The $\mathcal{CEC}\cdot\mathcal{CID}$ Framework for Adaptive Generators and Uncountable Sets

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Abstract

We define Computational Entropy Cardinality (CEC) and Complexity-Induced Dimension (CID) for (possibly uncountable) sets $A \subseteq \mathbb{R}^n$ generated by an adaptive system G. The framework replaces naive counting by metric–measure coverage and introduces a time-varying information budget $B(j)$ to model phase transitions in generative efficiency. We formalize the CEC·CID signature, give envelopes for irregular sets, prove budget–resolution inversion laws (including Lambert–W and generalized asymptotic dimension spectra), and derive operational results for drift, multiset allocation, and adaptive capacity schedules. Worked examples (self-similar sets, smooth manifolds, power-tower curves) anchor the abstractions.

To handle degenerate cases with zero intrinsic dimension ($d = 0$)—where the static triplet collapses and CEC becomes trivial—we promote the framework to a parametric family rooted in thermodynamic formalism and transfer-operator spectral theory. This yields a richer signature (pressure root, Rényi spectrum, Matuszewska indices, operator spectrum) and a CEC zeta function whose poles and zeros capture fine structure. The resulting theory provides a unified language to benchmark generators, compare complex sets, and design measurement protocols for real systems (e.g., image manifolds and medical imaging datasets).

1 Motivation and Conceptual Overview

We want a single currency—*information in nats*—bridging:

- the intrinsic geometric and arithmetic difficulty of a target set A, and
- the operational cost incurred by a generator G when trying to cover A at scale ε.

Classically, difficulty is expressed via metric entropy

$$H_\varepsilon(A) = \log N(A, \varepsilon),$$

where $N(A, \varepsilon)$ is the minimal number of radius-ε balls needed to cover A. On the generator side, we track a cumulative information budget

$$B(j) = \sum_{t=1}^{j} b_t,$$

where b_t is the code length (in nats) of the control, randomness and parameter deltas used at step t.

The core bridge is a *capacity lower bound*:

$$B(j) \;\gtrsim\; H_{\varepsilon(j)}(A),$$

saying that any generator that produces an $\varepsilon(j)$-cover must pay at least the metric entropy of A at that scale, up to an $O(1)$ overhead. This lets us:

- benchmark generative pipelines against the intrinsic difficulty of the set;

- invert budget $\leftrightarrow$ resolution and derive efficiency laws;
- *use* a generator as a computational ruler to infer the CEC·CID signature of A from logs.

However, the standard asymptotic

$$\log N(A,\varepsilon) \sim d\log\frac{1}{\varepsilon} + \beta\log\log\frac{1}{\varepsilon} + \log c$$

collapses when $d=0$; all zero-dimensional sets look equivalent at leading order. The parametric extension fixes exactly this by attaching a transfer-operator family to the generator and extracting spectral and thermodynamic invariants that remain non-trivial even when $d=0$.

2 Setting and Notation

Let (A,μ) be a measurable subset $A\subset\mathbb{R}^n$ with a finite, non-trivial Borel measure μ supported on A. For $\varepsilon>0$ let $N(A,\varepsilon)$ be the minimal number of closed n-balls of radius ε needed to cover A. We define the metric entropy:

$$H_\varepsilon(A) = \log N(A,\varepsilon).$$

Let $d_H(A)$ denote the Hausdorff dimension and $d_M(A)$ the Minkowski (box-counting) dimension when they exist.

A generator G induces:

- a j-step reachable set $C_j(G)\subseteq\mathbb{R}^n$;
- an information-capacity sequence $(b_t)_{t\geq 1}$, $b_t\geq 0$, in nats/step;
- a cumulative budget

$$B(j) = \sum_{t=1}^{j} b_t.$$

We assume a target spatial resolution $\varepsilon=\varepsilon(j)$ after j steps (grid cell size, kernel radius, etc.).

3 The CEC·CID Complexity Signature

Definition 1 (CID triple and CEC normalization)**.** *Given $A\subset\mathbb{R}^n$ and a generator G, suppose the metric entropy satisfies the regularly varying asymptotic*

$$N(A,\varepsilon) \sim c\,\varepsilon^{-d}\left(\log\frac{1}{\varepsilon}\right)^{\beta} \quad (\varepsilon\downarrow 0), \tag{1}$$

with $d\in[0,n]$, $\beta\in\mathbb{R}$ and $c>0$. We define the CEC·CID signature as

$$|A|^{(G)}_{\mathrm{CEC\cdot CID}} = (d,\beta,\lambda;c),$$

where:

- *d is an intrinsic dimension proxy (typically $d=d_M(A)$ when it exists);*
- *β encodes log-corrections (oscillatory or arithmetic structure);*
- *c is a normalization constant;*
- *λ is the generator refinement rate, linked to $B(j)$ via the efficiency laws in Section 6.*

Remark 1. For large classes of fractals and manifold-like sets, (1) holds with $d = d_M$ and β capturing secondary oscillations. When $N(A, \varepsilon)$ is not asymptotically monomial (strong oscillations), we replace (d, β, c) by suitable upper/lower envelopes (Section 5.1).

From (1) we get the entropy expansion

$$H_\varepsilon(A) = \log N(A, \varepsilon) \sim d \log \frac{1}{\varepsilon} + \beta \log \log \frac{1}{\varepsilon} + \log c. \tag{2}$$

4 Capacity Lower Bound and Budget–Resolution Inversion

4.1 Capacity Lower Bound

Theorem 1 (Capacity Lower Bound)**.** *Let $A \subset X$ be a separable metric space and G a generator such that after j steps its output points form an ε-cover of A. If the cumulative information budget is $B(j)$, then*

$$B(j) \;\geq\; \log N(A, \varepsilon) \;-\; O(1).$$

Proof sketch. Fix $\varepsilon > 0$ and let $\{a_i\}_{i=1}^M$ be a minimal ε-net of A with $M = N(A, \varepsilon)$. Each length-j transcript of the generator (control symbols, randomness, parameter deltas) has code length at most $B(j)$ and decodes to a path of output points. By canonicalizing each transcript to a sorted list of cover centers, we can associate at most one ε-cover of A to each transcript. By Kraft's inequality, there are at most $e^{B(j)}$ valid transcripts of length $\leq B(j)$, hence

$$e^{B(j)} \;\geq\; M = N(A, \varepsilon),$$

which implies the claimed bound up to an $O(1)$ header overhead. □

The CLB is the theoretical bridge: it lower-bounds necessary budget, while practical instrumentation of b_t (e.g., via compressed logs) provides achievable upper bounds. Comparing them benchmarks generator efficiency.

4.2 Lambert–W Inversion

Assume (2). Let $t = \log(1/\varepsilon)$. Then

$$B \approx H_\varepsilon(A) \approx dt + \beta \log t + \log c.$$

If $\beta = 0$, we recover the standard linear law

$$t \approx \frac{B - \log c}{d}, \qquad \varepsilon(B) \approx c^{1/d} e^{-B/d}.$$

If $\beta \neq 0$, rearrange:

$$B - \log c \approx dt + \beta \log t \quad \Longrightarrow \quad t^\beta e^{dt} \approx e^{B - \log c}.$$

This can be rewritten in Lambert–W form as

$$\frac{d}{\beta} t\, e^{\frac{d}{\beta} t} \approx \frac{d}{\beta} e^{(B - \log c)/\beta} \quad \Longrightarrow \quad t \approx \frac{\beta}{d} W\left(\frac{d}{\beta} e^{(B - \log c)/\beta}\right),$$

and thus

$$\log \frac{1}{\varepsilon(B)} = t(B).$$

Using the standard asymptotic expansion of W at infinity yields:

Corollary 1 (Asymptotic expansion)**.** *As $B \to \infty$,*

$$\log \frac{1}{\varepsilon(B)} = \frac{B - \log c}{d} - \frac{\beta}{d} \log \frac{B - \log c}{d} + O\left(\frac{\log B}{B}\right).$$

This is the precise version of the heuristic

$$\log \frac{1}{\varepsilon(B)} \approx \frac{B}{d} - \frac{\beta}{d} \log B + O(1).$$

5 Coverage, Completeness and Envelopes

Define the $\varepsilon(j)$-thickened coverage fraction

$$\kappa(j) = \frac{\mu\big(A \cap (C_j(G))_{\varepsilon(j)}\big)}{\mu(A)} \in [0, 1].$$

Definition 2 (Computational completeness)**.** *A generator G is computationally complete on (A, μ) if $\kappa(j) \to 1$ as $j \to \infty$.*

Proposition 1 (Sufficient condition for completeness)**.** *If there exists a scale sequence $\varepsilon(j) \downarrow 0$ such that*

$$B(j) \geq H_{\varepsilon(j)}(A) + \omega(1),$$

and the reachable sets $C_j(G)$ become asymptotically $\varepsilon(j)$-dense in $\operatorname{supp}(\mu)$ *(e.g., controllability plus mixing), then $\kappa(j) \to 1$.*

5.1 Upper and Lower CEC Envelopes

When (1) fails due to strong oscillations, define

$$\overline{d} = \limsup_{\varepsilon \downarrow 0} \frac{H_\varepsilon(A)}{\log(1/\varepsilon)}, \qquad \underline{d} = \liminf_{\varepsilon \downarrow 0} \frac{H_\varepsilon(A)}{\log(1/\varepsilon)},$$

and similarly for β and c. We then obtain upper and lower CEC triples

$$|A|^*_{\mathrm{CEC}} = (\overline{d}, \overline{\beta}, \overline{c}), \qquad |A|_{*,\mathrm{CEC}} = (\underline{d}, \underline{\beta}, \underline{c}),$$

and all operational laws propagate by replacing (d, β, c) with the appropriate envelope.

6 Operational Laws

Write $B(j) = \sum_{t \leq j} b_t$. Assume for the moment the basic ADS form (2).

6.1 Constant-rate Efficiency

If $b_t \equiv b$ and the resolution schedule is exponentially decaying $\varepsilon(j) \asymp e^{-\lambda j}$, then combining $B(j) = bj$ with (2) yields

$$b \;\geq\; d\,\lambda, \tag{3}$$

with equality $b = d\lambda$ being the critical (tight) regime. In that regime, solving $B(j) \approx H_\varepsilon(A)$ gives the step complexity

$$j_\varepsilon \;\asymp\; \frac{d}{\lambda} \log \frac{1}{\varepsilon} - \frac{\beta}{\lambda} \log\log \frac{1}{\varepsilon} + O(1).$$

6.2 Adaptive Schedules

For general $B(j)$ we can plug into the inversion formula.

Curriculum / log-boost schedule. If

$$b_t = \lambda_0 + \gamma \log(t+1),$$

then

$$B(j) = \lambda_0 j + \gamma \log((j+1)!) = \lambda_0 j + \gamma\big(j \log j - j + O(\log j)\big),$$

hence

$$\log \frac{1}{\varepsilon(j)} \approx \frac{\lambda_0}{d}\, j + \frac{\gamma}{d}\, j \log j - \frac{\beta}{d} \log j + O(1).$$

Past a characteristic $j_c \approx e^{\lambda_0/\gamma}$, the $j \log j$ term dominates and refinement accelerates.

Multigrid / burst schedule. If capacity bursts of weight Δw occur at times $t = 2^k$ on top of a baseline λ_0, we have

$$B(j) \approx \lambda_0 j + \Delta w\, \lfloor \log_2 j \rfloor,$$

so

$$\log \frac{1}{\varepsilon(j)} \approx \frac{\lambda_0}{d}\, j + \frac{\Delta w}{d} \lfloor \log_2 j \rfloor - \frac{\beta}{d} \log j + O(1),$$

giving a nearly-linear trend in j with log-periodic steps.

6.3 Drift Cost and Multiset Allocation

Let $T : A \to B$ be a measurable transport with resolution distortion $\Gamma_T(\varepsilon)$ such that achieving scale ε on A implies scale roughly $\varepsilon/\Gamma_T(\varepsilon)$ on B.

Proposition 2 (Incremental drift cost)**.** *At resolution $\varepsilon(j)$ for A, the extra number of steps required to maintain equivalent coverage on B satisfies*

$$\Delta j \ \gtrsim\ \frac{1}{b}\, \log \Gamma_T(\varepsilon(j)),$$

where b is a local average of b_t near j.

Now let $\{A_k\}$ be disjoint targets with

$$N(A_k, \varepsilon) \sim c_k\, \varepsilon^{-d_k} \big(\log(1/\varepsilon)\big)^{\beta_k},$$

and each sub-generator has refinement rate λ_k and receives j_k steps. For a relaxed coverage surrogate

$$\min_{j_k \ge 0} \sum_k j_k \quad \text{s.t.} \quad \sum_k \alpha_k \exp(d_k \lambda_k j_k) \ \ge\ T,$$

KKT conditions give

$$j_k^\star = \frac{1}{d_k \lambda_k} \big(\log \eta - \log \alpha_k\big)_+,$$

so cheaper arms (larger $d_k \lambda_k$) get more steps; in practice, one can use these to rank targets and then focus greedily on the best arm.

7 CEC Layer: From Counting to Measure

Definition 3 (CEC cardinality at scale)**.** *For $\varepsilon > 0$, define*

$$|A|_{\mathrm{CEC}}(\varepsilon) := \frac{N(A, \varepsilon)}{\varepsilon^{-n}} = c\, \varepsilon^{-(d-n)} \left(\log \frac{1}{\varepsilon}\right)^{\beta} (1 + o(1)).$$

Remark 2. For a compact C^1 submanifold $M^m \subset \mathbb{R}^n$ with smooth volume density, we expect $d = m$, $\beta = 0$, and $|M|_{\mathrm{CEC}}(\varepsilon) \to c$, where c encodes density and curvature corrections.

8 Generalized Asymptotic Dimension Spectrum

Power-tower and other iterated-exponential geometries break the single-log ADS model. We therefore allow a more general spectrum.

Definition 4 (Generalized ADS)**.** *In settings with more exotic scaling, we consider:*

1. ***Iterated-log expansion:***

$$H_\varepsilon(A) = dL + \sum_{k=1}^{m} \beta_k \log^{(k)} L + \log c + o(1),$$

where $L = \log(1/\varepsilon)$ *and* $\log^{(k)}$ *is the* k*-fold iterated logarithm.*

2. ***Stretched-exponential law:***

$$H_\varepsilon(A) = \alpha L^\gamma + \log c + o(1), \qquad 0 < \gamma < 1.$$

8.1 Corrected Inversion Formulas

Iterated-log case. From

$$B(j) \approx dL + \sum_{k=1}^{m} \beta_k \log^{(k)} L + \log c,$$

we obtain the leading-order inversion

$$L(j) \approx \frac{B(j)}{d} - \frac{\beta_1}{d} \log B(j) - \frac{\beta_2}{d} \log\log B(j) + O(1),$$

and hence $\varepsilon(j) = \exp(-L(j))$.

Stretched-exponential case. From

$$B(j) \approx \alpha L^\gamma + \log c,$$

we get

$$L(j) \approx \left(\frac{B(j)}{\alpha}\right)^{1/\gamma}, \qquad \varepsilon(j) \approx \exp\left\{-\left(\frac{B(j)}{\alpha}\right)^{1/\gamma}\right\}.$$

8.2 Empirical Model Selection

In practice, one fits $\log N(A, \varepsilon)$ across scales to:

- ADS (single (d, β));
- generalized ADS with iterated logs;
- stretched-exponential.

A simple Bayesian Information Criterion

$$\mathrm{BIC} = k \log n - 2 \log \hat{L}$$

(k = number of parameters, n = number of scales, $\hat{L}$ = maximized likelihood) helps avoid overfitting when choosing between these models.

9 Worked Examples

Example 1 (Middle-third Cantor set). Let $A = C \subset [0, 1]$ be the classical middle-third Cantor set. Then

$$d = \frac{\log 2}{\log 3}, \qquad \beta = 0, \qquad N(C, \varepsilon) \asymp \varepsilon^{-d}.$$

For constant $b_t \equiv b$ and exponential refinement $\varepsilon(j) \asymp e^{-\lambda j}$, the constant-rate law (3) gives $b \geq d\lambda$; at criticality,

$$j_\varepsilon \asymp \frac{d}{\lambda} \log \frac{1}{\varepsilon} + O(1).$$

Example 2 (Compact C^1 manifold). Let $M^m \subset \mathbb{R}^n$ be a compact C^1 submanifold with non-vanishing volume density. Then $d = m$, $\beta = 0$, c proportional to m-volume and curvature corrections, and

$$\log N(M, \varepsilon) \sim m \log \frac{1}{\varepsilon} + \log c.$$

From the inversion, for any budget $B(j)$,

$$\log \frac{1}{\varepsilon(j)} \approx \frac{B(j)}{m} + O(1),$$

so resolution scales linearly with budget with slope $1/m$.

Example 3 (Curve of limits / power-tower scaling). Consider

$$A = \{(a, y) : y = a^y,\ a_0 \leq a \leq a_1\},$$

with $0 < a_0 < a_1 < 1/e$. The geometry is effectively one-dimensional: $(d, \beta) = (1, 0)$, but $H_\varepsilon(A)$ may exhibit power-tower-influenced corrections at fine scales. In the simple ADS surrogate, $d = 1$, and for constant budget $B(j) = bj$ we get

$$\varepsilon(j) \approx e^{-B(j)},$$

while more refined behavior can be captured via generalized ADS if needed.

Example 4 (Orbit set / countable A). If A is a countable orbit $\{x_k\}$ of a discrete dynamical system, then $d = 0$ and ADS is degenerate. We then:

- report $d = 0$ for the static CEC·CID layer; and
- switch to listing or parametric analysis of generative cost per point (see the parametric section below).

10 Parametric CEC·CID for Degenerate ($d = 0$) Cases

The static (d, β, c) signature is powerful for $d > 0$ but collapses for zero-dimensional sets: a single point, a sparse sequence, and a complicated discrete spectrum all share $d = 0$ and often $\beta = 0$. To recover discriminative power we promote the framework to a parameterized family of invariants rooted in thermodynamic formalism and transfer-operator spectral theory.

10.1 Parametrization Scaffold

The minimal parametric scaffold uses three components.

Metric family (d_ρ). We replace the standard Euclidean metric by a log-cylinder metric that magnifies fine-scale differences. For integers $m, n \geq 1$,

$$d_\rho(m, n) = \Big|\log m - \log n\Big|^\rho, \qquad \rho > 0.$$

This makes multiplicative gaps salient and is well-suited to growth-complexity questions even when $d = 0$.

Potential φ_t. We endow each primitive element or cycle γ of the generator with a cost $\mathrm{cost}(\gamma)$ (information length, energy, time, or a composite), and define a one-parameter family of potentials

$$\varphi_t(\gamma) = -t \cdot \mathrm{cost}(\gamma), \qquad t \in \mathbb{R},$$

where t plays the role of inverse temperature or inverse complexity.

Transfer operator L_t. We construct a Ruelle transfer operator family $(L_t)_{t\in\mathbb{R}}$ acting on a suitable function space over the code space. For f and a state x,

$$(L_t f)(x) = \sum_{\gamma:\gamma x \text{ admissible}} e^{\varphi_t(\gamma)} f(\gamma x).$$

This operator counts weighted primitive completions and encodes the combinatorics of the generator.

10.2 Parametric Invariants

Definition 5 (Parametric CEC·CID signature)**.** *Given a generator G for a set A with suspected $d = 0$ in the static layer, the parametric signature is*

$$\mathrm{Sig}_{\mathrm{Param}}(G, A) = \Big\{ t^*,\ \{D_q\}_{q\in\mathbb{R}},\ (\underline{\gamma}, \overline{\gamma}),\ \mathrm{spec}(L_t) \Big\},$$

where:

- t^* *is the pressure root (thermodynamic dimension);*
- $\{D_q\}_{q\in\mathbb{R}}$ *is the Rényi entropy dimension spectrum;*
- $(\underline{\gamma}, \overline{\gamma})$ *are Matuszewska indices for slowly varying parts;*
- $\mathrm{spec}(L_t)$ *is the spectrum (eigenvalues, resonances) of L_t near t^*.*

More concretely:

Pressure root t^*. Let $P(t) = \log \rho(L_t)$, where ρ is the spectral radius. The Bowen parameter t^* is the solution of

$$P(t^*) = 0.$$

This t^* acts as an effective dimension replacing d in the degenerate regime.

Rényi entropy spectrum D_q. Let μ be the Gibbs measure at t^* and $\{B_i\}$ an ε-cover. Define

$$D_q = \frac{1}{q-1} \lim_{\varepsilon\to 0} \frac{\log \sum_i \mu(B_i)^q}{\log \varepsilon}.$$

The shape of the curve $q \mapsto D_q$ distinguishes monofractal from multifractal behavior even among sets with $d = 0$.

Matuszewska indices $(\underline{\gamma}, \overline{\gamma})$. If $N(A, \varepsilon) \sim L(1/\varepsilon)$ with L slowly varying, the Matuszewska indices quantify the lower and upper growth exponents of $L(x)$, separating, for example, logarithmic vs. iterated-log growth. All such variations are collapsed by $d = 0$ in the static layer.

Operator spectrum. Eigenvalues and resonances of L_t near t^* provide a fine-grained fingerprint of cyclic and transient structures of the generative process, including arithmetic periodicities invisible to (1).

10.3 Why This Resolves the $d = 0$ Problem

For $d = 0$, the static (d, β, c) has essentially no resolution: a single point, a sparse sequence of primes, and a Martin–Löf random subset of $\mathbb{N}$ can all map to $(0, 0, c)$. In contrast:

- t^* can be non-zero and vary continuously with complexity;
- the Rényi spectrum D_q can be non-constant, revealing multifractality;
- Matuszewska indices discriminate different flavors of sub-polynomial growth;
- $\mathrm{spec}(L_t)$ encodes arithmetic and dynamical structure of primitive concatenations.

10.4 Minimal Implementation Recipe

For a generator G and set A with suspected $d = 0$:

1. **Identify primitives.** Decompose G into irreducible elements or primitive cycles γ.
2. **Define cost and potential.** Assign a cost $\mathrm{cost}(\gamma)$ and set $\varphi_t(\gamma) = -t \cdot \mathrm{cost}(\gamma)$.
3. **Construct L_t.** Build a matrix or operator representation of L_t on the primitive space.
4. **Compute pressure and root.** Estimate $\rho(L_t)$, form $P(t) = \log \rho(L_t)$, and solve $P(t^*) = 0$.
5. **Calculate further invariants.** From the Gibbs measure at t^*: compute D_q; estimate $(\underline{\gamma}, \overline{\gamma})$ from fluctuations of $N(A, \varepsilon)$; and obtain leading eigenvalues of L_t near t^*.
6. **Report the parametric signature.** Present $\{t^*, \{D_q\}, (\underline{\gamma}, \overline{\gamma}), \mathrm{spec}(L_t)\}$ as the refined invariant.

10.5 Connection to Zeta Functions

When the primitive structure is multiplicative, we can define a CEC zeta function:

$$\zeta_{\mathrm{CEC}}(s) = \prod_{\gamma \text{ primitive}} \left(1 - e^{-s \cdot \mathrm{cost}(\gamma)}\right)^{-1} = \det(I - L_s)^{-1}.$$

The leading pole is at $s = t^*$; other poles and zeros encode resonance structure. For classical zeta-type systems this reduces to familiar Dirichlet/Euler products, but here the construction is anchored in the generative cost of G.

10.6 Worked Example: Sparse Binary Sequence

Example 5 (Sparse binary set). Let

$$A = \{2^{-n} : n \in S\},$$

where $S \subset \mathbb{N}$ is a sparse subset. We have $d_H(A) = 0$.

- **Static CEC·CID:** we obtain $(d = 0, \beta = 0, c = |S|)$ at best, which fails to distinguish different complexities of S.
- **Parametric layer:**
  - *Primitives:* bits deciding membership in S.
  - *Potential:* φ_t weighting each bit by its algorithmic cost (e.g., $-t$ per bit).
  - *Transfer operator:* L_t becomes an operator counting admissible binary strings. The pressure $P(t)$ and its root t^* vary with the complexity class of S: e.g., t^* smaller for a computable S, larger for a Martin–Löf random S.
  - *Rényi spectrum & indices:* the shape of D_q and the Matuszewska indices distinguish Salem–Spencer-type sets from primes or random sets.

11 Experimental Protocols and Reproducibility

The framework is intended to be empirically testable. A typical protocol logs two CSVs:

- `coverings.csv` with columns $(\varepsilon, \hat{N}, \text{set}, \text{generator})$, where $\hat{N}$ is the greedy ε-net size;
- `budget_trace.csv` with $(j, B(j), \varepsilon(j), \text{generator})$.

11.1 Drift Cost Validation

Setup.

- Target A: middle-third Cantor set with $(d, \beta, c) = (\log 2/\log 3, 0, 1)$.
- Transport $T(x) = x^p$ on $[0,1]$ with $p \in \{1.5, 2, 3, 5\}$; Lipschitz distortion $\Gamma_T = p$.
- Budgets: constant $b \in \{\log 2, 1\}$ nats/step.
- Target scales: $\varepsilon \in \{10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}\}$.

Prediction vs. measurement. The predicted extra steps are

$$\Delta j_{\text{pred}} = \frac{d \log \Gamma_T}{b}.$$

Measured extra steps come from the thresholds

$$\Delta j_{\text{meas}} = \left\lceil \frac{d \log(1/\varepsilon) + d \log \Gamma_T + \log c}{b} \right\rceil - \left\lceil \frac{d \log(1/\varepsilon) + \log c}{b} \right\rceil.$$

In experiments, $(\Delta j_{\text{meas}}, \Delta j_{\text{pred}})$ lies on the $y = x$ diagonal up to ± 1 due to ceiling effects, confirming tightness of the lower bound in practice.

11.2 Multiset Allocation Validation

Setup.

- A_1: Cantor-like set with $d_1 = \log 2/\log 3$, refinement rate $v_1 = 1$.
- A_2: unit interval with $d_2 = 1$, refinement rate $v_2 = 0.8$.
- Exponents $a_k = d_k v_k$, weights α_k, global surrogate target T.

The continuous KKT solution yields

$$j_k^\star = \frac{1}{a_k}(\log \eta - \log \alpha_k)_+,$$

and the empirical allocation minimizing total steps for a fixed coverage target matches this up to integer rounding, validating the allocation law.

11.3 Image Manifold / Medical Dataset Use Case (Blueprint)

For an embedded image manifold (e.g., MNIST or brain MRI slices embedded into $\mathbb{R}^d$ by a fixed encoder):

1. Choose scales ε and compute greedy ε-nets to estimate $N(A, \varepsilon)$.
2. Regress $\log N(A, \varepsilon)$ on $\log(1/\varepsilon)$ and $\log\log(1/\varepsilon)$ to estimate (d, β, c), and test generalized ADS if residuals show structure.
3. Implement a generator G (grid-based, diffusion-like, or GAN-style) and log $B(j)$, $\varepsilon(j)$.
4. Compare the observed budget–resolution curve to the predicted laws; diagnose inefficiencies (slack vs. the CEC·CID lower bound) and drift costs under data augmentations.

For cancer datasets and synthetic augmentation pipelines, the same protocol applies, with A representing the latent manifold of healthy and tumorous images, and G representing a generative model G_1 or G_2. The CEC·CID parameters and measured efficiency curves provide a principled way to compare "quality per budget" across models.

12 Summary and Glossary

Ready-to-use laws.

- Covering/entropy:
$$N(A, \varepsilon) \sim c\,\varepsilon^{-d}(\log(1/\varepsilon))^{\beta}, \quad H_\varepsilon \sim d\log(1/\varepsilon) + \beta\log\log(1/\varepsilon).$$
- Capacity lower bound: $B(j) \geq H_{\varepsilon(j)}(A)$.
- Constant-rate efficiency: $b \geq d\lambda,$ critical if $b = d\lambda$.
- Step complexity (critical regime): $j_\varepsilon \asymp \frac{d}{\lambda}\log(1/\varepsilon) - \frac{\beta}{\lambda}\log\log(1/\varepsilon) + O(1)$.
- Adaptive inversion: $\log(1/\varepsilon(j)) = B(j)/d - (\beta/d)\log(B(j)/d) + O(1)$ with Lambert–W refinement.
- Completeness: if $B(j) \geq H_{\varepsilon(j)} + \omega(1)$ and reachability/mixing hold, then $\kappa(j) \to 1$.
- Drift cost: $\Delta j \gtrsim b^{-1}\log\Gamma_T(\varepsilon(j))$.
- Union allocation: $j_k^\star \propto (d_k\lambda_k)^{-1}$ for normalized exponents.

Minimal glossary.

- d (CID): intrinsic dimension proxy (Minkowski/Hausdorff when stable).
- β: logarithmic correction (arithmetic structure, oscillations).
- c: normalization (density / volume constant).
- λ: refinement rate linked to $B(j)$; for $B(j) = bj$, critical regime $b = d\lambda$.
- $B(j)$: cumulative information budget; controls phase transitions.
- t^*: pressure root (thermodynamic dimension) in the parametric layer.
- D_q: Rényi entropy spectrum of the Gibbs measure.
- $(\underline{\gamma}, \overline{\gamma})$: Matuszewska indices of slow variation.
- $\zeta_{\text{CEC}}(s)$: CEC zeta function $\prod_\gamma(1 - e^{-s\,\text{cost}(\gamma)})^{-1}$.