1. Introduction

§1.1 Introduction

Let X be a mathematical space. For example, X could be the unit interval $[0, 1]$, a circle, a torus, or something far more complicated like a Cantor set. Let $T : X \to X$ be a function that maps X into itself.

Let $x \in X$ be a point. We can repeatedly apply the map T to the point x to obtain the sequence:

$$\{x, T(x), T(T(x)), T(T(T(x))), \ldots, ...\}.$$

We will often write $T^n(x) = T(\cdots(T(T(x))))$ (n times). The sequence of points $x, T(x), T^2(x), \ldots$ is called the *orbit* of x.

We think of applying the map T as the passage of time. Thus we think of $T(x)$ as where the point x has moved to after time 1, $T^2(x)$ is where the point x has moved to after time 2, etc.

Some points $x \in X$ return to where they started. That is, $T^n(x) = x$ for some $n > 1$. We say that such a point x is *periodic* with *period* n.

By way of contrast, points may move move densely around the space X. (A sequence is said to be dense if (loosely speaking) it comes arbitrarily close to every point of X.)

If we take two points x, y of X that start very close then their orbits will initially be close. However, it often happens that in the long term their orbits move apart and indeed become dramatically different. This is known as *sensitive dependence on initial conditions*, and is popularly known as *chaos*.

In general, for a given dynamical system T it is impossible to understand the orbit structure of *every* orbit. Ergodic theory takes a more qualitative approach: we aim to describe the long term behaviour of a typical orbit, at least in the case when T satisfies a technical condition called 'measure-preserving'.

To make the notion of 'typical' precise, we need to use measure theory. Roughly speaking, a measure is a function that assigns a 'size' to a given subset of X. One of the simplest measures is Lebesgue measure on $[0, 1]$; here the measure of an interval $[a, b] \subset [0, 1]$ is just its length $b - a$.

Let $T : [0, 1] \to [0, 1]$ and fix a subinterval $[a, b] \subset [0, 1]$. Let $x \in [0, 1]$. What is the frequency with which the orbit of x hits the set $[a, b]$? Recall that the characteristic function χ_A of a subset A is defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Then the number of times the first n points of the orbit of x hits $[a, b]$ is given by

$$\sum_{j=0}^{n-1} \chi_{[a,b]}(T^j(x)).$$

Thus the proportion of the first n points in the orbit of x that lie in $[a, b]$ is equal to

$$\frac{1}{n}\sum_{j=0}^{n-1} \chi_{[a,b]}(T^j(x)).$$

Hence the frequency with which the orbit of x lies in $[a, b]$ is given by

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \chi_{[a,b]}(T^j(x))$$

(assuming of course that this limit exists!).

One of the main results of the course, namely Birkhoff's ergodic theorem, tells us that when T is ergodic (a technical property—albeit an important one—that we won't define here) then for 'most' orbits the above frequency is equal to the measure of the interval $[a, b]$. In the case of Lebesgue measure, this means that:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \chi_{[a,b]}(T^j(x)) = b - a, \text{ for almost all } x \in X.$$

(Here 'almost all' is the technical measure-theoretic way of saying 'most'.)

One way of looking at Birkhoff's ergodic theorem is the following: the time average of a typical point $x \in X$ (i.e. the frequency with which its orbit lands in a given subset) is equal to the space average (namely, the measure of that subset).

In this course, we develop the necessary background that builds up to Birkhoff's ergodic theorem, together with some illuminating examples. We also study en route some interesting diversions to other areas of mathematics, notably number theory.

§1.2 Introducing the doubling map

Let $X = [0, 1]$ denote the unit interval. Define the map $T : X \to X$ by:

$$T(x) = 2x \bmod 1 = \begin{cases} 2x & \text{if } 0 \le x < 1/2 \\ 2x - 1 & \text{if } 1/2 \le x \le 1 \end{cases}$$

('mod 1' stands for 'modulo 1' and means 'ignore the integer part'; for example 3.456 mod 1 is .456).

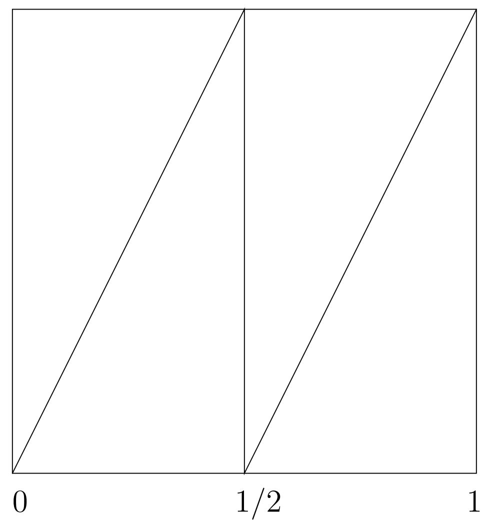

Figure 1.1: The graph of the doubling map $x \mapsto 2x \bmod 1$

Exercise 1.1
By sketching the orbit in the above diagram, indicate that $7/15$ *is periodic. Try sketching the orbits of some points near* $7/15$.

In §1.1 we mentioned in passing that we will be interested in a technical condition called 'measure-preserving'. We can illustrate this property here. Fix an interval $[a, b]$ and consider the set

$$T^{-1}[a, b] = \{x \in [0, 1] \mid T(x) \in [a, b]\}.$$

One can easily check that

$$T^{-1}[a, b] = \left[\frac{a}{2}, \frac{b}{2}\right] \cup \left[\frac{a+1}{2}, \frac{b+1}{2}\right],$$

so that $T^{-1}[a, b]$ is the union of two intervals, each of length $(b-a)/2$. Hence the length of $T^{-1}[a, b]$ is equal to $b - a$, which is the same as the length of $[a, b]$.

§1.3 Leading digits

The leading digit of a number $n \in \mathbb{N}$ is the digit (between 1 and 9) that appears at the leftmost-end of n when n is written in base 10. Thus, the leading digit of 4629 is 4, etc.

Consider the sequence 2^n:

$$1, 2, 4, 8, 16, 32, 64, 128, \ldots$$

and consider the sequence of leading digits:

$$1, 2, 4, 8, 1, 3, 6, 1, \ldots.$$

Exercise 1.2
By writing down the sequence of leading digits for 2^n for $n = 1, 2, \ldots,$ something large of your choosing, try guessing the frequency with which the digit 1 appears as a leading digit. (Hint: it isn't 3/10ths.) Do the same for the digit 2. Can you guess the frequency with which the digit r appears?

We will study this problem in greater detail later in the course...