PMATH 446/646 Commutative Algebra, Assignment 1 Due Sept 14, 11:30 pm

**1:** True or False: Let $R$ be a ring, and $R \to Q$ is a surjective map of abelian groups. Is it true that $Q$ has an induced ring structure if and only if it has an induced $R$-module structure?

**2:** Short answer: Explain why the following modules are or are not free:

(a) $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ as a $\mathbb{Z}$-module.

(b) $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ as a $\mathbb{Z}/5\mathbb{Z}$-module.

(c) $\mathbb{C}$ as an $\mathbb{R}[x]$-module, with the action $x \cdot z = iz$.

(d) $\mathbb{C}[x]$ as a $\mathbb{C}[x^2]$-module by the usual multiplication.

**3:** Consider a polynomial $f(x) = f_0 x^n + \cdots + f_n \in R[x]$.

(a) Prove that the structure of an $R[x]/(f(x))$-module on an abelian group $M$ is equivalent to:

- An $R$-module structure and
- An $R$-module homomorphism $\phi : M \to M$ which satisfies $\sum_{i=0}^{n} f_i \phi^i(m) = 0$ for all $m \in M$.

(b) Prove that the structure of a $\mathbb{C}$-vector space on an abelian group $M$ is equivalent to the structure of an $\mathbb{R}$-vector space together with an $\mathbb{R}$-linear map $I : M \to M$ satisfying $I^2 = -1$.

(c) Recall that $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$. Prove that if $M = \mathbb{F}_2 \oplus \mathbb{F}_2$ is a faithful module over a commutative ring $R$ which we can write as a quotient of $\mathbb{Z}[x]$, then $R \cong \mathbb{F}_2$, $\mathbb{F}_2 \oplus \mathbb{F}_2$, $\mathbb{F}_2[x]/(x^2)$, or $\mathbb{F}_4$.

(d) Bonus: Prove Q3(3) for $R$ an arbitrary commutative ring.

**4:** Recall that we have defined a $\mathbb{C}[x]$-module $\mathbb{C} * A^n$ for any $n \times n$ matrix $A$ by the action $x \cdot v = Av$. (We'll abuse notation and let $A : \mathbb{C}^n \to \mathbb{C}^n$ denote the linear map of multiplication by $A$.)

(a) Show that the linear map $B : \mathbb{C} * A^n \to \mathbb{C} * A'^n$ defines a module homomorphism if and only if $A'B = BA$ as matrices.

(b) Show that if $B$ is invertible, then $B$ defines a module isomorphism if and only if $B^{-1}A'B = A$.

(c) Show that $A$ is diagonalizable if and only if we have an isomorphism of $\mathbb{C}[x]$-module $\mathbb{C} * A^n \cong \mathbb{C} * [z_1]^1 \oplus \cdots \oplus \mathbb{C} * [z_n]^1$ for $z_i \in \mathbb{C}$.