

# RSMAR: An iterative method for range-symmetric linear systems

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joint work with Jia-Jun Fan and Fang Wang

Beijing Normal University at Zhuhai, on 03/30/2024

## Two main references

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- A. Montoison, D. Orban, and M. A. Saunders.  
MINARES: An iterative solver for symmetric linear systems.  
arXiv:2310.01757, 2023.
- Y. Liu, A. Milzarek, and F. Roosta.  
Obtaining pseudo-inverse solutions with MINRES.  
arXiv:2309.17096, 2023.

# Outline

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- ① Preliminaries
- ② GMRES-type methods for singular range-symmetric systems
- ③ Symmetric systems
- ④ Numerical experiments
- ⑤ Summary

# The pseudoinverse solution

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- $\mathbf{Ax} = \mathbf{b}$ ,  $\mathbf{A} \in \mathbb{R}^{m \times n}$ ,  $\mathbf{b} \in \mathbb{R}^m$ .

Consistent if  $\mathbf{b} \in \text{range}(\mathbf{A})$ , otherwise, inconsistent.

- $\mathbf{A}^\dagger$ : the Moore–Penrose inverse of  $\mathbf{A}$
- $\mathbf{A}^\dagger \mathbf{b}$ : the pseudoinverse solution

$\mathbf{Ax} = \mathbf{b}$	$\text{rank}(\mathbf{A})$	$\mathbf{A}^\dagger \mathbf{b}$
consistent	$= n$	unique solution
consistent	$< n$	unique minimum 2-norm solution
inconsistent	$= n$	unique least-squares (LS) solution
inconsistent	$< n$	unique minimum 2-norm LS solution

# Range-symmetric systems

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- range-symmetric  $\mathbf{A} \in \mathbb{R}^{n \times n}$ :

$$\text{range}(\mathbf{A}) = \text{range}(\mathbf{A}^\top).$$

- Fact I:

$$\mathbf{A} = \mathbf{U} \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{U}^\top.$$

( $\mathbf{C}$  is invertible and  $\mathbf{U}$  is orthogonal.)

- Fact II:

$$\mathbf{A}^\dagger = \mathbf{A}^D = \mathbf{U} \begin{bmatrix} \mathbf{C}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{U}^\top.$$

- Fact III:

$$\begin{aligned} \mathbf{A}^\dagger \mathbf{b} + \text{null}(\mathbf{A}) &= \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}^\top \mathbf{A} \mathbf{x} = \mathbf{A}^\top \mathbf{b}\} \\ &= \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}^2 \mathbf{x} = \mathbf{A} \mathbf{b}\}. \end{aligned}$$

# Krylov subspaces

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- $\mathbf{x}_0 \in \mathbb{R}^n$ ,  $\mathbf{r}_0 = \mathbf{b} - \mathbf{A}\mathbf{x}_0$ ,
- $\mathcal{K}_k(\mathbf{A}, \mathbf{r}_0)$ : the  $k$ th Krylov subspace

$$\mathcal{K}_k(\mathbf{A}, \mathbf{r}_0) := \text{span}\{\mathbf{r}_0, \mathbf{A}\mathbf{r}_0, \dots, \mathbf{A}^{k-1}\mathbf{r}_0\}.$$

- $\ell$ : the **grade** of  $\mathbf{r}_0$  with respect to  $\mathbf{A}$ , i.e.,  $\ell$  satisfies

$$\dim \mathcal{K}_k(\mathbf{A}, \mathbf{r}_0) = \begin{cases} k, & \text{if } k \leq \ell, \\ \ell, & \text{if } k \geq \ell + 1. \end{cases}$$

- *If  $\mathbf{b} \notin \text{range}(\mathbf{A})$ , there is at most one LS solution in  $\mathbf{x}_0 + \mathcal{K}_{\ell-1}(\mathbf{A}, \mathbf{r}_0)$ , and if  $\mathbf{b} \in \text{range}(\mathbf{A})$ , at most one solution in  $\mathbf{x}_0 + \mathcal{K}_{\ell}(\mathbf{A}, \mathbf{r}_0)$ .*

# GMRES for singular range-symmetric systems

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- The  $k$ th approximate solution at step  $k$  of GMRES:

$$\mathbf{x}_k := \operatorname{argmin}_{\mathbf{x} \in \mathbf{x}_0 + \mathcal{K}_k(\mathbf{A}, \mathbf{r}_0)} \|\mathbf{b} - \mathbf{A}\mathbf{x}\|.$$

- For singular range-symmetric  $\mathbf{A}$  we have [BW97]:
  - (i) If  $\mathbf{b} \in \operatorname{range}(\mathbf{A})$ , then for all  $0 \leq k \leq \ell - 1$ ,  $\mathbf{x}_k$  is not a solution, and

$$\mathbf{x}_\ell = \mathbf{A}^\dagger \mathbf{b} + (\mathbf{I} - \mathbf{A}^\dagger \mathbf{A}) \mathbf{x}_0,$$

the orthogonal projection of  $\mathbf{x}_0$  onto the solution set

$$\mathbf{A}^\dagger \mathbf{b} + \operatorname{null}(\mathbf{A}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b}\}.$$

- (ii) If  $\mathbf{b} \notin \operatorname{range}(\mathbf{A})$ , then  $\mathbf{x}_{\ell-1}$  is a least squares solution.

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[BW97] P. N. Brown and H. F. Walker. *GMRES on (nearly) singular systems*. SIMAX, 1997.

## A lifting strategy [LMR23]

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- $\mathbf{r}_{\ell-1} := \mathbf{b} - \mathbf{A}\mathbf{x}_{\ell-1}$ .
- If  $\text{range}(\mathbf{A}) = \text{range}(\mathbf{A}^\top)$  and  $\mathbf{b} \notin \text{range}(\mathbf{A})$ , then the lifted vector,

$$\tilde{\mathbf{x}}_{\ell-1} := \mathbf{x}_{\ell-1} - \frac{\mathbf{r}_{\ell-1}^\top (\mathbf{x}_{\ell-1} - \mathbf{x}_0)}{\mathbf{r}_{\ell-1}^\top \mathbf{r}_{\ell-1}} \mathbf{r}_{\ell-1},$$

is the orthogonal projection of  $\mathbf{x}_0$  onto the least squares solution set  $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}^\top \mathbf{A}\mathbf{x} = \mathbf{A}^\top \mathbf{b}\}$ , i.e.,

$$\tilde{\mathbf{x}}_{\ell-1} = \mathbf{A}^\dagger \mathbf{b} + (\mathbf{I} - \mathbf{A}^\dagger \mathbf{A})\mathbf{x}_0.$$

- By using  $\mathbf{x}_0 = \mathbf{0}$ , we obtain the pseudoinverse solution  $\mathbf{A}^\dagger \mathbf{b}$ .



## GMRES for (skew-)symmetric systems

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- “(skew-)symmetric”  $\in$  “range-symmetric”
- For symmetric  $\mathbf{A}$ , if  $\mathbf{b} \notin \text{range}(\mathbf{A})$ , then  $\mathbf{x}_{\ell-1}$  is a least squares solution, but not necessarily the pseudoinverse solution [CPS11].
- For skew-symmetric  $\mathbf{A}$ , i.e.,  $\mathbf{A}^\top = -\mathbf{A}$ , if  $\mathbf{b} \notin \text{range}(\mathbf{A})$ , then

$$\mathbf{r}_{\ell-1}^\top (\mathbf{x}_{\ell-1} - \mathbf{x}_0) = 0.$$

This implies that if  $\mathbf{x}_0 \in \text{range}(\mathbf{A})$  then the  $(\ell - 1)$ th GMRES iterate

$$\mathbf{x}_{\ell-1} = \mathbf{A}^\dagger \mathbf{b}.$$

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[CPS11] S.-C. T. Choi, C. C. Paige, and M. A. Saunders. *MINRES-QLP: A Krylov subspace method for indefinite or singular symmetric systems*. SISC, 2011.

# Summary of GMRES-type methods

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- For simplicity, we set  $\mathbf{x}_0 = \mathbf{0}$ .

Method	Minimization property at step $k$
GMRES	$\mathbf{x}_k := \operatorname{argmin}_{\mathbf{x} \in \mathcal{K}_k(\mathbf{A}, \mathbf{b})} \ \mathbf{b} - \mathbf{Ax}\ $
RRGMRES	$\mathbf{x}_k^R := \operatorname{argmin}_{\mathbf{x} \in \mathcal{K}_k(\mathbf{A}, \mathbf{Ab})} \ \mathbf{b} - \mathbf{Ax}\ $
DGMRES	$\mathbf{x}_k^D := \operatorname{argmin}_{\mathbf{x} \in \mathcal{K}_k(\mathbf{A}, \mathbf{Ab})} \ \mathbf{A}(\mathbf{b} - \mathbf{Ax})\ $

Consistent case:  $\mathbf{x}_\ell = \mathbf{x}_\ell^R = \mathbf{x}_\ell^D = \mathbf{A}^\dagger \mathbf{b}$

Inconsistent case:  $\tilde{\mathbf{x}}_{\ell-1} = \mathbf{x}_{\ell-1}^R = \mathbf{x}_{\ell-1}^D = \mathbf{A}^\dagger \mathbf{b}$

- How about  $\mathbf{x}_k^A := \operatorname{argmin}_{\mathbf{x} \in \mathcal{K}_k(\mathbf{A}, \mathbf{b})} \|\mathbf{A}(\mathbf{b} - \mathbf{Ax})\|$ ?

MINARES [MOS23] for symmetric systems

# RSMAR for range-symmetric systems

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- RSMAR generates an approximation

$$\mathbf{x}_k^A := \operatorname{argmin}_{\mathbf{x} \in \mathcal{K}_k(\mathbf{A}, \mathbf{b})} \|\mathbf{A}(\mathbf{b} - \mathbf{Ax})\|,$$

which is well defined for range-symmetric systems.

- For range-symmetric  $\mathbf{A}$ , if  $\mathbf{b} \in \operatorname{range}(\mathbf{A})$ , then  $\mathbf{x}_\ell^A = \mathbf{x}_\ell$ , and if  $\mathbf{b} \notin \operatorname{range}(\mathbf{A})$ , then  $\mathbf{x}_{\ell-1}^A = \mathbf{x}_{\ell-1}$ .

In other words, for range-symmetric systems, GMRES and RSMAR terminate with the same (least squares) solution.

- RSMAR for  $\mathbf{Ax} = \mathbf{b}$  “=” GMRES for  $\mathbf{Ay} = \mathbf{Ab}, \mathbf{y} = \mathbf{Ax}$ :

$$\min_{\mathbf{x} \in \mathcal{K}_k(\mathbf{A}, \mathbf{b})} \|\mathbf{A}(\mathbf{b} - \mathbf{Ax})\| = \min_{\mathbf{y} \in \mathcal{K}_k(\mathbf{A}, \mathbf{Ab})} \|\mathbf{Ab} - \mathbf{Ay}\|.$$

- For inconsistent systems,  $\|\mathbf{r}_{\ell-1}\| \neq 0$ , but  $\|\mathbf{Ar}_{\ell-1}\| = 0$ .

# Implementation I (inspired by simpler GMRES)

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- Arnoldi process for  $\mathcal{K}_k(\mathbf{A}, \mathbf{A}\mathbf{b})$ :

$$\begin{aligned}\hat{\beta}_1 \hat{\mathbf{v}}_1 &= \mathbf{A}\mathbf{b}, \quad \mathbf{A}\hat{\mathbf{V}}_k = \hat{\mathbf{V}}_{k+1}\hat{\mathbf{H}}_{k+1,k}, \quad \hat{\mathbf{V}}_k^\top \hat{\mathbf{V}}_k = \mathbf{I}_k, \\ \mathcal{K}_k(\mathbf{A}, \mathbf{A}\mathbf{b}) &= \text{span}\{\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \dots, \hat{\mathbf{v}}_k\}.\end{aligned}$$

- GMRES:  $\min_{\mathbf{y} \in \mathcal{K}_k(\mathbf{A}, \mathbf{A}\mathbf{b})} \|\mathbf{A}\mathbf{b} - \mathbf{A}\mathbf{y}\| = \min_{\hat{\mathbf{z}} \in \mathbb{R}^k} \|\hat{\beta}_1 \mathbf{e}_1 - \hat{\mathbf{H}}_{k+1,k} \hat{\mathbf{z}}\|.$   
 $\mathbf{y}_k = \mathbf{A}\mathbf{x}_k^{\mathbf{A}} = \hat{\mathbf{V}}_k \hat{\mathbf{z}}_k$  with  $\hat{\mathbf{z}}_k = \underset{\hat{\mathbf{z}} \in \mathbb{R}^k}{\text{argmin}} \|\hat{\beta}_1 \mathbf{e}_1 - \hat{\mathbf{H}}_{k+1,k} \hat{\mathbf{z}}\|.$
- From  $\mathcal{K}_k(\mathbf{A}, \mathbf{b}) = \text{span}\{\mathbf{b}, \hat{\mathbf{v}}_1, \dots, \hat{\mathbf{v}}_{k-1}\}$ , we have

$$\mathbf{x}_k^{\mathbf{A}} = \begin{bmatrix} \mathbf{b} & \hat{\mathbf{V}}_{k-1} \end{bmatrix} \mathbf{z}_k,$$

where  $\mathbf{z}_k$  solves

$$\mathbf{A} \begin{bmatrix} \mathbf{b} & \hat{\mathbf{V}}_{k-1} \end{bmatrix} \mathbf{z} = \hat{\mathbf{V}}_k \begin{bmatrix} \hat{\beta}_1 \mathbf{e}_1 & \hat{\mathbf{H}}_{k,k-1} \end{bmatrix} \mathbf{z} = \hat{\mathbf{V}}_k \hat{\mathbf{z}}_k.$$

## Implementation II (inspired by RRGMR)

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- Arnoldi process for  $\mathcal{K}_k(\mathbf{A}, \mathbf{b})$ :

$$\beta_1 \mathbf{v}_1 = \mathbf{b}, \quad \mathbf{A} \mathbf{V}_k = \mathbf{V}_{k+1} \mathbf{H}_{k+1,k}, \quad \mathbf{V}_k^\top \mathbf{V}_k = \mathbf{I}_k, \\ \mathcal{K}_k(\mathbf{A}, \mathbf{b}) = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}.$$

- The subproblem:

$$\min_{\mathbf{x} \in \mathcal{K}_k(\mathbf{A}, \mathbf{b})} \|\mathbf{A}(\mathbf{b} - \mathbf{A}\mathbf{x})\| \\ = \min_{\mathbf{z} \in \mathbb{R}^k} \|\beta_1 \mathbf{H}_{k+2,k+1} \mathbf{e}_1 - \mathbf{H}_{k+2,k+1} \mathbf{H}_{k+1,k} \mathbf{z}\|.$$

- Two QR factorizations are required:

$$\mathbf{H}_{k+1,k} = \mathbf{Q}_{k+1} \begin{bmatrix} \mathbf{R}_k \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{H}_{k+2,k+1} \mathbf{Q}_{k+1} \begin{bmatrix} \mathbf{I}_k \\ \mathbf{0} \end{bmatrix} = \tilde{\mathbf{Q}}_{k+2} \begin{bmatrix} \tilde{\mathbf{R}}_k \\ \mathbf{0} \end{bmatrix}.$$

# Symmetric systems

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- GMRES for symmetric systems “ $\Leftrightarrow$ ” MINRES
- RSMAR for symmetric systems “ $\Leftrightarrow$ ” MINARES [MOS23]
- The MINARES implementation is based on the Arnoldi relation  $\mathbf{A}\mathbf{V}_k = \mathbf{V}_{k+1}\mathbf{H}_{k+1,k}$ , and thus can be viewed as a short recurrence variant of RSMAR-II.
- We derive a new implementation for MINARES, which is based on  $\mathbf{A}\hat{\mathbf{V}}_k = \hat{\mathbf{V}}_{k+1}\hat{\mathbf{H}}_{k+1,k}$  and can be viewed as a short recurrence variant of RSMAR-I.

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[MOS23] A. Montoison, D. Orban, and M. A. Saunders. *MINARES: An iterative solver for symmetric linear systems*. arXiv:2310.01757, 2023.

# Numerical experiments

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- A boundary value problem

$$\begin{cases} \Delta u + d \frac{\partial u}{\partial x} = f, & \text{in } \Omega := [0, 1] \times [0, 1], \\ u(x, 0) = u(x, 1), & \text{for } 0 \leq x \leq 1, \\ u(0, y) = u(1, y), & \text{for } 0 \leq y \leq 1, \end{cases}$$

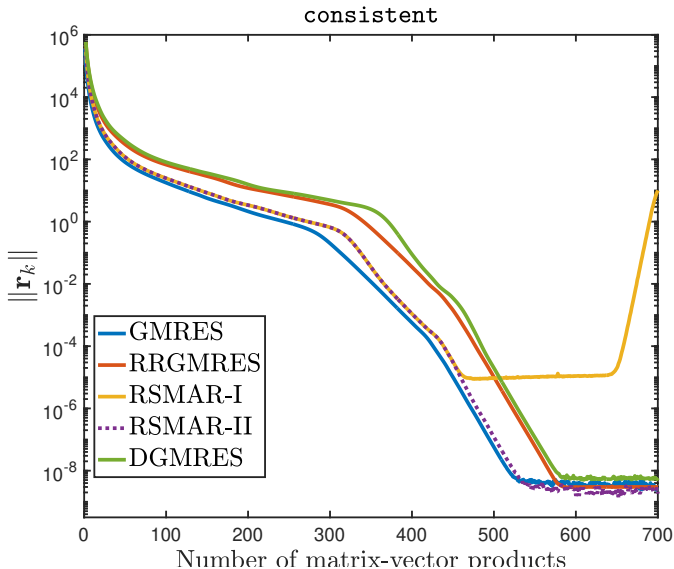
where  $d$  is a constant and  $f$  is a given function.

- FD discretization yields a singular range-symmetric  $\mathbf{A}$ :

$$\mathbf{A} = \begin{bmatrix} \mathbf{T}_m & \mathbf{I}_m & & \mathbf{I}_m \\ \mathbf{I}_m & \ddots & \ddots & \\ & \ddots & \ddots & \mathbf{I}_m \\ \mathbf{I}_m & & \mathbf{I}_m & \mathbf{T}_m \end{bmatrix}, \quad \mathbf{T}_m = \begin{bmatrix} -4 & \alpha_+ & & \alpha_- \\ \alpha_- & \ddots & \ddots & \\ & \ddots & \ddots & \alpha_+ \\ \alpha_+ & & \alpha_- & -4 \end{bmatrix},$$

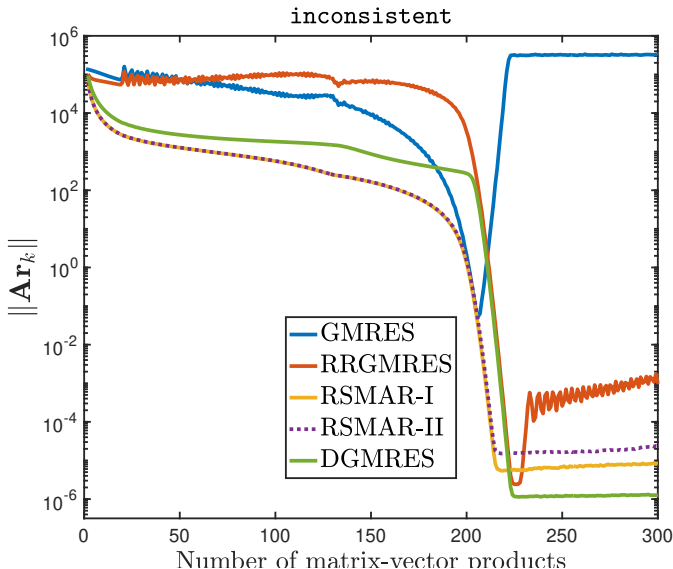
where  $m = 100$ ,  $h = 1/m$ ,  $\alpha_{\pm} = 1 \pm dh/2$ , and  $d = 10$ .

# Convergence history for a consistent system

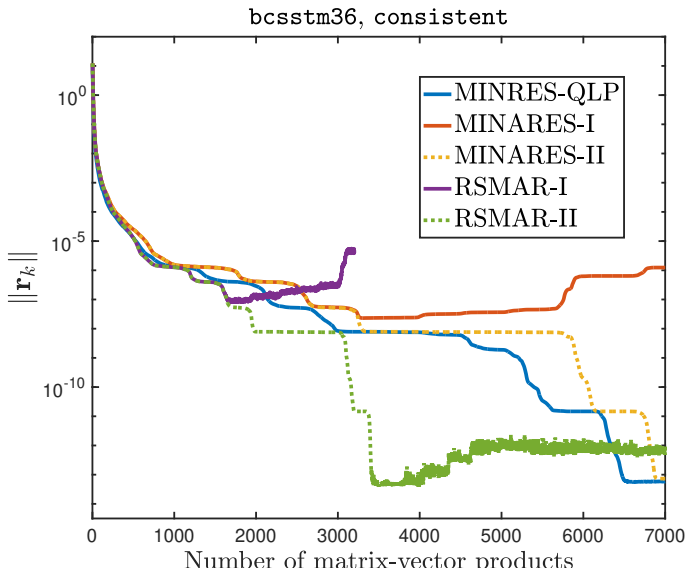




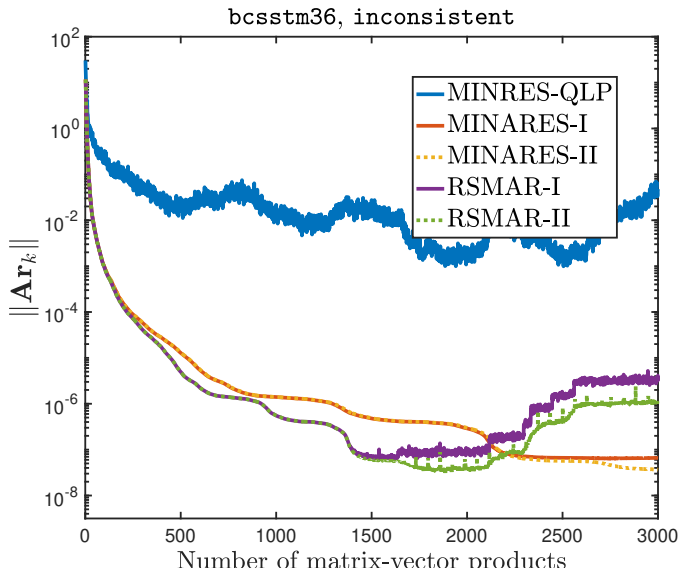
# Convergence history for an inconsistent system



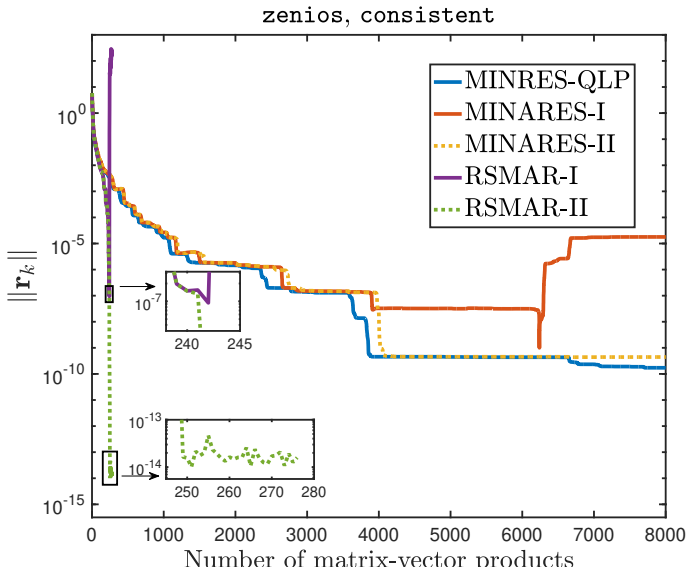
## More numerical results



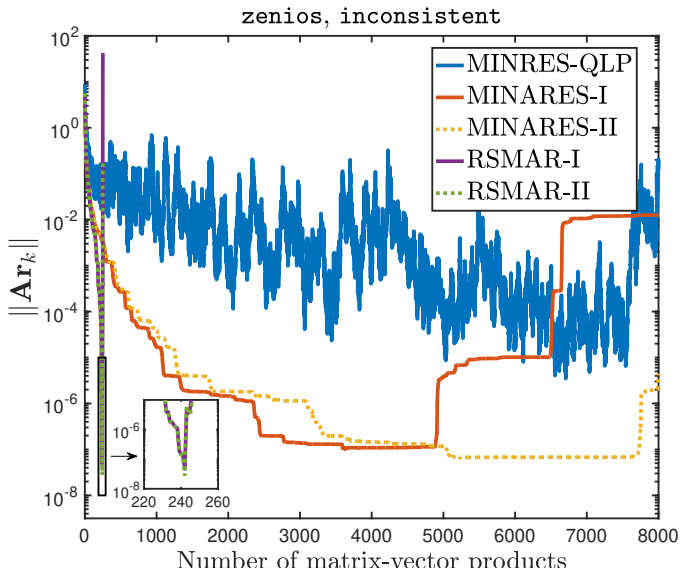
## More numerical results



## More numerical results

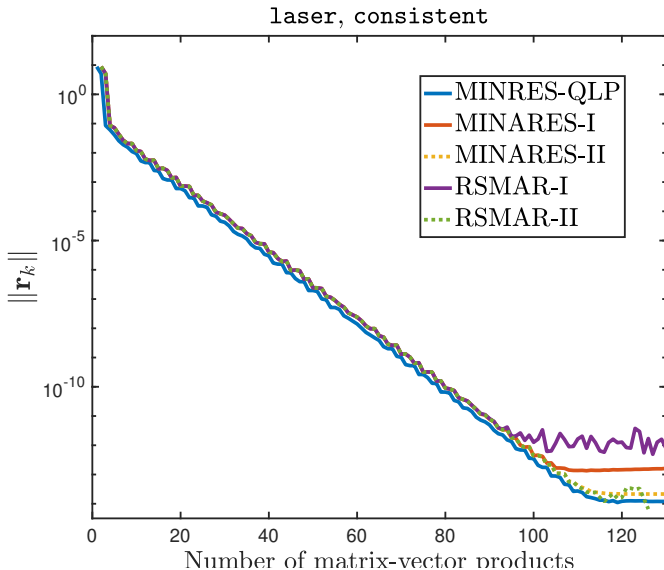


# More numerical results

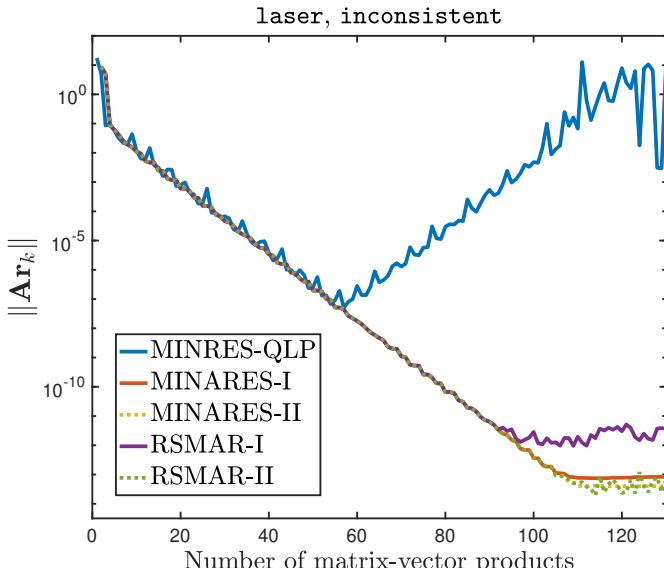


## More numerical results

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## More numerical results



## Summary

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- RSMAR completes the family of Krylov subspace methods based on the Arnoldi process for range-symmetric linear systems.
- By minimizing the  $\mathbf{A}$ -residual norm  $\|\mathbf{A}\mathbf{r}_k\|$  (which always converges to zero for range-symmetric  $\mathbf{A}$ ), RSMAR can be applied to solve any range-symmetric systems.
- We have shown that in exact arithmetic, RSMAR and GMRES both determine the pseudoinverse solution if  $\mathbf{b} \in \text{range}(\mathbf{A})$ , and terminate with the same least squares solution if  $\mathbf{b} \notin \text{range}(\mathbf{A})$ .
- A lifting strategy can be used to obtain the pseudoinverse solution when the reached least squares solution is not.



## Summary

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- Our numerical experiments show that on singular inconsistent range-symmetric systems, RSMAR outperforms GMRES, RRGMR, and DGMRES, and should be the preferred method in finite precision arithmetic.
- As for the implementation for RSMAR, RSMAR-II is better than RSMAR-I in finite precision arithmetic.
- Possible research directions:
  - (1) preconditioning techniques
  - (2) stopping criteria
  - (3) performance for linear discrete ill-posed problems

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# Manuscript and MATLAB codes

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- Kui Du, Jia-Jun Fan, and Fang Wang.

Obtaining the pseudoinverse solution of singular range-symmetric linear systems with GMRES-type methods.  
arXiv:2401.11788, 2024.

- MATLAB codes are available at  
<https://kuidu.github.io/code.html>
- The slides are available at  
<https://kuidu.github.io/talk.html>