Operator Analysis

Period: 2010, winter term
Lecturer: Yamagami Shigeru

Synopsis:
Graph theory is known to have vast applications in combinatorial problems. Turning viewpoints into its analytical aspect, we will be often faced with manipulating linear operators. In this course, mainly working with finite graphs, the spectral analysis is performed for graphs having smaller operator norms, through which we will get a good experience in mathematical classification problems.

Prerequisite:
Set theory (basic notions and terminology)
Linear algebra (undergraduate level)
Analysis (limit arguments in euclidean spaces)

Reference:
F.M. Goodman, P. de la Harpe and V. Jones, Coxeter Graphs and Towers of Algebras, Springer, 1989.

Further Reading:
J. Fröhlich and T. Kerler, Quantum Groups, Quantum Categories and Quantum Field Theory, Lec. Notes in Math. 1542(1993), Springer.
M. Reed and B. Simon, Functional Analysis I, Academic Press, 1981.

1. Spectral Properties of Hermitian Matrices

Given a square matrix $A = (a_{ij})$, let $\sigma(A)$ be the set of eigenvalues of A, which is referred to as the **spectrum** of A. There are several characterizations of $\sigma(A)$: (i) $\sigma(A) = \{\lambda \in \mathbb{C}; \det(\lambda I - A) = 0\}$, (ii) $\sigma(A) = \{\lambda \in \mathbb{C}; \lambda I - A \text{ is not invertible}\}$ and so on. The **spectral radius** of A is then defined to be

$$r(A) = \max\{|\lambda|; \lambda \in \sigma(A)\}.$$

The **operator norm** of A is, by definition,

$$\|A\| = \sup\{\|A\xi\|; \xi \in \mathbb{C}^n, \|\xi\| = 1\}.$$

Here $\|\xi\| = \sqrt{(\xi|\xi)}$ denotes the ordinary inner product norm of ξ. The norm $\|A\|$ is characterized as the largest constant satisfying

$$\|A\xi\| \leq \|A\|\,\|\xi\| \quad \text{for any } \xi.$$

In other words,

$$\|A\| = \sup\{\|A\xi\|/\|\xi\|; 0 \neq \xi \in \mathbb{C}^n\}.$$

Example 1.1. Let $A = (\delta_{ij} d_j)$ be a diagonal matrix. Then

$$\|A\| = \max\{|d_j|; j \geq 1\}.$$

Problem 1. Show that

$$\|A\| = \sup\{\|A\xi\|; \xi \in \mathbb{C}^n, \|\xi\| \leq 1\} = \sup\{|(\xi|A\eta)|; \|\xi\| \leq 1, \|\eta\| \leq 1\}.$$

Problem 2. Compute the norm for diagonal matrices.

Proposition 1.2. *We have the inequality $r(A) \leq \|A\|$, i.e.,*

$$\sigma(A) \subset \{\lambda \in \mathbb{C}; |\lambda| \leq \|A\|\}.$$

Proof. Let $\lambda \in \sigma(A)$ with ξ an eigenvector. Then

$$|\lambda|\|\xi\| = \|A\xi\| \leq \|A\|\,\|\xi\|$$

implies that $|\lambda| \leq \|A\|$. □

Remark . Here is a formula which refines the above estimate. (A proof can be found in any standard text on functional analysis.)

$$r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}.$$

If we define the **hermitian conjugate** A^* of a square matrix $A = (a_{jk})$ by $(A^*)_{jk} = \overline{a_{kj}}$, it is characterized by the relation

$$(A^*\xi|\eta) = (\xi|A\eta) \quad \text{for } \xi, \eta \in \mathbb{C}^n.$$

Proposition 1.3. *For any square matrix A, we have*

$$\|A^*\| = \|A\|, \qquad \|A\|^2 = \|A^*A\|.$$

Problem 3. Check these norm identities.

A matrix A is **hermitian** if $A^* = A$. A matrix U is **unitary** if $UU^* = U^*U = I$, where $I = (\delta_{j,k})$ denotes the unit matrix. (I stands for identity.)

A matrix A is said to be **normal** if $AA^* = A^*A$. Normal matrices constitute a class which includes hermitian and unitary ones.

Proposition 1.4.

(i) *For a hermitian matrix A, $\sigma(A) \subset \mathbb{R}$.*
(ii) *For a unitary matrix U, $\sigma(U) \subset \{z \in \mathbb{C}; |z| = 1\}$.*

Problem 4. Determine the spectrum of a matrix A which is hermitian and unitary at the same time.

Proposition 1.5. *Let A be a normal matrix.*

(i) *If $A\xi = \lambda\xi$ with $\xi \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$, then $A^*\xi = \overline{\lambda}\xi$.*
(ii) *If $A\xi = \lambda\xi$ and $A\eta = \mu\eta$ with $\lambda \neq \mu$, then $(\xi|\eta) = 0$.*

Proof. (i) This follows from

$$(A^*\xi - \overline{\lambda}\xi | A^*\xi - \overline{\lambda}\xi) = (A\xi - \lambda\xi | A\xi - \lambda\xi) = 0.$$

(ii) is a consequence of

$$\lambda(\xi|\eta) = (A^*\xi|\eta) = (\xi|A\eta) = \mu(\xi|\eta).$$

□

Theorem 1.6. *Let $\{\lambda_1, \dots, \lambda_n\}$ be the eigenvalue list (including multiplicity) of a normal matrix $A = (a_{ij})$. Then we can find a unitary matrix U satisfying*

$$A = U^* \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{pmatrix} U.$$

Proof. Let ξ_1 be a normalized eigenvector of A of eigenvalue λ_1. By Gram-Schmidt's orthogonalization, we can find an orthonormal basis $(\xi_1, \dots, \xi_n)$ so that

$$(A\xi_1, \dots, A\xi_n) = (\xi_1, \dots, \xi_n) \begin{pmatrix} \lambda_1 & * \\ 0 & B \end{pmatrix}$$

with B a square matrix of size $n-1$. By repeating the same procedure to B, we arrive at a unitary matrix U such that UAU^* is upper triangular. Since an upper triangular matrix is normal if and only if it is diagonal, we are done. □

Corollary 1.7. *For a normal matrix A, we have $\|A\| = r(A)$.*

Problem 5. Investigate what we can say for the converse implication.

Problem 6. Check that an upper triangular matrix is normal if and only if it is diagonal.

Let A be a hermitian matrix and set

$$\lambda_{max} = \max\{\lambda; \lambda \in \sigma(A)\},$$
$$\lambda_{min} = \min\{\lambda; \lambda \in \sigma(A)\}.$$

Proposition 1.8. *We have the following expressions.*

$$\lambda_{max} = \max\{(\xi|A\xi); \|\xi\| = 1\},$$
$$\lambda_{min} = \min\{(\xi|A\xi); \|\xi\| = 1\}.$$

Proof.

$$\lambda_{min}(|\xi_1|^2+\cdots+|\xi_n|^2) \leq \lambda_1|\xi_1|^2+\cdots+\lambda_n|\xi_n|^2 \leq \lambda_{max}(|\xi_1|^2+\cdots+|\xi_n|^2).$$

□

Proposition 1.9 (Variational Principle)**.** *Let A be a hermitian matrix. If a unit vector ξ attains the maximal value λ_{max} of the function $(\xi|A\xi)$, then $A\xi = \lambda_{max}\xi$.*

Proof. Use the quadratic inequality

$$(\xi + t\eta|(\lambda_{max}I - A)(\xi + t\eta)) \geq 0$$

for any $t \in \mathbb{R}$, together with the assumption $(\xi|(\lambda_{max}I - A)\xi) = 0$. □

Problem 7. Compute λ_{max} and λ_{min} for the real symmetric matrix

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

For a future reference, we recall permutation matrices here. Given a permutation $\sigma \in S_n$ of degree n, the associated **permutation matrix** P_σ is defined by

$$(P_\sigma)_{ij} = \delta_{i,\sigma(j)} = \delta_{\sigma^{-1}(i),j}.$$

The permutation matrix is orthogonal and satisfies $P_\sigma P_\tau = P_{\sigma\tau}$ for $\sigma, \tau \in S_n$.

Problem 8. Check the multiplicativity property of permutation matrices.

Problem 9. Compute the spectrum of permutation matrices.

2. Graphs and Adjacency Matrices

The Euler's solution to the seven bridges problem of Königsberg is known to be the birth of the notion of graph: A connected unoriented graph allows an Euler trail if and only if the number of vertices of odd degree is less than three (the number of odd-degree vertices being always even for unoriented graphs).

A **graph** is a quadruplet (Γ, X, s, t), where Γ and X are sets with $s, t : \Gamma \to X$ maps. We call Γ the set of edges, X the set of vertices, whereas s and t are referred to as the source and target maps respectively.

A graph is simply represented by the edge set when there arise no confusions.

Given vertices x, $y \in X$, set

$$ {}_x\Gamma = \{\gamma \in \Gamma; t(\gamma) = x\}, \quad \Gamma_y = \{\gamma \in \Gamma; s(\gamma) = y\}, $$
$$ {}_x\Gamma_y = \{\gamma \in \Gamma; s(\gamma) = y, t(\gamma) = x\}. $$

A graph is **weakly finite** if ${}_x\Gamma_y$ is a finite set for any ordered pair of elements (x, y) in X, **locally finite** if both of ${}_x\Gamma$ and Γ_x are finite sets for any $x \in X$, and **finite** if both of Γ and X are finite sets. Given a weakly finite graph Γ, its **adjacency matrix** is defined to be $\{|{}_x\Gamma_y|\}_{x,y\in X}$.

Two graphs Γ, Γ' are said to be **isomorphic** if we can find bijections $\phi : \Gamma \to \Gamma'$ and $\phi^{(0)} : X \to X'$ satisfying

$$ t(\phi(\gamma)) = \phi^{(0)}(t(\gamma)), \quad s(\phi(\gamma)) = \phi^{(0)}(s(\gamma)) $$

for any $\gamma \in \Gamma$.

A graph Γ is a **subgraph** of a graph Γ' if $\Gamma \subset \Gamma'$, $X \subset X'$, $s = s'|_\Gamma$ and $t = t'|_\Gamma$. By abuse of terminology, a graph isomorphic to a subgraph is also referred to as a subgraph.

Example 2.1. Cyclic permutations and circle graphs.

By an involution of a graph Γ, we shall mean a bijection $\Gamma \ni \gamma \mapsto \gamma^{-1} \in \Gamma$ such that $t(\gamma^{-1}) = s(\gamma)$ and $(\gamma^{-1})^{-1} = \gamma$. An **unoriented graph** is, by definition, a graph Γ which is furnished with an involution satisfying $\gamma^{-1} = \gamma$ for $\gamma \in \Gamma$ satisfying $s(\gamma) = t(\gamma)$.

Two square matrices are said to be **equivalent** if we can find a permutation matrix P such that $PAP^{-1} = B$.

Proposition 2.2. *There is a one-to-one correspondance between isopmorphism classes of finite graphs and equivalence classes of* $\mathbb{N}$*-valued square matrices.*

There is a one-to-one correspodance between isomorphism classes of finite unoriented graphs and equivalence classes of $\mathbb{N}$*-valued symmetric matrices.*

Let G be a (at most) countable group with $1 \notin S$ a set of generators. Then the associated **Cayley graph** $\Gamma(G, S)$ is defined by $X = G$ and $\Gamma = \{(g, ga); g \in G, a \in S\}$ with $t(g, ga) = g$ and $s(g, ga) = ga$.

If the generator set S further satisfies $S^{-1} = S$, i.e., $a \in S$ if and only if $a^{-1} \in S$, then $\Gamma(G, S)$ is unoriented with respect to the involution $(g, ga)^{-1} = (ga, g)$.

For the cyclic group C_n with a a generator, $\Gamma(C_n, \{a\})$ is an oriented n-gon and $\Gamma(C_n, \{a, a^{-1}\})$ an n-gon.

Problem 10. Depict the oriented graph $\Gamma(\mathbb{Z}, \{1\})$ and the unoriented graph $\Gamma(\mathbb{Z}, \{\pm 1\})$.

Problem 11. Depict the graph $\Gamma(F_2, \{a^{\pm 1}, b^{\pm 1}\})$. Here F_2 denotes the free group generated by a two-element set $\{a, b\}$

Given a finite graph Γ with A the incidence matrix, we define a linear operator $A : \ell^2(X) \to \ell^2(X)$ by

$$A\delta_x = \sum_{y \in X} A_{y,x}\delta_y.$$

By the **norm** $\|\Gamma\|$ of Γ, we shall mean the one for the linear operator A and the **spectrum** $\sigma(\Gamma)$ of Γ is, by definition, the spectrum of A.

Here is a visual interpretation of eigenvector equations such as $A\xi = \lambda\xi$: Look at a vertex x of the graph. Then

$$\lambda\xi(x) = \sum_y A_{x,y}\xi(y)$$

with $A_{x,y}$ the number of edges from y to x.

Example 2.3.

(i) The spectrum of cyclic permutations (oriented circle graphs).

$$\sigma(\Gamma) = \{e^{2\pi ik/n}; 0 \leq k \leq n-1\}.$$

(ii) The spectrum of unoriented circle graphs.

$$\sigma(\Gamma) = \{2\cos(2\pi k/n); 0 \leq k \leq n-1\}.$$

(iii) The spectrum and the norm of oriented lines (Jordan blocks).

(iv) The spectrum of unoriented lines.

$$\sigma(\Gamma) = \{2\cos(\pi k/(n+1)); 1 \leq k \leq n\}.$$

Problem 12. Check the case (i) and (iii).

Problem 13. Compute the spectrum of the complete graph K_n of n vertices.

For the spectral analysis of $\Gamma(F_2, \{a^{\pm 1}, b^{\pm 1}\})$, we need some machinery of free probability theory.

A **path** in a graph is a finite sequence $\gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ of edges satisfying

$$s(\gamma_j) = t(\gamma_{j+1}) \quad \text{for } 1 \leq j \leq n-1.$$

The number n is called the length of the path. We set $s(\gamma) = s(\gamma_n)$ and $t(\gamma) = t(\gamma_1)$.

A graph is **connected** if, given two vertices x, y, we can find a path γ such that $x = t(\gamma)$ and $y = s(\gamma)$.

Remark . Notice that a graph consisting of one vertex is connected if and only if the edge set is non-empty.

Problem 14. Compute the spectrum and the norm for connected unoriented graphs of radial shape.

In what follows, we shall exclusively deal with unoriented graphs and the adjective ‘unoriented’ will be omitted unless otherwise stated.

3. Perron-Frobenius Theorems

We shall describe basic results in Perron-Frobenius theory. To make the access easier, we restrict ourselves to the case of symmetric matrices.

A symmetric matrix $A = (a_{ij})$ is said to be **non-negative** if $a_{ij} = a_{ji} \geq 0$ for all i, j. A non-negative symmetric matrix A is said to be reducible if we can find a permutation matrix P such that

$$PAP^{-1} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

in a non-trivial manner, i.e., there is a finite non-empty proper subset X of $\Gamma^{(0)}$ such that $(PAP^{-1})_{xy} = (PAP^{-1})_{yx} = 0$ for $x \in X$ and $y \notin X$. Otherwise, we call it an **irreducible** matrix.

Proposition 3.1. *The following conditions for a symmetric matrix A of non-negative entries are equivalent.*

(i) *The matrix A is irreducible.*

(ii) *For any $1 \leq i, j \leq n$, we can find an integer $k \geq 1$ such that $(A^k)_{ij} > 0$.*

(iii) *Let Γ be the graph associated to the symmetric matrix whose (i, j)-th component is set to be either 1 or 0 according to $A_{ij} > 0$ or $a_{ij} = 0$. Then Γ is connected.*

Problem 15. Check the reducibility of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Problem 16. Interpret the condition $(A^k)_{ij} > 0$ graphically.

Definition 3.2. For a matrix A of non-negative entries, an eigenvector of eigenvalue $r(A)$ (the spectral radius of A) with non-negative components is called a **Perron eigenvector**. Note that $r(A) = \|A\|$ for a symmetric A.

Theorem 3.3. *Any symmetric matrix A of non-negative entries admits a Perron eigenvector: we can find a vector $\eta \neq 0$ satisfying $\eta_j \geq 0$ and $A\eta = \|A\|\eta$.*

Proof. Since A is hermitian, we can find a unit eigenvector ξ of eigenvalue λ such that $|\lambda| = \|A\|$. Let the unit vector $\eta = (\eta_j)$ be defined by $\eta_j = |\xi_j|$. Then

$$\|A\| = |\lambda| = |(\xi|A\xi)| \leq (\eta|A\eta) \leq \lambda_{max} \leq \|A\|$$

implies $\lambda_{max} = \|A\|$ and $\lambda_{max} = (\eta|A\eta)$. By the variational principle, η is a Perron eigenvector of A. $\square$

Theorem 3.4. *Let A be an irreducible symmetric matrix of non-negative entries.*

(i) *Perron eigenvector is unique up to scalar multiplication and all of its components are strictly positive.*
(ii) *Any eigenvector of maximal modulus eigenvalue is proportional to a Perron eigenvector.*
(iii) *Any eigenvector of A with non-negative components is a Perron eigenvector.*

Proof. (i) Let $\eta = (\eta_1, \dots, \eta_n)$ be a Perron eigenvector and assume that $\eta_i > 0$ for some i. Since A is irreducible, we can find N such that $(A^N)_{ji} > 0$ for any j and then

$$\|A\|^N \eta_j = (A^N \eta)_j = \sum_k (A^N)_{jk} \eta_k \geq (A^N)_{ji} \eta_i > 0.$$

(ii) If there exists an eigenvector which is not proportional to a given Perron eigenvector, we can find a eigenvector ξ admitting a zero component. Then, by the proof of the previous theorem, $|\xi|$ is a Perron eigenvector, which contradicts with the strict positivity of components of Perron eigenvectors.

(iii) Let ξ be an eigenvector of A with non-negative components. If its eigenvalue is different from $\|A\|$, then it is orthogonal to a Perron eigenvector. Again, by the strict positivity of Perron eigenvector, this is impossible. $\square$

Problem 17. Compute the Perron eigenvector of the matrix

$$\begin{pmatrix} e^a & e^b \\ e^b & e^{-a} \end{pmatrix}$$

for $a, b \in \mathbb{R}$.

Problem 18. Investigate the validity of uniquness in the case of reducible matrices.

The following simple observation is a key in the classification of connected graphs of smaller norm.

Lemma 3.5. *Let A and B be symmetric matrices of non-negative entries such that $a_{ij} \leq b_{ij}$ for any i, j. Then $\|A\| \leq \|B\|$.*

Furthermore, if A is irreducible and $A \neq B$, the strict inequality $\|A\| < \|B\|$ holds.

Proof. Let ξ be a normalized Perron eigenvector of A. Then we have

$$\|A\| = \|A\|(\xi|\xi) = (\xi|A\xi) \leq (\xi|B\xi) \leq \|B\|\|\xi\|^2 = \|B\|.$$

If we assume $\|A\| = \|B\|$ in addition, then $(\xi|B\xi) = \|B\|$ and the variational principle implies that ξ is a Perron eigenvector of B as well.

Now, from the irreducibility assumption of B, $\xi_j > 0$ for any j, which together with the relation $(B - A)\xi = \|B\|\xi - \|A\|\xi = 0$ shows that $B - A$ admits no positive entries, i.e., $B - A = 0$. $\square$

Corollary 3.6. *Let Γ' be a finite graph and Γ be a connected proper subgrapah of Γ'. Then $\|\Gamma\| < \|\Gamma'\|$.*

Remark . Perron-Frobenius theorems are originally formulated for non-symmetric matrices, which correspond to adjacency matrices of oriented graphs. The norm-increasing principle then turns out to be applicable to oriented graphs as well. As another application, we remark here the asymptotic analysis of stochastic matrices and the google pagerank in evaluating websites.

4. Graphs of type A and D

The graph A_l ($l \geq 2$) is a linear graph of l vertices. Number the vertices from one terminal sequencially and let $(c_1, c_2, \dots, c_l)$ be an eigenvector of eigenvalue λ. Then the eigenequation is of the form

$$\begin{aligned} \lambda c_1 &= c_2, \\ \lambda c_j &= c_{j-1} + c_{j+1} \quad \text{for } 2 \leq j \leq l-1, \\ \lambda c_l &= c_{l-1}. \end{aligned}$$

If we introduce monic polynomials $\{P_n(\lambda)\}_{n\geq 0}$ by

$$\lambda P_n(\lambda) = P_{n-1}(\lambda) + P_{n+1}(\lambda), \quad P_0(\lambda) = 1, P_1(\lambda) = \lambda,$$

then the eigenequation is reduced to the signle equation $c_{l+1} = c_1 P_l(\lambda) = 0$, i.e., $\det(\lambda I - A_l) = P_l(\lambda) = 0$.

$$P_2 = \lambda^2 - 1, P_3 = \lambda^3 - 2\lambda, P_4 = \lambda^4 - 3\lambda^2 + 1, P_5 = \lambda^5 - 4\lambda^3 + 3\lambda.$$

Returning to the original eigenequation, the generic part of the recursive relation is solved in terms of the characteristic roots q, q^{-1} of the quadratic equation $t^2 + 1 = \lambda t$ satisfying $\lambda = q + q^{-1}$: $c_k = \alpha q^k + \beta q^{-k}$ ($1 \leq k \leq l$).

If we take the initial condition $(q+q^{-1})c_1 = c_2$ into account, $\alpha + \beta = 0$ and we can set

$$c_k = \frac{q^k - q^{-k}}{q - q^{-1}} \equiv [k]_q$$

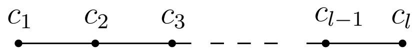
up to multiplicative constants. Here $[k]_q$ is a Laurent polynomial of q and can be evaluated at $q = \pm 1$: $[k]_{\pm 1} = \pm k$.

Finally, the terminal matching $(q + q^{-1})c_l = c_{l-1}$ is satisfied if and only if $[l+1]_q = 0$. By the exchange symmetry between q and q^{-1}, we may take $q = e^{\pi k i/(l+1)}$ ($1 \leq k \leq l$) with the associated eigenvalues given by $q + q^{-1} = 2\cos(k\pi/(l+1))$ ($1 \leq k \leq l$), i.e.,

$$P_l(\lambda) = \prod_{k=1}^{l} \left(\lambda - 2\cos\frac{k\pi}{l+1} \right).$$

The Perron eigenvector is obtained for the choice $q = e^{\pi i/(l+1)}$ with

$$\|A_l\| = q + q^{-1} = 2\cos\left(\frac{\pi}{l+1}\right), \quad c_k = \frac{\sin(k\pi/(l+1))}{\sin(\pi/(l+1))}.$$



Remark .

(i) The expression $[n]_q$ is referred to as q-integer, which is known to be a source of many interesting identities (so-called q-analogues). Here q is taken to be a kind of deformation parameter with the ordinary integal relations recovered by taking the limit $q \to 1$.

(ii) The polynomial $P_n(\lambda)$ is also referred to as the Chebyshev polynomial of second kind:

$$P_n(q+q^{-1}) = \frac{q^{n+1} - q^{-n-1}}{q - q^{-1}} \quad \text{with } q = e^{i\theta}.$$

Consider the graph D_l ($l \geq 4$), which is a graph with one edge and one vertex added to the graph A_{l-1} in a minimal way. Let $(a, b, c[n]_q, c[n-1]_q, \ldots, c[2]_q, c[1]_q)$ be an eigenvector of eigenvalue $q + q^{-1}$ ($l = n+2$). The choice of components automatically solves the most of eigenequation with the remaining part given by

$$(q+q^{-1})a = (q+q^{-1})b = c[n]_q, \quad (q+q^{-1})c[n]_q = a + b + c[n-1]_q,$$

which is solved by the equation

$$q^{2l-2} + 1 = 0$$

when $q + q^{-1} \neq 0$.

After explicit computations, we find that it is reasonable to split into two cases:

(i) The case of odd l: As solutions, we may take

$$q =^{k\pi i/(2l-2)}, \quad k = 1, 3, 5, \ldots, 2l-3.$$

Since l is supposed to be odd, $[2]_q = q + q^{-1} \neq 0$ for any of these and the eqigenequation is solved by

$$a = b = \frac{[n]}{[2]} c.$$

If $q + q^{-1} = 0$,

$$[n] = \frac{q^{-l+1}}{q - q^{-1}} \left((-1)^{l-2} - 1\right) \neq 0$$

is used to get

$$c = 0, \quad a + b = 0.$$

Consequently, we see

$$\sigma(D_l) = \{0\} \cup \{2 \cos \frac{k\pi}{2l-2}; k = 1, 3, \ldots, 2l-3\}$$

with all eigenvalues having multiplicity one.

(ii) The case of even l:

The solution

$$q =^{k\pi i/(2l-2)}, \quad k = 1, 3, 5, \dots, 2l-3.$$

of $q^{2l-2} + 1 = 0$ satisfies $q + q^{-1} = 0$ exactly when $k = l - 1$, whence each eigenvector is specified by

$$a = b = \frac{[n]}{[2]} c$$

for $k \neq l - 1$.

If $q + q^{-1} = 0$, $[n] = 0$ but $[n-1] = (-1)^{l/2}$ shows that the eigenspace is specified by

$$a + b + (-1)^{l/2} c = 0,$$

i.e., zero is an eigenvalue of multiplicity 2. Thus

$$\sigma(D_l) = \{2\cos\frac{k\pi}{2l-2}; k = 1, 3, \dots, 2l-3\},$$

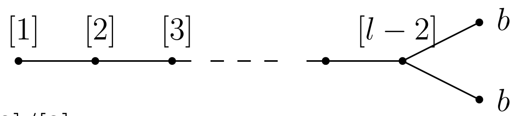
which contains 0.

In either case, we have

$$\det(\lambda I - D_l) = \lambda \prod_{k=1}^{l-1} \Big(\lambda - 2\cos\left(\frac{2k-1}{2l-2}\pi\right)\Big).$$

The Perron eigenvector is obtained if we choose $q = e^{\pi i/2(l-1)}$, which particularly implies

$$\|D_l\| = 2\cos\left(\frac{\pi}{2l-2}\right),$$



with $b = [l-2]/[2]$.

As a final series of graphs, consider a graph T_l, which is the graph obtained from A_l by adding one loop-edge at a terminal vertex. Let $([1]_q, [2]_q, \dots, [l]_q)$ be a Perron eigenvector. Then the matching condition arises at the loop-vertex:

$$(q + q^{-1})[l]_q = [l]_q + [l-1]_q.$$

In terms of the expression

$$[n]_q = q^{n-1} + q^{n-3} + \cdots + q^{-n+3} + q^{-n+1},$$

we see that the above condition is equivalent to

$$0 = q^l - q^{l-1} + q^{l-2} + \cdots + q^{-l} = q^{-l}\frac{1 - (-q)^{2l+1}}{1 - (-q)},$$

which has the solution (up to taking inverse in q)

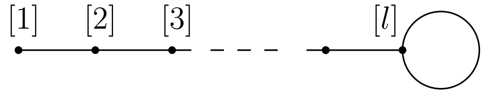
$$\frac{1}{2l+1}\pi, \quad \frac{3}{2l+1}\pi, \quad \dots, \quad \frac{2l-3}{2l+1}\pi, \quad \frac{2l-1}{2l+1}\pi,$$

i.e.,

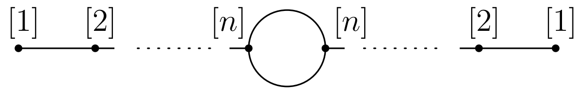
$$\det(\lambda I - T_l) = \prod_{k=1}^{l} \left(\lambda - 2\cos\left(\frac{2k-1}{2l+1}\pi\right)\right).$$

As a result, we know that $q = e^{\pi i/(2l+1)}$ for the choice of Perron eigenvector and the graph norm is computed by

$$\|T_l\| = 2\cos\left(\frac{\pi}{2l+1}\right).$$



Problem 19. Investigate the Perron eigenvalue of the following graph L_n. Show that $\lim_{n\to\infty} \|L_n\| = \frac{5}{2}$.

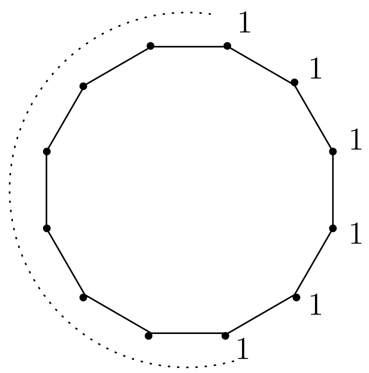


5. Graphs of Norm 2

We shall describe most of the graphs of norm 2. By adding one edge and one vertex to the graph of type A or D, there are series of graphs of norm 2.

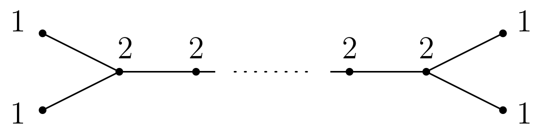
Example 5.1. Let $\widetilde{A}_l$ $(l \geq 2)$ be the loop graph of $l+1$ vertices.

It is immediate to see that the Perron eigenvector is $(1, 1, \ldots, 1)$ with an eigenvalue of 2.



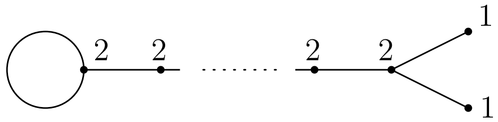
Problem 20. Compute the eigenpolynomial $\det(\lambda I - \widetilde{A}_l)$; use the relation $\widetilde{A}_l = C + C^*$ with C a cyclic permutation.

Example 5.2. Let $\widetilde{D}_l$ $(l \geq 4)$ be the I-shaped graph of $l+1$ vertices. Then the Perron eigenvector is $(1, 1, 1, 1, 2, 2, \ldots, 2)$ with an eigenvalue of 2.



By adding of one or two loop-edges to the graph of type A or D, we obtain the following.

Example 5.3. The extended tadpole graph $\widetilde{T}_l$ $(l \geq 2)$ of $l+1$ vertices.



Example 5.4. The double tadpole graph $\widehat{T}_l$ $(l \geq 1)$ of l vertices.

Problem 21. Compute the Perron eigenvalue of the graph obtained by adding one free edge to each vertex of $\widetilde{A}_l$.

If we allow infinite graphs, there are three more graphs of norm 2, which are limits of graphs of type A, D and T.

An infinite matrix $A = (a_{ij})$ is defined to be **locally finite** if

$$\{i; a_{i,j} \neq 0\} \quad \text{and} \quad \{i; a_{j,i} \neq 0\}$$

are finite sets for any j. A vector of infinitely many components $\xi = \{\xi_j\}$ is said to be essentially finite if we can find a finite subset F such that $\xi_j = 0$ unless $j \in F$.

An essentially finite vector is multiplied by a locally finite matrix, which is denoted by $A\xi$.

A graph is locally finite if and only if the associated matrix is locally finite. The norm of a locally finite graph is defined exactly as in the case of finite graphs.

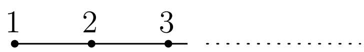
Lemma 5.5. *For a locally finite matrix A, its norm is a limit of those for finite submatrices. More precisely, given a finite subset F of indices, let A_F be the finite matrix with index set restricted to F. Then we have*

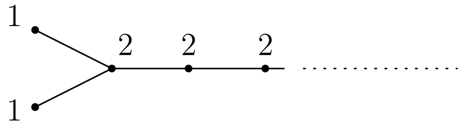
$$\|A\| = \lim_{F \to \Gamma^{(0)}} \|A_F\|.$$

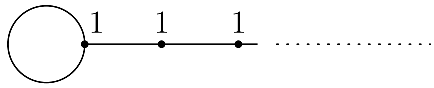
Corollary 5.6. *For locally finite graphs, $\Gamma \subset \Gamma'$ implies $\|\Gamma\| \leq \|\Gamma'\|$.*

Theorem 5.7. *Any connected locally finite infinite graph Γ contains the graph A_∞ as a subgraph and hence $\|\Gamma\| \geq 2$. The equality holds if and only if Γ is one of A_∞ (unilateral line), $\widetilde{A}_\infty$ (bilateral line), D_∞ or T_∞.*

For the proof of 'only if' part, we need the result for T-shaped graphs, which will be checked in the next section.



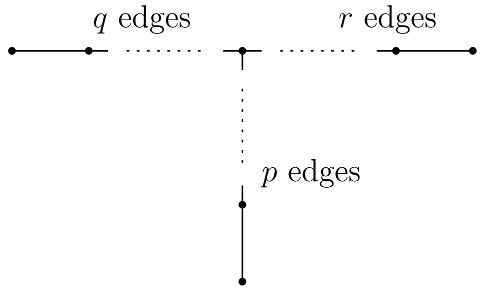




Problem 22. Show that a connected locally finite infinite graph contains A_∞ as a subgraph. Hint: By local finiteness the graph contains infinitely many vertices. Let x_j be an infinite sequence of vertices. Choose a path for each pair (x_j, x_{j+1}) of vertices. Concatenate them and remove redundant edges and vertices.

6. Graphs of type E

Let $T_{k,l,m}$ with $1 \leq k \leq l \leq m$ be the T-shaped graphs with lines of length k, l and m respectively ($T_{k,l,m}$ has $k+l+m$ edges). Since $T_{1,1,n} = D_{n+3}$, we focus on the other case.



Problem 23. Check the fact that $T_{k,l,m} \subset T_{k',l',m'}$ if and only if $k \leq k'$, $l \leq l'$ and $m \leq m'$.

We first seek for the graph of norm 2 among $T_{1,2,n-1}$'s. Let (a, b_1, b_2, c_j) be a Perron vector. As in the case of type A and D, we can set $c_j = j$ $(1 \leq j \leq n)$ for the longer line part. The eigenequation then takes the form

$$2a = n, \quad 2b_1 = b_2, \quad 2b_2 = b_1 + n, \quad 2n = a + b_2 + n - 1,$$

which has the solution

$$n = 6, a = 3, b_1 = 2, b_2 = 4.$$

Next we see for the graph of norm 2 among $T_{1,3,n-1}$. For the Perron eigenvector $(a, b, 2b, 3b, c_j = j)$, we have the equation

$$2a = n, \quad 6b = 2b + n, \quad 2n = a + 3b + n - 1$$

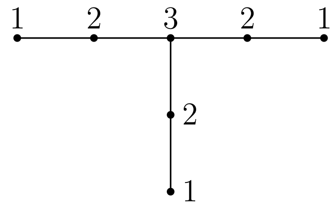
with the solution

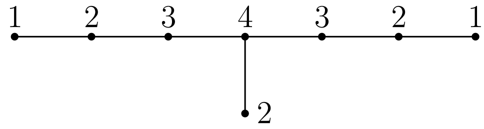
$$n = 4, a = 2, b = 1.$$

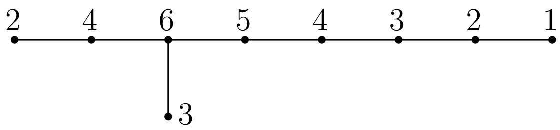
Similarly, the Perron eigenvector $(a, 2a, b, 2b, c_j = j)$ for $T_{2,2,n-1}$ admits an eigenvalue 2 if and only if

$$n = 3, a = b = 1.$$

It is conventional to denote these graphs by $\widetilde{E}_6 = T_{2,2,2}$, $\widetilde{E}_7 = T_{1,3,3}$ and $\widetilde{E}_8 = T_{1,2,5}$.







Theorem 6.1. *The graph $T_{k,l,m}$ has a norm ≤ 2 if and only if (k, l, m) is in the following list.*

(i)
$$(k, l, m) = (1, 2, 5), (1, 3, 3), (2, 2, 2).$$

(ii)
$$(k, l, m) = (1, 1, m \geq 1), (1, 2, 2), (1, 2, 3), (1, 2, 4).$$

Write $E_6 = T_{1,2,2}$, $E_7 = T_{1,2,3}$, $E_8 = T_{1,2,4}$.

Proof. The list in (i) has a norm of 2 as observed already and hence $T_{k,l,m}$ with $(k, l, m) > (2, 2, 2)$ or $(k, l, m) > (1, 3, 3)$ or $(k, l, m) > (1, 2, 5)$ (in the inclusion ordering) has a norm > 2. The remaining is in the list (ii). The graph E_l $(l = 6, 7, 8)$ has a norm smaller than 2 as a subgraph of $\widetilde{E}_l$. The graph D_l has a norm of $2\cos(\pi/(2l - 2))$. $\square$

Consider $T_{1,2,n-1}$ and let (a, b_1, b_2, c_j) be a Perron eigenvector with $c_j = (q^j - q^{-j})/(q - q^{-1})$ for $1 \leq j \leq n$ and an eigenvalue $q + q^{-1}$. This choice solves the eigenequation for the longest line. The remaining equations are

$$\begin{aligned}(q + q^{-1})a' &= q^n - q^{-n},\\ (q + q^{-1})b_1' &= b_2',\\ (q + q^{-1})b_2' &= b_1' + q^n - q^{-n},\\ (q + q^{-1})(q^n - q^{-n}) &= a' + b_2' + q^{n-1} - q^{-n+1}\end{aligned}$$

with $a' = (q - q^{-1})a$ and $b_j' = (q - q^{-1})b_j$. Under the condition that $(q - q^{-1})(q + q^{-1})(q^2 + 1 + q^{-2}) \neq 0$, these are equivalent to requiring

$$(q^5 - q^{-5})(q^n - q^{-n}) = (q + q^{-1})(q^3 - q^{-3})(q^{n-1} - q^{-n+1})$$

together with

$$a' = \frac{q^n - q^{-n}}{q + q^{-1}}, \quad b_1' = \frac{q^n - q^{-n}}{q^2 + q^{-2} + 1}, \quad b_2' = (q + q^{-1})b_1.$$

We shall write down explicitly for $n = 3, 4, 5$ (note $q - q^{-1} \neq 0$).

$n = 3$:

$$q^8 - q^4 + 1 = \frac{q^{12} + 1}{q^4 + 1} = 0.$$

$n = 4$:

$$q^{12} - q^6 + 1 = \frac{q^{18} + 1}{q^6 + 1} = 0.$$

$n = 5$:

$$q^{16} + q^{14} - q^{10} - q^8 - q^6 + q^2 + 1 = 0.$$

For $n = 3$ or $n = 4$, we see that the choice $q = e^{i\pi/12}$ or $q = e^{i\pi/18}$ gives positive components of Perron eigenvector, which particularly shows $\|E_6\| = 2\cos(\pi/12)$ and $\|E_7\| = 2\cos(\pi/18)$.

For the case $n = 5$, we may expect a similar conclusion: the last equation is a cyclotomic polynomial of q^2. Since

$\phi(2^a 3^b 5^c 7^d \cdots) = \phi(2^a)\phi(3^b)\phi(5^c)\phi(7^d)\cdots = 2^{a-1}{\cdot}23^{b-1}{\cdot}45^{c-1}{\cdot}67^{d-1}\cdots,$

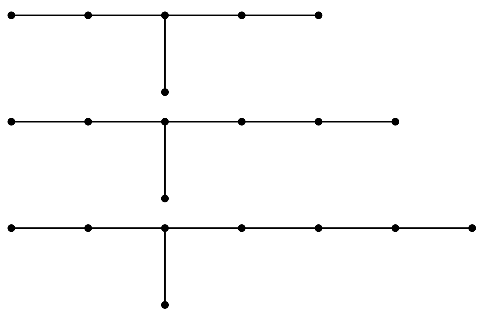
the cyclotomic polynomial of degree 8 is for $2^4 = 16$, $2^3 3 = 24$ or $2 \cdot 3 \cdot 5 = 30$. By an explicit computation based on the exclusion-inclusion principle,

$$\Phi_{16}(t) = \frac{t^{16} - 1}{t^8 - 1} = t^8 + 1,$$

$$\Phi_{24}(t) = \frac{(t^{24} - 1)(t^4 - 1)}{(t^{12} - 1)(t^8 - 1)} = t^8 - t^4 + 1,$$

$$\Phi_{30}(t) = \frac{(t^{30} - 1)(t^5 - 1)(t^3 - 1)(t^2 - 1)}{(t^{15} - 1)(t^{10} - 1)(t^6 - 1)(t - 1)} = t^8 + t^7 - t^5 - t^4 - t^3 + t + 1.$$

Comparing this with the equation for q, we conclude that q^2 is a 30-th primitive root of unity. In fact, for the choice $q = e^{i\pi/30}$, all the components of the relevant eigenvector is positive. Thus $\|E_8\| = 2\cos(\pi/30)$.



Problem 24.

(i) For integers m, n satisfying $1 \leq m \leq n$, show the identity

$$[m][n] = [n + m - 1] + [n + m - 3] + \cdots + [n - m + 3] + [n - m + 1].$$

(ii) Derive the identity

$$[5][n] - [2][3][n - 1] = [n + 4] - [n] - [n - 2]$$

and relate this with with the eqigenequation for $T_{1,2,n-1}$.

Problem 25. Investigate the Perron eigenvalue of the graph $T_{n,n,n}$ and show that

$$\lim_{n\to\infty} \|T_{n,n,n}\| = \frac{3}{\sqrt{2}}.$$

Problem 26. Show that $\|T_{k,l,m}\| < \frac{3}{\sqrt{2}}$.

7. Classification of Connected Graphs of Smaller Norm

Theorem 7.1. *Finite connected graphs of norm* 2 *are exactly one of the followings.*

(i) $\widetilde{A}_l$ $(l \geq 1)$.
(ii) $\widetilde{D}_l$ $(l \geq 3)$.
(iii) $\widetilde{T}_l$ $(l \geq 2)$.
(iv) $\widehat{T}_l$ $(l \geq 1)$.
(v) $\widetilde{E}_l$ $(l = 6, 7, 8)$.

Theorem 7.2. *Any connected graph of norm smaller than* 2 *is contained in a connected graph of norm* 2 *and isomorphic to one of the following graphs.*

(i) A_l $(l \geq 2)$.
(ii) D_l $(l \geq 3)$.
(iii) T_l $(l \geq 1)$.
(iv) E_l $(l = 6, 7, 8)$.

Let $\|\Gamma\| \leq 2$. Since $\|\widehat{T}_l\| = 2$ for $l \geq 1$, if Γ contains two or more loop edges, it must be $\widehat{T}_l$. Since $\|\widetilde{T}_l\| = 2$ for $2 \geq 1$, if Γ contains a triple vertex and a loop edge together, it should be $\widetilde{T}_l$.

So assume now that Γ contains no loop edges. Since $\|\widetilde{A}_l\| = 2$, Γ should not contain a proper closed circuit; Γ is $\widetilde{A}_l$ or a tree. Since $\|\widetilde{D}_4\| = 2$, Γ does not contain a quadruple vertice properly; $\Gamma = \widetilde{D}_4$ or Γ contains no quadruple vertices.

Since $\|\widetilde{D}_l\| = 2$ for $l \geq 5$, $\Gamma = \widetilde{D}_l$ if Γ contains two or more triple points and $\Gamma = T_{k,l,m}$ if Γ contains one triple vertex. If there is no triple vertex, $\Gamma = A_l$ has norm less than 2.

From $\|T_{2,2,2}\| = 2$, we see $\|T_{2,l,m}\| > 2$ for $(2, l, m) > (2, 2, 2)$ and are reduced to the case $T_{1,l,m}$. Again $\|T_{1,3,3}\| = 2$ shows that $\|T_{1,l,m}\| > 2$ for $(1, k, m) > (1, 3, 3)$.

Since $\|T_{1,2,5}\| = 2$, we see $\|T_{1,2,m}\| > 2$ for $m \geq 6$.

Finally $\|T_{1,1,m}\| = 2\cos(\pi/2(m + 2)) < 2$ for $m \geq 1$.

Problem 27. Investigate the norm of $T_{1,n,n}$ and show that $\lim_{n\to\infty} \|T_{1,n,n}\| = \sqrt{2 + \sqrt{5}}$. Note that $\sqrt{2 + \sqrt{5}} = \phi^{1/2} + \phi^{-1/2}$ with $\phi = (1 + \sqrt{5})/2$ (the Golden ratio).

Problem 28. Let C_n be the graph obtained from $\widetilde{A}_n$ by adding one more edge and vertex. Show that $\|C_n\| > \|C_{n+1}\|$ for $n \geq 1$ (decreasing!) and $\lim_{n\to\infty} \|C_n\| = \sqrt{2+\sqrt{5}}$.

Problem 29 (Challenging). Try to extend the classification results to oriented graphs.

8. Fusion Rule Algebras

Definition 8.1. A *-algebra $\mathbb{C}[S] = \sum_{s\in S} \mathbb{C}s$ with a distinguished countable basis S containing the unit element 1 is called an **fusion rule algebra** (or simply fusion algebra) if the following two conditions are satisfied.

(i) (Positivity) For x, y, $z \in S$, the coefficient N_{xyz} of 1 in xyz (i.e., $xyz = N_{xyz}1 + \dots$) is a non-negative integer.
(ii) (Duality) The $*$-operation makes S invariant globally (i.e., $x \in S$ implies $x^* \in S$) and satisfies $N_{x^*y} = \delta_{x,y}$.

Lemma 8.2. *Let $\mathbb{C}[S]$ be a fusion algebra.*

(i) *If N is extended to $\mathbb{C}[S]$ by $N(\sum_{s\in S} c(s)s) = c(1)$, then it is a tracial state in the sense that $N(ab) = N(ba)$, $N(a^*) = \overline{N(a)}$ for a, $b \in \mathbb{C}S$ and $N(1) = 1$. Moreover, we have*
$$N(c^*c) = \sum_{s\in S} |c(s)|^2.$$
(ii) *For x, $y \in S$,*
$$xy = \sum_{s\in S} N_{xys^*} s.$$
(iii) *For x, y, $z \in S$, we have*
$$N_{xyz} = N_{yzx}, \quad N_{xyz} = N_{z^*y^*x^*}.$$

In view of the second statement, we occasionally use the notation N_{xy}^z of structure constants for $N_{xyz^*} = N_{z^*xy}$.

Proof. (i) $N(ab) = N(ba)$ as well as the formula for $N(c^*c)$ is a consequence of the duality $N_{x^*y} = \delta_{x,y}$ and the relation $N(a^*) = \overline{N(a)}$ follows from $N(x) = \delta_{x,1}$ for $x \in S$.

(ii) If $xy = \sum_s c(s)s$ with $c(s) \in \mathbb{C}$, then the application of the tracial state N to xyz^* yields
$$N_{xyz^*} = \sum_{s\in S} c(s)N_{sz^*} = \sum_{s\in S} c(s)\delta_{z,s} = c(z).$$

(iii) is immediate from the trace property of N. □

Definition 8.3. Consider a *-algebra $\mathbb{C}[S]$ with a distinguished linear basis S containing the unit element 1 such that (i) the structure constants are non-negative reals, (ii) the set S is globally invariant under the *-operation and (iii) the coefficient of 1 in the expansion of s^*s

is strictly positive for any $s \in S$. From the previous lemma, fusion algebras fall into this class.

Definition 8.4. A multiplicative linear functional d on a *-algebra $\mathbb{C}[S]$ in the above definition is called a dimension function if $d(s) > 0$ and $d(s^*) = d(s)$ for any $s \in S$.

Proposition 8.5 (Sunder)**.** *Dimension function exists and is unique for a finite-dimensional *-algebra* $\mathbb{C}[S]$.

Proof. Let $C = \sum_{s \in S} s \in \mathbb{C}[S]$ and consider the linear operator on $\mathbb{C}[S]$ defined by the right multiplication of C. Then it is irreducible as a positive matrix (operator) by the property $s * s = c1 + \dots$ with $c > 0$. Let $\xi = \sum_{x \in S} \xi(x)x$ be its Perron-Frobenius eigenvector: $\xi(x) > 0$ for $x \in S$ and ξC is a positive multiple of ξ.

Since $s\xi \in \mathbb{C}[S]$ is again an eigenvector with non-negative coefficients, the uniqueness of Perron-Frobenius vector implies that $s\xi$ is proportional to ξ: let $d(s) > 0$ be defined by $s\xi = d(s)\xi$. Clearly this defines a multiplicative functional on $\mathbb{C}[S]$.

Let d' be another multiplicative functional satisfying $d'(s) > 0$ for $s \in S$. Then $\sum_{s \in S} d'(S)s$ is $\square$