4. More examples: the continued fraction map

§4.1 The continued fraction map

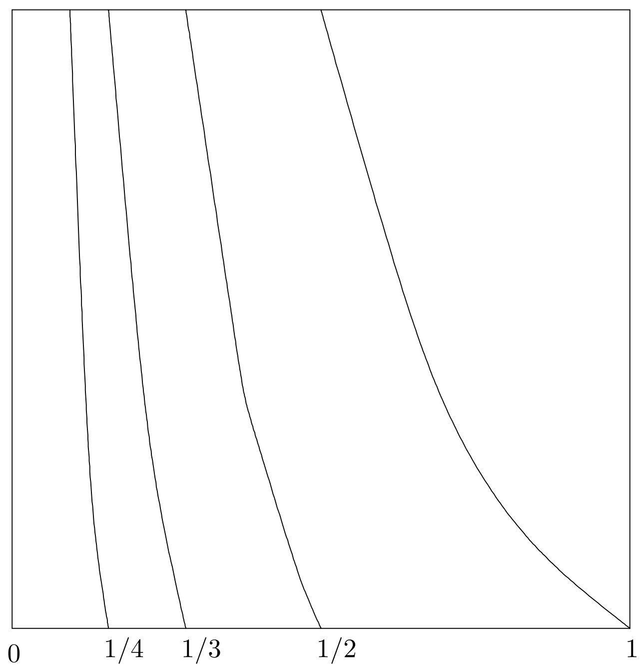

Figure 4.1: The graph of the continued fraction map

Every $x \in (0,1)$ can be expressed as a continued fraction:

$$x = \frac{1}{x_0 + \frac{1}{x_1 + \frac{1}{x_2 + \frac{1}{x_3 + \cdots}}}} \tag{4.1}$$

for $x_n \in \mathbb{N}$.

For example,

$$\begin{aligned} \frac{-1+\sqrt{5}}{2} &= \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \cdots}}}} \\ \frac{3}{4} &= \frac{1}{1 + \frac{1}{3}} \\ \pi &= 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292 + \cdots}}}} \end{aligned}$$

One can show that rational numbers have a *finite* continued fraction expansion (that is, the above expression terminates at x_n for some n). Conversely, it is clear that a finite continued fraction expansion gives rise to a rational number.

Thus each irrational $x \in (0,1)$ has an infinite continued fraction expansion of the form (4.1). Moreover, one can show that this expansion is unique. For brevity, we will sometime write (4.1) as $x = [x_0; x_1; x_2; \ldots]$.

Recall that in Lecture 2 we saw how the doubling map $x \mapsto 2x \bmod 1$ can be used to determine the base 2 expansion of x. Here we introduce a dynamical system that allows us to determine the continued fraction expansion of x.

We can read off the numbers x_i from the transformation $T : [0,1) \to [0,1)$ defined by $T(0) = 0$ and, for $0 < x < 1$,

$$T(x) = \begin{cases} 0 & \text{if } x = 0, \\ \left\{\frac{1}{x}\right\} = \frac{1}{x} \bmod 1 & \text{if } 0 < x < 1. \end{cases}$$

Then

$$x_0 = \left[\frac{1}{x}\right],\ x_1 = \left[\frac{1}{Tx}\right], \ldots, x_n = \left[\frac{1}{T^n x}\right].$$

Later in the course we will study the ergodic theoretic properties of the continued fraction map and use them to deduce some interesting facts about continued fractions.