PMATH 453/753 Functional Analysis, Assignment 4 Due Nov 14, 11:55 pm

1: A *Schauder basis* for a Banach space X is a sequence $\{e_n : n \geq 1\}$ such that for each $x \in X$, there are unique scalars $\{c_n\}$ such that $x = \lim_{n\to\infty} \sum_{i=1}^{n} c_i e_i$. For convenience, normalize so that $\|e_n\| = 1$ for $n \geq 1$.

(a) Show that $\varphi_n(x) = c_n$ is a linear functional.

(b) Define $S_n x = \sum_{i=1}^{n} c_i e_i$, and set $|||x||| = \sup_{n\geq 1} \|S_n x\|$. Prove $|||\cdot|||$ is a norm.

(c) Show that $(X, |||\cdot|||)$ is complete.

(d) Prove that the identity map T from $(X, |||\cdot|||)$ to $(X, \|\cdot\|)$ is an isomorphism. Hence deduce that $\sup_{n\geq 1} \|S_n\| = \|T^{-1}\|$ and that each φ_n is continuous.

2: Let X be a Banach space and let Y be a closed subspace. Show that if Y and X/Y are reflexive, then X is reflexive.

3:

(a) Let X be a Banach space. If Y is a weak-$*$ closed subspace of X^*, let $Y_\perp = \{x \in X : f(x) = 0 \text{ for all } f \in Y\}$. Show that $Y = (Y_\perp)^\perp$.

(b) Hence show that Y is a dual space, and that the weak-$*$ topology on Y as this dual space coincides with the weak-$*$ topology of X^* restricted to Y.

4: Let $(X, \mathcal{P})$ and $(Y, \mathcal{Q})$ be LCTVSs, and let $T : X \to Y$ be a linear map. Prove that the following are equivalent:

(1) T is continuous.

(2) T is continuous at 0.

(3) For each $q \in \mathcal{Q}$, there is a finite subset $F = \{p_1, \ldots, p_n\} \subset \mathcal{P}$ and $t_i \in \mathbb{R}^+$ so that $q(Tx) \leq \sum_{i=1}^{n} t_i p_i(x)$ for all $x \in X$.

5: If $(x_\lambda)_\lambda$ is a net in a Banach space X which converges weakly to x, find a sequence of points in the convex hull of the net converging to x in norm.
HINT: Show that the convex hull of the net intersects $b_{1/n}(x)$.

6:

(a) Let X be a Banach space, and let Y be a closed subspace of X^* which norms X. Show that a *sequence* in X which converges in the τ_Y topology must be bounded.

(b) Let H be a Hilbert space. Show that if a *sequence* $T_n \in \mathcal{B}(H)$ converges to T in the weak operator topology, then $\{T_n\}$ is bounded.
Note: The set of functionals determining this topology is not closed, so part (a) doesn't apply.

7: Let X be a Banach space and let X^* denote its Banach space dual. Consider $\mathcal{B}(X)$ as a LCTVS with respect to its weak operator topology (WOT) and its strong operator topology (SOT).

(a) Let $\varphi \in (\mathcal{B}(X), \mathrm{SOT})^*$ be a SOT-continuous linear functional.
Prove that there exist $x_1, \dots, x_n \in X$ and $\varphi_1, \dots, \varphi_n \in X^*$ such that

$$\varphi(T) = \sum_{i=1}^{n} \varphi_i(Tx_i), \quad T \in \mathcal{B}(X).$$

Hint: First use SOT continuity to find $x_1, \dots x_n \in X$ such that

$$|\varphi(T)| \leq \sum_i \|Tx_i\|, \quad T \in \mathcal{B}(X).$$

Then consider the linear functional $\psi : Z \to \mathbb{F}$ defined on the subspace

$$Z = \{(Tx_1, Tx_2, \dots, Tx_n) \mid T \in \mathcal{B}(X)\} \subseteq \underbrace{X \oplus_1 \dots \oplus_1 X}_{n \text{ copies}}$$

and given by

$$\psi((Tx_1, Tx_2, \dots, Tx_n)) = \varphi(T).$$

Show ψ is well defined and bounded. Consider a Hahn-Banach extension to all of $X \oplus_1 \dots \oplus_1 X$.

(b) Conclude that $(\mathcal{B}(X), \mathrm{SOT})$ and $(\mathcal{B}(X), \mathrm{WOT})$ have the same linear continuous functionals, and hence they have the same closed convex sets.

(c) Let $M_z \in \mathcal{B}(L^2(\mathbb{T}))$ be given by

$$M_z = zf, \quad f \in L^2(\mathbb{T}),$$

where $z(e^{i\theta}) = e^{i\theta}$. Show that $M_z^n \to 0$ in the WOT, but not with respect to the SOT. So, in general, these two topologies differ even though they have the same continuous linear functionals.

8: (Warning: This problem uses some measure theory - feel free to use any results you like from PMATH 450-451 without proof. Just be clear about what you are using.) Consider the Banach space $C_0(\mathbb{R})$ equipped with the uniform norm $\|\cdot\|_\infty$. Recall from measure theory that the Riesz-Markov-Kakutani Representation (RMKR) theorem linearly identifies $C_0(\mathbb{R})^*$ with the $M(\mathbb{R})$, the vector space of complex Borel measures μ on $\mathbb{R}$. The pairing is given by

$$\langle f, \mu \rangle = \int_{\mathbb{R}} f(x) d\mu(x) \quad (f \in C_0(\mathbb{R}), \mu \in M(\mathbb{R})).$$

Note that μ is in general a *complex measure*, and the polar decomposition of complex measures always allows us to write a complex measure μ as $d\mu = hd|\mu|$ for a unique Borel measurable $h : \mathbb{R} \to \mathbb{C}$ satisfying $|h| = 1$ and a unique finite positive measure $|\mu|$ (called the total variation of μ). So $\int_{\mathbb{R}} f d\mu = \int_{\mathbb{R}} f h d|\mu|$, and the latter integral is the usual integral with respect to a positive measure. We also have

$$\|\mu\|_{M(G)} := |\mu|(\mathbb{R}) = \sup_{f \in C_0(\mathbb{R}), \|f\|_\infty = 1} |\langle f, \mu \rangle|,$$

so the identification is isometric. From now on we simply identify $M(\mathbb{R})$ with $C_0(\mathbb{R})^*$, and consider the weak-$*$ topology $\tau_{C_0(\mathbb{R})}$ on $M(\mathbb{R})$.

Let $\mu, \nu \in M(\mathbb{R})$. Their *convolution product* $\mu \star \nu \in M(\mathbb{R})$ is the unique complex Borel measure defined by the pairing

$$\langle f, \mu \star \nu \rangle := \int_{\mathbb{R}} \int_{\mathbb{R}} f(x + y) d\mu(x) d\nu(y) \quad (f \in C_0(\mathbb{R})).$$

The duality $M(\mathbb{R}) = C_0(\mathbb{R})^*$ guarantees that the above pairing uniquely determines a measure $\mu \star \nu \in M(\mathbb{R})$.

(a) Show that $\mu \star \nu = \nu \star \mu$ and $\|\mu \star \nu\| \leq \|\mu\|\|\nu\|$. So $M(\mathbb{R})$ is a Banach algebra.

(b) Let $f \in L^1(\mathbb{R})$. Consider the complex Borel measure μ_f given by

$$d\mu_f(x) = f(x)dx.$$

Show that $\|\mu_f\|_{M(\mathbb{R})} = \|f\|_1$. So we can identify $L^1(\mathbb{R}) \subset M(\mathbb{R})$ as a closed subspace via the isometric linear map $f \mapsto \mu_f$. This allows us to consider the weak-$*$ topology on $L^1(G)$ as well.

(c) Let $f, g \in L^1(\mathbb{R})$ and $\nu \in M(\mathbb{R})$. Show that $\mu_{f \star g} = \mu_f \star \mu_g$ (where $f \star g \in L^1(\mathbb{R})$ is the usual convolution) and $\nu \star \mu_f = \mu_{(\nu \star f)} \in L^1(\mathbb{R})$, where $\nu \star f \in L^1(\mathbb{R})$ is given (almost everywhere) by

$$(\nu \star f)(x) = \int_{\mathbb{R}} f(x - y) d\nu(y)$$

Thus $L^1(\mathbb{R})$ is a norm-closed ideal in $M(\mathbb{R})$.

(d) Suppose we have a net $\mu_\lambda \to \mu \in M(\mathbb{R})$ in the weak-$*$ topology. Let $f \in L^1(\mathbb{R})$. Prove that $\mu_\lambda \star f \to \mu \star f \in L^1(\mathbb{R})$ with respect to the *weak-$*$ topology.*

(e) Let $S : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ be a linear map which satisfies $S(f \star g) = S(f) \star g$ for each $f, g \in L^1(\mathbb{R})$. In assignment 3, you showed that S is bounded. Prove that there exists a unique $\mu \in M(\mathbb{R})$ such that $S(f) = \mu \star f$.

Hint: Consider the sequence $f_n = 2n\chi_{[-1/n, 1/n]} \in L^1(\mathbb{R})$. This is an approximate identity: Argue that $f_n \star g \to g$ in the L^1-norm for any $g \in L^1(\mathbb{R})$. On the other hand $\|f_n\|_1 = 1$ for all n and S is bounded, so you can use Alaoglu's theorem to find a weak-$*$ limit $\mu \in M(\mathbb{R})$ of a subnet of $(S(f_n))_n \subset L^1(\mathbb{R}) \subset M(\mathbb{R})$. This is the μ you want!