1: For each of the following Banach spaces X, compute $\mathrm{Ext}(\overline{b_1(X)})$ and compute the *norm closed* convex hull of these extreme points.

(a) $X = C([0,1],\mathbb{R})$. Is this a dual space?

(b) $X = C([0,1],\mathbb{C})$.

(c) $X = \ell^p$, $1 < p < \infty$. **Hint:** Think about the case of equality in Minkowski's inequality.

2: It can be shown (Problem 2.7) that every $T \in B(\ell^1)$ has a matrix $[t_{ij}]$ such that the columns are uniformly bounded (in ℓ^1). Show that $T = S^*$ for some $S \in B(c_0)$ if and only if the rows of the matrix are in c_0.

3: Let V be a bounded operator on a Hilbert space $\mathcal{H}$.

(a) Show that V is an isometry if and only if $V^*V = I$.

(b) Show that the following are equivalent: (i) V is unitary, (ii) $V^* = V^{-1}$, and iii) V is an isometry and $VV^* = V^*V$.

4: If $A, B, C, D \in B(\mathcal{H})$, then

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is a bounded operator on the Hilbert space $\mathcal{H} \oplus \mathcal{H}$ given by

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Ax + By \\ Cx + Dy \end{pmatrix}.$$

(a) Find T^* (the Hilbert space adjoint).

(b) If $B = C = 0$, find $\|T\|$.

(c) If A is invertible, show that T is invertible if and only if $D - CA^{-1}B$ is invertible.

Hint: Find X and Y to factor

$$T = \begin{pmatrix} A & 0 \\ C & I \end{pmatrix} \begin{pmatrix} I & Y \\ 0 & X \end{pmatrix}.$$

5:

(a) If $E = E^2 \in B(X)$ is compact, show that E is finite rank.

(b) Show that if $f \in C[0,1]$, then the multiplication operator M_f on $L^2(0,1)$ given by $(M_f\xi)(x) = f(x)\xi(x)$ is not compact unless $f = 0$.

6: Let V be the Volterra operator on $L^2(0,1)$,

$$Vf(x) = \int_0^x f(t)\,dt.$$

(a) Express V^* as an integral operator.

(b) Suppose that f is an eigenvector of VV^*. Show that f is C^∞ and satisfies a second order ODE with boundary conditions.

(c) Hence diagonalize VV^* and compute $\|V\|$.