p=1/3).

19. Let S denote the event of spam message, and O denote the event of a message containing "opportunity". Based on the given information, we can know that $p(O \mid S)$ = 175/1000 = 0.175, $p(S)$ = 1000/1400 = 0.7143, p(non-spam message) = 400/1400 = 0.2857. By applying *Bayer's Theorem*, we can obtain that,

$$p(S \mid O) = \frac{p(O \mid S)p(S)}{p(O \mid S)p(S) + p(O \mid \bar{S})p(\bar{S})} = \frac{0.175 \times 0.7143}{0.175 \times 0.7143 + 0.05 \times 0.2857} \approx 0.8974$$

Since 0.8974<0.9, the incoming message would not be rejected as spam.