On a characterization of Cauchy-Stieltjes transforms of compactly supported complex measures and its applications

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Synopsis

Cauchy-Stieltjes transform is one of key notions in analysis of quantum probability and has been investigated from various points of views.

We shall here give a pragmatic characterization of the range of the transform applied to compactly supported complex measures in the real line, which enables us to identify certain holomorphic functions with Cauchy-Stieltjes transforms.

As a simple application of the characterization, we work out a spectral analysis of linear functionals of some polynomial hypergroups parametrized by $0 < r \leq 1/2$.

When $r = 1/2l$ with l a positive integer, it is isomorphic to the commutative algebra of radial functions on the free group of l generators and the spectral properties of Haagerup's positive definite functions are then described explicitly, which turns out to be useful in establishing Plancherel type formulae.

Cauchy-Stieltjes Transform

Given a complex measure $\mu(dt)$ on $\mathbb{R}$,
its **Cauchy-Stieltjes transform** $C(z)$ is a holomorphic function of $z = x + iy \in \mathbb{C} \setminus \mathbb{R}$ $(y \neq 0)$ defined by

$$C(z) = \int \frac{1}{t-z}\,\mu(dt) = \int \frac{1}{t-x-iy}\,\mu(dt).$$

From

$$\left|\frac{y}{t-x-iy}\right| \leq 1 \quad (y \neq 0), \lim_{y\to\pm\infty} \frac{y}{t-x-iy} = i$$

and

$$\lim_{y\to 0} \frac{y}{t-x-iy} = \begin{cases} i & (t = x) \\ 0 & (t \neq x) \end{cases},$$

we have $|yC(x+iy)| \leq |\mu|(\mathbb{R})$ and

$$\lim_{y\to\pm\infty} yC(x+iy) = i\mu(\mathbb{R}), \quad \lim_{y\to 0} yC(x+iy) = i\mu(\{x\}).$$

When μ is a signed measure, C is **real** in the sense that

$$\overline{C(z)} = C(\overline{z}) \quad (z \in \mathbb{C} \setminus \mathbb{R}).$$

Theorem (Stieltjes Inversion Formula)

The complex measure μ is recovered from its Cauchy-Stieltjes transform $C(z)$ as a Radom measure on $C_c(\mathbb{R})$ by

$$2\pi i\mu(dx) = \lim_{y\to+0}\Big(C(x+iy) - C(x-iy)\Big)\,dx.$$

The support $[\mu]$ of μ can be read off from $C(z)$ easily:
For $x \in \mathbb{R}$,
$x \notin [\mu] \iff C(z)$ is analytic in a neighborhood of $x \in \mathbb{C}$.

Given a holomorphic function $\varphi(z)$ of $z \in \mathbb{C} \setminus \mathbb{R}$, its **support** is

$$[\varphi] = \{c \in \overline{\mathbb{C}}; \varphi(z) \text{ is not analytic at } z = c\}.$$

Thus $[C]$ is the closure of $[\mu] \subset \mathbb{R}$ in the Riemann sphere $\overline{\mathbb{C}}$.

Example

For a Dirac measure $\delta(t-c)$ at $c \in \mathbb{R}$,

$$C(z) = \frac{1}{c-z}, \quad [C] = \{c\},$$

$$\begin{aligned} 2\pi i \delta(x-c)\,dx &= \lim_{y\to+0}\left(\frac{1}{c-x-iy} - \frac{1}{c-x+iy}\right)dx \\ &= \lim_{y\to+0}\frac{2iy}{(c-x)^2+y^2}\,dx \end{aligned}$$

Observation:
If C is the Cauchy-Stieltjes transform of a complex measure, isolated singularity of $C(z)$ at a real point must be a simple pole because higher order poles give rise to derivatives of $\delta(x-c)$.

Theorem (cf. Akhierzer, §3.1)

A real holomorphic function $C(z)$ of $z \in \mathbb{C} \setminus \mathbb{R}$ is the Cauchy-Stieltjes transform of a positive (non-zero) finite measure if and only if the following conditions are satisfied.

1. *Im* $C(z) > 0$ (*Im* $z > 0$) *and*
2. $\{yC(iy); y \geq 1\}$ *is a bounded set in* $\mathbb{C}$.

From the expression

$$C(x+iy) = \int \frac{t-x}{(t-x)^2+y^2}\,\mu(dt) + i\int \frac{y}{(t-x)^2+y^2}\,\mu(dt),$$

the conditions are clearly necessary and non-trivial is sufficiency, which can be deduced from the integral representation of $C(z)$ satisfying the positivity condition (i).

Theorem

A holomorphic function $C(z)$ of $z \in \mathbb{C} \setminus \mathbb{R}$ is the Cauchy-Stieltjes transform of a compactly supported complex measure if and only if the following conditions are satisfied.

1. *$C(z)$ is analytically extended to a neighborhood of $\infty \in \overline{\mathbb{C}}$ so that $C(\infty) = 0$.*
2. *The limit $\lim_{y \to +0} \big(C(x+iy) - C(x-iy)\big)\, dx$ exists as a complex Radon measure (as a linear functional) on $C_c(\mathbb{R})$.*

Notice that a complex measure on $\mathbb{R}$ is determined by the integration on $C_c(\mathbb{R})$ in the second condition.
The first condition implies that $C(x+iy) - C(x-iy)$ is integrable as a function of $x \in \mathbb{R}$ for any $0 \neq y \in \mathbb{R}$ and the limit in the second condition can be strengthened to functions in $C_0(\mathbb{R})$.

Example

For the Dirac mass at $c \in \mathbb{R}$, $C(z) = \frac{1}{c-z}$ and

$$C(x+iy)dx = \frac{c-x+iy}{(c-x)^2+y^2}dx$$

converges for $y \to \pm 0$ as a linear functional on $C_c^1(\mathbb{R})$ but not as a Radon measure on $C_c(\mathbb{R})$.

However, the combination

$$C(x+iy) - C(x-iy) = \frac{2iy}{(c-x)^2+y^2}$$

is an integrable function of $x \in \mathbb{R}$ for each $y > 0$ and the complex measure $(C(x+iy) - C(x-iy))dx$ on $\mathbb{R}$ converges to $2\pi\delta(x-c)$ as a linear functional on $C_c(\mathbb{R})$ for $y \to +0$.

$F = \langle f_1, \cdots, f_l \rangle$: a free group of l-generators.

$F_n = \{g \in F; |g| = n\}$: the set of words of length n.

$F_1 = \{f_1^{\pm 1}, \cdots, f_l^{\pm 1}\}$ and $|F_n| = 2l(2l-1)^{n-1}$ for $n \geq 1$.

$\mathbb{C}F = \sum_{g \in G} \mathbb{C}g$: the algebraic group algebra ($g^* = g^{-1}$).

Radial elements $h_n = h_n^* \in \mathbb{C}F$ are defined to be

$$h_n = \frac{1}{|F_n|} \sum_{g \in F_n} g \quad \text{for } n \geq 0$$

with $h_0 = e$ the unit element in $\mathbb{C}F$, which satisfy

$$h_1 h_n = \frac{1}{2l} h_{n-1} + \frac{2l-1}{2l} h_{n+1} = h_n h_1$$

and give a linear basis of a commutative *-subalgebra

$$\sum_{n \geq 0} \mathbb{C}h_n \subset \mathbb{C}F.$$

Polynomial Hypergroups

Given a real $1 \neq r \in \mathbb{R}$, let $\mathcal{A} = \sum_{n \geq 0} \mathbb{C} h_n$ be a universal *-algebra with h_0 the unit element and $h_n = h_n^*$ satisfying

$$h_1 h_n = r h_{n-1} + (1-r) h_{n+1} \quad (n \geq 1),$$

which is *-isomorphic to the polynomial algebra $\mathbb{C}[t]$ by the correspondence $h_n \leftrightarrow P_n(t)$.

Here $P_n(t)$ is a polynomial of degree n specified by the recurrence relation

$$t P_n(t) = r P_{n-1}(t) + (1-r) P_{n+1}(t)$$

with the initial condition $P_0(t) = 1$ and $P_1(t) = t$.

Generally a *-algebra with a distinguished basis $\{h_n = h_n^*\}_{n\geq 0}$ with h_0 the unit element is called a (hermitian) **hypergroup** if, in the expansion $h_j h_k = \sum_{n\geq 0} c_{j,k,n} h_n$, coefficients satisfy

$$c_{j,k,n} \geq 0, \quad \sum_n c_{j,k,n} = 1 \quad \text{and} \quad c_{j,k,0} > 0 \iff j = k.$$

For the *-algebra $\mathcal{A}$, coefficients are computed easily to see that

$$\mathcal{A} = \sum_{n\geq 0} \mathbb{C} h_n \text{ is a hypergroup} \iff 0 < r \leq \frac{1}{2}.$$

Note that $0 < r = 1/2l \leq 1/2$ for radial elements in $\mathbb{C}F$.

The hypergroup $\mathcal{A}$ is completed to a Banach *-algebra with respect to the ℓ^1-norm

$$\|\sum_{n\geq 0}\alpha_n h_n\|_1 = \sum_{n\geq 0}|\alpha_n|.$$

Let A be the accompanied commutative C*-algebra.
In the case of radial elements, we see that $A \subset C^*(F)$.

Under the correspondence
a linear functional $\phi : \mathcal{A} \to \mathbb{C} \iff$ a sequence $(\phi_n = \phi(h_n))$,

1. ϕ is ℓ^1-bounded $\iff (\phi_n)$ is bounded.
2. A multiplicative functional $\phi \iff c = \phi(h_1) \in \mathbb{C}$.
3. A *-homomorphism $\phi \iff c = \phi(h_1) \in \mathbb{R}$.
4. A bounded *-homomorphism $\phi \iff c = \phi(h_1) \in [-1, 1]$.

The spectrum $\sigma(h_1)$ of h_1 in A is therefore $[-1, 1]$ and

$$A = C^*(h_0, h_1) \cong C([-1, 1]).$$

Analytic Functionals

Definition

A linear functional $\phi : \mathcal{A} \to \mathbb{C}$ is **analytic** if the generating function $\phi(z) = \sum_{n\geq 0} \phi_n z^n$ converges at some $0 \neq z \in \mathbb{C}$.

Any C*-bounded functional ϕ is analytic: $\|a\| \leq \|a\|_1$ $(a \in \mathcal{A})$, whence (ϕ_n) is bounded and $\sum_{n\geq 0} \phi_n z^n$ converges for $|z| < 1$. Moreover ϕ is represented by a complex Radon measure μ on $[-1, 1]$ in such a way that

$$\phi(h_n) = \int_{[-1,1]} P_n(t)\, \mu(dt).$$

The C*-boundedness of a given sequence $(\phi(h_n))$ is then equivalent to finding μ.

Thanks to the recurrence relation, the generating function

$$P(z,t) = \sum_{n \geq 0} z^n P_n(t)$$

is expressed by

$$P(z,t) = \frac{1 - r - rzt}{1 - r - zt + rz^2}$$

and we see that

$$\begin{aligned} \phi(z) = \sum_{n \geq 0} \phi(h_n) z^n &= \int_{[-1,1]} \frac{1 - r - rzt}{1 - r - zt + rz^2} \, \mu(dt) \\ &= r\phi_0 + \frac{r^2 z^2 - (1-r)^2}{z} \int_{[-1,1]} \frac{1}{t - rz - (1-r)/z} \, \mu(dt). \end{aligned}$$

Introduce a new variable by $w = rz + (1-r)/z \iff$

$$z = \frac{w - \sqrt{w^2 - 4r(1-r)}}{2r} \sim \frac{1-r}{w} \ (w \to \infty).$$

so that

$$\int_{[-1,1]} \frac{1}{t-w}\,\mu(dt) = \frac{z}{r^2z^2 - (1-r)^2}(\phi(z) - r\phi_0).$$

Thus finding μ is reduced to the validity of the Stieltjes inversion formula for the right hand side.

Notice here that, given a complex Radon measure μ on $[-1,1]$, the left hand side is analytic in $1/w$ and vanishes at $w = \infty$, which in turn determines an analytic function $\phi(z)$ of z around $z = 0$ by equating it with the right hand side.

Recall that the Cauchy-Stieltjes transform

$$C(w) = \int_{[-1,1]} \frac{1}{t-w}\, \mu(dt)$$

of μ is holomorphic on $\overline{\mathbb{C}} \setminus [-1,1]$ and vanishes at $w = \infty$.
The Stieltjes inversion formula: In the weak* topology, we have

$$2\pi i \mu(dt) = \lim_{\epsilon \to +0} \Big(C(t+i\epsilon) - C(t-i\epsilon) \Big)\, dt.$$

Example

When $\phi(z) \equiv 1$, μ is the Kesten measure:

$$C(w) = \frac{(2r-1)w + \sqrt{w^2 - 4r(1-r)}}{2r(1-w^2)}$$

$$\iff \mu(dt) = \frac{1}{2\pi r} \frac{\sqrt{4r(1-r) - t^2}}{1-t^2}\, dt.$$

Functionals of Geometric Series

Given $v \in \mathbb{C}$, let

$$\phi(z) = \frac{1}{1 - vz} = \sum_{n=0}^{\infty} v^n z^n \Longleftrightarrow \phi_n = \phi(h_n) = v^n \ (n \geq 0).$$

Here comes into the Joukowsky transform of v:

$$c_r(v) = r\frac{1}{v} + (1 - r)v.$$

The "Cauchy-Stieltjes transform" $C(w)$ of ϕ then takes the form

$$\frac{v^{-1} - v}{2} \frac{(2r - 1)w + \sqrt{w^2 - 4r(1 - r)}}{(1 - w^2)(c_r(v) - w)} + \frac{1}{c_r(v) - w},$$

which is reduced at $v = 0$ to

$$C(w) = \frac{(2r - 1)w + \sqrt{w^2 - 4r(1 - r)}}{2r(1 - w^2)}.$$

Remark

1. If $\mathcal{A} \subset \mathbb{C}F$ consists of radial elements ($r = 1/2l$) and $0 < v < 1$, ϕ is the restriction of the Haagerup's positive definite function (a state on the group algebra) to $\mathcal{A} \subset \mathbb{C}F$.
2. When $v = 0$, C is the Cauchy-Stieltjes transform of the Kesten measure.
3. H. Yoshida found that

$$-C(w) = \cfrac{1}{w - v - \cfrac{r(1-v^2)}{w + rv - \cfrac{r(1-r)}{w - \cfrac{r(1-r)}{w - \ddots}}}}.$$

Theorem

The functional ϕ of geometric series is C-bounded if and only if υ falls into one of the following cases.*

1. $|\upsilon| < \sqrt{r/(1-r)}$ *or* $\upsilon = \pm\sqrt{r/(1-r)}$,

$$\mu_c(dt) = \frac{\upsilon^{-1} - \upsilon}{2\pi} \frac{\sqrt{4r(1-r) - t^2}}{(1-t^2)(c_r(\upsilon) - t)}\, dt.$$

2. $\upsilon \in \mathbb{R}$ *and* $\sqrt{r/(1-r)} < |\upsilon| \le 1$ ($c_r(\upsilon) \in \mathbb{R}$ *and* $2\sqrt{r(1-r)} < |c_r(\upsilon)| \le 1$ *then*),

$$\mu(dt) = \mu_c(dt) + \omega(\upsilon)\delta(t - c_r(\upsilon)),\ \omega(\upsilon) = \frac{1 - c_r(\upsilon^2)}{1 - c_r(\upsilon)^2}.$$

A C-bounded ϕ is positive if and only if* $-1 \le \upsilon \le 1$.

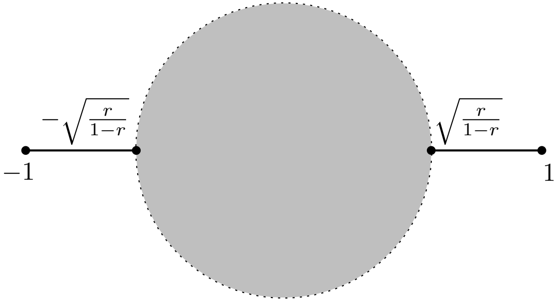

Figure: υ-region of C*-boundedness

Corollary (Free Poisson as a normalized limit)

In terms of the normalized parameter $s = \dfrac{t - v}{\sqrt{r(1 - v^2)}}$*, the normalized distribution converges to* $\nu(ds)$ *under the limit* $(r, v) \to (0, 0)$ *keeping* $\gamma = v/\sqrt{r} \in \mathbb{R}$ *constant:*

1. *For* $-1 \le \gamma \le 1$*,*

$$\nu(ds) = \frac{\sqrt{4 - (s+\gamma)^2}}{2\pi(1 - \gamma s)}\, ds.$$

2. *For* $\pm\gamma \ge 1$*,*

$$\nu(ds) = \frac{\sqrt{4 - (s+\gamma)^2}}{2\pi(1 - \gamma s)}\, ds + \left(1 - \frac{1}{\gamma^2}\right) \delta(s - 1/\gamma).$$

Sketch of the Proof

Introduce the regular spectrum σ_r by

$$\sigma_r = [-2\sqrt{r(1-r)}, 2\sqrt{r(1-r)}] \subset [-1,1]$$

and observe that

$$\sqrt{w^2 - 4r(1-r)} = w\sqrt{1 - \frac{4r(1-r)}{w^2}}$$

vanishes at $w = \infty$ and is holomorphic on $\overline{\mathbb{C}} \setminus \sigma_r$.
The same property holds for the Kesten part

$$\frac{(2r-1)w + \sqrt{w^2 - 4r(1-r)}}{1 - w^2}.$$

The possible singularity of $C(w)$ other than $\pm 2\sqrt{r(1-r)}$ is

$$c_r(v) = \frac{r}{v} + (1-r)v.$$

Recall some properties of the Joukowski transform $c_r(v)$ of v:

1. Invariance under the inversion $v \mapsto \dfrac{r}{1-r}\dfrac{1}{v}$
   with critical fixed points
   $$v = \pm\sqrt{\frac{r}{1-r}} \iff c_r(v) = \pm 1.$$
2. Both $|v| < \sqrt{\frac{r}{1-r}}$ and $|v| > \sqrt{\frac{r}{1-r}}$ (including $v = \infty$) are biholomorphically mapped onto a slit sphere $\overline{\mathbb{C}} \setminus \sigma_r$.
3. Circle $|v| = \rho \neq \sqrt{\frac{r}{1-r}}$ is mapped to an ellipse of vertices
   $\frac{r}{\rho} + (1-r)\rho$, $\frac{r}{\rho} - (1-r)\rho$,
   which shrinks to the segment σ_r when ρ approaches $\sqrt{\frac{r}{1-r}}$.
4. $-1 \leq c_r(v) \leq 1$ and $|v| > \sqrt{\frac{r}{1-r}}$
   $$\iff v \in \mathbb{R} \text{ and } \sqrt{\frac{r}{1-r}} < |v| \leq 1.$$

Singularity criterion of $c_r(v)$:

$$\lim_{w\to c_r(v)}(c_r(v)-w)C(w)=\begin{cases}0 & (|v|<\sqrt{r/(1-r)})\\ \omega(v) & (|v|>\sqrt{r/(1-r)})\end{cases}.$$

Combined with properties of Joukowski transform,
C is holomorphic on $\overline{\mathbb{C}}\setminus[-1,1]$ if and only if the following alternatives hold:

1. $|v|\le\sqrt{\frac{r}{1-r}}$. Then C is holomorphic on $\overline{\mathbb{C}}\setminus\sigma_r$.
2. $\sqrt{\frac{r}{1-r}}<\pm v\le 1 \iff 2\sqrt{r(1-r)}<\pm c_r(v)\le 1$.
   Then C is meromorphic on $\overline{\mathbb{C}}\setminus\sigma_r$ with $c_r(v)$ the unique simple pole.

The characterization of Cauchy-Stieltjes transform is applied to the following cases:

1. $|v| < \sqrt{r/(1-r)}$. A continuous measure on σ_r appears.
2. $\sqrt{r/(1-r)} < \pm v \leq 1$. Adding to the continuous measure on σ_r, an atomic measure appears at $c_r(v) \in [-1,1] \setminus \sigma_r$.
3. $|v| = \sqrt{r/(1-r)}$ but $v \neq \pm\sqrt{r/(1-r)}$. $C(w)$ is not a Cauchy-Stieltjes transform.
4. In the critical case $v = \pm\sqrt{r/(1-r)} \iff c_r(v) = \pm 2\sqrt{r(1-r)}$, the limit measure appears from (1) or (2).

The critical case: $c_r(v) = \pm c$ with $c = 2\sqrt{r(1-r)}$.

$$\frac{\sqrt{w^2 - 4r(1-r)}}{c_r(v) - w} = -\sqrt{\frac{w \pm c}{w \mp c}}.$$

We may assume $c_r(v) = c \iff v = \sqrt{\frac{r}{1-r}}$ by symmetry.

By an obvious estimate

$$\left|\frac{t + is + c}{t + is - c}\right| \leq \frac{|t + c| + |s|}{|s|},$$

$$s\left|\sqrt{\frac{t + is + c}{t + is - c}}\right| \leq \sqrt{|s|(|t + c| + |s|)}$$

is bounded for a bounded (t, s) and converges to 0 as $s \to 0$ for any $t \in \mathbb{R}$.

From another estimate

$$\left|\frac{t+is+c}{t+is-c}\right| \leq \frac{|t+c|+|s|}{|t-c|},$$

$\left|\sqrt{\frac{w+c}{w-c}}\right|$ is majorized uniformly in $0<|s|\leq 1$ by a locally integrable function $\sqrt{(|t+c|+1)/|t-c|}$ of $t\in\mathbb{R}$ and

$$\lim_{s\to\pm 0}\sqrt{\frac{t+is+c}{t+is-c}} = \begin{cases} \sqrt{\frac{t+c}{t-c}} & (t>c) \\ \mp i\sqrt{\frac{t+c}{c-t}} & (-c<t<c) \\ \sqrt{\frac{-c-t}{c-t}} & (t<-c) \end{cases},$$

whence

$$\lim_{s\to+0}\left(\sqrt{\frac{t+is+c}{t+is-c}}-\sqrt{\frac{t-is+c}{t-is-c}}\right)dt = -2i\sqrt{\frac{t+c}{c-t}}dt$$

as a Radon measure (supported by σ_r) on $C_c(\mathbb{R})$.

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