MATH 1107 Solution

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CSE-①

KHULNA UNIVERSITY OF ENGINEERING & TECHNOLOGY
B.Sc. Engineering 1st Year 1st Term Examination, 2018
Department of Computer Science and Engineering
MATH 1107
Differential and Integral Calculus

TIME: 3 hours FULL MARKS: 210

N.B. i) Answer **ANY THREE** questions from each section in separate scripts.
ii) Figures in the right margin indicate full marks.

SECTION A

(Answer **ANY THREE** questions from this section in Script A)

1. a) Define continuity of a function. A function $f(x)$ is defined as follows: (15)

$$f(x)=\begin{cases} x+\sqrt{2}a\sin x \ \text{for}\ 0\le x<\pi/4 \\ 2x\cot x+b \ \text{for}\ \pi/4\le x<\pi/2 \\ a\cos 2x-b\sin x \ \text{for}\ \pi/2\le x\le \pi \end{cases}$$

If $f(x)$ is continuous in the interval $0\le x\le\pi$, find the values of a and b.

b) State Rolle's theorem. Verify Rolle's theorem for $f(x)=2x^3+x^2-4x-2$ over $[-\sqrt{2},\sqrt{2}]$. (12)

c) Evaluate: $\lim\limits_{x\to 0}\left(\frac{1}{x}-\frac{1}{xe^x}\right)$. (08)

2. a) State Liebnitz's theorem. If $y=e^{m\sin^{-1}x}$, then show that (13)

$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+m^2)y_n=0.$$

b) Find the relative extreme values (maxima/minima) of the following function using second derivative test: $f(x)=x^3-6x^2+9x+5$. (12)

c) Define subtangent and subnormal. Prove that for the curve $by^2=(x+a)^3$, the square of the subtangent varies as the subnormal. (10)

3. a) State Euler's theorem. If $u=f(x^2+2yz, y^2+2zx)$, prove that (14)

$$(y^2-zx)\frac{\partial u}{\partial x}+(x^2-yz)\frac{\partial u}{\partial y}+(z^2-xy)\frac{\partial u}{\partial z}=0$$

b) Define tangent and normal. Find the equation of tangent and normal of the curve $x^3+y^3=3axy$ at $\left(\frac{3a}{2},\frac{3a}{2}\right)$, where $a>0$. (12)

c) Find $\frac{dy}{dx}$, when $y=x^{\cos^{-1}x}+(\sin x)^{\log x}$. (09)

4. a) Find the nth derivative of $y=\frac{1}{x^2-5x+6}+\cos 2x\cos x$ (12)

b) Find the Taylor series with reminder of $\sin x$ about $(x-\pi/2)$. (12)

c) Find the radius of curvature at the origin of the curve (11)
$3x^4-2y^4+5x^2y+2xy-2y^2+4x=0$.

SECTION B

(Answer **ANY THREE** questions from this section in Script B)

5. a) Calculate $\int\frac{dx}{(x^2+1)\sqrt{x^2-4}}$. (12)

b) Calculate $\int\sin^{-1}\left(\sqrt{\frac{x}{2+x}}\right)dx$. (13)

c) Calculate $\int(4x+15)\sqrt{x^2+6x+10}\,dx$. (10)

6\. a) Obtain a reduction formula for $\int \sin^m x \cos^n x \, dx$. (11)

b) If $U_n = \int_0^1 x^n \tan^{-1} x dx$, then prove that $(n+1)U_n + (n-1)U_{n-2} = \frac{\pi}{2} - \frac{1}{n}$. (12)

c) Evaluate $\int_0^{\infty} \frac{x dx}{(1+x)(1+x^2)}$. (12)

7\. a) Evaluate $\int_0^{\pi} x \log \sin x \, dx$. (12)

b) Evaluate $\int_0^{\frac{\pi}{4}} \frac{x dx}{1+\cos 2x + \sin 2x}$. (12)

c) Evaluate $\int_0^2 x(8-x^3)^{1/3} dx$. (11)

8\. a) Evaluate: $\lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2-1^2}} + \frac{1}{\sqrt{n^2-2^2}} + \ldots\ldots + \frac{1}{\sqrt{n^2-(n-1)^2}}\right]$. (09)

b) Define Gamma and Beta function. Find the relation between Gamma and Beta function. (15)

c) Using double integral to find the area of common portion of the curves $y^2 = 12x$ and $x^2 = 12y$. (11)

1 Ques : (a) Define continuity of a function. A function $f(x)$ is defined as follows:

$$f(x) = \begin{cases} x + \sqrt{2}\, a \sin x & \text{for } 0 \le x < \frac{\pi}{4} \\ 2x \cot x + b & \text{for } \frac{\pi}{4} \le x < \frac{\pi}{2} \\ a \cos 2x - b \sin x & \text{for } \frac{\pi}{2} \le x \le \pi \end{cases}$$

If $f(x)$ is contineous in the interval $0 \le x \le \pi$, find the values of a and b.

(b) State Rolle's theorem. Verify Rolle's theorem for $f(x) = 2x^3 + x^2 - 4x - 2$ over $[-\sqrt{2}, \sqrt{2}]$

(c) Evaluate $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{x e^x} \right)$.

Ans: (a) A function is said to be contineous at $x = c$ provided that the following rules are satisfied:

1) $f(c)$ is defined

2) $\lim_{x \to c} f(x)$ exist

3) $f(c) = \lim_{x \to c} f(x)$

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If a function abide by these rules, then it's called the continuety of a function.

(b) Rolle's Theorem: Let $f(x)$ be real valued function in interval $[a,b]$ such that,

(i) $f(x)$ is contineous in the closed interval $a \leq x \leq b$

(ii) $f'(x)$ exist in the open interval $a < x < b$

(iii) $f(a) = f(b)$

then there exist at least one value of x (say x_0) between a and b such that $f'(x_0) = 0$

Now $f(x) = 2x^3 + x^2 - 4x - 2$ over $[-\sqrt{2}, \sqrt{2}]$

Now, $f(-\sqrt{2}) = 2\cdot(-\sqrt{2})^3 + (-\sqrt{2})^2 - 4(-\sqrt{2}) - 2$

$= -4\sqrt{2} + 2 + 4\sqrt{2} - 2 = 0$

and $f(\sqrt{2}) = 2(\sqrt{2})^3 + (\sqrt{2})^2 - 4(\sqrt{2}) - 2$

$= 4\sqrt{2} + 2 - 4\sqrt{2} - 2$

$= 0$

$\therefore f(-\sqrt{2}) = f(\sqrt{2}) = 0$

So $f(x)$ is contineous in $[-\sqrt{2}, \sqrt{2}]$

According to Rolle's theorem,

$f'(x_0) = 6x_0^2 + 2x_0 - 4 = 0$

$$\Rightarrow 6x_0^2 + 6x_0 - 4x_0 - 4 = 0$$

$$\Rightarrow 6x_0(x_0 + 1) - 4(x_0 + 1) = 0$$

$$\Rightarrow (6x_0 - 4)(x_0 + 1) = 0$$

$$\Rightarrow x_0 = \frac{4}{6} = \frac{2}{3} \quad \text{or} \quad x_0 = -1$$

Thus there exist ~~atleast~~ two points $x_0 = -1$ and $x = \frac{2}{3}$ in $(-\sqrt{2}, \sqrt{2})$ respecting such that $f(-1) = 0$ and $f\left(\frac{2}{3}\right) = 0$.

Thus Rolle's theorem is verified.

(c) Now,

$$\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{xe^x} \right)$$

$$= \lim_{x \to 0} \frac{e^x - 1}{xe^x}$$

$$= \lim_{x \to 0} \frac{e^x - 1}{x} \cdot \lim_{x \to 0} \frac{1}{e^x}$$

$$= 1 \cdot \frac{1}{e^0}$$

$$= 1 \cdot 1$$

$$= 1 \text{ [Ans]}$$

If $f(x)$ is contineous in the interval $0 < x \leq \pi$

then, $\lim\limits_{x \to \frac{\pi}{4}^-} f(x) = f\left(\frac{\pi}{4}\right)$

$\Rightarrow \lim\limits_{x \to \frac{\pi}{4}^-} \left(x + \sqrt{2}\, a \sin x\right) = 2 \cdot \frac{\pi}{4} \cot \frac{\pi}{4} + b$

$\Rightarrow \lim\limits_{h \to 0} \left\{\left(\frac{\pi}{4} - h\right) + \sqrt{2}\, a \sin\left(\frac{\pi}{4} - h\right)\right\} = \frac{\pi}{2} \cdot 1 + b$

$\Rightarrow \frac{\pi}{4} + \sqrt{2}\, a \sin \frac{\pi}{4} = \frac{\pi}{2} + b$

$\Rightarrow \frac{\pi}{4} + a = \frac{\pi}{2} + b \Rightarrow a - b = \frac{\pi}{4}$ —— (I)

again, $\lim\limits_{x \to \frac{\pi}{2}^-} f(x) = f\left(\frac{\pi}{2}\right)$

$\cancel{\Rightarrow \lim\limits_{x \to \left(\frac{\pi}{2} - h\right)} \left\{a \cos 2\left(\frac{\pi}{2} - h\right) - b \sin\left(\frac{\pi}{2} - h\right)\right\}}$

$\Rightarrow \lim\limits_{x \to \frac{\pi}{2} - h} 2 \cdot \left(\frac{\pi}{2} - h\right) \cot\left(\frac{\pi}{2} - h\right) + b = a \cos 2 \cdot \frac{\pi}{2} - b \sin \frac{\pi}{2}$

$\Rightarrow \lim\limits_{h \to 0} 2\left(\frac{\pi}{2} - h\right) \cot\left(\frac{\pi}{2} - h\right) + b = a \cos \pi - b \sin \frac{\pi}{2}$

$\Rightarrow \pi \cdot 0 + b = -a - b$

$\Rightarrow a = -2b$ —— (II)

$\therefore -2b - b = \frac{\pi}{4}$ [From (I)]

$\Rightarrow b = -\frac{\pi}{12}$ [Ans]

$\therefore a = \frac{\pi}{6}$ [Ans]

(2.)

(a)

Given that,

$y = e^{m\sin^{-1}x}$

Diff. both sides w.r.t. x

$y_1 = e^{m\sin^{-1}x} \cdot \frac{m}{\sqrt{1-x^2}}$

$\Rightarrow \sqrt{1-x^2}\, y_1 = my$

$\Rightarrow (1-x^2)\, y_1^2 = m^2 y^2$

Again Diff. both sides w.r.t. x

$(1-x^2)\, 2y_1y_2 - 2xy_1^2 = 2yy_1m^2$

$\Rightarrow (1-x^2)y_2 - xy_1 - m^2y = 0$

Diff. n times using leibnitz theorem

$$(1-x^2)y_{n+2} + {}^nC_1(-2x)y_{n+1} + {}^nC_2(-2)y_n - \{xy_{n+1} + {}^nC_1(1)y_n - m^2y_n\} - m^2y_n = 0$$

$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-n+n+m^2)y_n = 0$

$\Rightarrow (1-x^2)\, y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$

(b) Given that,

$$f(x) = x^3 - 6x^2 + 9x + 5$$

$$\therefore f'(x) = 3x^2 - 12x + 9$$

$$f''(x) = 6x - 12$$

for maxima and minima, $f'(x) = 0$

$$\Rightarrow 3(x^2 - 4x + 3) = 0$$

$$\Rightarrow x = 1, 3$$

Now,

$f''(1) = 6 - 12 = -6 < 0$. Therefore, ~~maxim~~ at point $x = 1$ maximum value exists.

And maximum value is: $f(1) = 1 - 6 + 9 + 5$

$$= 9$$

Again, $f''(3) = 18 - 12 > 0$; Therefore at point $x = 3$ minimum value exists.

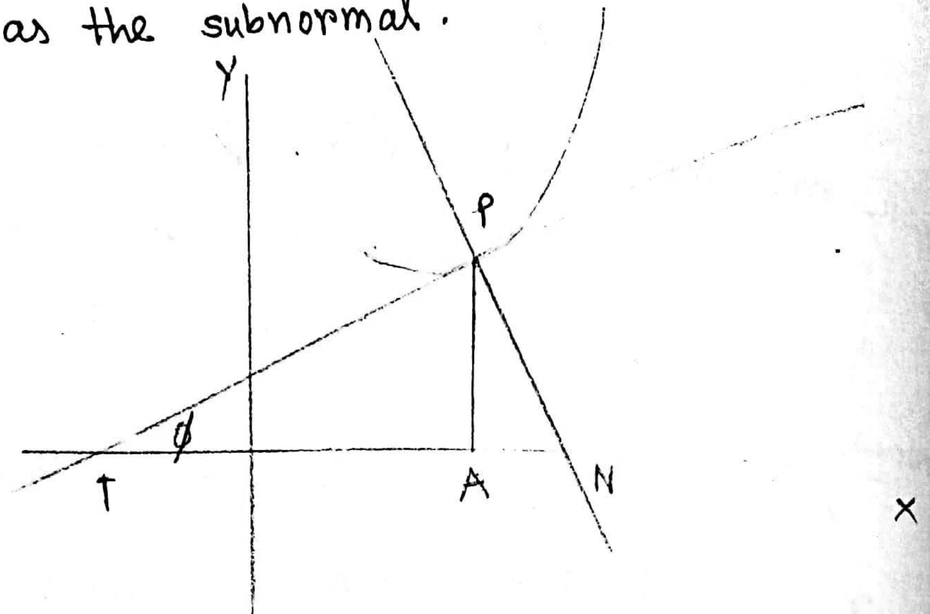
And minimum value is: $f(3) = 27 - 54 + 27 + 5$

$$= 5$$

$\therefore$ maximum value $= 9$

minimum " $= 5$

2. c) Define subtangent and subnormal. Prove that for the curve $by^2=(x+a)^3$ the squre of the subtangent varies as the subnormal.



Let $P=(x,y)$ be a point on a given curve with $A=(x,0)$ as projection onto the x-axis. Draw the tangent to the curve at P and let T be the point where this line intersects the x-axis. Then TA is defined to be the subtangent at P. Similarly, if normal to the curve at P intersects the x-axis at N then AN is called the subnormal. In this context, the lengths PT and PN are called the tangent and normal, not to be confused with the tangent line and the normal line which are called the tangent and normal

2nd Part

given eqn $by^2 = (x+a)^3$ — (1)

$\therefore 2byy_1 = 3(x+a)^2, \Rightarrow y_1 = \dfrac{3(x+a)^2}{2by}$ — (2)

$(\text{subtangent})^2 = \left(\dfrac{y}{y_1}\right)^2 = \dfrac{y^2}{y_1^2} = \dfrac{y^2}{9(x+a)^4/4b^2y^2}$; [by (2)]

$= \dfrac{4b^2y^4}{9(x+a)^4} = \dfrac{4(by^2)^2}{9(x+a)^4} = \dfrac{4(x+a)^6}{9(x+a)^4}$ [by (1)]

$= \dfrac{4}{9}(x+a)^2$

$(\text{subnormal}) = yy_1 = y \cdot \dfrac{3(x+a)^2}{2by}$; [by (2)]

$= \dfrac{3(x+a)^2}{2b}$

$\therefore \dfrac{(\text{subtangent})^2}{\text{subnormal}} = \dfrac{4(x+a)^2/9}{3(x+a)^2/2b} = \dfrac{8b}{27}$, is constant

$\therefore$ squre of subtangent varies as the subnormal.

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2018.

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4.

a) Find the nth derivative of $y = \frac{1}{x^2-5x+6} + \cos 2x . \cos x$.

b) Find the Taylor series with reminder of $\sin x$ about $(x-\frac{\pi}{2})$.

c) Find the Radius of curvature at the origin of the curve.

(a)

Given that

$$y = \frac{1}{x^2-5x+6} + \cos 2x . \cos x$$

$$\Rightarrow y = \frac{1}{(x-3)(x-2)} + \frac{1}{2}[\cos(2x+x) + \cos(2x-x)]$$

$$\Rightarrow y = \frac{1}{(x-3)(1)} + \frac{1}{(-1)(x-2)} + \frac{1}{2}(\cos 3x + \cos x)$$

$$\Rightarrow y = (x-3)^{-1} - (x-2)^{-1} + \frac{1}{2}[\cos 3x + \cos x] \text{ --- } (1)$$

differentiating equation (1) with respect to x:

$y_1 = (-1)(x-3)^{-2} - (-1)(x-2)^{-2} + \frac{1}{2}[-3\sin 3x - \sin x]$

$\Rightarrow y_1 = (-1)(x-3)^{-2} - (-1)(x-2)^{-2} + \frac{1}{2}$
$[3\cos(\frac{\pi}{2}+3x) + \cos(\frac{\pi}{2}+x)]$

$\Rightarrow y_2 = (-1)(-2)(x-3)^{-3} - (-1)(-2)(x-2)^{-3}$
$+ \frac{1}{2}[-3^2\sin(\frac{\pi}{2}+3x) - \sin(\frac{\pi}{2}+x)]$

$\Rightarrow y_2 = (-1)(-2)(x-3)^{-3} - (-1)(-2)(x-2)^{-3}$
$+ \frac{1}{2}[3^2\cos(2\pi+2+3x) + \cos(\frac{2\pi}{2}+x)]$

$\therefore y_3 = (-1)(-2)(-3)(x-3)^{-4} - (-1)(-2)(-3)$
$(x-2)^{-4} + \frac{1}{2}[-3^3\sin(\frac{2\pi}{2}+3x)$
$-\sin(\frac{2\pi}{2}+x)]$

$\Rightarrow y_3 = (-1)^3\, 3!\,(x-3)^{-3-1} - (-1)^3\, 3!\,(x-2)^{-3-1}$
$+ \frac{1}{2}[3^3\cos(\frac{3\pi}{2}+3x) + \cos(\frac{3\pi}{2}+x)]$

Simmillary if we go further.

$y_n = (-1)^n\, n!\,(x-3)^{-n-1} - (-1)^n\, n!$
$(x-2)^{-n-1}$
$+\frac{1}{2}[3^n\cos(\frac{n\pi}{2}+3x) + \cos(\frac{n\pi}{2}+x)]$

$$\therefore y_n = \frac{(-1)^n n!}{(x-3)^{n+1}} - \frac{(-1)^n n!}{(x-2)^{n+1}} + \frac{1}{2}\left[3^n \cos\left(\frac{n\pi}{2}+3x\right) + \cos\left(\frac{n\pi}{2}+x\right)\right]$$

8. (c)

$$3x^4 - 2y^4 + 5x^2y + 2xy - 2y^2 + 4x = 0 \quad \text{--- (i)}$$

Tangent equation in the main point

$x = 0$ (Actually y axis).

According to the method of Newton,

$$p = \lim_{\substack{x\to 0\\ y\to 0}} \frac{y^2}{2x} \quad \text{---(2)}$$

(i) $\div 2$

$$\frac{3x^3}{2} - 2y^2\frac{y^2}{2x} + \frac{5xy}{2} + y - 2\cdot\frac{y^2}{2x} + 2 = 0$$

$$\Rightarrow \lim_{x\to 0} \frac{3x^3}{2} - \lim_{y\to 0} 2y^2 \cdot \lim_{\substack{x\to 0\\ y\to 0}} \frac{y^2}{2x} + \lim_{\substack{x\to 0\\ y\to 0}} \frac{5xy}{2}$$

$$+ \lim_{y\to 0} - 2\lim_{\substack{x\to 0\\ y\to 0}} \frac{y^2}{2x} + 2 = 0$$

$\Rightarrow 0 - 0p + 0 + 0 - 2p + 2 = 0$

$\Rightarrow p = 1$

$\therefore p = 1$

(Ans)

(b)

According to Taylor's theorem we know that

$$f(x) = f(\tfrac{\pi}{2}) + (x - \tfrac{\pi}{2}) f'(\tfrac{\pi}{2}) + \frac{(x-\frac{\pi}{2})^2}{2!} f''(\tfrac{\pi}{2}) + \frac{(x-\frac{\pi}{2})^3}{3!} f'''(\tfrac{\pi}{2}) + \frac{(x-\frac{\pi}{4})^4}{4!} f^{iv}(\tfrac{\pi}{2}) + \dots + R_n.$$

Now,

$f(x) = \sin x \Rightarrow f(\frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1$

$f'(x) = \cos x \Rightarrow f'(\frac{\pi}{2}) = \cos(\frac{\pi}{2}) = 0.$

$f''(x) = -\sin x \Rightarrow f''(\frac{\pi}{2}) = -\sin\frac{\pi}{2} = -1$

$f'''(x) = -\cos x \Rightarrow f'''(\frac{\pi}{2}) = -\cos\frac{\pi}{2} = 0$

$f^{iv}(x) = \sin x \Rightarrow f^{iv}(\frac{\pi}{2}) = \sin\frac{\pi}{2} = 1$

Putting all the values in equation (1).

$$f(x) = 1 + 0 - \frac{(x-\frac{\pi}{2})^2}{2!} + 0 + \frac{(x-\frac{\pi}{2})^4}{4!} + \cdots$$

$$= 1 - \frac{(x-\frac{\pi}{2})^2}{2!} + \frac{(x-\frac{\pi}{2})^4}{4!} + \cdots + R_n.$$

Realizing that.

$$R_n = (-1)^{n+1} \frac{1}{(2n)!} (x-\frac{\pi}{2})^{2n}.$$

So,

$$\sin x = 1 - \frac{(x-\frac{\pi}{2})^2}{2!} + \frac{(x-\frac{\pi}{2})^4}{4!} + \cdots$$

$$\cdots + (-1)^{n+1} (x-\frac{\pi}{2})^{2n} \cdot \frac{1}{(2n)!}$$

Ans

Roll: 180707 (15)

(5) $\int \frac{dx}{(x^2+1)\sqrt{x^2-4}}$

Let, $x^2-4 = x^2 z^2$

$\Rightarrow x^2(1-z^2) = 4$ ---(1)

$\Rightarrow x^2 = \frac{4}{1-z^2}$

$\Rightarrow 2x\,dx = \frac{-4}{(1-z^2)^2}(-2z)$

$x\,dx = \frac{4z\,dz}{(1-z^2)^2}$

Again From eq (1) $\Rightarrow$

$z = \frac{\sqrt{x^2-4}}{x}$

$= \int \frac{4z\,dz}{x(1-z^2)\left(\frac{4}{1-z^2}+1\right) \cdot xz}$

$= 4\int \frac{dz}{\left(\frac{4}{1-z^2}\right)(1-z^2) \cdot \left(\frac{5-z^2}{1-z^2}\right)}$

$= \int \frac{dz}{5-z^2} = \int \frac{dz}{(\sqrt{5})^2-(z^2)}$

$= \frac{1}{2\sqrt{5}} \ln\left|\frac{\sqrt{5}+z}{\sqrt{5}-z}\right| + c$

$= \frac{1}{2\sqrt{5}} \ln\left|\frac{\sqrt{5}+\frac{\sqrt{x^2-4}}{x}}{\sqrt{5}-\frac{\sqrt{x^2-4}}{x}}\right| + c$

(6) $\int \sin^{-1}\sqrt{\frac{x}{2+x}}\,dx$

Let, $x = 2\tan^2\theta$

$\Rightarrow dx = 2\tan\theta \cdot \sec^2\theta\,d\theta$

$= \int \sin^{-1}\sqrt{\frac{2\tan^2\theta}{2+2\tan^2\theta}} \cdot 4\tan\theta \cdot \sec^2\theta\,d\theta$

$= 4\int \sin^{-1}\sqrt{\frac{\tan^2\theta}{\sec^2\theta}}\,\tan\theta \cdot \sec^2\theta\,d\theta$

$= \int \sin^{-1}\left(\frac{\sin\theta}{\cos\theta} \cdot \cos\theta\right)\tan\theta \cdot \sec^2\theta\,d\theta$

$$= \int \theta \cdot \tan\theta \sec^2\theta \, d\theta$$

Let $\tan\theta = z$ $\qquad \theta = \tan^{-1} z$

$\Rightarrow \sec^2\theta \, d\theta = dz$

$$= \int \tan^{-1}(z) \cdot z \cdot dz$$

$$= \tan^{-1} z \cdot z^2/2 - \int \frac{1}{1+z^2} \cdot z^2/2 \, dz$$

$$= z^2/2 \tan^{-1} z - \frac{1}{2} \int \frac{1+z^2-1}{1+z^2} \, dz$$

$$= \frac{z^2}{2} \tan^{-1} z - \frac{1}{2}\left[\int dz - \int \frac{dz}{1+z^2}\right]$$

$$= z^2/2 \tan^{-1} z - z/2 + \frac{1}{2} \tan^{-1} z \quad [\text{Here } z = \tan\theta]$$

5 (c) $\int (4x+15)\sqrt{x^2+6x+10} \, dx$

$x^2+6x+10 = z^2$

$\Rightarrow (2x+6)\,dx = 2z\,dz$

$(4x+12)\,dx = 4z\,dz$

$$= \int \left\{4/2\,(2x+6)+3\right\}\sqrt{x^2+6x+10} \, dx$$

$$= 2\int (2x+6)\sqrt{x^2+6x+10}\, dx + 3\int \sqrt{x^2+6x+10}\, dx$$

$$= 2 \cdot 2/3\,(x^2+6x+10)^{3/2} + 3 \cdot \int \sqrt{x^2+2\cdot 3x+(3)^2+1}\, dx$$

$$= 4/3\,(x^2+6x+10)^{3/2} + 3\int \sqrt{(x+3)^2+1^2}\, dx$$

$$= 4/3\,(x^2+6x+10)^{3/2} + 3\left[x/2\sqrt{(x+3)^2+1^2} + \frac{1}{2}\ln\left|\sqrt{(x+3)^2+1^2}+x\right|\right] + c$$

Question : 6 (a)

Obtain a reduction formula for $\int \sin^m x \cos^n x \, dx$

Solution:

$I = \int \sin^m x \cos^n x \, dx$

$= \int \sin^m x \cos^{n-1} \cos x \, dx$

$= \cos^{n-1} x \cdot \frac{\sin^{m+1} x}{m+1} + \int (n-1) \cos^{n-2} x \, (-) \sin x \, \frac{\sin^{m+1} x}{m+1} \, dx$

$= \cos^{n-1} x \cdot \frac{\sin^{m+1} x}{m+1} + \frac{(n-1)}{(m+1)} \int \sin^m x \, \frac{\sin^2 x}{\cancel{1 - \cos^2 x}} \cos^{n-2} x \, dx$

$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{(n-1)}{(m+1)} \int \sin^m x \, (1 - \cos^2 x) \cos^{n-2} x \, dx$

$I = \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{(n-1)}{(m+1)} \int \sin^m x \cos^{n-2} x$

$\quad - \frac{(n-1)}{(m+1)} \int \sin^m x \cos^n x \, dx$

$$I = \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{(n-1)}{(m+1)} I_{m,n-2} - \frac{(n-1)}{(m+1)} I.$$

$$I \left(\frac{m+1+n-1}{m+1} \right) = \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{(n-1)}{(m+1)} I_{m,n-2}$$

$$I = \frac{\cos^{n-1} x \sin^{m+1} x}{m+n} + \frac{(n-1)}{m+n} I_{m,n-2}.$$

$= \frac{1}{2}x$ ~~tod~~

6(b) $U_n = \int_0^1 x^{n-1} \; x\tan^{-1}x \, dx$

$= \left[x^{n-1} \int x\tan^{-1}x \, dx\right]_0^1 -$

$\int_0^1 \left\{(n-1)x^{n-2} \int x\tan^{-1}x \, dx\right\} dx$

here, $\int x\tan^{-1}x \, dx$

$= \tan^{-1}x \, \frac{x^2}{2} - \int \left\{\frac{1}{1+x^2} \cdot \frac{x^2}{2}\right\} dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x$

$\therefore U_n = \left[x^{n-1} \left\{\frac{x^2}{2}\tan^{-1}x - \frac{x}{2} + \frac{1}{2}\tan^{-1}x\right\}\right]_0^1$

$- \int_0^1 \left[(n-1)x^{n-2} \left\{\frac{x^2}{2}\tan^{-1}x - \frac{x}{2} + \frac{1}{2}\tan^{-1}x\right]\right. dx$

(19)

6/c. $\int_0^{\infty} \frac{x\,dx}{(1+x)(1+x^2)}$

$$\frac{x}{(1+x)(1+x^2)} \equiv \frac{A}{1+x} + \frac{Bx+C}{1+x^2}$$

$\Rightarrow x \equiv A(1+x^2) + (Bx+C)(1+x) \quad \cdots\cdots (1)$

$\Rightarrow x = A + Ax^2 + Bx + Bx^2 + C + Cx$

$\Rightarrow x \equiv x^2(A+B) + x(B+C) + A + C \quad \cdots\cdots (2)$

(i) নং এ $x = (-1)$,

$-1 \equiv A\{1 + (-1)^2\} + 0$

$\therefore A = -\frac{1}{2}$

$\therefore$ (ii) $\rightarrow$ $A+B = 0 \Rightarrow B = \frac{1}{2}$

$B + C = 1 \Rightarrow C = \frac{1}{2}$,

$$\therefore \int_0^{\infty} \frac{x\,dx}{(1+x)(1+x^2)} \equiv \int_0^{\infty} -\frac{1}{2}(1+x)^{-1} + \frac{\frac{1}{2}x + \frac{1}{2}}{1+x^2}$$

$$= \frac{1}{2}\left[\ln(1+x)\right]_0^{\infty} + \frac{1}{2}x + \frac{1}{2}\left[\tan^{-1}x\right]_0^{\infty}$$

$$= \frac{1}{2}\left\{\ln(1+\infty) - \ln(1+0)\right\} + \frac{1}{2}x + \frac{1}{2}\left[\tan^{-1}\infty - \tan^{-1}0\right]$$

$$= \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8} - \int_0^1 (n-1)\frac{x^n}{2}\tan^{-1}x\,dx +$$

$$\int_0^1 \frac{x}{2}(n-1)x^{n-2}\,dx - \int_0^1 (n-1)x^{n-2}\tan^{-1}x\,dx$$

$$= \frac{\pi}{4} - \frac{1}{2} - \frac{(n-1)}{2}U_n + \frac{(n-1)}{2}\int_0^1 x^{n-1}\,dx - \frac{(n-1)U_{n-2}}{2}$$

$$\Rightarrow U_n = \frac{\pi}{4} - \frac{1}{2} - \frac{(n-1)}{2}U_n + \frac{n-1}{2}\left[\frac{x^n}{n}\right]_0^1 - \left(\frac{n-1}{2}\right)U_{n-2}$$

$$\Rightarrow \left(1+\frac{n-1}{2}\right)U_n = \frac{\pi}{4} - \frac{1}{2} + \frac{n-1}{2n} - \left(\frac{n-1}{2}\right)U_{n-2}$$

$$\Rightarrow \left(\frac{2+n-1}{2}\right)U_n = \frac{\pi}{4} - \frac{1}{2} + \frac{n-1}{2n} - \frac{n-1}{2}U_{n-2}$$

$$\Rightarrow (n+1)U_n = \frac{\pi}{2} - 1 + \frac{n-1}{n} - (n-1)U_{n-2}$$

$$\Rightarrow (n+1)U_n + (n-1)U_{n-2} = \frac{\pi}{2} - 1 + 1 - \frac{1}{n}$$

$$= \frac{\pi}{2} - \frac{1}{n}$$

(21)

$$\therefore (n+1)U_n + (n-1)U_{n-2} = \frac{\pi}{2} - \frac{1}{n}$$

[Proved]

Roll: 80

Question : 7(a)

Evaluate $\int_0^{\pi} x \log \sin x \, dx$.

Solution :

$$\int_0^{\pi} x \log \sin x \, dx \quad \cdots\cdots\cdots (I)$$

$$= \int_0^{\pi} (\pi - x) \log \sin x \, dx \quad \cdots\cdots (II)$$

[by property 4].

(I) + (II)

$$= \pi \int_0^{\pi} \log \sin x \, dx$$

$$= 2\pi \int_0^{2 \cdot \pi/2} \log \sin x \, dx.$$

$$I = 2\pi \int_0^{\pi/2} \log \sin x \, dx \quad \text{[by property 6]}.$$

$\cdots\cdots\cdots$ (III)

$$I = 2\pi \int_0^{\pi/2} \log \sin(\pi/2 - x) \, dx.$$

$$I = 2\pi \int_0^{\pi/2} \log \cos x \, dx \quad \cdots\cdots (IV).$$

(III) + (IV)

$$2I = 2\pi \int_0^{\pi/2} \log \sin x + \log \cos x \, dx.$$

$$2I = 2\pi \int_0^{\pi/2} \log \sin x \cos x \, dx.$$

$$2I = 2\pi \int_0^{\pi/2} \log \frac{\sin 2x}{2} \, dx.$$

$$2I = 2\pi \int_0^{\pi/2} \log \sin 2x \, dx + 2\pi \int_0^{\pi/2} \log \tfrac{1}{2} \, dx$$

Now, put, $2x = z$

$2\,dx = dz.$

when, $x = \pi/2$, $z = \pi$

$x = 0$, $z = 0$.

$$2I = \frac{2\pi}{2} \int_0^{\pi} \log \sin z \, dz + 2\pi \cdot \frac{\pi}{2} \log \frac{1}{2}$$

$$2I = \pi \cdot 2 \int_0^{\pi/2} \log \sin x \, dx + \pi^2 \log \frac{1}{2}$$

$$2I = I + \pi^2 \log \frac{1}{2}$$

$$I = \pi^2 \log \frac{1}{2}$$

Question : 7 (b)

Evaluate $\int_0^{\pi/4} \frac{x\,dx}{1+\cos 2x+\sin 2x}$

Solution :

$$\int_0^{\pi/4} \frac{x\,dx}{1+\cos 2x+\sin 2x} \quad \cdots\cdots\cdots (I)$$

$$= \int_0^{\pi/4} \frac{(\pi/4 - x)}{1+\cos 2x+\sin 2x} \quad [\text{by property } 4].$$

$\cdots\cdots\cdots (II)$

(I) + (II)

$$= \frac{\pi}{4} \int_0^{\pi/4} \frac{dx}{1+\cos 2x+\sin 2x}$$

$$= \frac{\pi}{4} \int_0^{\pi/4} \frac{\sec^2 x\,dx}{1+\tan^2 x+1-\tan^2 x+2\tan x}$$

$$= \frac{\pi}{4} \int_0^{\pi/4} \frac{\sec^2 x\,dx}{2(1+\tan x)}$$

$$= \frac{\pi}{8} \int_0^{\pi/4} \frac{\sec^2 x\,dx}{1+\tan x}$$

Now, $1+\tan x = z$

$\sec^2 x\,dx = dz$

when, $x = \frac{\pi}{4}$, $z = 2$.

$x = 0$, $z = 1$.

$$= \frac{\pi}{8} \int_1^2 \frac{dz}{z}$$

$$= \frac{\pi}{8} \left[\ln z\right]_1^2$$

$$= \frac{\pi}{8} \ln 2$$

Question : 7 (c)

Evaluate $\int_0^2 x(8-x^3)^{1/3}\,dx$.

Solution :

$$\int_0^2 x\,(8-x^3)^{1/3}\,dx$$

put, $x = 2z$ | when, $x=2$, $z=1$

$dx = 2dz$. | when, $x=0$, $z=0$

$$= 8\int_0^1 z\,(1-z^3)^{1/3}\,dz$$

Again put $z = t^{1/3}$ | when, $z=1$, $t=1$

$dz = \frac{1}{3}t^{-2/3}\,dt$ | $z=0$, $t=0$

$$I = \frac{8}{3}\int_0^1 t^{1/3}(1-t^{3/3})^{1/3}\cdot t^{-2/3}\,dt.$$

$$I = \frac{8}{3}\int_0^1 t^{-1/3}(1-t)^{1/3}\,dt.$$

$$= \frac{8}{3}\int_0^1 t^{2/3-1}(1-t)^{\frac{4}{3}-1}\,dt.$$

$$\therefore \frac{8}{3}\cdot\frac{\Gamma\frac{2}{3}\,\Gamma\frac{4}{3}}{\Gamma\left(\frac{2}{3}+\frac{4}{3}\right)}$$

$$= \frac{8}{3}\,\frac{\Gamma\frac{2}{3}\cdot\frac{4}{3}\,\Gamma\frac{1}{3}}{\Gamma 2}$$

$$= \frac{32}{9}\,\frac{\Gamma\frac{2}{3}\,\Gamma\frac{1}{3}}{2\cdot\Gamma 1}$$

$$= \frac{16}{9}\,\Gamma\frac{1}{3}\,\Gamma\frac{2}{3}$$

$$= \frac{16\pi}{9\sin\frac{\pi}{3}}$$

Reflection formula

$$\Gamma z\,\Gamma 1-z = \frac{\pi}{\sin(\pi z)}$$

$$\Gamma\frac{1}{3}\,\Gamma\frac{2}{3} = \Gamma\frac{1}{3}\,\Gamma 1-\frac{1}{3}$$

$$= \frac{\pi}{\sin(\pi\cdot\frac{1}{3})}$$

a

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8(a)/ Evaluate: $\lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2-1^2}}+\frac{1}{\sqrt{n^2-2^2}}+\cdots+\frac{1}{\sqrt{n^2-(n-1)^2}}\right]$

Ans:

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2-1^2}}+\frac{1}{\sqrt{n^2-2^2}}+\cdots+\frac{1}{\sqrt{n^2-(n-1)^2}}\right]$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n-1}\frac{1}{\sqrt{n^2-r^2}}$$

$$=\cancel{\lim} = \lim_{n\to\infty}\sum_{r=1}^{n-1}\frac{1}{n\sqrt{1-\left(\frac{r}{n}\right)^2}}$$

$$=\int_0^1\frac{1}{\sqrt{1-x^2}}\,dx$$

$$=\left[\sin^{-1}x\right]_0^1$$

$$=\frac{\pi}{2}$$

Ans: $\frac{\pi}{2}$.

8(b)/Q.] Define Gamma and Beta function. Find the relation between Gamma and Beta function.

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Gamma function:

The integral $\int_0^\infty e^{-x} x^{n-1} dx$; $n>0$ is called Gamma function. It is denoted by Γn. It is also called 2nd Euleriah integral.

Beta function:

The integral $\int_0^1 x^{m-1} (1-x)^{n-1} dx$; $m,n>0$ is called Beta function. It is denoted by $B(m,n)$. It is also called 1st Euleriah integral.

Finding the relation:

We have, $\Gamma m = \int_0^\infty e^{-x} x^{m-1} dx$ --- (1)

$\therefore \Gamma m = \int_0^\infty e^{-\lambda y} (\lambda y)^{m-1} \lambda dy$

$= \lambda^m \int_0^\infty e^{-\lambda y} y^{m-1} dy$ — (2)

putting
$x = \lambda y$
$dx = \lambda dy$
$x = 0; \infty$
$y = 0, \infty$

multiplying both sides by $e^{-\lambda} \lambda^{n-1}$ and integrating it with w.r to λ within the limit 0 to ∞. We get,

$$\Gamma m \int_0^\infty e^{-\lambda} \lambda^{n-1} d\lambda = \int_0^\infty \left(\lambda^m e^{-\lambda} \lambda^{n-1} \int_0^\infty e^{-\lambda y} y^{m-1} dy\right) dy$$

(28)

$$\Gamma m \, \Gamma m = \int_0^\infty \left(\int_0^\infty e^{-\lambda(1+y)} \lambda^{m+n-1} d\lambda \right) y^{m-1} dy$$

$$= \int_0^\infty \frac{\Gamma m+n}{(1+y)^{m+n}} y^{m-1} dy$$

$$= \Gamma m+n \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy$$

$$\Rightarrow \frac{\Gamma m \, \Gamma n}{\Gamma m+n} = \beta(n,m) \quad \left[as,\ \beta(m,n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dy \right]$$

$$= \beta(m,n) \quad [as\ \beta(m,n) = \beta(n,m)]$$

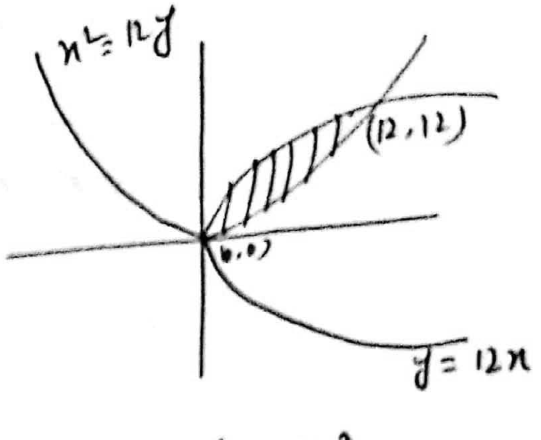
$$\therefore \beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

which is the relation between Beta and Gamma function.

3(c) using double integral to find the area of common portion of the curves $y^2 = 12x$ and $x^2 = 12y$. (29)

→ Given curves, $y^2 = 12x$ — (i)

$x^2 = 12y$ — (ii)



from (i) and (ii) we get,

$x = 0, 12$

when, $x=0$, $y=0$

when, $x=12$, $y=12$

so, (i) & (ii) intersects are (0,0) and (12,12)

Treating R as type I:

$$\text{Area } A = \iint_R dA = \int_{x=0}^{12} \int_{y=\frac{x^2}{12}}^{\sqrt{12x}} dy\,dx$$

$$= \int_{x=0}^{12} [y]_{\frac{x^2}{12}}^{\sqrt{12x}} dx = \int_{x=0}^{12} \left(\sqrt{12x} - \frac{x^2}{12}\right) dx$$

$$= \left[\frac{\sqrt{12}\,x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{3\times12}\right]_0^{12}$$

$= 48$ sq unit.

Or

Treating R as type II:

$$\text{Area } A = \iint_R dA = \int_{y=0}^{12} \int_{x=\frac{y^2}{12}}^{\sqrt{12y}} dx\,dy$$

(30)

$$= \int_{y=0}^{12} [x]_{\frac{y^2}{12}}^{\sqrt{12y}} \, dy$$

$$= \int_{y=0}^{12} \left[\sqrt{12y} - \frac{y^2}{12}\right] dy$$

$$= \left[\frac{\sqrt{12}\, y^{\frac{3}{2}}}{\frac{3}{2}} - \frac{y^3}{3\times 12}\right]_0^{12}$$

$= 48$ sq unit.

Ans: 48 sq unit.

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KHULNA UNIVERSITY OF ENGINEERING & TECHNOLOGY
B.Sc. Engineering 1st Year 1st Term Examination, 2017
Department of Computer Science and Engineering
MATH 1107
Differential and Integral Calculus

TIME: 3 hours

FULL MARKS: 210

N.B. i) Answer ANY THREE questions from each section in separate scripts.
ii) Figures in the right margin indicate full marks.

SECTION A
(Answer ANY THREE questions from this section in Script A)

1. a) Define limit and continuity of a function. A function $f(x)$ is defined as follows: (15)
$$f(x)=\begin{cases}3+2x & \text{for } -\frac{3}{2}<x\le 0\\ 3-2x & \text{for } 0<x\le \frac{3}{2}\end{cases}$$
Discuss the continuity and differentiability of $f(x)$ at any values of x.

b) Evaluate $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{e^{\tan x}-1}{e^{\tan x}+1}$. (10)

c) Find the differential coefficient of $(\tan x)^{\cot x}+(\cot x)^{\tan x}=0$. (10)

2. a) Find nth derivation of $y=\log(x+a)$. (13)

b) State Leibnitz's theorem. If $y=(\operatorname{cosec}^{-1}x)^2$ then find y_{n+2}. (12)

c) Verify Rolle's theorem for $f(x)=(x-2)(x-3)(x-4)$ at $(2,3)$. (10)

3. a) Define homogeneous function. If $u=\log r$ and $r^2=x^2+y^2+z^2$, then prove that $r^2\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}\right)=1$. (13)

b) Define maxima and minima of a function at a given point. Find the maximum and minimum values of $u=\dfrac{4}{x}+\dfrac{36}{y}$, where $x+y=2$. (12)

c) Find the equations of tangent and normal to the curve $y(x-2)(x-3)-x+7=0$ at the point where it cuts the x-axis. (10)

4. a) Find the radius of curvature at (x,y) on the curve $ay^2=x^3$. (10)

b) Find the asymptotes of $x^4-x^2y^2+x^2+y^2-a^2=0$. (15)

c) In the curve $x^p y^q=a^{p+q}$, show that subtangent at any point varies as the abscissa of the point. (10)

SECTION B
(Answer ANY THREE questions from this section in Script B)

5. Integrate any three of the followings: (35)

i) $\displaystyle\int\frac{x^3-3x}{x^2-4}dx$

ii) $\displaystyle\int\frac{x^2}{x^4+x^2-2}dx$

iii) $\displaystyle\int\frac{dx}{(1+x)\sqrt{1+x^2}}$

iv) $\displaystyle\int e^{2x}\left(\frac{1+\sin 2x}{1+\cos 2x}\right)dx$

6. Evaluate any three of the following definite integrals:

i) $\int_0^a \frac{dx}{a^2+x^2}$

ii) $\int_0^1 \frac{\log(1+x)}{1+x^2}dx$

iii) $\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx$

iv) $\int_0^\infty e^{-ax}\cos bx dx \ ; a>0$

7. a) Obtain the reduction formula for $\int \frac{dx}{(x^2+a^2)^{n/2}}$ and hence find the value of $\int \frac{dx}{(x^2+a^2) ...}$

b) Define Gamma function and Beta function. Prove that $\Gamma(n+1)=n\Gamma n$.

c) Evaluate $\lim_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\ldots+\frac{n}{n^2+n^2}\right]$.

8. a) i) Find the area under the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, x-axis and the coordinate $x=c$ an...

ii) Find the area between its latus-rectum.

b) Find the length of the curve $y^2=-4x$ from (0, 0) to (−1, 2).

1. ⓐ Define limit and continuity of a function.

A function $f(x)$ is defined as follows:

$$f(x) = \begin{cases} 3+2x & \text{for } -3/2 < x \le 0 \\ 3-2x & \text{for } 0 < x \le 3/2 \end{cases}$$

Discuss the continuity and differentiability of $f(x)$ at any value of x

<u>solution:</u>

$$L.H.L = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (3+2x) = 3 + 2.0 = 3$$

$$R.H.L = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (3-2x) = 3 - 2.0 = 3$$

As, $L.H.L = R.H.L$, limit exists and the

Limit value is 3.

Again, $f(0) = 3 + 2.0 = 3$

~~As, L.H.L = R.H.L = f~~

As, $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$ the

function is continuous.

(6) Evaluate, $\lim_{x \to \pi/2} \dfrac{e^{\tan x} - 1}{e^{\tan x} + 1}$

<u>solution:</u>

$$R.H.L = \lim_{x \to (\frac{\pi}{2}+h)} \frac{e^{\tan x} \cancel{+} 1}{e^{\tan x} + 1}$$

$$= \lim_{h \to 0} \frac{e^{\tan(\frac{\pi}{2}+h)} - 1}{e^{\tan(\frac{\pi}{2}+h)} + 1}$$

$$= \lim_{h \to 0} \frac{e^{-\cot h} - 1}{e^{-\cot h} + 1}$$

$$= \frac{e^{-\infty} - 1}{e^{-\infty} + 1} = -1$$

$$\therefore L.H.L = \lim_{x \to (\frac{\pi}{2}-h)} \frac{e^{\tan x} - 1}{e^{\tan x} + 1} = \lim_{h \to 0}$$

$$= \lim_{h \to 0} \frac{e^{\tan(\pi/2 - h)} - 1}{e^{\tan(\pi/2 - h)} + 1}$$

$$= \lim_{h \to 0} \frac{e^{\cot h} - 1}{e^{\cot h} + 1}$$

$$= \lim_{h \to 0} \frac{e^{\cot h}\left(1 - \frac{1}{e^{\cot h}}\right)}{e^{\cot h}\left(1 + \frac{1}{e^{\cot h}}\right)}$$

$$= \frac{1 - e^{-\coth}}{1 + e^{-\coth}}$$

$$= \frac{1 - e^{-\infty}}{1 + e^{-\infty}}$$

$$= \frac{1 - 0}{1 + 0}$$

$$= 1$$

as R.H.L $\neq$ L.H.L , the Limit value does not exist.

(e) Find the differential co-efficient of,

$$(\tan x)^{\cot x} + (\cot x)^{\tan x} = 0$$

solution:

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2.(a)

Q. Find the nth derivation of $y = \log(x+a)$

Soln:

$$y = \log(x+a)$$

$$y_1 = \frac{1}{x+a} = (-1)^{1+1}\frac{1!}{(x+a)^1}$$

$$y_2 = -\frac{1}{(x+a)^2} = (-1)^{2+1}\frac{2!}{(x+a)^2}$$

$$y_3 = \frac{1.3}{(x+a)^3} = (-1)^{3+1}\frac{3!}{(x+a)^3}$$

$$y_n = (-1)^{n+1}\frac{n!}{(x+a)^n}$$

2.(b)

Q. State Leibnitz's theorem, ~~If y= cosec x~~ If $y = (\operatorname{cosec}^{-1}x)^2$ then find y_{n+2}.

Soln: Leiblitz's theorem: If a function is product of two function's then the nth derivation of the function is.

$$(uv)_n = u_n v + {}^nc_1 u_{n-1} v_1 + {}^nc_2 u_{n-2} v_2 + \cdots + u v_n.$$

$$y = (\operatorname{cosec}^{-1} x)^2$$

$$\Rightarrow y_1 = 2\operatorname{cosec}^{-1} x \cdot \frac{1}{x\sqrt{x^2-1}}$$

$$\Rightarrow y_1\, x\sqrt{x^2-1} = 2\operatorname{cosec}^{-1} x$$

$$\Rightarrow y_1^2\, x^2(x^2-1) = 4(\operatorname{cosec}^{-1} x)^2$$

$$\Rightarrow y_1^2 x^4 - y_1^2 x^2 = 4y$$

$$\cancel{\Rightarrow 2y_1 y_2 x^4 + 4x^3 \quad 2y_1 y_2 x^3 + 2x}$$

$$\Rightarrow 2y_1 y_2 x^4 + 4x^3 y_1^2 - 2y_1 y_2 x^2 - 2x y_1^2 = 4y_1$$

$$\Rightarrow y_2 x^4 + 2y_1 x^3 - y_2 x^2 - y_1 x = 2$$

$$\Rightarrow y_2 (x^4 - x^2) = 2 + y_1(x - x^3)$$

n^{th} derivation on both sides

$$y_{n+2}(x^4 - x^2) + {}^nc_1\, y_{n+1}(4x^3 - 2x) + {}^nc_2\, y_n (12x^2 - 2)$$
$$+ {}^nc_3\, y_{n-1} \cdot 24x + {}^nc_4\, y_{n-2} \cdot 24 = y_{n+1}(x - x^3) + {}^nc_1\, y_n (1 - 3x^2)$$
$$+ {}^nc_2\, y_{n-1}(-)6x + {}^nc_3\, y_{n-2}(-)6$$

$$\therefore y_{n+2} = \frac{1}{x^4 - x^2}\Big[y_{n+1}(x - x^3) + {}^nc_1\, y_n (1 - 3x^2) - 6\,{}^nc_2\, y_{n-1}\, x$$
$$- 6\,{}^nc_3\, y_{n-2} - {}^nc_1\, y_{n+1}(4x^3 - 2x)$$
$$- {}^nc_2\, y_n (12x^2 - 2) - {}^nc_3\, y_{n-1}\, 24x + {}^nc_4\, y_{n-2} \cdot 24 \Big]$$

2.(c)

Q.: Varify Rolle's theorem for ~~f(x)~~

$f(x) = (x-2)(x-3)(x-4)$ at $(2,3)$

Solⁿ:

$f(x) = (x-2)(x-3)(x-4)$

so, $x = 2, 3, 4$

and, $f(2) = 0.(-1).(-2) = 0$

$f(3) = 0$

$f(4) = 0$

so $f(2) = f(3) = f(4)$

clearly $f(x)$ is continuous in $[2,3], [3,4], [2,4]$

now, $f(x) = (x-2)(x-3)(x-4)$

$= (x^2-5x+6)(x-4)$

$= x^3-9x^2-14x-30$.

$f'(x) = 3x^2-18x-14$

i.e. $f'(x)$ exists in $(2,3), (3,4), (2,4)$

According to Rolle's theorem. $f'(x_0) = 0$

$$f'(x_0) = 3x_0^2 - 18x_0 - 14 = 0$$

$$\therefore x_0 = \frac{-(-18) \pm \sqrt{(-18)^2 - 4.3(-14)}}{2.3}$$

$$= \frac{18 \pm \sqrt{492}}{6}$$

$$= \cancel{6.69}\ 6.69, -0.69$$

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3. (a) Define homogeneous function. If $u = \log r$ and $r = x^2+y^2+z^2$, then prove that $r^2\left(\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} + \frac{\delta^2 u}{\delta z^2}\right) = 1$

(b) Define maxima and minima of a function at a given point. Find the maximum and minimum values of $u = \frac{4}{x} + \frac{36}{y}$, where $x+y=2$

(c) Find the equation of tangent and normal to the curve $y(x-2)(x-3) - x + 7 = 0$ at the point where it cuts the x-axis.

Answer to the Question no-3

(a)

Given,

$$u = \log r$$

$$\Rightarrow u = \log \sqrt{x^2+y^2+z^2}$$

$$\frac{du}{dx} = \frac{1}{\sqrt{x^2+y^2+z^2}} \cdot \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2x$$

$$= \frac{x}{x^2+y^2+z^2}$$

$$\frac{d^2u}{dx^2} = \frac{(x^2+y^2+z^2)(1) - x(2x)}{(x^2+y^2+z^2)^2}$$

অনুরূপভাবে,

$$\frac{d^2u}{dy^2} = \frac{x^2+y^2+z^2-2y^2}{(x^2+y^2+z^2)^2}$$

$$\frac{d^2u}{dz^2} = \frac{x^2+y^2+z^2-2z^2}{(x^2+y^2+z^2)^2}$$

$$\underline{L.H.S} = r^2\left(\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} + \frac{\delta^2 u}{\delta z^2}\right)$$

$$= (x^2+y^2+z^2)\left\{\frac{-x^2+y^2+z^2-y^2+x^2+z^2-z^2+x^2+y^2+z^2}{(x^2+y^2+z^2)^2}\right\}$$

$$= (x^2+y^2+z^2)\left(\frac{1}{x^2+y^2+z^2}\right)$$

$$= 1$$

$$= R.H.S$$

$\therefore$ L.H.S = R.H.S (Proved)

(b)

Sol^n: Given,

$$u = \frac{4}{x} + \frac{36}{y}$$

$$\Rightarrow u = \frac{4}{x} + \frac{36}{2-x} \quad \text{--- (1)}$$

$$\Rightarrow \frac{du}{dx} = -\frac{4}{x^2} + \frac{36}{(2-x)^2}$$

Now,

$$\frac{du}{dx} = 0$$

$$\Rightarrow \frac{4}{x^2} = \frac{36}{(2-x)^2}$$

$$\Rightarrow 4(4-4x+x^2) = 36x^2$$

$$\Rightarrow 16-16x+4x^2 = 36x^2$$

$$\Rightarrow 32x^2+16x-16=0$$

$$\Rightarrow 2x^2+x-1=0$$

$$\Rightarrow 2x^2+2x-x-1=0$$

$$\Rightarrow (x+1)(2x-1)=0$$

$$\therefore x = -1, \frac{1}{2}$$

when, $x=-1$; $\frac{d^2u}{dx^2} = -8 + \frac{72}{27}$

$= -\frac{16}{3}$; which is negative

$\therefore$ The maximum value exists when $x=-1$

$\therefore$ The maximum value is $= -4 + \frac{36}{3} = -4+12$

$= 8$ (Ans.)

when,

$x=\frac{1}{2}$; $\frac{d^2u}{dx^2}=64+\frac{72\times 8}{27}$

$=\frac{256}{3}$, which is positive

$\therefore$ The ~~maximum~~ minimum value is $=-8+\frac{72}{3}$

$=32$

Ans: The maximum value is 8,

The minimum value is 32

3-(c)

Sol^n: The tangent cuts the x axis, when y is 0.

Putting $y=0$ in the given curve, we get

$-x+7=0$

$\therefore x=+7$

$\therefore$ we get the point $(7,0)$

Let $f(x,y)=y(x-2)(x-3)-x+7=0$

$=x^2y-5xy-x+6y+7$

Now, $f_x=2xy-5y-1$ $\qquad f_y=$

$\Rightarrow f_x=-1$ $\qquad =-20$

Now, the equation of tangent at $(7,0)$

$$(X-x)f_x + (Y-y)f_y = 0$$

$$\Rightarrow (X-7)(-1) + (Y-0)(-20) = 0$$

$$\therefore X - 20Y - 7 = 0$$

$\therefore$ The equation of normal is $\dfrac{X-x}{f_x} = \dfrac{Y-y}{f_y}$

$$\therefore 20X + Y - 140 = 0$$

$\therefore$ The eq^n of the tangent and normal are $X - 20Y - 7 = 0$ and $20X + Y - 140 = 0$ respectively.

(Ans.)

MATH-1107

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Roll - 1807085

4] ⓐ Find the radius of curvature at (x,y) on the curve $ay^2 = x^3$

Solve:

$$ay^2 = x^3$$

$$y^2 = \frac{x^3}{a}$$

$$y = \pm \frac{x^{3/2}}{\sqrt{a}}$$

$$y_1 = \pm \frac{3\sqrt{x}}{2\sqrt{a}}$$

$$y_2 = \pm \frac{3}{4\sqrt{ax}}$$

$\therefore$ radius of curvature $R = \frac{\{1+y_1^2\}^{3/2}}{y_2}$

$$= \pm \frac{\left(1+\frac{9}{4}\frac{x}{a}\right)^{3/2}}{\frac{3}{4\sqrt{ax}}}$$

$$= \pm \frac{4\left(1+\frac{9x}{4a}\right)^{3/2}\sqrt{ax}}{3}$$

$\therefore$ radius of curvature $R_{(x,y)} = \pm \frac{4\sqrt{ax}\left(1+\frac{9x}{4a}\right)^{3/2}}{3}$

⑤ (i) $\int \frac{x^3-3x}{x^2-4}\,dx$

$= \int \frac{x(x^2-3)}{x^2-4}\,dx$

$= \int \frac{x(x^2-4+1)}{x^2-4}\,dx$

$= \int \frac{x(x^2-4)}{x^2-4}\,dx + \int \frac{x}{x^2-4}\,dx$

$= \int x\,dx + \int \frac{x}{x^2-4}\,dx$

$= \frac{x^2}{2} + \frac{1}{2}\int \frac{dz}{z}$

$= \frac{x^2}{2} + \frac{1}{2}\ln z + c$

$= \frac{x^2}{2} + \frac{1}{2}\ln(x^2-4) + c$

Let,
$x^2-4=z$
$\Rightarrow 2x\,dx = dz$
$\Rightarrow x\,dx = \frac{dz}{2}$

(Ans).

(11) $\int \frac{x^2}{x^4+x^2-2}\,dx$

$= \int \frac{x^2}{(x^2)^2+2\cdot x^2\cdot \frac{1}{2}+\left(\frac{1}{2}\right)^2-2-\left(\frac{1}{2}\right)^2}\,dx$

$= \int \frac{x^2}{\left(x^2+\frac{1}{2}\right)^2-\frac{9}{4}}\,dx$

$= \int \frac{x^2}{\left(x^2+\frac{1}{2}+\frac{3}{2}\right)\left(x^2+\frac{1}{2}-\frac{3}{2}\right)}\,dx$

$= \int \frac{x^2}{(x^2+2)(x^2-1)}\,dx$

$= \int \frac{x^2}{(x^2+2)(x-1)(x+1)}\,dx.$

Let,

$\frac{x^2}{(x^2+2)(x-1)(x+1)} \equiv \frac{Ax+B}{x^2+2}+\frac{C}{(x-1)}+\frac{D}{(x+1)}$

$x^2 = (Ax+B)(x-1)(x+1) + C(x^2+2)(x+1) + D(x^2+2)(x-1)$

Put, $x=1$	Put, $x=-1$	Put, $x=0$,
$1 = C(3)(2)$	$1 = D(3)(-2)$	$0 = -B+2C-2D$
$\therefore C=\frac{1}{6}$	$\therefore D = -\frac{1}{6}$	$\Rightarrow 0 = -B+\frac{1}{3}+\frac{1}{3}$
		$B = \frac{2}{3}$

x^3:

$0 = A + C + D$

$\Rightarrow 0 = A + \frac{1}{6} - \frac{1}{6}$

$A = 0$

$$\therefore \int \frac{x^2}{(x^2+2)(x-1)(x+1)}dx = \int \frac{\frac{2}{3}}{x^2+2}dx + \int \frac{\frac{1}{6}}{x-1}dx - \int \frac{\frac{1}{6}}{x+1}dx$$

$$= \int \frac{2}{3}\frac{1}{x^2+2}dx + \frac{1}{6}\int \frac{1}{x-1}dx - \frac{1}{6}\int \frac{1}{x+1}dx$$

$$= \frac{2}{3}\times\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + \frac{1}{6}\ln(x-1) - \frac{1}{6}\ln(x+1) + C$$

(Ans)

(iii) $\int \frac{dx}{(1+x)\sqrt{1+x^2}}$

Let, $1+x = \frac{1}{z}$

$\Rightarrow x = \frac{1}{z} - 1$

$\therefore dx = -\frac{1}{z^2}$

$\therefore I = -\int \frac{dz}{z^2(1/z)\sqrt{1-(\frac{1}{z}-1)^2}}$

$= -\int \frac{dz}{z\sqrt{1-(\frac{1}{z^2}-2/z+1)}}$

$= -\int \frac{dz}{z\sqrt{2/z-(1/z^2)}}$

$= -\int \frac{dz}{\sqrt{2z-1}}$

$= -\frac{1}{2}\int \frac{2\,dz}{\sqrt{2z-1}}$

$= -\frac{1}{2} \times 2 \cdot \sqrt{2z-1}$

$= -\sqrt{2z-1}$

$= -\sqrt{\frac{2}{1+x}-1}$

$= -\sqrt{\frac{1-x}{1+x}}$

(Ans).

(iv) $\int e^{2x}\dfrac{(1+\sin 2x)}{(1+\cos 2x)}\,dx$

Solution: Let, $I = \int \dfrac{e^{2x}(1+\sin 2x)}{(1+\cos 2x)}\,dx$

And, $2x = z$

$dx = \frac{1}{2}dz.$

$\therefore I = \frac{1}{2}\int e^{z}\dfrac{(1+\sin z)}{1+\cos z}\,dz.$

$= \frac{1}{2}\int e^{z}\left(\dfrac{1}{1+\cos z}+\dfrac{\sin z}{1+\cos z}\right)dz$

$= \frac{1}{2}\int e^{z}\left(\dfrac{1}{2\cos^2 z/2}+\dfrac{2\sin z/2\,\cos z/2}{2\cos^2 z/2}\right)dz$

$= \frac{1}{2}\int e^{z}\left(\frac{1}{2}\sec^2 z/2 + \tan z/2\right)dz.$

$= \frac{1}{2}e^{z}.\tan z/2 + c.$

$= \frac{1}{2}e^{2x}.\tan x + c$

(Ans).

Ans. to the Question No. 6 (MATH 2017)

(i) $I = \int_0^a \frac{dx}{a^2+x^2}$

Let,

$x = a\tan\theta$; $\theta = \tan^{-1}\left(\frac{x}{a}\right)$

$dx = a\sec^2\theta \, d\theta$

When, $x = 0 \longrightarrow \theta = 0$

$x = a \longrightarrow \theta = \frac{\pi}{4}$

$$I = \int_0^{\pi/4} \frac{a\sec^2\theta}{a^2+a^2\tan^2\theta} \, d\theta$$

$$I = \frac{1}{a}\int_0^{\pi/4} \frac{\sec^2\theta}{1+\tan^2\theta} \, d\theta$$

$$I = \frac{1}{a}\int_0^{\pi/4} \frac{\sec^2\theta}{\sec^2\theta} \, d\theta$$

$$I = \frac{1}{a}\int_0^{\pi/4} d\theta$$

$$I = \frac{1}{a}\left[\theta\right]_0^{\pi/4}$$

$$I = \frac{1}{a}\left[\frac{\pi}{4} - 0\right]$$

$$I = \frac{\pi}{4a}$$

Ans. to the Question No. 6 (MATH 2017)

(ii) $I = \int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$

Let, $x = \tan\theta$; $\theta = \tan^{-1} x$

$dx = \sec^2\theta\, d\theta$

When $x = 0 \longrightarrow \theta = 0$

$x = 1 \longrightarrow \theta = \frac{\pi}{4}$

$$I = \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{1+\tan^2\theta} \sec^2\theta\, d\theta$$

$$I = \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta \text{ ——— } (1)$$

$$I = \int_0^{\pi/4} \log\left\{1 + \tan\left(\frac{\pi}{4}-\theta\right)\right\} d\theta$$

$$I = \int_0^{\pi/4} \log\left(1 + \frac{\tan\frac{\pi}{4} - \tan\theta}{1+\tan\frac{\pi}{4}\cdot\tan\theta}\right) d\theta$$

$$I = \int_0^{\pi/4} \log\left(\frac{1+\tan\theta+1-\tan\theta}{1+\tan\theta}\right) d\theta$$

$$I = \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right) d\theta$$

$$I = \int_0^{\pi/4} \left\{\log 2 - \log(1+\tan\theta)\right\} d\theta$$

$$I = \int_0^{\pi/4} \log 2\, d\theta - \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta$$

$$I = \log 2 \cdot [\theta]_0^{\pi/4} - I \qquad \text{[From (1)]}$$

$$2I = \left(\frac{\pi}{4} - 0\right) \log 2$$

$$I = \frac{\pi}{8} \log 2$$

Ans. to the Question No. 6 (MATH 2017)

(iii) $$I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx \quad \text{—— (I)}$$

$$I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\{\cos(\pi - x)\}^2}\, dx$$

$$I = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x} \quad \text{—— (II)}$$

(I) + (II) $\Rightarrow$ $$2I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx + \int_0^{\pi} \frac{\pi \sin x - x \sin x}{1+\cos^2 x}\, dx$$

$$2I = \int_0^{\pi} \frac{x\sin x + \pi \sin x - x \sin x}{1+\cos^2 x}\, dx$$

$$2I = \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx$$

$\cancel{2I = \pi \int_{-1}^{1}}$

Let, $\cos x = z$; $x = \cos^{-1}(z)$

$-\sin x\, dx = dz$

When $x = 0 \longrightarrow z = 1$

$x = \pi \longrightarrow z = -1$

$$2I = \pi \int_1^{-1} \frac{(-1)dz}{1+z^2}$$

$$I = \frac{\pi}{2} \int_{-1}^{1} \frac{dz}{1+z^2}$$

$$= \frac{\pi}{2} \cdot \left[\tan^{-1} z/1\right]_{-1}^{1}$$

$$I = \frac{\pi}{2} \cdot \left[\tan^{-1}(1) - \tan^{-1}(-1)\right]$$

$$I = \frac{\pi}{2}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right]$$

$$I = \frac{\pi^2}{4}$$

Ans. to the Question No. 6 (MATH 2017)

(iv) $I = \int_0^\infty e^{-ax} \cos bx \, dx \quad ; a>0$

$$I = \cos bx \int_0^\infty e^{-ax} dx - \int_0^\infty \left(\frac{d}{dx}(\cos bx) \int_0^\infty e^{-ax} dx \right) dx$$

$$I = \cos bx \cdot \frac{e^{-ax}}{-a} - \int_0^\infty \left(-\sin bx \cdot (b) \cdot \frac{e^{-ax}}{-a} \right) dx$$

$$I = \frac{e^{-ax} \cdot \cos bx}{-a} - \frac{b}{a} \int_0^\infty e^{-ax} \cdot \sin bx \, dx$$

$$I = \frac{-e^{-ax} \cos bx}{a} - \frac{b}{a} \left[\sin bx \int_0^\infty e^{-ax} dx - \int_0^\infty \left(\frac{d}{dx}(\sin bx) \int_0^\infty e^{-ax} dx \right) dx \right]$$

$$I = \frac{-e^{-ax} \cos bx}{a} - \frac{b}{a} \left[\sin bx \cdot \frac{e^{-ax}}{-a} - \int_0^\infty \left(b \cdot \cos bx \cdot \frac{e^{-ax}}{-a} \right) dx \right]$$

$$I = \frac{-e^{-ax} \cos bx}{a} + \frac{b}{a^2} \cdot \frac{e^{-ax} \sin bx}{1} - \frac{b^2}{a^2} \cdot \int_0^\infty e^{-ax} \cos bx \, dx$$

$$I = \frac{-e^{-ax} \cos bx}{a} + \frac{b}{a^2} \cdot e^{-ax} \sin bx - \frac{b^2}{a^2} I$$

$$\left(1 + \frac{b^2}{a^2}\right) I = \left(-\frac{e^{-ax} \cos bx}{a} + \frac{b}{a^2} \cdot e^{-ax} \sin bx \right)$$

$$I = \frac{a^2}{a^2+b^2} \left(-\frac{e^{-ax} \cos bx}{a} + \frac{b \cdot e^{-ax} \cdot \sin bx}{a^2} \right)$$

$$I = \frac{1}{a^2+b^2} \left(-a e^{-ax} \cdot \cos bx + b \cdot e^{-ax} \sin bx \right)$$

(52)

$$I = \frac{1}{a^2+b^2}\left[\left(b e^{-ax}\sin bx - a \cdot e^{-ax}\cos bx\right)\right]_0^\infty$$

$$I = \frac{1}{a^2+b^2}\left[(0-0) - (0-a)\right]$$

$$I = \frac{a}{a^2+b^2}$$

7. (a)

Q. Obtain the reduction formula for $\int \frac{dx}{(x^2+a^2)^{n/2}}$ and hence find the value of $\int \frac{dx}{(x^2+a^2)^{5/2}}$.

Sol^n: Let $I_n = \int \frac{dx}{(x^2+a^2)^{n/2}} = \int (x^2+a^2)^{-n/2} dx$

$$= \int (x^2+a^2)^{-n/2} \cdot 1\, dx$$

$$= (x^2+a^2)^{-n/2} \cdot x - \int -\frac{n}{2}(x^2+a^2)^{\frac{-n-2}{2}} \cdot 2x \cdot x\, dx$$

$$= \frac{x}{(x^2+a^2)^{n/2}} + n\int \frac{x^2}{(x^2+a^2)^{\frac{n+2}{2}}} dx$$

$$= \frac{x}{(x^2+a^2)^{n/2}} + n\int \frac{a^2+x^2-a^2}{(x^2+a^2)^{\frac{n+2}{2}}} dx$$

$$= \frac{x}{(x^2+a^2)^{n/2}} + n\int \frac{dx}{(x^2+a^2)^{n/2}} - na^2\int \frac{dx}{(x^2+a^2)^{\frac{n+2}{2}}}$$

$$I_n = \frac{x}{(x^2+a^2)^{n/2}} + nI_n - na^2 I_{n+2} \quad \text{——— (I)}$$

Replacing n by $n-2$

$$I_{n-2} = \frac{x}{(x^2+a^2)^{\frac{n-2}{2}}} + (n-2)I_{n-2} - (n-2)a^2 I_n$$

$$\Rightarrow (n-2)a^2 I_n = \frac{x}{(x^2+a^2)^{\frac{n-2}{2}}} + (n-2)I_{n-2} - I_{n-2}$$

$$\Rightarrow I_n = \frac{x}{(n-2)a^2(x^2+a^2)^{\frac{n-2}{2}}} + \frac{(n-3)I_{n-2}}{(n-2)a^2} \quad \text{——— (II)}$$

This is the reduction formula.

Now, $\displaystyle\int \frac{dx}{(x^2+a^2)^{5/2}} = I_5$

~~From equ^n (I)~~

$$\cancel{I_1 = \frac{x}{(x^2+a^2)^{1/2}} + I_1 - a^2 I_3}$$

$$\cancel{\therefore I_3 = \frac{x^2}{\sqrt{x^2+a^2}}}$$

from equ^n (II)

$$I_5 = \frac{x}{(5-2)\,a^2\,(x^2+a^2)^{\frac{5-2}{2}}} + \frac{(5-3)\,I_{n-2}}{(5-2)\,a^2}$$

$$= \frac{x}{3a^2(x^2+a^2)^{3/2}} + \frac{2\,I_3}{3a^2}$$

$$I_3 = \frac{x}{(3-2)\,a^2\,(x^2+a^2)^{\frac{3-2}{2}}} + 0$$

$$\therefore I_5 = \frac{x}{3a^2(x^2+a^2)^{3/2}} + \frac{2x}{3a^4(x^2+a^2)^{1/2}} + c$$

Ans.

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7.(b)

Q. Define Gamma function and Beta function.

Prove that $\Gamma(n+1) = n\Gamma n$

Solⁿ:

Gamma function: The integral $\int_0^\infty e^{-x} x^{n-1} dx$; $n>0$ is called Gamma function. It is denoted by Γn. It is also called 2nd Eulerian Integral.

Beta fuction: The integral $\int_0^1 x^{m-1} (1-x)^{n-1} dx$; $m,n>0$ is called Beta function. It is denoted by $\beta(m,n)$. It is also called 1st Eulerian Integral.

We have, $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$ ——— (1)

So, $\Gamma n+1 = \int_0^\infty e^{-x} x^{n+1-1} dx$

$= \int_0^\infty e^{-x} x^n dx$

$= \left[-x^n e^{-x}\right]_0^\infty - \int_0^\infty n x^{n-1} (-) e^{-x} dx$

$= 0 + n\int_0^\infty e^{-x} x^{n-1} dx$

$= n\Gamma n$

(proved)

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7.(c)
Q. Evaluate $\lim_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\cdots+\frac{n}{n^2+n^2}\right]$

Sol^n:

$$\lim_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\cdots+\frac{n}{n^2+n^2}\right]$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{n}{n^2+r^2}$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n\left(1+\frac{r^2}{n^2}\right)}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+\left(\frac{r}{n}\right)^2}$$

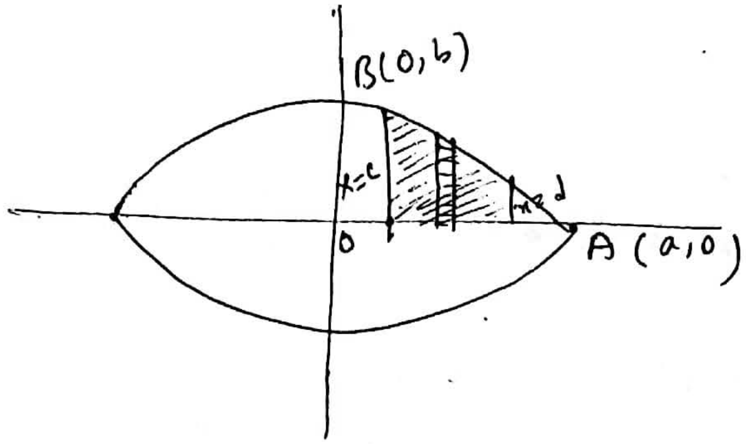
$$=\int_0^1\frac{1}{1+x^2}\,dx$$

$$=\left[\tan^{-1}x\right]_0^1$$

$$=\frac{\pi}{4}-0$$

$$=\frac{\pi}{4}$$

8(a)



Here , $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\Rightarrow \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$

$\Rightarrow y = \frac{b}{a}\sqrt{a^2-x^2}$ [1st quadrant]

Area of ellipse :

$$A = \int_c^d y\,dx = \frac{b}{a}\int_c^d \sqrt{a^2-x^2}\,dx$$

$$= \frac{b}{a}\left[\frac{x\sqrt{x^2-a^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_c^d$$

$$= \frac{b}{a}\left[\frac{d\sqrt{a^2-d^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{d}{a} - \frac{c\sqrt{a^2-c^2}}{2} - \frac{a^2}{2}\sin^{-1}\frac{c}{a}\right]$$

$$= \frac{b}{2a}\left[d\sqrt{a^2-d^2} - c\sqrt{a^2-c^2} + a^2\left(\sin^{-1}\frac{d}{a} - \sin^{-1}\frac{c}{a}\right)\right]$$

CSE-1 Jami

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KHULNA UNIVERSITY OF ENGINEERING & TECHNOLOGY

B.Sc. Engineering 1st Year 1st Term Examination, 2016

Department of Computer Science and Engineering

MATH 1107

Differential and Integral Calculus

TIME: 3 hours FULL MARKS: 210

N.B. i) Answer **ANY THREE** questions from each section in separate scripts.
ii) Figures in the right margin indicate full marks.

SECTION A

(Answer **ANY THREE** questions from this section in Script A)

1\. a) Define limit of a function. Discuss the continuity and differentiability of the function $f(x)=|x-2|+|x+4|$ at $x=-4$. (15)

b) Differentiate $x^{\sin x}$ with respect to $(\sin x)^x$. (08)

c) If $y=e^{a\sin^{-1}x}$ then find $y_{n+2}(x)$. (12)

2\. a) State Rolle's theorem. Expand $\sin(m\sin^{-1}x)$ in ascending powers of x. (15)

b) Define maxima and minima of a function. Given $xy=4$, find the maximum and minimum values of $4x+9y$. (10)

c) If $u=e^{xyz}$, then prove that $\frac{\partial^3 u}{\partial x\partial y\partial z}=(1+3xyz+x^2y^2z^2)e^{xyz}$. (10)

3\. a) Define homogeneous function. If $u=F(y-z,z-x,x-y)$, then prove that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$. (15)

b) Evaluate $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$. (10)

c) Find $\frac{dy}{dx}$, $\sin x^{\cos y}+\cos x^{\sin y}=0$ (10)

4\. a) Find where the tangent is parallel and perpendicular to the x-axis for the curve $ax^2+2hxy+by^2=1$. (15)

b) Find the equations of the normal at $\theta=\pi/2$ to the curve $x=a(\theta+\sin\theta)$, $y=a(1+\cos\theta)$. (10)

c) Find the radius of curvature of the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ at the point where $y=x$ cuts it. (10)

SECTION B

(Answer **ANY THREE** questions from this section in Script B)

5\. Integrate any three of the followings: (35)

i) $\int\frac{1}{x^4-1}dx$

ii) $\int\frac{1}{(x^2+1)\sqrt{x^2+4}}dx$

iii) $\int\frac{4x-1}{\sqrt{6x^2+6x+1}}dx$

iv) $\int e^x\frac{(1+\cos x)}{(1+\sin x)}dx$

$\frac{1}{(x^2)^2-}$

$\frac{1}{x^2-a^2}$

6\. Evaluate any three of the followings: (35)

i) $\int_0^{\pi/2}\frac{dx}{1+\sqrt{\cot x}}$

ii) $\int_{\sqrt{2}}^{2}\frac{dx}{x^2\sqrt{x^2-1}}$

iii) $\int_0^{\pi/2}\ln(\sin x)dx$

iv) $\lim_{n\to\infty}\left[\frac{1}{n^4+1}+\frac{2^3}{n^4+2^4}+\frac{3^3}{n^4+3^4}+\dots+\frac{1}{2n}\right]$

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7. a) Obtain the reduction formula for $\int e^{ax} \sin^n x dx$. (13)

b) Define Beta function and Gamma function. Prove that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. (12)

c) Find the area of a loop of the curve $a^2y^2 = x^2(a^2 - x^2)$. (10)

8. a) Find the volume and the surface area of the solid generated by revolving the cardioide $r = a(1-\cos\theta)$ about the initial line. (15)

b) Find the length of the perimeter of the astroide $x^{2/3} + y^{2/3} = a^{2/3}$. (14)

c) Define proper and improper integral with examples. (06)

1 (a) $f(x) = |x-2| + |x+4|$ (58)

find out the continuity & differentiality of $f(x)$ at $x = -4$.

$\therefore f(-4) = |-4-2| + |-4+4|$

$= 6.$

$\therefore \lim_{x \to (-4)^+} |x-2| + |x+4|$ [Here $x > 0$]

$= \lim_{x \to (-4)^+} x-2+x+4$

$= \lim_{x \to (-4)^+} 2x+2$

$= (-4 \times 2 + 2)$

$= -8+2$

$= -6$ Ans.

(b)

$y = x^{\sin x}$

$$\frac{dy}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \ln x \cos x\right)$$

$z = (\sin x)^x$

$$\frac{dz}{dx} = (x\cot x + \ln \sin x)(\sin x)^x$$

$$\therefore \frac{dy}{dz} = \frac{x^{\sin x}\left(\frac{\sin x}{x} + \ln x . \cos x\right)}{(\sin x)^x (x\cot x + \ln \sin x)}$$

(c)

$y = e^{a\sin^{-1}x}$

$$y_1 = e^{a\sin^{-1}x} \cdot \frac{a}{\sqrt{1-x^2}}$$

$\Rightarrow (1-x^2)\, y_1^2 = y^2 a^2$

$\Rightarrow (1-x^2)\, 2y_1 y_2 = 2y y_1 a^2 + 2x y_1^2$

$\Rightarrow (1-x^2)\, y_2 - x y_1 = a^2 y$

$\Rightarrow (1-x^2)\, y_{n+2} - 2x y_{n+1} \times {}^nc_1 + {}^nc_2 (-2) y_n$

$- (x y_{n+1} + {}^nc_1 y_n) - a^2 y_n = 0$

$\therefore (1-x^2) y_{n+2} - (2n+1)\, y_{n+1} \times x - (n^2+a^2) y_n = 0.$

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2(a) Stade Rolle's theorem. Expand $\sin(m\sin^{-1}x)$ in asceending powers of x.

0

(i) Rolle's theorem: If $f(x)$ is continuous in the closed interval $[a,b]$, $f'(x)$ exists in the open interval (a,b) and $f(a)=f(b)$ then there exists at least one point say x_0, with $a<x_0<b$ such that $f'(x_0)=0$

(ii)

$$y=f(x)=\sin(m\sin^{-1}x) \quad \cdots\cdots (i)$$

$$\sqrt{1-x^2}\,y_1 = m\cos(m\sin^{-1}x) \quad \cdots\cdots (ii)$$

$$(1-x^2)\,y_2 - xy_1 + m^2 y = 0 \quad \cdots\cdots (iii)$$

$$(1-x^2)\,y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0 \quad \cdots\cdots (iv)$$

putting $x=0$ in (1), (ii), (iii), (iv). we get $(y)_0$, i.e, $f(0)=0$; $(y_1)_0$ i.e $f'(0)=m$; $(y_2)_0$, i.e, $f''(0)=0$

$$(y_{n+2})_0 = (n^2-m^2)(y_n)_0 \quad \cdots\cdots (v)$$

Putting, $n=2,4,6,\ldots$ in (5), we get $(y_4)_0=(y_6)_0=\cdots=0$, i.e; $f^{iv}(0)=f^{vi}(0)=0$

putting $n = 1, 3, 5, \ldots$ in (5), we get;

$(y_3)_0$, i.e; $f'''(0) = m(1^2 - m^2)$; $f^{v}(0) = m(1^2 - m^2)(3^2 - m^2)$ e.t.

Since $\sin(m\sin^{-1}x) = f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots$

we get the expansion of $\sin(m\sin^{-1}x)$ on substituting

the values of $f(0), f'(0), \ldots$

$$\sin(m\sin^{-1}x) = 0 + mx + \frac{x^2}{2!}\cdot 0 - \frac{m(m^2-1^2)}{3!}x^3 + \ldots$$

$$= mx - \frac{m(m^2-1^2)}{3!}x^3 + \frac{m(m^2-1^2)(m^2-3^2)}{5!}x^5 - \ldots$$

2) (b) Define maxima and minima of a function. Given $xy = 4$, find the maximum and minimum values of $4x + 9y$.

Here,

$$xy = 4$$

$$\therefore y = \frac{4}{x} \qquad \text{and} \qquad f(x) = 4x + 9y$$

$$\therefore f(x) = 4x + 9\frac{4}{x}$$

$$= 4x + 36x^{-1}$$

$$f'(x) = 4 - 36x^{-2}$$

$$f''(x) = 72x^{-3} = \frac{72}{x^3}$$

For maximum and minimum,

$$f'(x) = 0$$

$$4 - 36x^{-2} = 0$$

$$\Rightarrow 4 = \frac{36}{x^2}$$

$$\Rightarrow x^2 = 9$$

$$\therefore x = \pm 3$$

Now,

When, $x = 3$ then, $f''(3) = \frac{72}{(3)^3} = \frac{8}{3} > 0$ $\therefore$ There is a minimum

When, $x = -3$ then, $f''(-3) = \frac{72}{(-3)^3} = -\frac{8}{3} < 0$ $\therefore$ There is a maximum

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When,

$x = -3$, $f(-3) = 4(-3) + \frac{36}{(-3)} = -24$

$x = 3$, $f(3) = 4(3) + \frac{36}{3} = 24$

at point $x = -3$ the maximum value is -24

and at point $x = 3$ the minimum value is 24

2(c) If $u = e^{xyz}$, Then prove that $\frac{\partial^3 u}{\partial x \partial y \partial z} = (1+3xyz+x^2y^2z^2).e^{xyz}$

Solu^n:

Here, $u = e^{xyz}$

$$\frac{\partial u}{\partial z} = e^{xyz}.xy$$

$$\frac{\partial^2 u}{\partial y.\partial z} = e^{xyz}.x^2yz + e^{xyz}.x$$

$$\therefore \frac{\partial^3 u}{\partial x.\partial y.\partial z} = e^{xyz}.x^2y^2z^2 + e^{xyz}.2xyz + e^{xyz}.xyz + e^{xyz}$$

$$= e^{xyz}(1+3xyz+x^2y^2z^2)$$

3(a) Define homogeneous function. If $u = f(y-z, z-x, x-y)$ then prove that, $\frac{\delta u}{\delta x} + \frac{\delta u}{\delta y} + \frac{\delta u}{\delta z} = 0$.

<u>Homogeneous functions</u> Homogeneous function is one with multiplicative ~~behaviour~~ scaling behaviour. It mean if all its arguments are ~~multiplicated~~ multiplied by a factor, then its value is multiplied by some power of this factor.

<u>Example:</u>

a homogeneous ~~fac~~ function of two variables x and y is a real valued function that satisfies the condition $f(\alpha x, \alpha y) = \alpha^k f(x, y)$ for some constant k and all real numbers α. The constant k is called the degree of homogeneity.

3 (b) Evaluate $\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x}$

Let,

$$y = \lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x}$$

$$\Rightarrow \ln y = \lim_{x\to 0} \left(\frac{\ln(\tan x) - \ln(x)}{x}\right)$$

Applying L'Hospital rule,

$$= \lim_{x\to 0} \left(\frac{\frac{1}{\tan x}\cdot \sec^2 x - \frac{1}{x}}{1}\right)$$

$$= \lim_{x\to 0} \left(\frac{1}{\sin x\cdot \cos x} - \frac{1}{x}\right)$$

$$= \lim_{x\to 0} \left(\frac{2}{\sin 2x} - \frac{1}{x}\right) \quad [\text{form } \infty - \infty]$$

$$= \lim_{x\to 0} \frac{2x - \sin 2x}{(x \sin 2x)}$$

Againg Applying L'Hospital rule,

$$= \lim_{x\to 0} \frac{2 - 2\cos 2x}{\sin 2x + 2x\cos 2x} \quad \left[\text{form } \frac{0}{0}\right]$$

Applying L'Hospital for the third time,

$$= \lim_{x\to 0} \frac{4\sin 4x}{2\cos 2x - 4x\sin 2x + 2\cos 2x}$$

$$= \frac{0}{2-0+2} = \frac{0}{4} = 0$$

$$\therefore y = e^0 = 1 \text{ Ans}$$

3(e) Find $\frac{dy}{dx}$, $\sin x^{\cos y} + \cos x^{\sin y} = 0$

Let,

$P = \sin x^{\cos y}$

$\Rightarrow \ln P = \cos y \ln(\sin x)$

$\Rightarrow \frac{1}{P}\frac{dP}{dx} = (-\sin y)\frac{dy}{dx}\ln(\sin x) + \cos y \frac{1}{\sin x}\cdot\cos x$

$\Rightarrow \frac{dP}{dx} = \sin x^{\cos y}\left(\tan x \cos y - \ln(\sin x)\sin y\frac{dy}{dx}\right)$ --- (i)

And,

$Q = \cos x^{\sin y}$

$\Rightarrow \ln Q = \sin y \ln(\cos x)$

$\Rightarrow \frac{1}{Q}\frac{dQ}{dx} = \cos y \ln(\cos x)\frac{dy}{dx} + \sin y \frac{1}{\cos x}(-\sin x)$

$\Rightarrow \frac{dQ}{dx} = (\cos x)^{\sin y}\left\{\cos y \ln(\cos x)\frac{dy}{dx} - \tan x \sin y\right\}$ —— (ii)

(i) + (ii)

$\frac{dQ}{dx} + \frac{dP}{dx} = 0$

$\Rightarrow \cos x^{\sin y}\cos y \ln(\cos x)\frac{dy}{dx} - \cos x^{\sin y}\tan x \sin y$

$+ \sin x^{\cos y}\tan x \cos y - \sin x^{\cos y}\ln(\sin x)\sin y\frac{dy}{dx} = 0$

$\Rightarrow \left\{\cos x^{\sin y}\cos y \ln(\cos x) - \sin x^{\cos y}\ln(\sin x)\sin y\right\}\frac{dy}{dx}$

$= -\left\{\sin x^{\cos y}\tan x \cos y - \cos x^{\sin y}\tan x \sin y\right\}$

P.T.O.

$$\therefore \frac{dy}{dx} = \frac{\cos x^{\sin y} \cdot \tan x \cdot \sin y - \sin x^{\cos y} \tan x \cos y}{\cos x^{\sin y} \cos y \ln(\cos x) - \sin x^{\cos y} \ln(\sin x) \sin y}$$

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Ans

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(a) $ax^2 + 2hxy + by^2 - 1 = 0$ —— (1)

$a \cdot 2x + 2hy + 2hx\frac{dy}{dx} + b2y\frac{dy}{dx} = 0$

$\frac{dy}{dx}\{2hx - 2by\} = -2ax - 2hy$

$\frac{dy}{dx}(hx - by) = -(ax + by)$.

$\frac{dy}{dx} = \frac{-(ax+by)}{(hx-by)}$

if the tangent is parallel.

$\frac{dy}{dx} = 0$

$\therefore \frac{-(ax+by)}{hx-by} = 0$

$ax + by = 0$

$\boxed{ax = -by}$

if the tangent is perpendicular

$\frac{dy}{dx} = \frac{1}{0}$

$\frac{-(ax+by)}{hx-by} = \frac{1}{0}$

$\boxed{hx = by}$

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(b)

$x = a(\theta + \sin\theta)$ —— (I)

$y = a(1 + \cos\theta)$ —— (II)

$\frac{dx}{d\theta} = a(1 + \cos\theta)$ —— (III)

$\frac{dy}{d\theta} = a(0 - \sin\theta)$ —— (IV)

$\theta = \pi/2$

$x = a(\pi/2$

(III) $\div$ (IV)

$\frac{dy}{dx} = \frac{-\sin\theta}{1 + \cos\theta}$

$y = a(1 + 0)$

$y = a$

$x = a(\frac{\pi}{2}$

$= -\frac{\cancel{2}\sin\theta/2\, \cancel{\cos\theta/2}}{\cancel{2}\cos^2\theta/2}$

$$\boxed{\frac{dy}{dx} = \tan\theta/2}$$

$\frac{dy}{dx} = \tan(\pi/4) = 1$

$\therefore$ Equation of normal.

$y - a = 1\{x - (\frac{a\pi}{2} + a)\}$

$y - a = x - \frac{\pi a}{2} + a$

$$\boxed{x - y + 2a - \frac{\pi a}{2} = 0}$$

(c) $\sqrt{x}+\sqrt{y}=\sqrt{a}$

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find the radius of curvature of the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ at the point where $y=x$ cuts it.

5.

i. $\int \frac{dx}{x^4-1}$

$\Rightarrow \int \frac{dx}{(x^2+1)(x^2-1)}$

$= \int \frac{dx}{2(x^2-1)} + \frac{dx}{(-2)(x^2+1)}$

$= \frac{1}{2}\int \frac{dx}{x^2-1} + \int \frac{dx}{(-2)(x^2+1)}$

$= \frac{1}{2} \times \frac{1}{2} \ln \frac{x-1}{x+1} - \frac{1}{2} \tan^{-1}x + c$

$= \frac{1}{2} \left\{ \frac{1}{2} \ln \frac{x-1}{x+1} - \tan^{-1}x \right\} + c$

ii. $\int \frac{dx}{(x^2+1)\sqrt{x^2+4}}$

Let,

$x = \frac{1}{z}$;

$dx = -\frac{1}{z^2} dz$.

Again,

$1+4z^2 = p^2$

$4z^2 = p^2-1$

$\therefore z^2 = \frac{p^2-1}{4}$

$\therefore 2z\,dz = \frac{2p}{4} dp$.

$= \int \frac{-dz}{z^2(1/z^2+1)\sqrt{1/z^2+4}}$

$= -\int \frac{z\,dz}{(1+z^2)\sqrt{1+4z^2}}$

$= -\frac{1}{4}\int \frac{p\,dp}{\left\{1+\frac{p^2-1}{4}\right\}p}$

$= -\frac{1}{4}\int \frac{4\,dp}{4+p^2-1}$

$= -\int \frac{du}{3+p^2}$

$= -\frac{1}{\sqrt{3}} \tan^{-1} \frac{p}{\sqrt{3}} + c$

$= -\frac{1}{\sqrt{3}} \tan^{-1} \frac{1}{\sqrt{3}} \sqrt{(x^2+4)/x^2} + c$

iii. $\int \frac{4x-1}{\sqrt{6x^2+6x+1}}\,dx$

$= \int \frac{\frac{1}{3}(12x+6)-3}{\sqrt{6x^2+6x+1}}\,dx$

$= \frac{1}{3}\int \frac{(12x+6)\,dx}{\sqrt{6x^2+6x+1}} - \int \frac{3\,dx}{\sqrt{6x^2+6x+1}}$

$= \frac{1}{3} \times 2\sqrt{6x^2+6x+1} - \frac{3}{\sqrt{6}}\int \frac{dx}{\sqrt{x^2+x+\frac{1}{6}}}$

$= \frac{2}{3}\sqrt{6x^2+6x+1} - \frac{3}{\sqrt{6}}\int \frac{dx}{\sqrt{x^2+x+\frac{1}{4}-\frac{1}{4}+\frac{1}{6}}}$

$= \frac{2}{3}\sqrt{6x^2+6x+1} - \frac{3}{\sqrt{6}}\int \frac{dx}{\sqrt{(x+\frac{1}{2})^2-\frac{1}{12}}}$

$= \frac{2}{3}\sqrt{6x^2+6x+1} - \frac{3}{\sqrt{6}} \times \frac{1}{2\times\left(\frac{1}{\sqrt{12}}\right)} \ln \frac{(x+\frac{1}{2})-\frac{1}{\sqrt{12}}}{(x+\frac{1}{2})+\frac{1}{\sqrt{12}}} + c$

iv. $\int e^x\left(\frac{1+\cos x}{1+\sin x}\right)dx$

$= \int \frac{e^x}{1+\sin x} + \frac{e^x \cos x}{1+\sin x}\,dx$

$= \int e^x\left(\frac{1}{(\sin x/_2+\cos x/_2)^2} + \right.$ ——

Answer to question- 6

Evaluate the followings:

(i) $\int_0^{\pi/2} \frac{dx}{1+\sqrt{\cot x}}$

$= \int_0^{\pi/2} \frac{dx}{1+\sqrt{\frac{\cos x}{\sin x}}} = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$

$\Rightarrow$ let $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx \cdots$ (i)

$\Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\sin(\pi/2-x)}\, dx}{\sqrt{\sin(\pi/2-x)}+\sqrt{\cos(\pi/2-x)}}$ $\left[\begin{array}{c}\text{using} \\ \text{4 th} \\ \text{Rule}\end{array}\right]$ $\cdots\cdots$ (ii)

$\Rightarrow$ add (ii) with (i)

$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}\, dx}{\sqrt{\sin x}+\sqrt{\cos x}} + \int_0^{\pi/2} \frac{\sqrt{\cos x}\, dx}{\sqrt{\sin x}+\sqrt{\cos x}}$

$\therefore I = \frac{1}{2} \int_0^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$

$= \frac{1}{2} [x]_0^{\pi/2}$

$= \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{4}$ Ans

$\boxed{11}$ $\displaystyle\int_{\sqrt{2}}^{2} \frac{dx}{x^2\sqrt{x^2-1}}$

$$= \int_{1/\sqrt{2}}^{\sqrt{3}/2} \frac{z\,dz}{x(1-z^2)^2 \cdot \frac{1}{(1-z^2)} \cdot \sqrt{\frac{1}{1-z^2}-1}}$$

$$= \int_{1/\sqrt{2}}^{\sqrt{3}/2} \frac{z\,dz}{x\,\frac{z}{\sqrt{(1-z^2)}} \cdot (1-z^2)}$$

$$= \int_{1/\sqrt{2}}^{\sqrt{3}/2} \frac{dz}{\frac{1}{\sqrt{1-z^2}} \cdot \sqrt{1-z^2}} \qquad \left[\because x^2 = \frac{1}{1-z^2}\right]$$

$$= \left[z\right]_{1/\sqrt{2}}^{\sqrt{3}/2}$$

$$= \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{3}-\sqrt{2}}{2}$$

Ans

we put,

$x^2 - 1 = x^2 z^2$

$\Rightarrow x^2(1-z^2) = 1$

$\Rightarrow x^2 = \frac{1}{1-z^2}$

$\Rightarrow 2x\,dx = \frac{-1}{(1-z^2)^2}(-2z)$

$\Rightarrow dx = \frac{z}{x(1-z^2)^2}$

when $x = 2, \sqrt{2}$

$z = \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}$

$\boxed{111}$ $\int_0^{\pi/2} \ln(\sin x)\, dx$

Let, $I = \int_0^{\pi/2} \ln(\sin x)\, dx \quad \cdots\cdots (i)$

$= \int_0^{\pi/2} \ln\{\sin(\pi/2 - x)\}\, dx$ [using Rule no 4]

$= \int_0^{\pi/2} \ln \cos x \, dx \quad \cdots\cdots (ii)$

add (i) with (ii)

$$2I = \int_0^{\pi/2} \{\ln(\sin x) + \ln(\cos x)\}\, dx$$

$$= \int_0^{\pi/2} \ln(\sin x \cos x)\, dx$$

$$= \int_0^{\pi/2} \ln(2\sin x \cos x/2)\, dx$$

$$= \int_0^{\pi/2} \ln(\cancel{2}\sin 2x/2)\, dx$$

$$= \int_0^{\pi/2} \ln(\sin 2x)\, dx - \ln 2 \int_0^{\pi/2} dx \quad \cdots\cdots (iii)$$

$= \cancel{\int_0^{\pi/2} \ln}$ let $J = \int_0^{\pi/2} \ln(\sin 2x)\, dx$

$$= \frac{1}{2}\int_0^{\pi} \ln(\sin t)\, dt$$

we put
$t = 2x$
$\therefore dt = 2dx$
$\therefore dx = \frac{1}{2} dt$
$x = \pi/2,\ 0$
$t = \pi,\ 0$

Here x, t are dummy variables

But $\int_0^{2a} f(x)\, dx = \int_0^{a} \{f(x) + f(2a-x)\}\, dx$. The

Then

$$J = \frac{1}{2}\int_0^{\pi/2} \{\ln(\sin x) + \ln \sin(\pi - x)\}\, dx$$

$$= \frac{1}{2}\int_0^{\pi/2} 2\ln(\sin x)\, dx$$

$$J = \int_0^{\pi/2} \ln(\sin x)\, dx = I$$

from equ^n (iii)

$$2I = I - \ln 2 \int_0^{\pi/2} dx$$

$$\therefore I = -\ln 2 \cdot \frac{\pi}{2}$$

Ans

IV $\lim_{n\to\infty}\left[\frac{1}{n^4+1}+\frac{2^3}{n^4+2^4}+\frac{3^3}{n^4+3^4}+\cdots+\frac{1}{2n}\right]$

$$= \lim_{n\to\infty}\left[\frac{1^3}{n^4+1^4}+\frac{2^3}{n^4+2^4}+\frac{3^3}{n^4+3^4}+\cdots+\frac{n^3}{n^4+n^4}\right]$$

$$= \cancel{\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{r^3/n^3}{\quad}}$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{r^3}{n^4+r^4}$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{r^3}{n^4\left(1+\frac{r^4}{n^4}\right)}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{r^3/n^3}{1+r^4/n^4}$$

Replacing $\lim_{n\to\infty}\frac{1}{n}$ by dx, $\frac{r}{n}$ by x

Σ by $\int_0^1$

$$= \int_0^1 \frac{x^3}{1+x^4}\,dx$$

$$= \frac{1}{4}\int_0^1 \frac{4x^3}{1+x^4}\,dx$$

$$= \frac{1}{4}\left[\ln(1+x^4)\right]_0^1$$

$\left\{ \int \frac{f'(x)}{f(x)}\,dx = \ln(f(x)) \right.$

$$= \frac{1}{4}\left[\ln 2 - \ln 1\right]$$

$$= \frac{1}{4}\left[\ln 2 - 0\right]$$

$$= \frac{1}{4}\ln 2$$

Ans

1

Roll : 1807096

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Solution to the Question no. 7 – MATH 1107 (2016)

7. (a) Obtain the reduction formula for $\int e^{ax} \sin^n x\, dx$.

Solution: We put, $I_n = \int e^{ax} \sin^n x\, dx$

$$\Rightarrow I_n = \sin^n x \frac{e^{ax}}{a} - n \int \sin^{n-1} x \cdot \cos x \cdot \frac{e^{ax}}{a}\, dx$$

$$\Rightarrow I_n = \frac{e^{ax} \sin^n x}{a} - \frac{n}{a} \int (\sin^{n-1} x \cos x) e^{ax}\, dx$$

$$\Rightarrow I_n = \frac{e^{ax} \sin^n x}{a} - \frac{n}{a} \left[\sin^{n-1} x \cos x \int e^{ax} dx - \int \left\{ \frac{d}{dx} (\sin^{n-1} x \cos x) \int e^{ax} dx \right\} dx \right.$$

$$\Rightarrow I_n = \frac{e^{ax} \sin^n x}{a} - \frac{n}{a} \left[\sin^{n-1} x \cos x \frac{e^{ax}}{a} \right.$$

$$- \int \left\{ (n-1) \sin^{n-2} x \cdot \cos^2 x - \sin^n x \right\} \frac{e^{ax}}{a} dx$$

$$\Rightarrow I_n = \frac{e^{ax}\sin^n x}{a} - \frac{n}{a}\left[\frac{e^{ax}\sin^{n-1}x\cos x}{a}\right.$$

$$- \frac{(n-1)}{a}\int e^{ax}\sin^{n-2}x\,(1-\sin^2 x)\,dx$$

$$\left. + \frac{1}{a}\int e^{ax}\sin^n x\,dx\right]$$

$$\Rightarrow I_n = \frac{e^{ax}\sin^n x}{a} - \frac{n}{a}\left[\frac{e^{ax}\sin^{n-1}x\cos x}{a}\right.$$

$$- \frac{(n-1)}{a}\int e^{ax}\sin^{n-2}x\,dx + \frac{(n-1)}{a}\int e^{ax}\sin^n x$$

$$\left. + \frac{1}{a}\int \sin^n x\,dx\right]$$

$$\Rightarrow I_n = \frac{e^{ax}\sin^n x}{a} - \frac{n}{a}\left[\frac{e^{ax}\sin^{n-1}x\cos x}{a}\right.$$

$$\left. - \frac{(n-1)}{a} I_{n-2} + \left(\frac{n}{a} - \frac{1}{a} + \frac{1}{a}\right) I_n\right]$$

$$\Rightarrow I_n = \frac{e^{ax}\sin^n x}{a} - \frac{n}{a^2}e^{ax}\sin^{n-1}x\cos x + \frac{n(n-1)}{a^2} I_{n-2}$$

$$- \frac{n^2}{a^2} I_n$$

$$\Rightarrow \left(1+\frac{n^2}{a^2}\right) I_n = \frac{e^{ax}\sin^n x}{a} - \frac{n}{a^2} e^{ax}\sin^{n-1}x\cos x + \frac{n(n-1)}{a^2} I_{n-2}$$

$$\Rightarrow \left(\frac{a^2+n^2}{a^2}\right) I_n = \frac{e^{ax}\sin^n x}{a^2}(a\sin x - n\cos x) + \frac{n(n-1)}{a^2} I_{n-2}$$

$$\Rightarrow I_n = \frac{e^{ax}\sin^{n-1}x}{a^2+n^2}(a\sin x - n\cos x) + \frac{n(n-1)}{a^2+n^2} I_{n-2}$$

$$\therefore \cancel{\int e^{ax}\sin^n x\,dx} = \frac{e^{ax}\sin^{n-1}x}{a^2+n^2} + \frac{n(n-1)}{a^2+n^2}\int e^{ax}\sin^{n-2}x\,dx$$

$$\therefore \int e^{ax}\sin^n x\,dx = \frac{e^{ax}\sin^{n-1}x}{a^2+n^2}(a\sin x - n\cos x) + \frac{n(n-1)}{a^2+n^2}\int e^{ax}\sin^{n-2}x\,dx$$

This is the required reduction formula.

(Answer)

7.(b) Define Beta function and Gamma function. Prove that $\Gamma(1/2) = \sqrt{\pi}$.

Solution:

Beta Function: The integral $\int_0^1 x^{m-1}(1-x)^{n-1} dx$

$m, n > 0$ is called Beta function. It is denoted by $\beta(m, n)$. It is also called 1st Eulerian integral.

Gamma Function: The integral $\int_0^\infty e^{-x} x^{n-1} dx$

$n > 0$ is called Gamma Function. It is denoted by Γn. It is also called 2nd Eulerian integral.

Prove $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$:

We have, $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ ——— (1)

Put $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\,d\theta$

$x = 0,\ \theta = 0$ and $x = 1,\ \theta = \frac{\pi}{2}$

$$\therefore \frac{\Gamma m\,\Gamma n}{\Gamma m+n} = \int_0^{\frac{\pi}{2}} \sin^{2m-2}\theta\cos^{2n-2}\theta\,.\,2\sin\theta\cos\theta\,d\theta$$

$$\left[\because \beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}\right]$$

$$= 2\int_0^{\pi/2} \sin^{2m-2}\theta\cos^{2n-2}\theta\,.\,2\sin\theta\cos\theta\,d\theta$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sin^{2m-2}\theta\cos^{2n-2}\theta\,.\,2\sin\theta\cos\theta\,d\theta = \frac{\Gamma m\,\Gamma n}{2\,\Gamma m+n}$$

——— (2)

Put $m = n = \frac{1}{2}$ in equation (2), we get

$$\int_0^{\pi/2} d\theta = \frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{2\Gamma 1}$$

$$\Rightarrow \frac{\left(\Gamma\frac{1}{2}\right)^2}{2} = \frac{\pi}{2} \quad [\because \Gamma 1 = 1]$$

$$\Rightarrow \left(\Gamma\frac{1}{2}\right)^2 = \pi$$

$$\therefore \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

(Proved)

7.(c) Find the area of a loop of the curve $a^2y^2 = x^2(a^2-x^2)$

Solution: Given curve is,

$$a^2y^2 = x^2(a^2-x^2) \quad \text{--- (1)}$$

Since by putting $-x$ for x and $-y$ for y

in equation (1), the equation remain unchanged, so the curve is symmetrical about both the axes.

If $y=0$ then $x^2(a^2-x^2)=0$

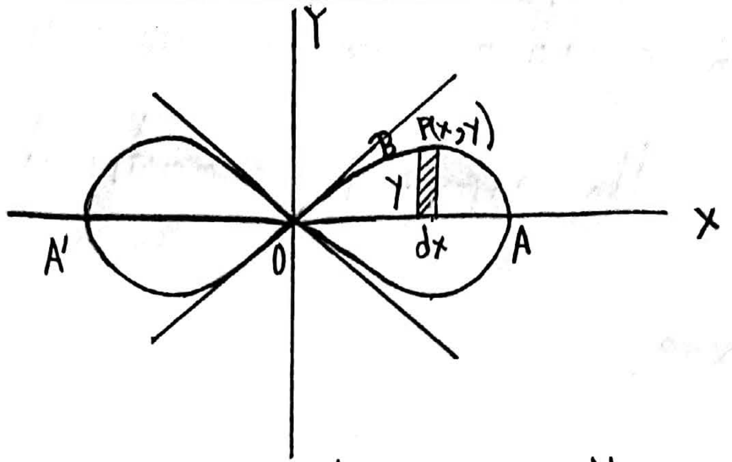
$$\Rightarrow x = 0, -a, a$$

Hence the curve passes through the points $A'(-a,0)$, $O(0,0)$ and $A(a,0)$.

But for $x>a$ and $x<-a$ then $y^2 = -ve$

$\Rightarrow y$ is imaginary.

So the curve does not exist in the region $x>a$ and $x<-a$.



If A is the required area then

$$A = 4 \times \text{area } OAB = 4\int_0^a y\,dx = 4\int_0^a \frac{x\sqrt{a^2-x^2}}{a}\,dx$$

[From (1)]

We put, $x = a\sin\theta$ then $dx = a\cos\theta\, d\theta$

Limits: If $x = 0$ then $\theta = 0$

If $x = a$ then $\theta = \frac{\pi}{2}$

$$A = \frac{4}{a}\int_0^{\frac{\pi}{2}} a\sin\theta\sqrt{a^2\cos^2\theta}\,.\,a\cos\theta\, d\theta$$

$$= 4a^2\int_0^{\frac{\pi}{2}} \sin\theta\cos^2\theta\, d\theta$$

$$= 4a^2 \frac{\Gamma(1)\,\Gamma\left(\frac{3}{2}\right)}{2\,\Gamma\left(\frac{5}{2}\right)}$$

$\left[\Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right)\right]$

$$= 4a^2 \cdot \frac{1\,\Gamma\frac{3}{2}}{2\cdot\frac{3}{2}\,\Gamma\frac{3}{2}} = \frac{4a^2}{3} \text{ square unit.}$$

$[\Gamma(n+1) = n\Gamma(n)\ \&\ \Gamma(1) = 1]$

(Answer)

8.a) Find the volume and the surface area of the solid generated by revolving the cardioide $r=a(1-\cos\theta)$ about the initial line.

Solⁿ: 1st part,

$$r=a(1-\cos\theta) \quad \text{—} (1)$$

The curve no (1) is symmetrical with the primary line.

If $r=0$, then $0=a(1-\cos\theta) \Rightarrow \theta=0$

If $r=2a$, then $2a=a(1-\cos\theta) \Rightarrow \theta=\pi$

$\therefore$ Upper half of the cardioide stay in between $\theta=0, \theta=\pi$

V is the volume of the solid generated by revolving the cardioide $r=a(1-\cos\theta)$

$$\therefore V=\int_0^{\pi}\frac{2}{3}\pi r^3\sin\theta\,d\theta=\frac{2\pi}{3}\int_0^{\pi}a^3(1-\cos\theta)^3\sin\theta\,d\theta \quad [\text{by } (1)]$$

$$=\frac{2\pi a^3}{3}\int_0^{\pi}(2\sin^2\theta/2)^3\,2\sin\theta/2\,.\cos\theta/2\,d\theta$$

$$=\frac{32\pi a^3}{3}\int_0^{\pi}\sin^7\theta/2\,\cos\theta/2\,d\theta$$

$$=\frac{32\pi a^3}{3}\,2\int_0^{\frac{\pi}{2}}\sin^7 t\cos t\,dt\;;\quad \text{when } \frac{\theta}{2}=t$$

$$=\frac{64\pi a^3}{3}\cdot\frac{\Gamma(4)\,\Gamma(1)}{2\,\Gamma(5)}=\frac{64\pi a^3\,\Gamma(4)\,1}{3.2.4\,\Gamma(4)}=\frac{8\pi a^3}{3}.$$

2nd part,

$$r = a(1-\cos\theta) \quad \text{——} \ (1)$$

$\therefore$ A is the surface of the upper side of primary line for the half ~~of~~ of the cardioide θ is from 0 to π.

$$\therefore A = 2\int_0^{\pi} \frac{1}{2} r^2 d\theta$$

$$= \int_0^{\pi} a^2(1-\cos\theta)^2 d\theta \quad [\text{by eqn } 1]$$

$$= a^2 \int_0^{\pi} \left(2\sin^2\frac{\theta}{2}\right)^2 d\theta$$

$$= 4a^2 \int_0^{\pi} \sin^4\frac{\theta}{2}\, d\theta$$

Let, $\frac{\theta}{2} = t \quad \therefore d\theta = 2dt$

If $\theta = 0$ then, $t = 0$

If $\theta = \pi$ then, $t = \frac{\pi}{2}$

$$\therefore A = 4a^2 2\int_0^{\frac{\pi}{2}} \sin^4 t\, dt$$

$$= \frac{8a^2\,\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2.\Gamma(3)}$$

$$= \frac{8a^2 . \frac{3}{2} . \frac{1}{2}\sqrt{\pi}\sqrt{\pi}}{2.2.1} = \frac{3\pi a^2}{2} \text{ squre unit.}$$

8(b) Find the length of the perimeter of the astroide $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$

Solⁿ Given that, $x^{2/3}+y^{2/3}=a^{2/3}$ ——— (1)

Since in (1) we write $-x$ for x and $-y$ for y, then the equation remain unchanged, so the curve (1) is symmetrical about both the axes.

when, $y=0$ (1) $\Rightarrow x^{2/3}=a^{2/3}$ $\therefore x^2=a^2 \Rightarrow x=\pm a$

$\therefore A(a,0),\ A'(-a,0),\ B(0,a)\ \&\ B'(0,-a)$

Differentiating (1) w.r.to x we get,

$$\frac{2}{3}x^{-\frac{1}{3}}+\frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx}=0$$

$$\Rightarrow y^{-\frac{1}{3}}\frac{dy}{dx}=-x^{-\frac{1}{3}}$$

$$\therefore \frac{dy}{dx}=-\frac{x^{-1/3}}{y^{-1/3}}=-\frac{y^{1/3}}{x^{1/3}}$$

Now, $\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+\frac{y^{2/3}}{x^{2/3}}}=\sqrt{\frac{x^{2/3}+y^{2/3}}{x^{2/3}}}=\sqrt{\frac{a^{2/3}}{x^{2/3}}}=\sqrt{\left(\frac{a^{1/3}}{x^{1/3}}\right)^2}=\frac{a^{1/3}}{x^{1/3}}$

Now, the length, $S=4\int_0^a\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$$=4\int_0^a\frac{a^{1/3}}{x^{1/3}}\,dx$$

$$=4a^{1/3}\left[\frac{x^{2/3}}{2/3}\right]_0^a=\frac{4}{2}\cdot 3a^{1/3}\left(a^{2/3}-0\right)$$

$$=6a$$

$\therefore S=6a$ (Ans).

8(c) Define proper and improper integral with example.

Sol^n:

Proper Integral: If the lower limit a, upper limit b are finite and the integrand $f(x)$ is continuous in the interval $a \le x \le b$, then $\int_a^b f(x)\,dx$ is called a proper integral. Every proper integral is convergent.

Example $\int_0^{\frac{\pi}{2}} \frac{dx}{1+2\cos x}$, $\int_0^{\pi} \frac{dx}{a-\cos x}$, $a>1$

Improper Integral: If the limit or limits of integral be infinite, that is interval of $f(x)$ is infinite but the integrand $f(x)$ is continuous in the interval, or if the interval is finite and the integrand $f(x)$ is discontinuous at one or more points in the interval, then this type of integral is called improper integral.

Example: $\int_0^{\infty} \frac{x\,dx}{x^4+1}$, $\int_{-\infty}^{3} \frac{dx}{\sqrt{7-x}}$