Propositional Logic

CSE 191, Class Note 01
Propositional Logic
Computer Sci & Eng Dept
SUNY Buffalo

Discrete Mathematics

What is **Discrete Mathematics**?

- In Math 141-142, you learn continuous math. It deals with continuous functions, differential and integral calculus.
- In contrast, discrete math deals with mathematical topics in a sense that it analyzes data whose values are separated (such as integers: integer number line has gaps).
- Here is a very rough comparison between continuous math and discrete math: consider an analog clock (one with hands that continuously rotate, which shows time in continuous fashion) vs. a digital clock (which shows time in discrete fashion).

Course Topics

This course provides some of the mathematical foundations and skills that you will need in your further study of computer science and engineering. These topics include:

- Logic (propositional and predicate logic)
- Logical inferences and mathematical proof
- Counting methods
- Sets and set operations
- Functions and sequences
- Introduction to number theory and Cryptosystem
- Mathematical induction
- Relations
- Introduction to graph theory

By definition, computers operate on discrete data (binary strings). So, in some sense, the topics in this class are more relavent to CSE major than calculus.

The Foundations: Logic and Proof

- The rules of logic specify the precise meanings of mathematical statements.
- It is the basis of the correct mathematical arguments, that is, the proofs.
- It also has important applications in computer science:
  - to verify that computer programs produce the correct output for all possible input values.
  - To show algorithms always produce the correct results.
  - To establish the security of systems.
  - ...

Propositional Logic

Definition

A proposition is a declarative statement.

- It must be either TRUE or FALSE.
- It cannot be both TRUE and FALSE.
- We use T to denote TRUE and F to denote FALSE.

Example of propositions:

Example of propositions:	Example of non-propositions:
John loves CSE 191.	Does John love CSE 191?
2+3=5.	$2+3$.
2+3=8.	Solve the equation $2+x=3$.
Sun rises from West.	$2+x>8$.

Negation operator

Definition:

Suppose p is a proposition.

- The negation of p is $\neg p$.
- Meaning of $\neg p$: p is false.

Example:

John does not love CSE191.

Note that $\neg\, p$ is a new proposition generated from p.

- We have generated one proposition from another proposition.
- So we call $\neg$ (the symbol we used to generate the new proposition) the negation operator.

Logic operators

In general, we can define logic operators that transform one or more propositions to a new proposition.

- Negation is a unitary operator since it transforms one proposition to another.
- We will see a few binary operators shortly. They transform two propositions to a new proposition.

Truth table

How can we formally specify an operator (e.g., the negation operator)?

- One possibility is to use a truth table.
- The truth table lists all possible combinations of the values of the operands, and gives the corresponding values of the new proposition.

Example: Truth table for negation:

p	$\neg p$
T	F
F	T

Conjunction

Now we introduce a binary operator: conjunction $\wedge$, which corresponds to and:

- $p \wedge q$ is true if and only if p and q are both true.

Example:

Alice is tall AND slim.

Truth table for conjunction:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

Another binary operator is disjunction $\vee$, which corresponds to or, (but is slightly different from common use.)

- $p \vee q$ is true if and only if p or q (or both of them) are true.

Example:

Alice is smart OR honest.

Truth table for disjunction:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Implication

Yet another binary operator implication $\rightarrow$: $p \rightarrow q$ corresponds to p implies q.

Example:

If this car costs less than $10000, then John will buy it.

Truth table for implication:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Note that when p is F, $p \rightarrow q$ is always T.

Bidirectional implication

Another binary operator bidirectional implication $\leftrightarrow$: $p \leftrightarrow q$ corresponds to p is T if and only if q is T.

Example:

A student gets A in CSE 191 if and only if his weighted total is $\geq$ 95%.

Truth table for bidirectional implication:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Terminology for implication.

Because implication statements play such an essential role in mathematics, a variety of terminology is used to express $p \to q$:

- "if p, then q".
- "q, if p".
- "p, only if q".
- "p implies q".
- "p is sufficient for q".
- "q is necessary for p".
- "q follows from p".

Terminology for implication.

Example

Proposition p: Alice is smart.
Proposition q: Alice is honest.

- $p \to q$.
  - That Alice is smart is sufficient for Alice to be honest.
  - "Alice is honest" if "Alice is smart".
- $q \to p$:
  - That Alice is smart is necessary for Alice to be honest.
  - "Alice is honest" only if "Alice is smart".

Exclusive Or operator

Truth table for Exclusive Or $\oplus$

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

- Actually, this operator can be expressed by using other operators: $p \oplus q$ is the same as $\neg (p \leftrightarrow q)$.
- $\oplus$ is used often in CSE. So we have a symbol for it.

Number of binary logic operators

We have introduced 5 binary logic operators. Are there more?

Fact: There are totally 16 binary logic operators.

To see this:

- For any binary operator, there are 4 rows in its truth table.
- The operator is completely defined by the T/F values in the 3rd column of its truth table.
- Each entry in the 3rd column of the truth table has 2 possible values (T/F).
- So the total number of different 3rd column (hence the number of different binary operators) is $2 \times 2 \times 2 \times 2 = 16$.

Most of other 11 binary operators are not used often, so we do not have symbols for them.

Precedence of Operators

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Example:

$\neg\, p \wedge q$ means $(\neg\, p) \wedge q$

$p \wedge q \rightarrow r$ means $(p \wedge q) \rightarrow r$

- When in doubt, use parenthesis.

Translating logical formulas to English sentences

Using the above logic operators, we can construct more complicated logical formulas. (They are called compound propositions.)

Example

Proposition p: Alice is smart.
Proposition q: Alice is honest.

$\neg p \wedge q$: Alice is not smart but honest.

$p \vee (\neg p \wedge q)$: Either Alice is smart, or she is not smart but honest.

$p \rightarrow \neg q$: If Alice is smart, then she is not honest.

Translating logical formulas from English sentences

We can also go in the other direction, translating English sentences to logical formulas:

- Alice is either smart or honest, but Alice is not honest if she is smart:
  $(p \vee q) \wedge (p \to \neg q)$.
- That Alice is smart is necessary and sufficient for Alice to be honest:
  $(p \to q) \wedge (q \to p)$.
  (This is often written as $p \leftrightarrow q$).

Tautology and Logical equivalence

Definitions:

- A compound proposition that is always True is called a tautology.
- Two propositions p and q are logically equivalent if their truth tables are the same.
- Namely, p and q are logically equivalent if $p \leftrightarrow q$ is a tautology.
- If p and q are logically equivalent, we write $p \equiv q$.

Examples of Logical equivalence

Example:

Look at the following two compound propositions: $p \to q$ and $q \vee \neg p$.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$\neg p$	$q \vee \neg p$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

- The last column of the two truth tables are identical. Therefore $(p \to q)$ and $(q \vee \neg p)$ are logically equivalent.
- So $(p \to q) \leftrightarrow (q \vee \neg p)$ is a tautology.
- Thus: $(p \to q) \equiv (q \vee \neg p)$.

Example:

By using truth table, prove $p \oplus q \equiv \neg (p \leftrightarrow q)$.

De Morgan law

We have a number of rules for logical equivalence. For example:

De Morgan Law:

$$\neg(p \wedge q) \equiv \neg p \vee \neg q \quad (1)$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q \quad (2)$$

The following is the truth table proof for (1). The proof for (2) is similar.

p	q	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

p	q	$\neg p \vee \neg q$
T	T	F
T	F	T
F	T	T
F	F	T

Distributivity

Distributivity

$$
\begin{aligned}
p \vee (q \wedge r) &\equiv (p \vee q) \wedge (p \vee r) \qquad (1)\\
p \wedge (q \vee r) &\equiv (p \wedge q) \vee (p \wedge r) \qquad (2)
\end{aligned}
$$

The following is the truth table proof of (1). The proof of (2) is similar.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Contrapositives

Contrapositives

The proposition $\neg q \to \neg p$ is called the Contrapositive of the proposition $p \to q$. They are logically equivalent.

$$p \to q \equiv \neg q \to \neg p$$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$\neg q \to \neg p$
T	T	T
T	F	F
F	T	T
F	F	T

Logic Equivalences

Equivalence	Name
$p \wedge T \equiv p, \quad p \vee F \equiv p$	Identity laws
$p \vee T \equiv T, \quad p \wedge F \equiv F$	Domination laws
$p \vee p \equiv p, \quad p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \vee q) \equiv \neg p \wedge \neg q$ $\neg(p \wedge q) \equiv \neg p \vee \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv T, \quad p \wedge \neg p \equiv F$	Negation laws

Logic Equivalences

Logical Equivalences Involving Conditional Statements

$p \rightarrow q \equiv \neg p \vee q$
$p \rightarrow q \equiv \neg q \rightarrow \neg p$
$p \vee q \equiv \neg p \rightarrow q$
$p \wedge q \equiv \neg(p \rightarrow \neg q)$
$\neg(p \rightarrow q) \equiv p \wedge \neg q$
$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$
$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$
$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$
$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

Logical Equivalences Involving Biconditional Statements

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Prove equivalence

By using these laws, we can prove two propositions are logical equivalent.

Example 1: Prove $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$.

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{DeMorgan} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{DeMorgan} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{Distributivity} \\
&\equiv \mathsf{F} \vee (\neg p \wedge \neg q) && \text{Because } \neg p \wedge p \equiv \mathsf{F} \\
&\equiv \neg p \wedge \neg q && \text{Because } \mathsf{F} \vee r \equiv r \text{ for any } r
\end{aligned}
$$

Prove equivalence

Example 2: Prove $(p \wedge q) \rightarrow (p \vee q) \equiv \mathsf{T}$.

$$
\begin{aligned}
(p \wedge q) \rightarrow (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) && \text{Substitution for } \rightarrow \\
&\equiv (\neg p \vee \neg q) \vee (p \vee q) && \text{DeMorgan} \\
&\equiv (\neg p \vee p) \vee (\neg q \vee q) && \text{Commutativity and Associativity} \\
&\equiv \mathsf{T} \vee \mathsf{T} && \text{Because } \neg p \vee p \equiv \mathsf{T} \\
&\equiv \mathsf{T}
\end{aligned}
$$

Prove equivalence

Example 3:

Prove $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$.

Note that, by "Substitution for $\to$", we have: RHS = $\neg\, p \vee (q \wedge r)$.
So, we start from the LHS and try to get this proposition:

$$\begin{aligned}
(p \to q) \wedge (p \to r) &\equiv (\neg\, p \vee q) \wedge (\neg\, p \vee r) && \text{Substitution for } \to \text{, twice} \\
&\equiv \neg p \vee (q \wedge r) && \text{Distribution law} \\
&\equiv p \to (q \wedge r) && \text{Substitution for } \to
\end{aligned}$$

More examples.

Example: (Page 35, problem 10 (a)

Prove that: $[\neg\, p \wedge (p \vee q)] \to q$ is a tautology.

- By using truth table.
- By using logic equivalence laws.

Example: (Page 35, problem 10 (b)

Prove that: $[(p \to q) \wedge (q \to r)] \to [p \to r]$ is a tautology.

- By using truth table.
- By using logic equivalence laws.

We will show these examples in class.

Solving logic puzzles by using propositional logic

Example:

There are two types of people on an island:

- Knight: Always tell truth.
- Knave: Always lie

A says: “B is a knight.”
B says: “Two of us are opposite types.”
Determine the types of A and B.

We can describe the puzzle by the following propositions:

p: A is a knight, tells truth.
$\neg p$: A is a knave, lies.
q: B is a knight, tells truth.
$\neg q$: B is a knave, lies.

Solution

Suppose p=T:

- A tells truth: “B is a knight”. So B tells truth.
- B said: “Two of us are opposite types.”. So A and B are different types.
- This is false, because both A and B are knights.

Suppose p=F:

- A lies. So B is a knave. So B lies.
- B said: “Two of us are opposite types.”. So A and B are the same type.
- This holds and we get conclusion: Both A and B are knaves.

More Examples:

Example

There are two rooms: A and B. Each room has a sign.

- Sign at room A: "There is a lady in room A, and a tiger in room B."
- Sign at room B: "There is a lady in one room, and a tiger in another room."

Assume that:

- Exactly one sign is true and another sign is false.
- Exactly one thing (lady or tiger) in each room.

Determine which room contains what?

We will discuss solution in class.

More Examples:

Example (Page 24, Problem 34)

Five friends (Abby, Heather, Kevin, Randy and Vijay) have access to an on-line chat room. We know the following are true:

1. Either K or H or both are chatting.
2. Either R or V but not both are chatting.
3. If A is chatting, then R is chatting.
4. V is chatting if and only if K is chatting.
5. If H is chatting, then both A and K are chatting.

Determine who is chatting.

We will discuss solution in class.