Numerical Linear Algebra Assignment 6

Exercise 1. (Shufang Xu, 10 points)

Consider the stationary iterative method

$$\mathbf{x}^{(m)} = \mathbf{R}\mathbf{x}^{(m-1)} + \mathbf{c}.$$

Assume $\mathbf{R} \in \mathbb{C}^{n\times n}$ and the spectral radius $\rho(\mathbf{R}) = 0$. For any given $\mathbf{c} \in \mathbb{R}^n$ and $\mathbf{x}^{(0)} \in \mathbb{C}^n$, prove that the nth iterate $\mathbf{x}^{(n)}$ is the solution of $(\mathbf{I} - \mathbf{R})\mathbf{x} = \mathbf{c}$.

Exercise 2. (Zhihao Cao, 10 points)

Consider the stationary iterative method

$$\mathbf{x}^{(m)} = \mathbf{R}\mathbf{x}^{(m-1)} + \mathbf{c}.$$

Assume that $\|\mathbf{R}\|_2 < 1$. Prove that

$$\|\mathbf{x}^{(m)} - \mathbf{x}_*\|_2 \le \frac{\|\mathbf{R}\|_2^m}{1 - \|\mathbf{R}\|_2}\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|_2,$$

where $\mathbf{x}_*$ is the solution of $(\mathbf{I} - \mathbf{R})\mathbf{x} = \mathbf{c}$.

Exercise 3. (Shufang Xu, 10 points)

Consider the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ with

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}.$$

Discuss the convergence of Jacobi's method and Gauss–Seidel method for this linear system.

Exercise 4. (10 points)

Prove that $0 < \omega < 2$ is required for the convergence of SOR(ω) for all starting vectors.

Exercise 5. (Programming, 10 points)

Construct a linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ with a strictly row diagonally dominant 15×15 matrix $\mathbf{A}$. Plot the convergence history of Jacobi's, Gauss–Seidel, SOR(0.5), and SSOR(0.5) methods in one figure. You must use Matlab's `semilogy`: the x-axis is the iteration index m, and the y-axis is $\|\mathbf{x}^{(m)} - \mathbf{A}^{-1}\mathbf{b}\|_2$. For each method, stop at the 30th iteration.

Additional Exercise 1.

Let $\mathbf{A} \in \mathbb{C}^{m\times m}$. Prove that $\lim_{k\to\infty} \mathbf{A}^k = \mathbf{0} \;\Leftrightarrow\; \rho(\mathbf{A}) < 1 \;\Leftrightarrow\; \sum_{k=0}^{\infty} \mathbf{A}^k = (\mathbf{I} - \mathbf{A})^{-1}$.

Additional Exercise 2.

Let $\|\cdot\|$ be a given induced matrix norm. Let $\mathbf{A} \in \mathbb{C}^{m\times m}$. Prove that $\lim_{k\to\infty} \|\mathbf{A}^k\|^{\frac{1}{k}} = \rho(\mathbf{A})$.

Additional Exercise 3.

Let $\|\cdot\|$ be a given induced matrix norm. Assume that $\mathbf{A} \in \mathbb{C}^{m\times m}$ and $\|\mathbf{A}\| < 1$. Prove that $\mathbf{I} - \mathbf{A}$ is nonsingular and

$$\|(\mathbf{I} - \mathbf{A})^{-1}\| \le \frac{1}{1 - \|\mathbf{A}\|}.$$