1:

(a) Assuming AC, show (‡) if $f : X \to Y$ is surjective, there is a function $g : Y \to X$ such that $f \circ g = id_Y$.

(b) Show that property (‡) implies AC.

2: Let $h : X \to Y$ be a continuous bijection of X onto Y. Suppose that X is compact and Y is Hausdorff. Prove that h is a homeomorphism.

3: In a topological space X, say that a net $(x_\lambda)_{\lambda\in\Lambda}$ has x as a *cluster point* if for each $\lambda_0 \in \Lambda$ and each open neighbourhood $U \ni x$, there is some $\lambda \geq \lambda_0$ so that $x_\lambda \in U$. Prove that x is a cluster point of this net if and only if there is a subnet with limit x. Be explicit about use of the Axiom of Choice.

4: Let V be the vector space of complex-valued functions on $\mathbb{R}$. Put a topology τ on V with a subbase given by $U_{t,a,r} = \{f \in V : |f(t) - a| < r\}$ for $t \in \mathbb{R}$, $a \in \mathbb{C}$, $r > 0$.

(a) Show that a net $(f_\lambda)_{\lambda\in\Lambda}$ in V converges if and only if $\lim f_\lambda(t)$ exists for each $t \in \mathbb{R}$, i.e. pointwise convergence. (See Example 1.4.8.) In particular, $\varepsilon_t(f) := f(t)$ is continuous on V for each $t \in \mathbb{R}$.

(b) Let $E = \{f \in V : f(t) = 2 \text{ except for finitely many } t \in \mathbb{R}\}$. Construct a net in E with limit 0.

(c) Show that no sequence of points in E can converge to 0.

5: The *one-point compactification* of X. Let (X, τ) be LCH, but not compact. Form the space $X^* = X \cup \{p\}$. Define a topology τ^* by specifying that $\{p\}$ is closed, $\tau^* \cap X = \tau$, and open neighbourhoods of p have the form $\{p\} \cup (X \setminus K)$ for K compact in X. Prove that X^* is compact and Hausdorff.

6: The *Stone–Čech compactification* of X. Let (X, τ) be LCH, but not compact. For each $f \in C^b_{\mathbb{R}}(X)$, let $I_f = [-\|f\|_\infty, \|f\|_\infty]$. Then the product space $Y = \prod I_f$ is compact and Hausdorff. Define $\varepsilon : X \to Y$ by $\varepsilon(x)(f) = f(x)$.

(a) Show that ε is continuous.

(b) Define $\beta X = \overline{\varepsilon(X)}$. Observe that βX is compact Hausdorff and X is dense in βX.

(c) Prove that if K is compact and Hausdorff and $g : X \to K$ is continuous, then there is a (unique) continuous function $\tilde{g} : \beta X \to K$ such that $\tilde{g} \circ \varepsilon = g$.

7: *(Arzela–Ascoli Theorem).* Let X be compact Hausdorff. A subset $\mathcal{F} \subseteq C(X)$ is *equicontinuous* at $x \in X$ if for each $\varepsilon > 0$, there is an open neighbourhood $U_\varepsilon \ni x$ such that $U_\varepsilon \subseteq f^{-1}(b_\varepsilon(f(x)))$ for all $f \in \mathcal{F}$. It is equicontinuous if it is equicontinuous at every point $x \in X$. Prove that $\mathcal{F}$ is a compact subset of $C(X)$ if and only if it is closed, bounded and equicontinuous. Use the following outline.

(a) First show that if $\mathcal{F}$ is compact, then it is closed, bounded and equicontinuous.

(b) $C(X)$ is a metric space, so use sequential compactness. Fix a sequence (f_n) in $\mathcal{F}$. Let $\{x_j : j \geq 1\}$ be a countable dense subset of X. Use boundedness and the diagonal technique to extract a subsequence $(f_{n_i})_{i\geq 1}$ such that $\lim_i f_{n_i}(x_j) = a_j$ exists for all $j \geq 1$.

(c) Fix $x \in X$. For $\varepsilon = \frac{1}{3n}$, let U_n be the open set from equicontinuity at x. Choose $x_{j_n} \in U_n$ and find I_n so that for $i \geq I_n$, $|f_{n_i}(x_{j_n}) - a_{j_n}| < \frac{1}{3n}$. Show that for any $y \in U_n$ that $|f_{n_i}(y) - a_{j_n}| < \frac{1}{n}$ for $i \geq I_n$.

(d) Deduce that $(f_{n_i}(x))_{i\geq 1}$ is Cauchy. Call the limit $f(x)$.

(e) Deduce that $f^{-1}(b_{1/n}(f(x)))$ contains U_n, whence f is continuous.

8: The *Moore plane* is $\Gamma = \{(x, y) : x \in \mathbb{R}, y \geq 0\}$ where for each (x, y) with $y > 0$, the usual open balls $B_r((x, y))$ of radius $r \leq y$ are open; and for $(x, 0) \in R = \{(x, 0) : x \in \mathbb{R}\}$, the sets $\{(x, 0)\} \cup B_r((x, r))$ are open.

(a) Show that this determines a Hausdorff topology.

(b) Show that Γ is separable, but that R with the induced topology is not.

(c) Show that Γ is first countable, but not second countable.

(d) Show that $Q = \{(x, 0) : x \in \mathbb{Q}\}$ and $I = \{(x, 0) : x \notin \mathbb{Q}\}$ are disjoint closed sets, but if $Q \subset U$ and $I \subset V$, where U and V are open, then $U \cap V \neq \varnothing$, i.e. Γ is not normal.

(e) Show that if $C \subset \Gamma$ is closed and $p \notin C$, there is a continuous function $f : \Gamma \to [0, 1]$ so that $f|_C = 0$ and $f(p) = 1$. (Γ is completely regular.)