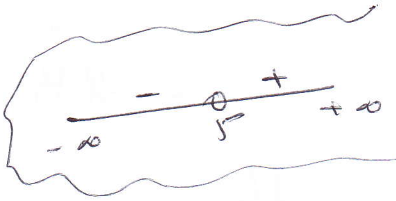
Ex3/TD2 : X : v.a.r.c | f_X : sa fct de densité.

$$f_X(x) = K e^{-|x-5|} \quad , \quad x \in \mathbb{R}$$

• f_X : fct de densité $\Rightarrow \begin{cases} f_X(x) \geq 0 \equiv K \geq 0 \\ \int_{\mathbb{R}} f_X(x)\, dx = 1 \end{cases}$

• $$|x-5| = \begin{cases} x-5 & \text{si } x \geq 5 \\ -x+5 & \text{si } x < 5 \end{cases}$$



Donc :

$$f_X(x) = \begin{cases} K e^{-(-x+5)} & \text{si } x < 5 \\ K e^{-(x-5)} & \text{si } x \geq 5 \end{cases}$$

$$\Rightarrow f_X(x) = \begin{cases} K e^{x-5} & \text{si } x < 5 \\ K e^{-x+5} & \text{si } x \geq 5 \end{cases}$$

① Determinons la constante K :

$$\int_{\mathbb{R}} f_X(x)\, dx = \int_{-\infty}^{+\infty} f_X(x)\, dx = \int_{-\infty}^{+\infty} K e^{-|x-5|} dx = 1$$

$$= \int_{-\infty}^{5} K e^{x-5} dx + \int_{5}^{+\infty} K e^{-x+5} dx = 1$$

$$= K \int_{-\infty}^{5} e^{x-5} dx + K \int_{5}^{+\infty} e^{-x+5} dx = 1$$

$$= K \int_{-\infty}^{5} e^{x} e^{-5} dx + K \int_{5}^{+\infty} e^{-x} . e^{5} dx = 1$$

$$= K e^{-5} \int_{-\infty}^{5} e^{x} dx + K e^{5} \int_{5}^{+\infty} e^{-x} dx = 1$$

$$= K . e^{-5} \left[e^{x} \right]_{-\infty}^{5} + K e^{5} \left[-e^{-x} \right]_{5}^{+\infty} = 1$$

$$\int_{\mathbb{R}} f_X(x)\,dx = Ke^{-5}\Big(e^{5} - \underbrace{\lim_{x\to -\infty} e^{x}}_{0}\Big) + Ke^{5}\Big(\underbrace{\lim_{x\to +\infty} -e^{-x}}_{0} - (-e^{-5})\Big) = 1$$

$$= Ke^{-5}e^{5} + Ke^{5}e^{-5} = 1$$

$$= Ke^{-5+5} + Ke^{5-5} = 1$$

$$= K\underbrace{e^{0}}_{1} + K\underbrace{e^{0}}_{1} = 1$$

$$= 2K = 1 \Rightarrow \boxed{K = \frac{1}{2}}$$

Alors :

$$\boxed{f_X(x) = \begin{cases} \frac{1}{2}e^{x-5} & \text{si } x < 5 \\ \frac{1}{2}e^{-x+5} & \text{si } x \geq 5 \end{cases}}$$

② Calculons la fct de répartition F_X de X :

$$F_X(x) = \int_{-\infty}^{x} f(t)\,dt$$

1ier cas si $x < 5$:

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{2}e^{t-5}\,dt$$

$$= \frac{1}{2}\int_{-\infty}^{x} e^{t-5}\,dt = \frac{1}{2}\int_{-\infty}^{x} e^{t}\cdot e^{-5}\,dt$$

$$= \frac{1}{2}e^{-5}\int_{-\infty}^{x} e^{t}\,dt$$

$$= \frac{1}{2}e^{-5}\left[e^{t}\right]_{-\infty}^{x}$$

$$= \frac{1}{2}e^{-5}\Big(e^{x} - \underbrace{\lim_{t\to -\infty} e^{t}}_{0}\Big)$$

$$= \frac{1}{2}e^{-5}e^{x} = \boxed{\frac{1}{2}e^{x-5}}$$

2ième cas si $x \geq 5$:

$$F_X(x) = \int_{-\infty}^{5} \frac{1}{2}e^{t-5}\,dt + \int_{5}^{x} \frac{1}{2}e^{-t+5}\,dt$$

$$= \frac{1}{2}\int_{-\infty}^{5} e^{t}\cdot e^{-5}\,dt + \frac{1}{2}\int_{5}^{x} e^{-t}\cdot e^{5}\,dt$$

$$= \frac{1}{2}e^{-5}\int_{-\infty}^{5} e^{t}\,dt + \frac{1}{2}e^{5}\int_{5}^{x} e^{-t}\,dt$$

$$= \frac{1}{2}e^{-5}\left[e^{t}\right]_{-\infty}^{5} + \frac{1}{2}e^{5}\left[-e^{-t}\right]_{5}^{x}$$

$$= \frac{1}{2}e^{-5}\Big(e^{5} - \underbrace{\lim_{t\to -\infty} e^{t}}_{0}\Big) + \frac{1}{2}e^{5}\left(-e^{-x} + e^{-5}\right)$$

$$= \frac{1}{2}\underbrace{e^{-5}e^{5}}_{1} + \frac{1}{2}\underbrace{e^{5}e^{-5}}_{1} - \frac{1}{2}e^{5}e^{-x}$$

$$= \underbrace{\frac{1}{2} + \frac{1}{2}}_{} - \frac{1}{2}e^{5}e^{-x}$$

$$= \boxed{1 - \frac{1}{2}e^{-x+5}}$$

Alors: $F_X(x)=\begin{cases}\frac{1}{2}e^{x-5} & \text{si } x<5\\ 1-\frac{1}{2}e^{5-x} & \text{si } x\geq 5\end{cases}$

③ Calculons l'espérance Math (Moyenne):

$$E(X)=\int_{-\infty}^{+\infty} x f(x)\,dx=\int_{-\infty}^{5}\frac{1}{2}x e^{x-5}\,dx+\int_{5}^{+\infty}\frac{1}{2}x e^{-x+5}\,dx$$

$$=\frac{1}{2}\underbrace{\int_{-\infty}^{5} x e^{x-5}\,dx}_{I_1}+\frac{1}{2}\underbrace{\int_{5}^{+\infty} x e^{-x+5}\,dx}_{I_2}$$

$$=\frac{1}{2}I_1+\frac{1}{2}I_2.$$

<u>I_1:</u> $I_1=\int_{-\infty}^{5} x e^{x-5}\,dx$ (Intégration Par Partie).

$\begin{cases}u(x)=x\\ v'(x)=e^{x-5}\end{cases}$ $\begin{cases}u'(x)=1\\ v(x)=e^{x-5}\end{cases}$

$\int e^{x-5}dx=\int e^{x}e^{-5}$
$=e^{-5}\int e^{x}dx=e^{-5}e^{x}+$
$=\boxed{e^{x-5}+c}$

$$I_1=\left[x e^{x-5}\right]_{-\infty}^{5}-\int_{-\infty}^{5}1.\,e^{x-5}\,dx$$

$$=\underbrace{\Big[(5\underbrace{e^{5-5}}_{1})-\underbrace{\lim_{x\to-\infty}xe^{x-5}}_{0}\Big]}_{=5}-\left[e^{x-5}\right]_{-\infty}^{5}$$

$$=5-\Big(\underbrace{e^{5-5}}_{1}-\underbrace{\lim_{x\to-\infty}e^{x-5}}_{0}\Big)$$

$$I_1=5-1=\boxed{4}$$

<u>I_2:</u> $I_2=\int_{5}^{+\infty} x e^{-x+5}\,dx$ (Intégration Par Partie).

$\begin{cases}u(x)=x\\ v'(x)=e^{-x+5}\end{cases}$ $\begin{cases}u'(x)=1\\ v(x)=-e^{-x+5}\end{cases}$

$\int e^{-x+5}dx=\int e^{-x}.e^{5}d$
$=e^{5}\int e^{-x}dx=e^{5}(-e^{-x})$
$=\boxed{-e^{-x+5}+c}$

$$I_2 = \left[-x^{-x+5}\right]_0^{+\infty} - \int_0^{+\infty} -e^{-x+5}\,dx = \left[-xe^{-x+5}\right]_5^{+\infty} + \int_5^{+\infty} e^{-x+5}\,dx$$

$$= \left[\lim_{x\to+\infty}(-xe^{-x+5}) - (-5e^{-5+5})\right] + \left[-e^{-x+5}\right]_5^{+\infty}$$

$$= \underbrace{\left[\;0 \;-\; 5\underbrace{e^{0}}_{=1}\;\right]}_{5} + \underbrace{\left[\left(\underbrace{\lim_{x\to+\infty}(-e^{-x+5})}_{=0}\right) - \left(\underbrace{-e^{-5+5}}_{=-1}\right)\right]}_{1}$$

$$= 5 + 1 = \boxed{6}$$

Donc : $E(X) = \frac{1}{2}(4) + \frac{1}{2}(6) = 2 + 3 = \boxed{5}$

<u>Calculons la Variance :</u>

$$Var(X) = E(X^2) - (E(X))^2$$

$$= \underbrace{\int_{-\infty}^{+\infty} x^2 f(x)\,dx}_{?} - \underbrace{(E(X))^2}_{\checkmark}$$

$$\int_{-\infty}^{+\infty} x^2 f(x)\,dx = \int_{-\infty}^{5} x^2 \frac{1}{2} e^{x-5}\,dx + \int_5^{+\infty} x^2 \frac{1}{2} e^{-x+5}\,dx$$

$$= \frac{1}{2}\underbrace{\int_{-\infty}^{5} x^2 e^{x-5}\,dx}_{=J_1} + \frac{1}{2}\underbrace{\int_5^{+\infty} x^2 e^{-x+5}\,dx}_{=J_2}$$

<u>J_1 :</u>

$J_1 = \int_{-\infty}^{5} x^2 e^{x-5}\,dx \Longrightarrow$ (Intégration Par Partie)

$U(x) = x^2 \qquad U'(x) = 2x$

$V'(x) = e^{x-5} \qquad V(x) = e^{x-5}$

$$J_1 = \left[x^2 e^{x-5}\right]_{-\infty}^{5} - \int_{-\infty}^{5} 2x e^{x-5}\,dx$$

$$= \left[x^2 e^{x-5}\right]_{-\infty}^{5} - 2\int_{-\infty}^{5} x e^{x-5}\,dx \rightarrow \text{Déjà calculer } (I_1 \text{ ds le calcul de la Moy})$$

$$J_1 = \left[(25)\underbrace{(e^{5-5})}_{=1} - \underbrace{\lim_{x\to-\infty} x^2 e^{x-5}}_{=0}\right] - 2(4) = 25 - 8 = \boxed{17}$$

$\underline{I_2}$:

$$J_2 = \int_5^{+\infty} x^2 e^{-x+5}\, dx \Rightarrow \text{(Integration Par Partie)}.$$

$$U(x) = x^2 \qquad U'(x) = 2x$$
$$V'(x) = e^{-x+5} \qquad V(x) = -e^{-x+5}$$

$$J_2 = \left[x^2\left(-e^{-x+5}\right)\right]_5^{+\infty} - \int_5^{+\infty} 2x\left(-e^{-x+5}\right) dx$$

$$= \left[x^2\left(-e^{-x+5}\right)\right]_5^{+\infty} - 2\int_5^{+\infty} x\left(-e^{-x+5}\right) dx$$

$$= \left[x^2\left(-e^{-x+5}\right)\right]_5^{+\infty} + 2\int_5^{+\infty} x\, e^{-x+5}\, dx$$

Déjà calculer (I_2 ds calcul de Moy

$$= \left[\underbrace{\lim_{x\to+\infty} x^2\left(-e^{-x+5}\right)}_{=0} - (5)^2\left(\underbrace{-e^{-5+5}}_{=-1}\right)\right] + 2\,(6)$$

$$= 0 + 25 + 12 = \boxed{37}$$

Alors :

$$E(X^2) = \frac{1}{2}(17) + \frac{1}{2}37 = \frac{54}{2} = \boxed{27}$$

$$Var(X) = E(X^2) - \left(E(X)\right)^2$$
$$= 27 - (5)^2 = 27 - 25 = \boxed{2}.$$