DCIT105
Mathematics for IT Professionals

Session 3 – Propositional Logic

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Session Outline

The key topics to be covered in the session are as follows:

- Logic
- Logical Operation
  - Negation
  - Conjunction
  - Disjunction
  - Implication
  - Bi-implication
- Precedence of Logical Operators
- Truth tables
- Bitwise Operations
- Applications
- Propositional Equivalence

Topic Three

PROPOSITIONAL LOGIC

Reference

- **Chapter 1**

Kenneth H. Rosen, "Discrete Mathematics and its Applications", Seventh edition, McGraw Hill, 2012

Logic

- The rules of logic give precise meaning to mathematical statements
- Logic rules are used to distinguish between valid and invalid mathematical arguments
- Logic has numerous applications to computer science:
  - Design of computer circuits
  - Construction of computer programs
  - Verification of the correctness of the program
- Two types of logic:
  - Propositional logic
  - Predicate logic

Introduction

- A **proposition** is a declarative sentence that is either TRUE or FALSE, but not both
- Examples- **propositions**:

1. Washington, D.C., is the capital of the USA	(TRUE)
2. Toronto is the capital of Canada	(FALSE)
3. $1 + 1 = 2$	(TRUE)
4. $2 + 2 = 3$	(FALSE)

- Examples- **not propositions:**

1. What time is it ?	(Not a Declarative Sentence)
2.. Read this carefully	(Not a Declarative Sentence)
3. $x + 1 = 2$	(Neither TRUE nor FALSE)
4. $x + y = z$	(Neither TRUE nor FALSE)

Introduction

- The area of the logic that deals with propositions is called *propositional logic* or *propositional calculus*

- **Notation:**
  - Propositional variables are denoted conventionally by the letters *p, q, r, s, …*
  - If the truth value of a proposition is TRUE, it is denoted by T
  - If the truth value of a proposition is FALSE, it is denoted by F

Introduction

- **Compound propositions** are the declarative statements that are constructed by combining one or more existing propositions using *logical operators*.

- The logical operators are:
  - Negation (NOT) denoted by the symbol, ¬
  - Conjunction (AND) denoted by the symbol, ∧
  - Disjunction (OR) denoted by the symbol, ∨

Negation

- The *negation* of a proposition *p*, denoted by ¬*p* is the statement "It is not the case that *p*", and is read as "**not *p***."
- The truth value of ¬*p*, is the opposite of the truth value of *p*.

- Example:

  p = Michael's PC runs Linux

  ¬*p* = It is not the case that Michael's PC runs Linux

  or

  Michael's PC does not run Linux

- Negation is applicable to only single proposition

Truth table for Negation

P	***¬p***
T	F
F	T

Conjunction

- The *conjunction* of two propositions *p*, *q*, denoted by $p \wedge q$ is the proposition "***p* and *q***".
- The truth value of $p \wedge q$ is TRUE only when *p, q* are both TRUE and otherwise it is FALSE
- Conjunction is applicable to two or more propositions

- Example:

  p = Daniel can read well

  q = Daniel can write well

  $p \wedge q$ = Daniel can read well and Daniel can write well or Daniel can read and write well

Truth table for Conjunction

P	*q*	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

- The *disjunction* of two propositions *p*, *q*, denoted by *p*∨*q* is the proposition "***p* or *q***".
- The truth value of *p*∨*q* is FALSE only when *p, q* are both FALSE and otherwise it is TRUE
- Disjunction is applicable to two or more propositions

- Example:

  p = Daniel can read well

  q = Daniel can write well

  p∨*q* = Daniel can read well or Daniel can write well or Daniel can read or write well

Truth table for Disjunction		
P	***q***	***p*∨*q***
T	T	T
T	F	T
F	T	T
F	F	F

Exclusive OR

- The connective '**or**' in a *disjunction* corresponds to **inclusive or**
- The **exclusive or** of two propositions p, q, denoted by $p \oplus q$ is the proposition "***p* or *q* (but not both)**".
- The truth value of $p \oplus q$ is TRUE when exactly one of p, q is TRUE, and is FALSE otherwise
- 'Exclusive or' is applicable to two or more propositions
- Example:

p = Daniel can read well q = Daniel can write well

$p \oplus q$ = Daniel can read well or Daniel can write well or Daniel can either read or write well but not both

Truth table for Exclusive or		
P	***q***	***p⊕q***
T	T	F
T	F	T
F	T	T
F	F	F

Implication

Conditional Statement (or Implication):

- The *conditional statement* $p \rightarrow q$ is the proposition "**if *p*, then *q***", read as "*p* implies *q*".
- The truth value of $p \rightarrow q$ is FALSE when *p* is TRUE and *q* is FALSE, otherwise it is TRUE
- In $p \rightarrow q$,

  p is called *hypothesis* or *antecedent* or *premise*

  and

  q is called the *conclusion* or *consequence*

Truth table for Implication

P	***q***	***p→q***
T	T	T
T	F	F
F	T	T
F	F	T

Implication

- Examples:
  1) "If I am elected, then I will lower taxes" It is only when the politician is elected but does not lower taxes that voters can say that the politician has broken the campaign pledge.

  2) "If you get 100%, then you will get an A" It is only when you get 100%, but the professor does not give you an A, you feel the statement is FALSE

Truth table for Implication		
P	***q***	***p→q***
T	T	T
T	F	F
F	T	T
F	F	T

Implication

- Exercise (What did you observe?):
  1) "If Juan has a smartphone, then 2 + 3 = 5"
  2) "If Juan has a smartphone, then 2 + 3 = 6"

- we consider conditional statements of a more general sort than we use in English

- The **if-then** construct used in programming languages is different from that used in logic

Truth table for Implication

P	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Implication Variants

Given a conditional statement $p \rightarrow q$, we can form three new conditional statements, namely

1) The ***converse*** of $p \rightarrow q$, denoted by $q \rightarrow p$
2) The ***contrapositive*** of $p \rightarrow q$, denoted by $\neg q \rightarrow \neg p$
3) The ***inverse*** of $p \rightarrow q$, denoted by $\neg p \rightarrow \neg q$

- The contrapositive will have the same truth value as $p \rightarrow q$
- The converse and inverse will have same truth values

Equivalence of implication variants

"When two compound propositions always have the same truth value we call them **equivalent**"

So, $p \rightarrow q$ and $\neg q \rightarrow \neg p$ are equivalent and

$q \rightarrow p$ and $\neg p \rightarrow \neg q$ are equivalent

- Example: "The home team wins whenever it is raining"
  (hint: Identify the Hypothesis and conclusion)

  Contrapositive : "If the home team does not win, then it is not raining"

  Converse : "If the home team wins, then it is raining"

  Inverse : "If it is not raining, then the home team does not win."

Bi-implication

Biconditional Statement (Bi-implication):

- The *biconditional statement* , denoted by $p \leftrightarrow q$ is the proposition " ***p* if and only if *q***" (p iff q).
- $p \leftrightarrow q$ is TRUE when p and q have the same truth values, and is FALSE otherwise.
- $p \leftrightarrow q$ has exactly the same truth value as $(p \rightarrow q) \wedge (q \rightarrow p)$
- Example:

  p = " You can take the flight"

  q = " You buy a ticket"

Truth table for Bi-implication		
P	***q***	***p*↔*q***
T	T	T
T	F	F
F	T	F
F	F	T

p ↔ q = "You can take the flight if and only if you buy a ticket."

Truth Tables of Compound Propositions

- Example: Construct truth table for the compound proposition;

$$(p \vee \neg q) \rightarrow (p \wedge q)$$

Truth table for $(p \vee \neg q) \rightarrow (p \wedge q)$					
P	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Precedence of logical operators

- In the absence of proper parenthesization, compound propositions are evaluated according to the precedence of the operators

- Examples:

  $\neg p \wedge q$ is same as $(\neg p) \wedge q$

  $p \wedge q \vee r$ is same as $(p \wedge q) \vee r$

  $p \vee q \rightarrow r$ is same as $(p \vee q) \rightarrow r$

Precedence of logical operators	
Operator	**Precedence**
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Logic and Bit operations

- Computers represent information using **Bit**s
- Bit means **B**inary Digi**t**, which is either 0 or 1.
- Bits can be used to represent truth values with 1 for TRUE and 0 for FALSE
- **Boolean variable** is a variable having a value of either TRUE or FALSE, and hence ***can be represented using a bit***

Truth value	Bit
T	1
F	0

Logic and Bit operations

- Computer bit operations correspond to the logical operators;
  ¬ (NOT)
  ∧ (AND)
  ∨ (OR)
  ⊕ (XOR)

Bit Operations						
x	***y***	***¬ x***	***¬ y***	***x∨y***	***x∧y***	***x⊕ y***
0	0	1	1	0	0	0
0	1	1	0	1	0	1
1	0	0	1	1	0	1
1	1	0	0	1	1	0

Bit strings and operations

- A bit string is a sequence of zero and/or one bits
- The length of a string is the number of bits in it
  - e.g. 101010011 is a bit string of length nine
- Computers manipulate the information by performing operations on the bit strings
- The symbols $\vee$, $\wedge$, and $\oplus$ represent the **bitwise OR**, **bitwise AND**, and **bitwise XOR** operations, respectively

- Example: For the strings 01 1011 0110 and 11 0001 1101
  bitwise OR gives 11 1011 1111
  bitwise AND gives 01 0001 0100
  bitwise XOR gives 10 1010 1011

Applications

Translating English sentences:

- Helps in removing ambiguity, analysing and determining the truth values and manipulating them using rules of inference

- Examples:

  1) *English sentence*: "You can access the Internet (a) from campus only if you are a computer science major (c) or you are not a freshman ($\neg f$)".

  Logical expression: $(c \vee \neg f) \rightarrow a$

  2) *English sentence*: "You cannot ride the roller coaster ($\neg q$) if you are under 4 feet tall (r) and not older than 16 years old ($\neg s$)."

  Logical expression: $(r \wedge \neg s) \rightarrow \neg q$

Applications

System Specification:

- System and software engineers take requirements in natural language and produce precise and unambiguous specifications that can be used as the basis for system development
- Example:

  1) *Specification* : "The automated reply cannot be sent when the file system is full"

  q : $\neg p$

  Logical expression: $q \rightarrow \neg p$

Applications

Boolean Searches:

- Search over large collections of information, for example Web search, employ techniques from propositional logic and hence called Boolean searches
- The connective AND is used to match records that contain both search terms, the connective OR is used to match one or both search terms, and the connective NOT is used to exclude a particular search term
- Examples:

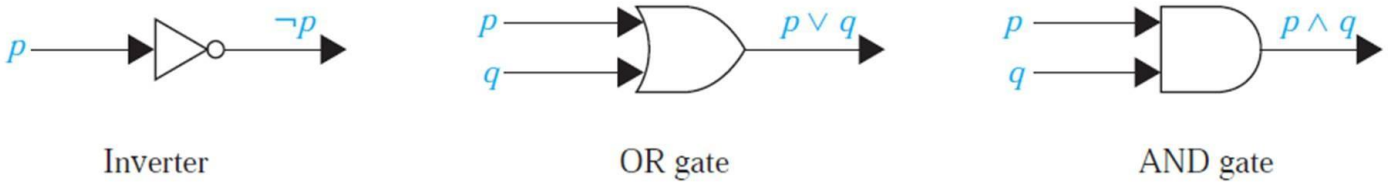
  Search for web pages matching NEW AND MEXICO AND UNIVERSITIES

  Search for web pages matching (NEW AND MEXICO OR ARIZONA) AND UNIVERSITIES.

Applications

Logic Circuits:

- Propositional logic can be applied to the design of computer hardware (*logic circuits* or *digital circuits*)
- A **logic circuit** (or **digital circuit**) receives input signals $p_1, p_2, \ldots, p_n$, each bit [either 0 (OFF) or 1 (ON)], and produces output signals $s_1, s_2, \ldots, s_n$, each bit respectively.
- Complicated digital circuits are constructed from the three basic circuits, called **gates**



Inverter OR gate AND gate

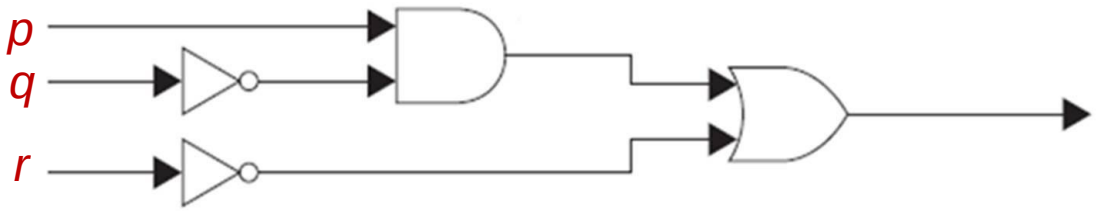
Applications

Logic Circuits:

- Given a circuit built from the basic logic gates and the inputs to the circuit, we determine the output by tracing through the circuit
- Example:

  Determine the output of the following digital circuit when

  $p = 1, q = 0$, and $r = 1$



Answer: $(p \wedge \neg q) \vee \neg r$

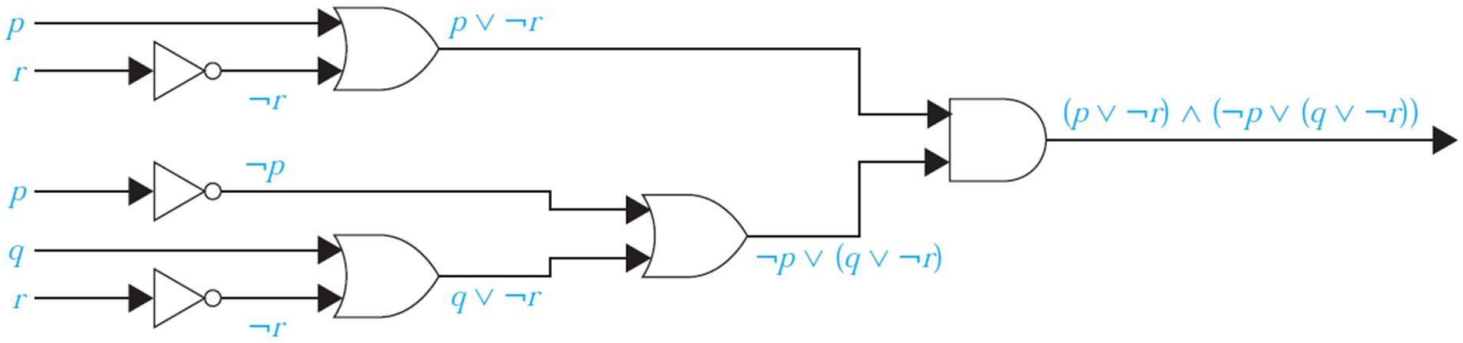
$= 1$

Applications

Logic Circuits:

- Example: Build a digital circuit that produces the output $(p \vee \neg r) \wedge (\neg p \vee (q \vee \neg r))$ when given input bits p, q, and r

Answer:



Propositional equivalence

- Tautology:
- *A compound proposition that is always* TRUE, no matter what the truth values of the propositional variables that occur in it, is called a *tautology*.
- Contradiction:
- *A compound proposition that is always* FALSE, no matter what the truth values of the propositional variables that occur in it, is called a *contradiction*.
- Contingency:
- A compound proposition that is neither a tautology nor a contradiction is called a *contingency*.

Propositional equivalence

- Example:

  Given a proposition p, the compound proposition $p \vee \neg p$ is a tautology, because, it is always TRUE.

  Similarly, $p \wedge \neg p$ is a contradiction, because, it is always false

- Tautologies and contradictions are often important in mathematical reasoning

Propositional equivalence

Logical equivalence:

- Compound propositions that have the same truth values in all possible cases are called **logically equivalent**.

or in other words,

- The compound propositions p and q are called *logically equivalent* if $p \leftrightarrow q$ is a tautology.
- The notation $p \equiv q$ (or sometimes $p \Leftrightarrow q$) is used to denote that p and q are logically equivalent.

Propositional equivalence

Logical equivalence:

- The equivalence of two compound statements can be determined by constructing a truth table and checking whether the columns giving their truth values are same
- Example: Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

Truth Tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

- Note: $\neg(p \vee q) \equiv \neg p \wedge \neg q$ is one of the two **De Morgan's** Laws

Propositional equivalence

Logical equivalence:

- Example: Show that $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.

Truth Tables for $\neg p \vee q$ and $p \rightarrow q$.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

- Note: To establish a logical equivalence of two compound propositions involving three different propositional variables p, q, and r using a truth table, we need 8 rows for combinations of truth values as TTT, TTF, TFT, TFF, FTT, FTF, FFT, and FFF

Propositional equivalence

Logical equivalence:

- Example: Show that $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

  (This is the *distributive law* of disjunction over conjunction)

A Demonstration That $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ Are Logically Equivalent.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Propositional equivalence

Important Logical equivalences:

Here, **T** denotes the compound proposition that is always TRUE and **F** denotes the compound proposition that is always FALSE

De Morgan's laws can be extended to any number of propositions

For example,

$$\neg(p_1 \vee p_2 \vee \cdots \vee p_n) \equiv (\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n)$$

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Propositional equivalence

Logical Equivalences Involving Conditional Statements.

$p \rightarrow q \equiv \neg p \vee q$

$p \rightarrow q \equiv \neg q \rightarrow \neg p$

$p \vee q \equiv \neg p \rightarrow q$

$p \wedge q \equiv \neg(p \rightarrow \neg q)$

$\neg(p \rightarrow q) \equiv p \wedge \neg q$

$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$

$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$

$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$

$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$

$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Propositional equivalence

- The logical identities we already know like distributive law, commutative law, De Morgan's law etc., may be used in constructing new logical equivalences and proving tautology, without constructing truth tables

- Problems:

1) Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent
2) Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent
3) Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Thank you