Homework 6 Exercises:

6.1 ~~1, 8, 17, 23, 32, 44~~

1. a) There are **5850** (18*325=5850) different ways to pick representatives based on the pattern.
   b) There are **343** (18+325=343) different ways to pick one representative based on the pattern.

8. There are **15600** (26*25*24=15600) different three-letter initials with none of the letters repeated.

17. There are **1321368961** (128^5-127^5=1321368961) strings of five ASCII characters contain the character “@” at least one.

23. a) There are **128** (□(999-100)/7□= 128) positive integers that can are divisible by 7.
    b) There are 450 (□(999-100)/2□+1=450, we need to add one since 100, the minimum value, can be divided by 2) positive even integers. Because we have 900 numbers in the range, the left **450** (900-450=450) are positive odd integers.
    c) There are **9** (9*1*1) positive integers that have the same three decimal digits.
    d) There are 225 (□(999-100)/4□+1=225, we need to add one since 100, the minimum value, can be divided by 4) positive integers that can be divided by 4. Because we have 900 numbers in the range, the left **675** (900-225=675) are positive integers that are not divisible by 4.
    e) There are 225 (□(999-100)/4□+1=225, we need to add one since 100, the minimum value, can be divided by 4) positive integers that can be divided by 4 and 300 (□(999-100)/3□+1 =300, we need to add one since 999, the maximum value, can be divided by 3) positive integers that can be divided by 3. Since there are some integers that can be divided by both 3 and 4 (eg. 252), we need to remove these “duplicates,” the number of them are 75 (we use 12 for 12 is the lcm of 3 and 4, a number that are divisible by 3 and 4 must be a multiple of 12). Therefore, there are **450** (225+300-75=450) positive numbers that can be divisible by either 3 or 4.
    f) There are **450** (900-450=450, according to the previous answer) positive integers that are not divisible by either 3 or 4.
    g) There are **275** (300-75=275, the number of integers that are divisible by 3 - the number of integers that are divisible by both 3 and 4) positive integers that can be divided by 3 but not by 4.
    h) There are **75** (according to e)) positive integers that can be divided by 3 and by 4.

32. a) There are **208827064576** (26^8=208827064576) strings of eight uppercase English letters if repeated letters are allowed.
    b) There are **62990928000** (26*25*24*23*22*21*20*19=62990928000) strings of eight uppercase English letters if repeated letters are not allowed.
    c) There are **8031810176** (26^7=8031810176) strings of eight uppercase English letters if repeated letters are allowed and the strings start with *X*.
    d) There are **2422728000** (25*24*23*22*21*20*19=2422728000) strings of eight uppercase English letters if repeated letters are not allowed and the strings start with *X*.
    e) There are **308915776** (26^6=308915776) strings of eight uppercase English letters if repeated letters are allowed and the strings start and end with *X*.
    f) There are **308915776** (26^6=308915776) strings of eight uppercase English letters if repeated letters are allowed and the strings start with *BO*.

g) There are **456976** (26^4=456976) strings of eight uppercase English letters if repeated letters are allowed and the strings start and end with *BO*.

h) There are **617374576** (308915776*2-456976=617374576, the number of strings start with *BO* * the number of strings end with *BO*, which are the same, - the number of strings start and end with *BO*) strings of eight uppercase English letters if repeated letters are allowed and the strings start or end with *BO*.

44. There are **1260** ((10*9*8*7)/4=1260, because the rotation of the people are considered as the repeated arrangement and the number of such rotation is 4 for each. For example, the arrangement *ABCD* (four different people) is the same with *BCDA*, *CDAB*, and *DABC*) ways to arrange the seats according to the rule in the question.

6.2 ~~3, 8, 11, 16, 18, 27, 35~~

3. a) **3** socks since as the third one are taken out, by the Pigeonhole Principle, there are at least 2 socks have the same color among the three socks.

b) **14** socks since as the fourteenth socks are taken out, almost 12 brown socks have already been taken out, that is, in the case of all brown socks are taken out, the drawer can picked out at least 2 black socks when the 14th has been picked out.

8. Since |*S*|>|*T*|, which means the number of elements in *S* is larger than that in *T*, by the Pigeonhole Principle, there are at least two elements in *S* correspond to a same value in *T*, as required, *f*(*s*1) and *f*(*s*2), where *s*1 and *s*2 are elements in *S*, have the same value, or in other words, they map the same value that in *T*.

11. To ensure the midpoint has integer coordinates, suppose there are two points, (*A*, *B*, *C*) and (*I*, *J*, *K*), that is, to ensure that the coordinates of the midpoint, (*A*+*I*)/2, (*B*+*J*)/2, and (*C*+*K*)/2, are integers, *A* and *I*, *B* and *J*, and *C* and *K*, should have the same parity: odd and odd, or even and even. There are 8 combinations within a point: (odd, odd, odd), (odd, odd, even), (odd, even, even), (odd, even, odd), (even, even, even), (even, even, odd), (even, odd, odd), and (even, odd, even). A midpoint has integer coordinates, for example, the midpoint of (*A*, *B*, *C*) and (*I*, *J*, *K*) that are both (even, even, odd). Since we have nine distinct points in the space, by the Pigeonhole Principle, there are at least two points have the same pattern of parity, which means, their midpoint will have integer coordinates.

16. We can pair up these 8 numbers in the set as the sum of each pair is 16: (1, 15), (5, 11), (13, 3), and (7, 9). As we select the fifth number, by the Pigeonhole Principle, there are at least one pair of these numbers can be formed, which guarantee that the sum is 16. Therefore, we must select **5** numbers from the set to guarantee that at least one pair of these numbers add up to 16.

18. a) In the case of the first eight students that are 4 male students and 4 female students, by the Pigeonhole Principle, at least one gender (at most as well) the ninth student's gender corresponds to, that is, add one to one of the gender. Therefore, the class must have 5 male students or 5 female students.

b) We can list the possible outcome of their gender (Male #, Female #): (9,0), (8,1), (7,2), (6,3), (5,4), (4,5), (3,6), (2,7), (1,8), and (0,9). We can observe that each case satisfies one of the conditions—at least 3 male students or at least seven female students ((9,0) to (3,6) meet condition 1 and (2,7) to (0,9) meet condition 2). Thus, we can prove that the class must have at least 3 male students or at least seven female students.

27. Let A be one of the ten people in the group. The other 9 people must have either six enemies or 4 friends of A. **I**. In the case of six enemies of A, by the Generalized Pigeonhole Principle, in the other 5 people, there are either three or more friends of B, or three or more enemies of B. If any two of the friends are friends, then they can form a group of three mutual friends with B; or, they will form a group of three mutual enemies. Thus, in the group of six people, there are either three mutual friends or three mutual enemies. If there are three mutual enemies, they can form four mutual enemies with A; or, there are three mutual friends. **II**. In the case of four friends of A, if any two of these four people are friends, they can form a group of three mutual friends with A; otherwise, they will form a group of four mutual enemies by themselves. We have proved the first condition in the question. Now suppose there are either six friends of A or four enemies of A. This case is basically the reverse of the above, thus we can conclude there are either four mutual friends or three mutual enemies, which proves the condition 2. Hence, in a group of 10 people, there are either three mutual friends/enemies or four mutual enemies/friends.

35. By the Generalized Pigeonhole Principle, we need at least **18** rooms ($\lceil 677/38 \rceil = 18$).

6.3 2, 4, 9, 13, 17, 25, 41

2.