STAT 111: Introduction to Statistics and Probability I

Lecture 5: Set Theory and Counting Processes

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Presentation outline

1. Introduction
2. Definition of sets
3. Types, Operations and Properties of sets
4. Use of Venn Diagrams
   - Two sets problems
   - Three sets problems
   - Inclusion-Exclusion criteria
5. Permutations (Unrestricted and Restricted)
6. Combinations(Unrestricted and Restricted)

Purpose and Learning outcomes

- The purpose of this lesson is to equip students with knowledge and skills in set theory and counting processes; the concept and applications.
- By the end of this lesson, students should be able to
  1. Define and explain basic concepts in set theory: sample space,events and outcomes of an experiment.
  2. Define and apply various sets operations.
  3. Use Venn diagrams to illustrate set operations (two and three sets problems)
  4. Use counting techniques to solve various counting problems such as permutations and combinations (restricted and unrestricted)

Random experiment

- A random experiment is any activity in which the results cannot be predicted with certainty.
- In other words; a set of actions, the result of which cannot be predicted with certainty.
- eg. roll of a die, toss a coin, selecting a ball at random from a box containing 3 balls (marked a, b, c)

Outcome, Events and Sample Space

In describing an experiment, we use the following terms; Outcome, Events and Sample space

- **Outcome**: An outcome is a result of an experiment
- **Event**: An event is any collection of outcomes.
- **Sample space**: The sample space of a given experiment is a set of S that contains all possible outcomes of the experiment.
- **Elements**: Element of a set are members that comprise the sample space. eg $A = \{a, b, c, d\}$, This means that a, b, c and d are elements of the set A.
- **Subset**: if Set $B = \{a, b\}$, then we say that B $\subset$ A. because all members or every element of set B is in set A

Discrete and Continuous Sample space

- A sample space is textbfdiscrete if it contains a finite or countable infinite set of outcomes. eg. $S = \{1, 2, 3, ...\}$.
- A sample space is continuous if it contains an interval (either finite or infinite) of real numbers. eg. $S = \{0 \leq x \leq \infty\}$.
- Example : All the following describe a continuous sample space EXCEPT

  A. $S = \{(x, y) : x \geq 0, y \geq 0\}$

  B. $S = \{(a, b) : 1 \leq a \leq b, 1 \leq b \leq 6\}$

  C. $S = \{k : k, is, odd, or, k = 4, 6\}$

  D. $S = \{x : x = 1, 2, 3, or, x \geq 8\}$.

Class Exercise

1. What is meant by a random experiment?
2. An experiment consists of tossing two dice.
   (a) Find the sample space S.
   (b) Find the event A that the sum of the dots on the dice equals 7
   (c) Find the event B that the sum of the dots on the the dice is greater than 12
   (d) Find the event C that the sum of the dots on the dice is greater than 12.

2. Consider the following experiment of tossing a coin three times.

- Find the sample space S
- Find the event A of two consecutive heads
- Find the event B of first head occuring
- Find the event C of one head or a tail.
- Solution: the sample space when three coins are tossed three times $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$
- continue...

Assigning Probabilities to events

- We can assign probabilities to events A, B and C.
- Probability of an event $A = \frac{n(A)}{n(S)}$
- $A = \{HHH, HHT, THH\}$ therefore n(A)=3, p(A)=3/8
- $B = \{HHH, HHT, HTH, HTT\}$, n(B)=4, P(A)=4/8=1/4
- $C = \{HHT, HTH, HTT, THH, THT, TTH\}$ n(C)=6, P(C)==n(C)/n(S)=6/8=3/4

Algebra of Sets

- Most often we are interested in a collection of related outcomes from a random experiment.
- We can also be interested in describing new events from combinations of existing events.
- In such situations as stated above, we rely on basic set operations such as Equality, Union, Intersections, Complement, Null, and Disjoint to form or describe other events of interest.

Set Operations

- Equality: Two sets A and B are equal ; $A = B$, if and only if $A \subset B$ and $B \subset A$
  eg. $A = \{1, 3, 7, 9\}$, $B = \{3, 1, 9, 7\}$
  Thus A = B
- Complementation: Suppose $A \subset S$. The complement of set A, denoted by $\bar{A}$ or A' is the set containing all elements in S but not in A.
- Union: The union of sets A and B denoted by$A \cup B$, is the set containing all elements in either A or B or both.
  $A \cup B = \{a : a \in A, or, a \in B\}$.

- Intersection: The intersection of sets A and B denoted by $A \cap B$, is the set containing all elements in both A and B.

$$A \cap B = \{a : a \in A, and, a \in B\}$$

- Null Sets: The set containing non element. it is denoted by $\emptyset$. Note that $\bar{S} = \emptyset$
- Disjoint sets: Two sets A and B are called disjoint or mutually exclusive if they contain common element, thus, if

$$A \cap B = \emptyset$$

- This means that A and B cannot occur together.
- In general we say two or more events are mutually exclusive if no pair of them has any outcome in common.

- The definitions of the union and intersection of two sets can be extended to any number of sets as follows:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup ... \cup A_n$$

$$= \{a : a \in A_1, or, a \in A_2, or, ...a \in A_n\}$$

.

- The above represents the events that at least one of the $A_i's$ occured.
- Similarly

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 ... \cap A_n$$

$$= \{a : a \in A_1 and, a \in A_2, ..., and a \in A_n\}$$

Example

- Given that

$$S = \{5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$$

$$A = \{6, 8, 10, 12\}$$

$$B = \{5, 7, 9, 11\}$$

$$C = \{13, 15, 6, 11\}$$

- Find (i) A' (ii)B' (iii) $A \cap B$
- Solution

  (i).

$$A' = \{5, 7, 9, 11, 13, 14, 15\}$$

  (ii).

$$B' = \{6, 8, 10, 12, 13, 14\}$$

  (iii).

$$A \cap B = \emptyset$$

Example 2

- let $A = \{5, 1, 0, -3, 6, 8, 9\}$, $B = \{4, 1, 2\}$,$C = \{4, 3, 6, 8\}$
- Find

$$(i)A \cup B(ii)B \cup C(iii)A \cap (B \cap C)$$

- Let $A_1 = \{n|2 \leq n <\}$ and $A_2 = \{n|5 < n < 155\}$, then

$$A_1 \cup A_2 = \{n|2 \leq n < 155\}$$

and

$$A_1 \cap A_2 = \{n|5 < n < 15\}$$

.

- Can also find

$$A_1' \cup A_2, and, A_1' \cup A_2'$$

1. A digital scale is used to provides weights to the nearest gram.
(a). What is the sample space for this experiment?
Let A denote the event that a weigh exceeds 11 grams, let B denote the event that a weight is less than or equal 15 grams, and C denote the event that a weigh is greater than or equal 8 grams and less than 12 grams.
Describe the followings events
(b) $A \cup B$ (c) $A \cap B$ (d) A' (e) $A \cup B \cup C$

(a) $S = \{0,1,2,3,...\}$

(b) $A \cup B = S = \{0,1,2,3,...\}$

(c) $A \cap B = \{12,13,14,15\}$

(d) $A' = \{0,1,2,3,4,...,11\}$

2. The sample space of an experiment is the real line expressed as

$S = \{v : -\infty < v < \infty\}$ Consider the event

$$A_1 = \{v : 0 \le v \le 1/2\}$$
$$A_2 = \{v : 1/2 \le v \le 3/4\}$$
$$A_i = \left\{v : 1 - \frac{1}{2^{i-1}} \le v \le 1 - \frac{1}{2^i}\right\}$$

Determine the events $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$

Exercises

3. Consider an experiment of tossing a coin repeatedly and of counting the number of tosses required until the first head appears. We define the events

$$A = \{k : k \text{ is odd}\}$$

$$B = \{k : 4 \le k \le 7\}$$

$$C = \{k : 1 \le k \le 10\}$$

where k is the number of tosses required until the first head (H) appears.
Determine the events $\bar{A}$, $\bar{B}$, $\bar{C}$, $A \cup B$, $B \cup C$, $A \cap B$, $A \cap C$, $B \cap C$ and $\bar{A} \cap B$

4. Consider the sample space in question 2 above. Consider the events

$$B_1 = \{v : v \le 1/2\}$$

$$B_2 = \{v : v \le 1/4\}$$

$$B_i = \{v : v \le 1/2^i\}$$

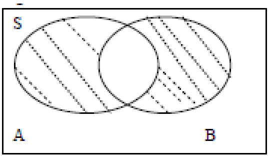
Determine the events $\bigcup_{i=1}^{\infty} B_i$ and $\bigcap_{i=1}^{\infty} B_i$

Venn Diagram

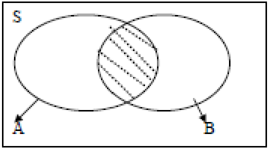
- Venn diagram is a graphical representation that is very useful in set operation.
- it is used to represent sample space and events in a sample space.
- The sample space is represented by the points in the rectangle.
- The events A, B and C
- shaded areas represent respectively,

$$A \cup B, A \cap B, A', A \cap B \cap C$$

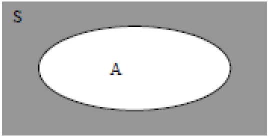
Understanding the shaded regions of the Venn diagram



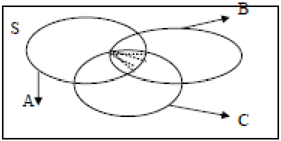
(a) Shaded area: $A \cup B$



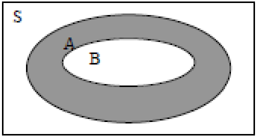
(b) Shaded area: $A \cap B$



(c) Shaded area: A'



(d) Shaded area: $A \cap B \cap C$



(e) $B \subset A$, Shaded region: $A \cap B'$

Understanding the shaded regions of the Venn diagram

The union and intersections operations also satisfy the following laws:

Commutative Laws:

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

Associative Laws:

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributive Laws:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

De Morgan's Laws:

$$(A \cup B)' = A' \cap B'$$
$$(A \cap B)' = A' \cup B'$$

Counting processes

- The counting method deals with counting the number of elements in sets.
- Counting problems occur in a variety of applications and can be studied using properties of sets.
- We begin with useful notational convention.
- The number of elements in a set S is denoted by $n(S)$
- $n(A \cap B)$, the number of elements in both A and B

Let A and B be sets. Then

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Proof

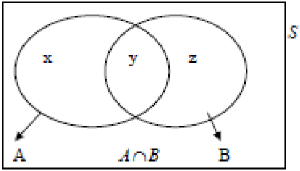


Fig.1.3

In fig. 1.3, A and B are any two events defined on the sample space S. Now you agred with me that

$n(A \cup B) = x + y + z \Rightarrow$

$n(A \cup B)$=(x+y)+(y+z)-y ....(1.1)

$$(x + y) = n(A)$$

It can be seen that $(y + z) = n(B)$ and

$$y = n(A \cap B)$$

It follows that (1.1) becomes

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Example 3

An investment club has 300 members, 150 of whom invest in either stocks or bonds. Suppose 104 invest in stocks and 83 in bonths. How many invest in both stocks and bonds?

- **Solution**
- Let $A = \{members, who, are, investing, in, stocks\}$
- $B = \{members, who, are, investing, in, bonds\}$ then
- $A \cup B = \{members, who, invest, in, either, stocks, and, bonds\}$
  $A \cap B = \{members, who, invest, in, either, stocks, and, bonds\}$
- We also know that $n(A) = 104, n(B) = 83, n(A \cup B) = 150$

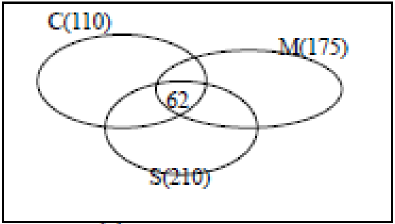
$$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 50$$

- Thus 50 members of the club invest in both stocks and bonds.
- This can be extended to more than 2 sets.

Example 4

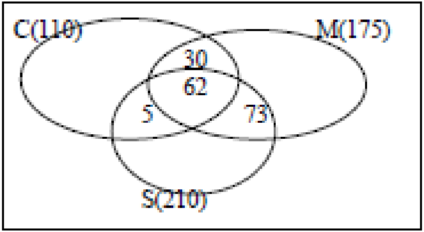
At level 200, University of Ghana, 270 arts students are survey about 3 courses: Mathematics (M), Statistics (S) and Computer science (C). it is found that 62 students take all 3 courses and that: 110 take C, 210 take S, 175 take M, 67 take C and S, 135 take S and M, 92 take C and M. How many students take non of the 3 courses.

Solution

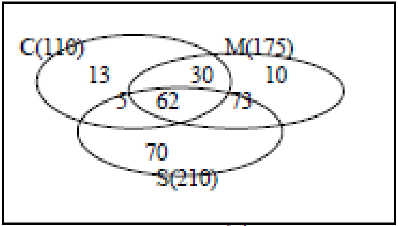


(a)

Solution to Example 4



(b)



(c)

Exercise 1

1. A group of faculty members at a small college operate a carpool to three kinds of activities baseball games, the opera, and the theater. Suppose there are 86 families in the carpool and that in a given month: 11 families attend none of the activities, 33 families go to baseball games, 35 families go to the opera, 39 families go to the theater, 14 families go to just baseball games, 17 families go to just the theater, 15 families go to just the opera.
   - How many familes go to all the three activities?
   - How many families go to baseball games and the opera but not to the theater?
   - How many families go to at least two activities?

Exercise 1 cont'd

1. The difference A-B, of two sets A and B is defined to be the set of all elements in A that are ot in B. Use the Venn diagram to illustrate the following sets:
   $(a) A - B (b)(A - B) \cap (B - A), (c)(A \cup B) - (A \cap B)(c) U - A$ where U denote the universal set.
2. A group of faculty members at a small college operate a carpool to three kinds of activities: baseball games, football games, and the dancing. Suppose there are 86 families in the carpool and that in a given month. 11 families attend none of the activities, 33 families go to baseball games, 35 families go to the football games, 39 families go the dancing, 14 families go to just baseball games, 17 familes go to just the dancing and 15 families go to just football games.
   (a) How many families go to all three activities?
   (b) How many families go to baseball games and the football games but not to the dancing?
   (c) How many families go to at least two activities?

Permutations and Combinations

By the end of this section, students should be able to compute basic counting processes involving combinations and permutations with or without restrictions.

1. Permutations (Unrestricted)
2. Permutations (Restricted/with constraints)
3. Combinations (Unrestricted)
4. Combinations (Resstricted/with constraints)

PERMUTATION AND COMBINATION

1. Permutation and Combinations are the various ways in which objects from a set may be selected generally, without replacement, to form subsets.
2. The selection of subsets is called **permutation** when the order of selection is a factor to be considered.
3. A **combination** is where the order is not a factor or of importance.

Usefulness of Combination and Permutations

The application and use of combination and permutations are varied

1. Design of computer chips involves consideration of possible permutations of input to output pins.
2. Field of molecular biology; sequencing of gens,virus, DNA , atoms, molecules are essentially permutation problems under certain constraints.
3. String searching algorithms may rely on combinatorics of words and characters. etc.

Permutation of distinct objects

- **Theorem** : The permutation of n distinct objects is n!.
- Where n! is called 'n' factorial, defined as the product of the first 'n' natural numbers.

$$n! = n(n-1)(n-2)...(3)(2)(1)$$

$$4! = 4x3x2x1 = 24$$

- Example: How many ways can we rearrange the word SHARP.
- The word SHARP has 5 distinct letters hence, the number of arrangement is thus 5!=5x3x2x1=120

Permutation of non-distinct objects

- **Theorem**: The number of distinct permutations of n objects of which n_1 are of one kind, n_2 are of a second kind, n_k are of k^{th}, and $n_1 + n_2 + n_k = n$ is
$$\frac{n!}{n_1!n_2!...n_k!}.$$
- Example 1: Find the number of rearrangements of the letters in the word DISTINCT.
- Solution: There are 8 letters. Both I and T are repeated 2 times. This means that the number of rearrangement of the word DISTINCT is
$$\frac{8!}{2!2!} = 10080$$
- Example 2: Find the number of rearrangement of the letters in the word CARRIER.
$$Solution : \frac{7!}{3!} = 7x6x5x4 = 840$$

In how many ways can the letters in the word STATISTICS be arranged?
Ans.50400

Permutation of subsets from a set

- In general if r objects are selected from a set of n distinct objects, any particular arrangement of these r objects is referred to as permutation.
- Theorem:
  The number of permutations of r objects selected from a set of n objects is denoted by
  $$_nP_r = \frac{n!}{(n-r)!}$$
- $_nP_r$ denotes an ordered arrangement of r objects from a set A containing n objects ($0 \leq r \leq n$) or
- A permutation of the elements of A taken r at a time, or
- The number of r elements permutations of a set containing n objects.

Illustration

4 men and 4 women are to be selected in a row of chairs numbered 1 through 8.

1. How many total arrangements are possible? (Ans: 8!=40320)
2. How many arrangements are possible if the men are required to sit in alternative chairs. (Ans: 2x4!x4!=1152)
   Example
3. The number of permutations of the four letters a, b, c and d, is 4!=24
4. What is the number of permutation if we take the four letters two at a time? Ans. ${}_4P_2 = \frac{4!}{2!} = 12$

Permutations with restrictions

As recap

- A permutation is an arrangement of a set of objects in an ordered way.
- An addition of some restrictions give rise to situation of permutation with restrictions.
- Ex. Suppose a class of 25 students a teacher wants to select three students for three different post of responsibility. In how many ways can it be done?
- Answer is ${}^{25}P_3 = 25!/22!$ ways in which the teacher can choose any three students.
- But if she puts a restriction, a student must have scored above 80% in a certain test?
- How can we deal with such types of permutations?

Permutations with restrictions

Some common permutation restrictions

1. Suppose we have n letters or items out of which k are of same kind and the rest are all different $= \frac{n!}{k!}$
2. Number of permutations of n items, taken k at a time, when we include a particular item in each arrangement = ${}^{n-1}P_{k-1} x k$
3. When a particular thing is fixed, number of permutation of n items out of which k number of items are taken at a time = ${}^{n-1}P_{k-1}$
4. The number of permutations of n items, taken k at a time when a particular thing is never taken $= {}^{n-1}P_k$

Further illustrations

- Consider a word 'YOURSELVES'.
- In how many ways the letter can be arranged if U and S always come together an U always preceeds S?
- Solution:
- The word YOURSELVES has 11 letters out of which S repeats two times and E repeats three times. The rest are all different.
- If the letters U and S come together, they are considered as one letter.
- The remaining 10 letters can rearrange themselves in $10!/(2!3!) = 302400$ ways.

- Find the number of ways in which five persons A, B, C,D and E sit around a round table such that:
  (1) There is no restriction
  (2) A and D must always sit together
  (3) C and E must not sit together.
- Solution: (1) Five persons can sit around a round table in (5-1)!=4!=24 ways
- (2) Since A and D sit together in all the possible arrangement, we have to consider them as one unit.ie (4-1)!=6. But A and D interchange their positions in two ways. 6x2=12
- The number of ways in which C and E must not sit together = Total number possible ways - Number of ways in which C and E can sit together = 24-12=12 ways.

COMBINATIONS

- In combinations unlike permutations, the order in which the elements are arranged is immaterial.
- In other words, it is a possible selection of a certain number of objects taken from a group with no regard given to order.
- For instance if we choose two letters from X, Y and Z we could write the letters as

$$XY, XZ, and, YZ$$

- Thus the order in which the letters are arranged is of no concern; thus, XY could be written as YX.

Definition: An unordered arrangements of r objects from a set A containing n objects ($r \leq n$) is called an r-element combination of A.
Or A combination of the elements of set A taken r at a time.
It is denoted by ${}^nC_r$

Theorem

The number of combinations of n distinct objects taken r at a time is

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$$

- In the National Lottery, 6 numbers are chosen from 49. You win if the 6 balls you pick match the six balls selected by the machine. how many ways can 6 balls be chosen from 49 is given by

$$^{49}C_6 = \frac{49!}{6!(49-6)!} = 13983816$$

Illustrations in Combinations 2

- If we have available seven men and need a working party of four men, how many different groups may we possibly select?
- Solution

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

, where n=7,r=4

$$\binom{7}{4} = {}^{7}C_{4} = \frac{7!}{4!(7-4)!} = \frac{7!}{4!3!}$$

$$= \frac{7x6x5}{3x2x1} = 35$$

Illustration 3

- An insurance firm has 15 members of staff in the account department. In how many ways can 5 staff be selected from the department? **Solution**. Ans. 30,240.
- In how many ways can two mathematics and three biology books be selected from eight mathematics and six biology books
- Solution : There $\binom{8}{2}$ possible ways to select two math books and $\binom{6}{3}$ possible ways to select 3 biology books.
- Therefore by the counting principle we have

$$\binom{6}{3} x \binom{6}{3} = 560$$

Try

1. There are 10 balls in a bag numbered from 1 to 10. Three balls are selected at random. How may different ways are there of selecting the three balls?
2. Answer: =120

Points to Note

By convention

1. 0!=1,
2. $\binom{n}{i} = 0$, whenever $i < 0$, $i > n$
3. $$\binom{n}{r} = \binom{n}{n-r}$$
4. A useful combinational identity
$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

Combinations with Restrictions

We consider some common restrictions on combinations as follows;

1. m number of combinations of n different objects taken r at a time when p particular objects are always included $= {}^{n-p}C_{r-p}$
2. Number of combination of n different objects, taken r at a time, when p particular objects are always to be excluded $= {}^{n-p}C_r$
3. When p particular objects are always included and q particular objects are always excluded ${}^{n-p-q}C_{r-p}$

Illustrations 1

- In how many ways can a cricket eleven be chosen out of 15 players? if
- (i) A particular player is always chosen
- (ii) A particular player is never chosen.
- Solution
- Ans.(i)Required number of ways ${}^{14}C_{10} = {}^{14}C_4 = 1365$
- (ii) A particular player is never chosen, it means that 11 players are selected out of 14 players.
- Required number of ways $= {}^{14}C_{11} = 364$

illustration 2

- If out 16 players 11 are to be selected. if the keeper and captain is always included in each selection and one player is injured. Find how many ways of selection?
- Solution.
- This relate to the third restriction. n=16, r=11, p=2
  $^{n-p-q}C_{r-p} = {}^{16-2-1}C_{11-2} = {}^{13}C_9 = 715$
- Grace belong to group of eight workers. How many ways can a team of four workers be selected if Grace must be on the team. ANS. $^7C_3$

Illustration 3

- Four students are to be chosen from a group of eight students for the school tennis team. Two members of the group Kofi and Ama, do not get along and cannot both be on the team. How many ways can the team be selected.
- **Solution**
  1. how many teams can kofi and Ama be on them = ${}^{6}C_{2}$
  2. How many total teams can be formed= ${}^{8}C_{4}$
  3. This means teams without kofi and Ama = ${}^{8}C_{4} - {}^{6}C_{2}$

Combination with multiple restrictions

- If we require to make multiple selections from seperate groups, the the multiplication principle dictates that we multiply the number of ways of performing each task.
- eg. There are seven women and four men in a workplace. how many groups of five can be chosen
  (a) without restrictions
  (b) containing three women and two men
  (c) containing at least one man
  (d) containing at most one man.

Combinations with multiple restrictions

- Solution: (a) ${}^{11}C_5 = 462$
- (b) ${}^{7}C_3 x {}^{4}C_2 = 210$
- (c) ${}^{11}C_5 - {}^{7}C_5 = 441$
- (d) ${}^{7}C_5 + {}^{4}C_1 x {}^{7}C_4 = 161$

Exercise

1. In a group of 2 camels, 3 goats and 10 sheep, in how many ways may one choose 6 animals if

1. There are no constraints in species
2. The two camels must be included
3. The two camels must be excluded
4. There must be at least 3 sheep
5. There must be at most 3 sheep
6. Kofi camel, Eric Goat, and Antwi sheep hate each other and they will not work in the same group. How many compatible committees are there?

2. A committee of 5 people is to be chosen from a group of 6 men and 4 women. How many committtees are possibel if.

- there are no restrictions?
- one particular person must be chosen on the committee?
- one particular woman must be excluded from the committee?

3. From a group of 5 women and 7 men, how many different committees consisting of 2 women and 3 men can be formed?. What if that two men are feuding and refuse to serve on the committee together.

THANK YOU FOR YOUR ATTENTION
ANY QUESTIONS?.