1: A filter on $\mathbb{N}$ is a non-empty collection $\mathcal{U}$ of non-empty subsets of $\mathbb{N}$ with the property that if $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$ and if $C \supset A$, then $C \in \mathcal{U}$.

(a) Use Zorn's lemma to show that there exist a maximal filter $\mathcal{U}$ of $\mathbb{N}$ containing all cofinite sets. (This is known as a *free ultrafilter.*)

(b) Show that for each $A \subset \mathbb{N}$, either $A \in \mathcal{U}$ or $A^c \in \mathcal{U}$.

(c) Show that there is no subsequence of $\mathbb{N}$ such that $\lim_{i\to\infty} x_{n_i}$ exists for all bounded sequences $x = (x_n)$.

(d) For $A \in \mathcal{U}$, let $n_A = \min\{k : k \in A\}$. Prove that n_A is a (cofinal) subnet of the sequence $1, 2, 3, \ldots$ with the property that $\lim_{A\in\mathcal{U}} x_{n_A}$ does exist for all bounded sequences $x = (x_n)$.

2: Let X be a locally compact Hausdorff space, and let βX be its Stone-Čech compactification. Prove that there is an isometric isomorphism of Banach spaces $\Phi : C(\beta X) \to C_b(X)$, such that $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\bar{f}) = \overline{\Phi(f)}$ for all $f, g \in C(\beta X)$. Here $\bar{f}(t) = \overline{f(t)}$ is the natural involution on continuous functions given by pointwise complex conjugation. (In the language of C*-algebras, one says that $C(\beta X)$ and $C_b(X)$ are isomorphic as C*-algebras. If you ran into a C*-algebraist on the street and asked them to define βX, they'd probably say it is *the* compact Hausdorff space βX such that the above isomorphism $C_b(X) \cong C(\beta X)$ of C*-algebras holds. Gelfand theory asserts that such a space βX exists and is unique up to homeomorphism.)

3: Let $(X, \|\cdot\|_X)$ be a normed vector space. If X is not complete, we can consider its completion, say Y, as a metric space. Remind yourself about this notion, and explain how the norm on X naturally induces a norm $\|\cdot\|_Y$ on Y, making $(Y, \|\cdot\|_Y)$ a Banach space.

4: Let $(V, \|\cdot\|)$ be a normed vector space. Prove that V is complete if and only if for every sequence (v_n) in V with $\sum_n \|v_n\| < \infty$, the series $\sum_n v_n$ *converges in* V.
HINT: For a Cauchy sequence (x_n), drop to a subsequence so $\sum_i \|x_{n_i} - x_{n_{i+1}}\| < \infty$.

5:

(a) Prove that no infinite dimensional Banach space has a countable basis *as a vector space* (i.e. a collection $\{e_n : n \geq 1\}$ so that every vector in X is a *finite* linear combination of $\{e_n : n \geq 1\}$).

(b) Let X be an infinite dimensional Banach space. Recursively construct a sequence of unit vectors x_n so that $\operatorname{dist}(x_n, \operatorname{span}\{x_i : i < n\}) > 1 - 2^{-n}$. Hence deduce that the unit ball of X is not compact.

6:

(a) Prove that $l_1^* = l_\infty$.

(b) Describe all infinite matrices $T = [t_{ij}]_{i,j=1}^\infty$ which act as bounded operators from l_1 to itself and find a formula for $\|T\|$.

7: Prove Proposition 2.3.9 that two Hilbert spaces are isomorphic if and only if they have the same dimension.

8: Let X, Y be Banach spaces and consider the tensor product vector space $X \otimes Y$. In general, there are many distinct norms that one can place on $X \otimes Y$, yielding very different Banach space completions of $X \otimes Y$. We shall call a norm $\|\cdot\|_\alpha$ on $X \otimes Y$ a *cross-norm* if

$$\|x \otimes y\|_\alpha = \|x\|_X \|y\|_Y \quad (x \in X, y \in Y).$$

Define $\|\cdot\|_\gamma : X \otimes Y \to [0, \infty)$ by

$$\|u\|_\gamma = \inf \left\{ \sum_{i=1}^{n} \|x_i\|_X \|y_i\|_Y : u = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y \right\}.$$

(a) Show that $\|\cdot\|_\gamma$ defines a cross-norm on $X \otimes Y$ (called the *projective tensor norm*). The resulting Banach space $X \otimes^\gamma Y := \overline{X \otimes Y}^{\|\cdot\|_\gamma}$ is called the projective tensor product of X and Y.

(b) Show that $\|\cdot\|_\gamma$ is the largest cross norm on $X \otimes Y$: $\|u\|_\alpha \le \|u\|_\gamma$ for any other cross norm $\|\cdot\|_\alpha$ on $X \otimes Y$.

(c) Show that the map

$$\delta_i \otimes \delta_j \mapsto \delta_{(i,j)}, \quad i, j \in \mathbb{N}$$

extends continuously to an isometric isomorphism $\ell^1(\mathbb{N}) \otimes^\gamma \ell^1(\mathbb{N}) \cong \ell^1(\mathbb{N} \times \mathbb{N})$. Here δ_i is the usual indicator function of $i \in \mathbb{N}$.

9: Let (X, d) be a metric space. Recall that

$$Lip(X, d) = \left\{ f : X \to \mathbb{C} \mid f \text{ is bounded } \& \ L(f) := \sup_{x \neq y \in X} \frac{|f(x) - f(y)|}{d(x, y)} < \infty \right\}$$

is the vector space of *bounded Lipschitz functions*, and comes equipped with the norm $\|f\|_{Lip} = \|f\|_\infty + L(f)$.

(a) For $z, x \in X$, set $f_z(x) = \min\{1, d(x, z)\}$. Show that $f_z \in Lip(X, d)$ and $\|f_z\|_{Lip} \le 2$.

(b) Let $x \in X$. Define $e_x : Lip(X, d) \to \mathbb{C}$ be given by $e_x(f) = f(x)$. Verify that $e_x \in Lip(X, d)^*$, and show that if X has bounded diameter, then there exists $c > 0$ such that

$$c d(x, y) \le \|e_x - e_y\| \le d(x, y), \quad x, y \in X$$

Hence, $d'(x, y) = \|e_x - e_y\|$ defines a metric on X which is equivalent to d. In other words, the Banach space $Lip(X, d)$ remembers information about the metric on X.

(c) On the other hand, $C_b(X)$ does not do such a good job at remembering the metric. Now let, for $x \in X$, $e_x \in C_b(X)^*$ be the evaluation functional $e_x(f) = f(x)$. Show, in this context, that

$$\|e_x - e_y\| = 2, \quad x \neq y \in X.$$

So we only recover the discrete metric on X in this case.