1: Let X, Y be Banach spaces and consider the tensor product vector space $X \otimes Y$. (For a refresher/introduction to tensor products of vector spaces, see e.g. `https://en.wikipedia.org/wiki/Tensor_product)` .

In general, there are many distinct norms that one can place on $X \otimes Y$, yielding very different Banach space completions of $X \otimes Y$. We shall call a norm $\|\cdot\|_\alpha$ on $X \otimes Y$ a *cross-norm* if

$$\|x \otimes y\|_\alpha = \|x\|_X \|y\|_Y \quad (x \in X, y \in Y).$$

Define $\|\cdot\|_\gamma : X \otimes Y \to [0, \infty)$ by

$$\|u\|_\gamma = \inf \left\{ \sum_{i=1}^n \|x_i\|_X \|y_i\|_Y : u = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y \right\}.$$

(a) Show that $\|\cdot\|_\gamma$ defines a cross-norm on $X \otimes Y$ (called the *projective tensor norm*). The resulting Banach space $X \otimes^\gamma Y := \overline{X \otimes Y}^{\|\cdot\|_\gamma}$ is called the projective tensor product of X and Y. **Hint:** Consider the natural linear map $X \otimes Y \to B(X^*, Y)$ given by

$$u = \sum_i x_i \otimes y_i \mapsto \tilde{u} \in B(X^*, Y); \quad \tilde{u}(\phi) = \sum_i \phi(x_i) y_i \quad \phi \in X^*.$$

Show that $u \mapsto \tilde{u}$ is well-defined, injective, and use the Hahn-Banach theorem to show that $\|\tilde{u}\|_{B(X^*,Y)} \le \|u\|_\gamma$.

(b) Show that $\|\cdot\|_\gamma$ is the largest cross norm on $X \otimes Y$: $\|u\|_\alpha \le \|u\|_\gamma$ for any other cross norm $\|\cdot\|_\alpha$ on $X \otimes Y$.

(c) Show that the map

$$\delta_i \otimes \delta_j \mapsto \delta_{(i,j)}, \quad i, j \in \mathbb{N}$$

extends continuously to an isometric isomorphism $\ell^1(\mathbb{N}) \otimes^\gamma \ell^1(\mathbb{N}) \cong \ell^1(\mathbb{N} \times \mathbb{N})$. Here δ_i is the usual indicator function of $i \in \mathbb{N}$.

2: Consider the Banach space $\ell^1(\mathbb{Z}) = \{f : \mathbb{Z} \to \mathbb{C} : \|f\|_1 = \sum_n |f(n)| < \infty\}$.

(a) Prove that $\ell^1(\mathbb{Z})$ is a commutative associative unital algebra when equipped with the *convolution product* $\ell^1(\mathbb{Z}) \times \ell^1(\mathbb{Z}) \ni (f, g) \mapsto f \star g \in \ell^1(\mathbb{Z})$ given by

$$(f \star g)(n) := \sum_{k \in \mathbb{Z}} f(k) g(n-k) \quad n \in \mathbb{Z}.$$

(b) Show that $\|f \star g\| \le \|f\|_1 \|g\|_1$. (i.e., $\ell^1(\mathbb{Z})$ is a *Banach algebra*). In particular, for each $g \in \ell^1(\mathbb{Z})$, the map $T_g(f) = f \star g$ belongs to $B(\ell^1(\mathbb{Z}))$. Prove that $\|T_g\| = \|g\|_1$.

(c) For $k \in \mathbb{Z}$, let $L_k : \ell^1(\mathbb{Z}) \to \ell^1(\mathbb{Z})$ be the translation operator $L_k f(n) = f(n-k)$. (L_k is clearly an invertible isometry). Let $S : \ell^1(\mathbb{Z}) \to \ell^1(\mathbb{Z})$ be a linear map such that $S(L_k f) = L_k(Sf)$ for all $f \in \ell^1(\mathbb{Z})$ and $k \in \mathbb{Z}$. Prove that S is bounded $\Leftrightarrow$ $S(f \star g) = S(f) \star g$ for all $f, g \in \ell^1(\mathbb{Z})$ $\Leftrightarrow$ there is a unique $h \in \ell^1(\mathbb{Z})$ such that $S = T_h$.

(d) Now consider the continuous analogue of the above setup: Consider $L^1(\mathbb{R})$ (with respect to Lebesgue measure). From measure theory (Fubini's theorem in particular), $L^1(\mathbb{R})$ is again a commutative Banach algebra when equipped with the convolution product

$$f \star g(x) = \int_{\mathbb{R}} f(y)g(x-y)\,dy.$$

Here $f, g \in L^1(\mathbb{R})$ and the above formula is understood as an almost everywhere equality. Note that $\|f \star g\|_1 \leq \|f\|_1 \|g\|_1$. Suppose $S : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ is a linear map such that $S(f \star g) = S(f) \star g$. Prove that S is automatically bounded. (In HW4 you'll see how to describe all such maps S.)

3: Let $T : H \to H$ be a linear map on a Hilbert space H such that $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in H$. Prove that $T \in B(H)$.

4: Let Y be a closed subspace of a Banach space X. Define $Y^{\perp} = \{\phi \in X^* : \phi(Y) = \{0\}\}$. Prove that there are isometric isomorphisms

$$Y^* \cong X^*/Y^{\perp} \quad \& \quad (X/Y)^* \cong Y^{\perp}.$$

5: Let $\phi \in C([0,1], \mathbb{R})^*$ be given by $\phi(f) = \int_0^1 f(t)\,dt$. Let $\tilde{\phi} \in B([0,1], \mathbb{R})^*$ be any Hahn-Banach extension of ϕ, where $B([0,1], \mathbb{R})$ is the Banach space of bounded functions $f : [0,1] \to \mathbb{R}$, equipped with the supremum norm. What is $\tilde{\phi}(\chi_{[0,0.5]})$? What are the possible values of $\tilde{\phi}(\chi_{\mathbb{Q}\cap[0,1]})$?

6: Let $Y, Z \subseteq X$ be closed subspaces of a Banach space X. We say Y, Z are *complementary subspaces* if $X = Y + Z$ and $Y \cap Z = \{0\}$.

(a) If Y, Z are complementary, show that there are Banach space isomorphisms $X \cong Y \oplus_1 Z$ and $Z \cong X/Y$.

(b) Show that Y has a complement Z iff there exists a bounded linear idempotent map $P = P^2 \in B(X)$ such that $P(X) = Y$ and $\ker P = Z$.

(c) Given a Banach space V denote by $\hat{V} = \{\hat{v} : v \in V\} \subseteq V^{**}$ the image V in V^{**} under the canonical isometric inclusion. Prove that there is continuous idempotent $P : X^{***} \to \widehat{X^*}$ given by

$$P\psi = \widehat{\psi|_X} \quad \psi \in X^{***}.$$

This shows that X^* is always complemented in X^{***}.