

Lecture 18: Multigrid



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1. Finite difference discretization of a BVP

- Consider the following 1-D Dirichlet boundary value problem

$$\begin{cases} -u''(x) = f(x), & x \in (0, 1) \\ u(0) = u(1) = 0 \end{cases}$$

- For $n \in \mathbb{N}$, let

$$h = \frac{1}{n+1}, \quad x_i = ih = \frac{i}{n+1}, \quad 0 \leq i \leq n+1.$$

- The finite difference method is: let $u_0^h = u_{n+1}^h = 0$, $f_i^h = f(x_i)$, $1 \leq i \leq n$, find

$$\mathbf{u}^h = [u_1^h \quad u_2^h \quad \cdots \quad u_n^h]^\top$$

such that

$$-\frac{u_{i+1}^h - 2u_i^h + u_{i-1}^h}{h^2} = f_i^h, \quad 1 \leq i \leq n.$$

- The FD system $\mathbf{A}_h \mathbf{u}^h = \mathbf{f}^h$:

$$\frac{1}{h^2} \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \begin{bmatrix} u_1^h \\ u_2^h \\ \vdots \\ u_{n-1}^h \\ u_n^h \end{bmatrix} = \begin{bmatrix} f_1^h \\ f_2^h \\ \vdots \\ f_{n-1}^h \\ f_n^h \end{bmatrix}.$$

2. Classical stationary iterative methods (Lecture 6)

- Given a starting vector $\mathbf{u}^{(0)}$,

$$\mathbf{u}^{(j)} = \mathbf{R}\mathbf{u}^{(j-1)} + \mathbf{c}, \quad j = 1, 2, \dots$$

- Jacobi's method
- Gauss–Seidel method
- Successive overrelaxation: $\text{SOR}(\omega)$
- Symmetric successive overrelaxation: $\text{SSOR}(\omega)$

2.1. Jacobi's method and its relaxation for the FD system

- The iteration matrix

$$\begin{aligned}\mathbf{R} = \mathbf{D}^{-1}(\mathbf{D} - \mathbf{A}) &= \begin{bmatrix} \frac{1}{2} & & & & \\ & \frac{1}{2} & & & \\ & & \ddots & & \\ & & & \frac{1}{2} & \\ & & & & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & 1 \\ & & & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{2} & 0 & \frac{1}{2} \\ & & & \frac{1}{2} & 0 \end{bmatrix}.\end{aligned}$$

- The relaxation of Jacobi's method:

$$\mathbf{R}(\omega) = (1 - \omega)\mathbf{I} + \omega\mathbf{R} = \mathbf{I} - \omega\mathbf{D}^{-1}\mathbf{A}.$$

- The eigenvalues of \mathbf{R} are given by

$$\lambda_k = \cos(k\pi h), \quad 1 \leq k \leq n,$$

and the corresponding eigenvectors are given by

$$\mathbf{v}_k = [\sin(k\pi h) \quad \sin(2k\pi h) \quad \cdots \quad \sin(nk\pi h)]^\top, \quad 1 \leq k \leq n.$$

The convergence of the Jacobi's method becomes worse for larger n since the spectral radius approaches 1 in this situation.

- The eigenvalues of $\mathbf{R}(\omega)$ are given by

$$\lambda_k(\omega) = 1 - \omega + \omega\lambda_k = 1 - \omega + \omega \cos(k\pi h), \quad 1 \leq k \leq n.$$

Note that relaxation does not lead to an improved convergence, since, in this case, the optimal relaxation parameter is $\omega_\star = 1$. (why?)

- $\mathbf{R}(\omega)$ and \mathbf{R} have the same eigenvectors.

2.2. What makes the convergence of Jacobi's method slow?

- Recall that the solution \mathbf{u} of the linear system is a fixed point, i.e.,

$$\mathbf{u} = \mathbf{R}\mathbf{u} + \mathbf{c}.$$

This leads to

$$\mathbf{u} - \mathbf{u}^{(j)} = \mathbf{R}(\mathbf{u} - \mathbf{u}^{(j-1)}) = \dots = \mathbf{R}^j(\mathbf{u} - \mathbf{u}^{(0)}).$$

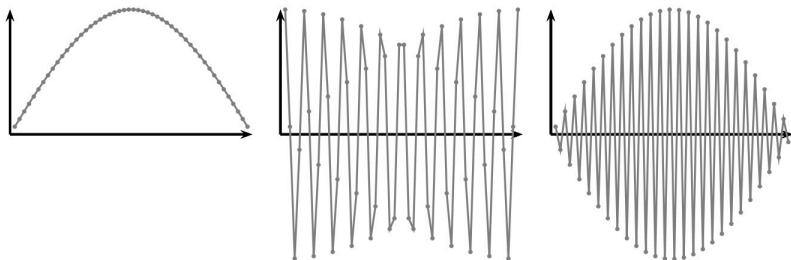
We expand $\mathbf{u} - \mathbf{u}^{(0)}$ in the basis consisting of the eigenvectors:

$$\mathbf{u} - \mathbf{u}^{(0)} = \sum_{k=1}^n \alpha_k \mathbf{v}_k.$$

This gives

$$\mathbf{u} - \mathbf{u}^{(j)} = \sum_{k=1}^n \alpha_k \lambda_k^j \mathbf{v}_k.$$

- The eigenvectors of $\mathbf{R}(\omega)$ with $n = 50$. From left to right: \mathbf{v}_1 , \mathbf{v}_{25} , and \mathbf{v}_{50} . Each graph shows the points $(ih, (\mathbf{v}_k)_i)$ for $1 \leq i \leq n$ linearly connected.



- If $|\lambda_k|$ is small then the component of $\mathbf{u} - \mathbf{u}^{(j)}$ in the direction of \mathbf{v}_k vanishes quickly.
- After only a few iterations, the error is dominated by those components in direction \mathbf{v}_k , where $|\lambda_k| \approx 1$.

- In particular, the error in direction \mathbf{v}_1 and direction \mathbf{v}_n is large, which means no matter how many steps in Jacobi's method we compute, the error will always contain both low-frequency and high-frequency eigenvectors.
- To avoid this, let us have another look at the relaxation of Jacobi's method. Choosing $\omega = 1/2$ yields the eigenvalues

$$\lambda_k(1/2) = (1 + \cos(k\pi h))/2, \quad 1 \leq k \leq n.$$

For large k this means that $\lambda_k(1/2)$ is very close to zero, while for small k we have $\lambda_k(1/2)$ very close to 1.

- Consider the error after j iterations

$$\mathbf{u} - \mathbf{u}^{(j)} = [\mathbf{R}(1/2)]^j (\mathbf{u} - \mathbf{u}^{(0)}) = \sum_{k=1}^n \alpha_k [\lambda_k(1/2)]^j \mathbf{v}_k.$$

The low-frequency eigenvectors dominate and the influence of the high-frequency eigenvectors tends to zero.

- The error, in a certain way, is “smoothed” during the process.
- A “smoother” error can be represented using a smaller n and this gives the idea of the two-grid method, as follows.
 - Compute j steps of the relaxation, resulting in an error

$$\boldsymbol{\varepsilon}^{(j)} = \mathbf{u} - \mathbf{u}^{(j)}$$

which is much “smoother” than $\boldsymbol{\varepsilon}^{(0)}$.

- We have $\mathbf{u} = \mathbf{u}^{(j)} + \boldsymbol{\varepsilon}^{(j)}$ and $\boldsymbol{\varepsilon}^{(j)}$ satisfies

$$\mathbf{A}\boldsymbol{\varepsilon}^{(j)} = \mathbf{A}(\mathbf{u} - \mathbf{u}^{(j)}) = \mathbf{b} - \mathbf{A}\mathbf{u}^{(j)} =: \mathbf{r}^{(j)}.$$

Hence, if we can solve $\mathbf{A}\boldsymbol{\varepsilon}^{(j)} = \mathbf{r}^{(j)}$ then the overall solution is given by $\mathbf{u} = \mathbf{u}^{(j)} + \boldsymbol{\varepsilon}^{(j)}$.

- Since we expect the error $\boldsymbol{\varepsilon}^{(j)}$ to be “smooth”, we will solve the equation $\mathbf{A}\boldsymbol{\varepsilon}^{(j)} = \mathbf{r}^{(j)}$ somehow on a coarser grid to save computational time and transfer the solution back to the finer grid.

3. Two-grid, V-cycle, and Multigrid

- Assume that we are given two grids: a fine grid X_h with n_h points and a coarse grid X_H with $n_H < n_h$ points. Associated with these grids are discrete solution spaces $V_h = \mathbb{R}^{n_h}$ and $V_H = \mathbb{R}^{n_H}$.
- We need a *prolongation* operator $\mathbf{I}_H^h : V_H \mapsto V_h$ which maps from coarse to fine and we need a *restriction* operator $\mathbf{I}_h^H : V_h \mapsto V_H$ which maps from fine to coarse.
- Suppose the coarse grid is given by

$$X_H = \{jH : 0 \leq j < n_H\}$$

with $n_H = 2^m + 1$, $m \in \mathbb{N}$, and $H = 1/(n_H - 1)$. Then the natural fine grid X_h would consist of X_H and all points in the middle between two points from X_H , i.e.,

$$X_h = \{jh : 0 \leq j < n_h\}$$

with $h = H/2$ and $n_h = 2^{m+1} + 1$.

- In this case we could define the prolongation and restriction operators as follows. The prolongation $\mathbf{v}^h = \mathbf{I}_H^h \mathbf{v}^H$ is defined by linear interpolation on the “in-between” points:

$$\begin{aligned} v_{2j}^h &:= v_j^H, & 0 \leq j \leq n_H - 1; \\ v_{2j+1}^h &:= \frac{v_j^H + v_{j+1}^H}{2}, & 0 \leq j \leq n_H - 2. \end{aligned}$$

In matrix form we have

$$\mathbf{I}_H^h = \frac{1}{2} \begin{bmatrix} 2 & & & & & \\ 1 & 1 & & & & \\ & 2 & & & & \\ & 1 & 1 & & & \\ & & \vdots & \vdots & \vdots & \\ & & & 1 & 1 & \\ & & & & 2 & \end{bmatrix}.$$

- For the restriction, $\mathbf{v}^H = \mathbf{I}_h^H \mathbf{v}^h$ we could use the natural inclusion, i.e., we could simply define $v_j^H := v_{2j}^h$, $0 \leq j \leq n_H - 1$.

We could, however, also use a so-called full weighting, which is given by

$$v_j^H = \frac{1}{4}(v_{2j-1}^h + 2v_{2j}^h + v_{2j+1}^h), \quad 0 \leq j \leq n_H - 1,$$

where we have implicitly set $v_{-1}^h = v_{n_h}^h = 0$. In matrix form (for the full weighting case) we have

$$\mathbf{I}_h^H = \frac{1}{4} \begin{bmatrix} 2 & 1 & & & & \\ & 1 & 2 & 1 & & \\ & & & 1 & \cdots & \\ & & & & \cdots & \\ & & & & \cdots & 1 \\ & & & & & 1 & 2 \end{bmatrix}.$$

For this case, we have $\mathbf{I}_H^h = 2(\mathbf{I}_h^H)^\top$.

- Our goal is to solve the finite difference system $\mathbf{A}_h \mathbf{u}^h = \mathbf{f}^h$ on the fine level, using the possibility of solving a system $\mathbf{A}_H \boldsymbol{\varepsilon}^H = \mathbf{r}^H$ on a coarse level.

Note that the matrix \mathbf{A}_h and \mathbf{A}_H only refer to interior nodes.

Hence, we delete the first and last columns and rows in the matrix representation of \mathbf{I}_H^h and \mathbf{I}_h^H . We still use \mathbf{I}_H^h and \mathbf{I}_h^H to denote the resulting matrices, i.e.,

$$\mathbf{I}_H^h = \frac{1}{2} \begin{bmatrix} 1 & & & & \\ 2 & & & & \\ 1 & 1 & & & \\ & \vdots & \vdots & \vdots & \\ & & & & 1 \end{bmatrix}, \quad \mathbf{I}_h^H = \frac{1}{4} \begin{bmatrix} 1 & 2 & 1 & \cdots & \\ & & 1 & \cdots & \\ & & & \cdots & \\ & & & & \cdots & 1 \end{bmatrix}.$$

We can prove that (Exercise)

$$\mathbf{A}_H = \mathbf{I}_h^H \mathbf{A}_h \mathbf{I}_H^h.$$

- As mentioned above, we will use an iterative method as a “smoother”. Recall that such a consistent iterative method for solving $\mathbf{A}_h \mathbf{u}^h = \mathbf{f}^h$ is given by

$$\mathbf{u}_{(j+1)}^h = \mathbf{S}_h(\mathbf{u}_{(j)}^h) := \mathbf{R}_h \mathbf{u}_{(j)}^h + \mathbf{c}^h,$$

where the solution \mathbf{u}^h of the linear system is a fixed point of $\mathbf{S}_h(\cdot)$, i.e., it satisfies

$$\mathbf{u}^h = \mathbf{S}_h(\mathbf{u}^h) := \mathbf{R}_h \mathbf{u}^h + \mathbf{c}^h.$$

- If we apply $\ell \in \mathbb{N}$ iterations of such a smoother with initial data $\mathbf{u}_{(0)}^h$, it is easy to see that the result has the form

$$\mathbf{S}_h^\ell(\mathbf{u}_{(0)}^h) = \mathbf{R}_h^\ell \mathbf{u}_{(0)}^h + \sum_{j=0}^{\ell-1} \mathbf{R}_h^j \mathbf{c}^h := \mathbf{R}_h^\ell \mathbf{u}_{(0)}^h + \mathbf{s}^h.$$

- Note that we can use any consistent method as the smoother $\mathbf{S}_h(\cdot)$.

Algorithm: Two-grid for $\mathbf{A}_h \mathbf{u}^h = \mathbf{f}^h$, $\text{TG}(\mathbf{A}_h, \mathbf{f}^h, \mathbf{u}_{(0)}^h, \ell_1, \ell_2)$

Input: $\mathbf{A}_h, \mathbf{f}^h, \mathbf{u}_{(0)}^h, \ell_1, \ell_2 \in \mathbb{N}$.

Output: Approximation to $\mathbf{A}_h^{-1} \mathbf{f}^h$.

1. Presmooth : $\mathbf{u}^h := \mathbf{S}_h^{\ell_1}(\mathbf{u}_{(0)}^h)$
 2. Get residual : $\mathbf{r}^h := \mathbf{f}^h - \mathbf{A}_h \mathbf{u}^h$
 3. Coarsen : $\mathbf{r}^H := \mathbf{I}_h^H \mathbf{r}^h$
 4. Solve : $\boldsymbol{\varepsilon}^H := \mathbf{A}_H^{-1} \mathbf{r}^H$
 5. Prolong : $\boldsymbol{\varepsilon}^h := \mathbf{I}_H^h \boldsymbol{\varepsilon}^H$
 6. Correct : $\mathbf{u}^h := \mathbf{u}^h + \boldsymbol{\varepsilon}^h$
 7. Postsmooth : $\mathbf{u}^h := \mathbf{S}_h^{\ell_2}(\mathbf{u}^h)$
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- The two-grid method can be seen as only one update of a new stationary iterative method.

Theorem 1

Assume that \mathbf{A}_H is invertible. Assume that $\mathbf{u}_{(j)}^h$ is the input vector and $\mathbf{u}_{(j+1)}^h$ is the resulting output vector of the two-grid method. Then, we have

$$\mathbf{u}_{(j+1)}^h = \mathbf{T}_h \mathbf{u}_{(j)}^h + \mathbf{d}^h \quad \text{and} \quad \mathbf{u}^h = \mathbf{T}_h \mathbf{u}^h + \mathbf{d}^h,$$

where the iteration matrix \mathbf{T}_h is given by

$$\mathbf{T}_h = \mathbf{R}_h^{\ell_2} \mathbf{T}_{h,H} \mathbf{R}_h^{\ell_1} \quad \text{with} \quad \mathbf{T}_{h,H} = \mathbf{I} - \mathbf{I}_H^h \mathbf{A}_H^{-1} \mathbf{I}_h^H \mathbf{A}_h.$$

Moreover, we have the error representation

$$\mathbf{u}_{(j+1)}^h - \mathbf{u}^h = \mathbf{T}_h (\mathbf{u}_{(j)}^h - \mathbf{u}^h) = \mathbf{T}_h^{j+1} (\mathbf{u}_{(0)}^h - \mathbf{u}^h),$$

showing that the method converges if the spectral radius $\rho(\mathbf{T}_h) < 1$.

Proof. We go through the two-grid method step by step. With the first step $\mathbf{u}_{(j)}^h$ is mapped to $\mathbf{S}_h^{\ell_1}(\mathbf{u}_{(j)}^h)$, which is the input to the second step. After the second and third steps we have

$$\mathbf{r}_{(j)}^H = \mathbf{I}_h^H(\mathbf{f}^h - \mathbf{A}_h \mathbf{S}_h^{\ell_1}(\mathbf{u}_{(j)}^h)),$$

which is the input for the fourth step, so that the results after the fourth and fifth steps become

$$\boldsymbol{\varepsilon}_{(j)}^h = \mathbf{I}_H^h \mathbf{A}_H^{-1} \mathbf{I}_h^H(\mathbf{f}^h - \mathbf{A}_h \mathbf{S}_h^{\ell_1}(\mathbf{u}_{(j)}^h)).$$

Applying steps 6 and 7 to this finally results in the new iteration

$$\begin{aligned} \mathbf{u}_{(j+1)}^h &= \mathbf{S}_h^{\ell_2} \left(\mathbf{S}_h^{\ell_1}(\mathbf{u}_{(j)}^h) + \mathbf{I}_H^h \mathbf{A}_H^{-1} \mathbf{I}_h^H(\mathbf{f}^h - \mathbf{A}_h \mathbf{S}_h^{\ell_1}(\mathbf{u}_{(j)}^h)) \right) \\ &= \mathbf{S}_h^{\ell_2} \left((\mathbf{I} - \mathbf{I}_H^h \mathbf{A}_H^{-1} \mathbf{I}_h^H \mathbf{A}_h) \mathbf{S}_h^{\ell_1}(\mathbf{u}_{(j)}^h) + \mathbf{I}_H^h \mathbf{A}_H^{-1} \mathbf{I}_h^H \mathbf{f}^h \right) \\ &= \mathbf{S}_h^{\ell_2} \left(\mathbf{T}_{h,H} \mathbf{S}_h^{\ell_1}(\mathbf{u}_{(j)}^h) + \mathbf{I}_H^h \mathbf{A}_H^{-1} \mathbf{I}_h^H \mathbf{f}^h \right). \end{aligned}$$

Note that the operator $\mathbf{S}_h(\cdot)$ is only affine and not linear. Define

$$\tilde{\mathbf{T}}_h(\cdot) := \mathbf{S}_h^{\ell_2}(\mathbf{T}_{h,H} \mathbf{S}_h^{\ell_1}(\cdot)), \quad \tilde{\mathbf{d}}^h := \mathbf{R}_h^{\ell_2} \mathbf{I}_H^h \mathbf{A}_H^{-1} \mathbf{I}_h^H \mathbf{f}^h.$$

By straightforward calculations, we have

$$\mathbf{u}_{(j+1)}^h = \tilde{\mathbf{T}}_h(\mathbf{u}_{(j)}^h) + \tilde{\mathbf{d}}^h,$$

and

$$\begin{aligned} \tilde{\mathbf{T}}_h(\mathbf{u}) &= \mathbf{S}_h^{\ell_2}(\mathbf{T}_{h,H} \mathbf{S}_h^{\ell_1}(\mathbf{u})) \\ &= \mathbf{S}_h^{\ell_2} \left(\mathbf{T}_{h,H} \left(\mathbf{R}_h^{\ell_1} \mathbf{u} + \sum_{j=0}^{\ell_1-1} \mathbf{R}_h^j \mathbf{c}^h \right) \right) \\ &= \mathbf{R}_h^{\ell_2} \mathbf{T}_{h,H} \mathbf{R}_h^{\ell_1} \mathbf{u} + \mathbf{R}_h^{\ell_2} \mathbf{T}_{h,H} \sum_{j=0}^{\ell_1-1} \mathbf{R}_h^j \mathbf{c}^h + \sum_{j=0}^{\ell_2-1} \mathbf{R}_h^j \mathbf{c}^h \\ &=: \mathbf{T}_h \mathbf{u} + \hat{\mathbf{d}}^h, \end{aligned}$$

which shows

$$\mathbf{u}_{(j+1)}^h = \mathbf{T}_h \mathbf{u}_{(j)}^h + \mathbf{d}^h \quad \text{with} \quad \mathbf{d}^h = \widehat{\mathbf{d}}^h + \widetilde{\mathbf{d}}^h.$$

Hence, the iteration matrix is indeed given by

$$\mathbf{T}_h = \mathbf{R}_h^{\ell_2} \mathbf{T}_{h,H} \mathbf{R}_h^{\ell_1}.$$

As $\mathbf{S}_h(\cdot)$ is consistent, i.e., $\mathbf{S}_h(\mathbf{u}^h) = \mathbf{u}^h$, we have

$$\begin{aligned} \mathbf{T}_h \mathbf{u}^h + \mathbf{d}^h &= \mathbf{T}_h \mathbf{u}^h + \widehat{\mathbf{d}}^h + \widetilde{\mathbf{d}}^h = \widetilde{\mathbf{T}}_h(\mathbf{u}^h) + \widetilde{\mathbf{d}}^h \\ &= \mathbf{S}_h^{\ell_2}((\mathbf{I} - \mathbf{I}_H^h \mathbf{A}_H^{-1} \mathbf{I}_h^H \mathbf{A}_h) \mathbf{u}^h) + \mathbf{R}_h^{\ell_2} \mathbf{I}_H^h \mathbf{A}_H^{-1} \mathbf{I}_h^H \mathbf{f}^h \\ &= \mathbf{S}_h^{\ell_2}(\mathbf{u}^h) - \mathbf{R}_h^{\ell_2} \mathbf{I}_H^h \mathbf{A}_H^{-1} \mathbf{I}_h^H \mathbf{f}^h + \mathbf{R}_h^{\ell_2} \mathbf{I}_H^h \mathbf{A}_H^{-1} \mathbf{I}_h^H \mathbf{f}^h \\ &= \mathbf{u}^h. \end{aligned}$$

This shows

$$\mathbf{u}_{(j+1)}^h - \mathbf{u}^h = \mathbf{T}_h(\mathbf{u}_{(j)}^h - \mathbf{u}^h) = \mathbf{T}_h^{j+1}(\mathbf{u}_{(0)}^h - \mathbf{u}^h). \quad \square$$

Proposition 2

Assume the following four conditions hold.

- (1) The matrix \mathbf{A}_h is symmetric and positive definite.
- (2) The prolongation and restriction operators are connected by

$$\mathbf{I}_H^h = \gamma(\mathbf{I}_h^H)^\top$$

with $\gamma > 0$.

- (3) The prolongation operator \mathbf{I}_H^h is injective.
- (4) The coarse grid matrix is given by $\mathbf{A}_H := \mathbf{I}_h^H \mathbf{A}_h \mathbf{I}_H^h$.

Then we have:

- (i) The coarse-grid correction operator $\mathbf{T}_{h,H}$ is an orthogonal projection with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathbf{A}_h}$.
- (ii) The range of $\mathbf{T}_{h,H}$ is $\langle \cdot, \cdot \rangle_{\mathbf{A}_h}$ -orthogonal to the range of \mathbf{I}_H^h .

Proof. We start by showing that \mathbf{A}_H is symmetric and positive definite. It is symmetric since

$$\mathbf{A}_H^\top = (\mathbf{I}_H^h)^\top \mathbf{A}_h^\top (\mathbf{I}_H^H)^\top = \gamma \mathbf{I}_h^H \mathbf{A}_h \frac{1}{\gamma} \mathbf{I}_H^h = \mathbf{I}_h^H \mathbf{A}_h \mathbf{I}_H^h = \mathbf{A}_H.$$

It is positive definite since we have for any $\mathbf{x} \neq \mathbf{0}$ that $\mathbf{I}_H^h \mathbf{x} \neq \mathbf{0}$ because of the injectivity of \mathbf{I}_H^h and hence

$$\mathbf{x}^\top \mathbf{A}_H \mathbf{x} = \mathbf{x}^\top \mathbf{I}_h^H \mathbf{A}_h \mathbf{I}_H^h \mathbf{x} = \frac{1}{\gamma} (\mathbf{I}_H^h \mathbf{x})^\top \mathbf{A}_h (\mathbf{I}_H^h \mathbf{x}) > 0.$$

This means in particular that the coarse grid correction operator

$$\mathbf{T}_{h,H} = \mathbf{I} - \mathbf{I}_H^h \mathbf{A}_H^{-1} \mathbf{I}_h^H \mathbf{A}_h$$

is well-defined. Next, let

$$\mathbf{Q}_{h,H} := \mathbf{I} - \mathbf{T}_{h,H} = \mathbf{I}_H^h \mathbf{A}_H^{-1} \mathbf{I}_h^H \mathbf{A}_h.$$

Actually, the mapping $\mathbf{Q}_{h,H}$ is a projection since we have

$$\begin{aligned}\mathbf{Q}_{h,H}^2 &= (\mathbf{I}_H^h \mathbf{A}_H^{-1} I_h^H \mathbf{A}_h)(\mathbf{I}_H^h \mathbf{A}_H^{-1} I_h^H \mathbf{A}_h) \\ &= \mathbf{I}_H^h \mathbf{A}_H^{-1} (I_h^H \mathbf{A}_h \mathbf{I}_H^h) \mathbf{A}_H^{-1} I_h^H \mathbf{A}_h \\ &= \mathbf{I}_H^h \mathbf{A}_H^{-1} I_h^H \mathbf{A}_h \\ &= \mathbf{Q}_{h,H}.\end{aligned}$$

It is also self-adjoint and hence an orthogonal projector:

$$\begin{aligned}\langle \mathbf{Q}_{h,H} \mathbf{x}, \mathbf{y} \rangle_{\mathbf{A}_h} &= \mathbf{x}^\top \mathbf{Q}_{h,H}^\top \mathbf{A}_h \mathbf{y} = \mathbf{x}^\top \mathbf{A}_h^\top (\mathbf{I}_h^H)^\top (\mathbf{A}_H^{-1})^\top (\mathbf{I}_H^h)^\top \mathbf{A}_h \mathbf{y} \\ &= \mathbf{x}^\top \mathbf{A}_h \left(\frac{1}{\gamma} \mathbf{I}_H^h \right) \mathbf{A}_H^{-1} \gamma \mathbf{I}_h^H \mathbf{A}_h \mathbf{y} \\ &= \mathbf{x}^\top \mathbf{A}_h \mathbf{Q}_{h,H} \mathbf{y} \\ &= \langle \mathbf{x}, \mathbf{Q}_{h,H} \mathbf{y} \rangle_{\mathbf{A}_h}.\end{aligned}$$

Then $\mathbf{T}_{h,H}$ is also an orthogonal projection with respect to $\langle \cdot, \cdot \rangle_{\mathbf{A}_h}$.

It remains to show that $\text{range}(\mathbf{T}_{h,H})$ and $\text{range}(\mathbf{I}_H^h)$ are orthogonal. This is true because we have

$$\begin{aligned}
 \langle \mathbf{T}_{h,H} \mathbf{x}, \mathbf{I}_H^h \mathbf{y} \rangle_{\mathbf{A}_h} &= \mathbf{x}^\top (\mathbf{I} - \mathbf{I}_H^h \mathbf{A}_H^{-1} \mathbf{I}_H^H \mathbf{A}_h)^\top \mathbf{A}_h \mathbf{I}_H^h \mathbf{y} \\
 &= \mathbf{x}^\top (\mathbf{I} - \mathbf{A}_h \mathbf{I}_H^h \mathbf{A}_H^{-1} \mathbf{I}_H^H) \mathbf{A}_h \mathbf{I}_H^h \mathbf{y} \\
 &= \mathbf{x}^\top (\mathbf{A}_h \mathbf{I}_H^h - \mathbf{A}_h \mathbf{I}_H^h \mathbf{A}_H^{-1} \mathbf{I}_H^H \mathbf{A}_h \mathbf{I}_H^h) \mathbf{y} \\
 &= \mathbf{x}^\top (\mathbf{A}_h \mathbf{I}_H^h - \mathbf{A}_h \mathbf{I}_H^h) \mathbf{y} \\
 &= 0
 \end{aligned}$$

for all \mathbf{x} and \mathbf{y} . □

- Let \mathbf{D}_h be the diagonal part of \mathbf{A}_h . We say that $\mathbf{S}_h(\cdot)$ has the *smoothing property* if there is a constant $\alpha > 0$ such that

$$\|\mathbf{R}_h \mathbf{v}^h\|_{\mathbf{A}_h}^2 \leq \|\mathbf{v}^h\|_{\mathbf{A}_h}^2 - \alpha \|\mathbf{A}_h \mathbf{v}^h\|_{\mathbf{D}_h^{-1}}^2, \quad \forall \mathbf{v}^h.$$

We say that the prolongation operator \mathbf{I}_H^h has the *approximation property* if there is a constant $\beta > 0$ such that

$$\min_{\mathbf{v}^H} \|\mathbf{v}^h - \mathbf{I}_H^h \mathbf{v}^H\|_{\mathbf{D}_h} \leq \beta \|\mathbf{A}_h \mathbf{v}^h\|_{\mathbf{D}_h^{-1}}, \quad \forall \mathbf{v}^h.$$

Theorem 3

Let the conditions of Proposition 2 be satisfied. Assume:

- (1) the smoothing process $\mathbf{S}_h(\cdot)$ has the smoothing property,*
- (2) the prolongation operator \mathbf{I}_H^h has the approximation property.*

Then, we have

$$\alpha \leq \beta$$

and for the iteration matrix \mathbf{T}_h of the two-grid method,

$$\|\mathbf{T}_h\|_{\mathbf{A}_h} < \sqrt{1 - \alpha/\beta} \text{ if } \alpha < \beta \text{ and } \|\mathbf{T}_h\|_{\mathbf{A}_h} = 0 \text{ if } \alpha = \beta.$$

Hence, as an iterative scheme, the two-grid method converges.

Proof. Since $\text{range}(\mathbf{T}_{h,H})$ is \mathbf{A}_h -orthogonal to $\text{range}(\mathbf{I}_H^h)$ we have

$$\langle \mathbf{v}^h, \mathbf{I}_H^h \mathbf{v}^H \rangle_{\mathbf{A}_h} = 0, \quad \forall \mathbf{v}^h \in \text{range}(\mathbf{T}_{h,H}), \quad \forall \mathbf{v}^H.$$

From this, we can conclude for all $\mathbf{v}^h \in \text{range}(\mathbf{T}_{h,H})$ and \mathbf{v}^H that

$$\begin{aligned}\|\mathbf{v}^h\|_{\mathbf{A}_h}^2 &= \langle \mathbf{v}^h, \mathbf{v}^h - \mathbf{I}_H^h \mathbf{v}^H \rangle_{\mathbf{A}_h} = \langle \mathbf{A}_h \mathbf{v}^h, \mathbf{v}^h - \mathbf{I}_H^h \mathbf{v}^H \rangle_2 \\ &= \langle \mathbf{D}_h^{-1/2} \mathbf{A}_h \mathbf{v}^h, \mathbf{D}_h^{1/2} (\mathbf{v}^h - \mathbf{I}_H^h \mathbf{v}^H) \rangle_2 \\ &\leq \|\mathbf{D}_h^{-1/2} \mathbf{A}_h \mathbf{v}^h\|_2 \|\mathbf{D}_h^{1/2} (\mathbf{v}^h - \mathbf{I}_H^h \mathbf{v}^H)\|_2 \\ &= \|\mathbf{A}_h \mathbf{v}^h\|_{\mathbf{D}_h^{-1}} \|\mathbf{v}^h - \mathbf{I}_H^h \mathbf{v}^H\|_{\mathbf{D}_h}.\end{aligned}$$

Going over to the infimum over all \mathbf{v}^H and using the approximation property leads to

$$\|\mathbf{v}^h\|_{\mathbf{A}_h} \leq \sqrt{\beta} \|\mathbf{A}_h \mathbf{v}^h\|_{\mathbf{D}_h^{-1}}, \quad \forall \mathbf{v}^h \in \text{range}(\mathbf{T}_{h,H}).$$

This is equivalent to

$$\|\mathbf{T}_{h,H} \mathbf{v}^h\|_{\mathbf{A}_h} \leq \sqrt{\beta} \|\mathbf{A}_h \mathbf{T}_{h,H} \mathbf{v}^h\|_{\mathbf{D}_h^{-1}}, \quad \forall \mathbf{v}^h.$$

Using this and the smoothing property then leads to

$$\begin{aligned}
 0 \leq \|\mathbf{R}_h \mathbf{T}_{h,H} \mathbf{v}^h\|_{\mathbf{A}_h}^2 &\leq \|\mathbf{T}_{h,H} \mathbf{v}^h\|_{\mathbf{A}_h}^2 - \alpha \|\mathbf{A}_h \mathbf{T}_{h,H} \mathbf{v}^h\|_{\mathbf{D}_h^{-1}}^2 \\
 &\leq \|\mathbf{T}_{h,H} \mathbf{v}^h\|_{\mathbf{A}_h}^2 - \frac{\alpha}{\beta} \|\mathbf{T}_{h,H} \mathbf{v}^h\|_{\mathbf{A}_h}^2 \\
 &= (1 - \alpha/\beta) \|\mathbf{T}_{h,H} \mathbf{v}^h\|_{\mathbf{A}_h}^2 \\
 &\leq (1 - \alpha/\beta) \|\mathbf{v}^h\|_{\mathbf{A}_h}^2,
 \end{aligned}$$

where the last inequality follows from the fact that $\mathbf{T}_{h,H}$ is an \mathbf{A}_h -orthogonal projection. This means, first of all, $\alpha \leq \beta$ and secondly

$$\|\mathbf{R}_h \mathbf{T}_{h,H}\|_{\mathbf{A}_h} \leq \sqrt{1 - \alpha/\beta}.$$

By the smoothing property, we have $\|\mathbf{R}_h\|_{\mathbf{A}_h} < 1$. Then we finally derive $\|\mathbf{T}_h\|_{\mathbf{A}_h} = 0$ if $\alpha = \beta$, and if $\alpha < \beta$,

$$\|\mathbf{T}_h\|_{\mathbf{A}_h} = \|\mathbf{R}_h^{\ell_2} \mathbf{T}_{h,H} \mathbf{R}_h^{\ell_1}\|_{\mathbf{A}_h} < \|\mathbf{R}_h \mathbf{T}_{h,H}\|_{\mathbf{A}_h} \leq \sqrt{1 - \alpha/\beta}. \quad \square$$

Algorithm: V-cycle for $\mathbf{A}_h \mathbf{u}^h = \mathbf{f}^h$, $\text{V-cycle}(\mathbf{A}_h, \mathbf{f}^h, \mathbf{u}_{(0)}^h, \ell_1, \ell_2, h_0)$

Input: $\mathbf{A}_h, \mathbf{f}^h, \mathbf{u}_{(0)}^h, \ell_1, \ell_2 \in \mathbb{N}, h_0$.

Output: Approximation to $\mathbf{A}_h^{-1} \mathbf{f}^h$.

1. Presmooth : $\mathbf{u}^h := \mathbf{S}_h^{\ell_1}(\mathbf{u}_{(0)}^h)$
2. Get residual : $\mathbf{r}^h := \mathbf{f}^h - \mathbf{A}_h \mathbf{u}^h$
3. Coarsen : $H := 2h, \quad \mathbf{r}^H := \mathbf{I}_h^H \mathbf{r}^h$
4. if $H = h_0$
 Solve $\mathbf{A}_H \boldsymbol{\epsilon}^H = \mathbf{r}^H$
 else
 $\boldsymbol{\epsilon}^H := \text{V-cycle}(\mathbf{A}_H, \mathbf{r}^H, \mathbf{0}, \ell_1, \ell_2, h_0)$
 end
5. Prolong : $\boldsymbol{\epsilon}^h := \mathbf{I}_H^h \boldsymbol{\epsilon}^H$
6. Correct : $\mathbf{u}^h := \mathbf{u}^h + \boldsymbol{\epsilon}^h$
7. Postsmooth : $\mathbf{u}^h := \mathbf{S}_h^{\ell_2}(\mathbf{u}^h)$

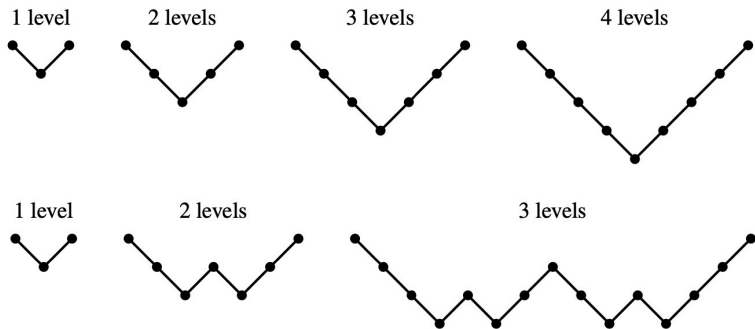
Algorithm: Multigrid for $\mathbf{A}_h \mathbf{u}^h = \mathbf{f}^h$, $\text{MG}(\mathbf{A}_h, \mathbf{f}^h, \mathbf{u}_{(0)}^h, \ell_1, \ell_2, \ell, h_0)$

Input: $\mathbf{A}_h, \mathbf{f}^h, \mathbf{u}_{(0)}^h, \ell_1, \ell_2, \ell \in \mathbb{N}, h_0$.

Output: Approximation to $\mathbf{A}_h^{-1} \mathbf{f}^h$.

1. Presmooth : $\mathbf{u}^h := \mathbf{S}_h^{\ell_1}(\mathbf{u}_{(0)}^h)$
2. Get residual : $\mathbf{r}^h := \mathbf{f}^h - \mathbf{A}_h \mathbf{u}^h$
3. Coarsen : $H := 2h, \quad \mathbf{r}^H := \mathbf{I}_h^H \mathbf{r}^h$
4. if $H = h_0$
 Solve $\mathbf{A}_H \boldsymbol{\epsilon}^H = \mathbf{r}^H$
 else
 $\boldsymbol{\epsilon}^H := \mathbf{0}$
 for $j = 1 : \ell$
 $\boldsymbol{\epsilon}^H := \text{MG}(\mathbf{A}_H, \mathbf{r}^H, \boldsymbol{\epsilon}^H, \ell_1, \ell_2, \ell, h_0)$
 end
 end
 end
5. Prolong : $\boldsymbol{\epsilon}^h := \mathbf{I}_H^h \boldsymbol{\epsilon}^H$
6. Correct : $\mathbf{u}^h := \mathbf{u}^h + \boldsymbol{\epsilon}^h$
7. Postsmooth : $\mathbf{u}^h := \mathbf{S}_h^{\ell_2}(\mathbf{u}^h)$

- Obviously, $\ell = 1$ leads to the V-cycle method. It is helpful to visualize the recursion in the following way, depending on the choice of ℓ and how many levels there are, meaning how many grids we use.



The recursion of multigrid with $\ell = 1$ (top) and $\ell = 2$ (bottom).

- To determine the iteration matrix \mathbf{M}_h of the multigrid method, we start with the iteration matrix of the two-grid method

$$\mathbf{T}_h = \mathbf{R}_h^{\ell_2} (\mathbf{I} - \mathbf{I}_H^h \mathbf{A}_H^{-1} \mathbf{I}_h^H \mathbf{A}_h) \mathbf{R}_h^{\ell_1}$$

and recall that the term \mathbf{A}_H^{-1} came from step 4 in two-grid cycle and hence has now to be replaced by ℓ steps of the multigrid method on grid X_H :

$$\begin{aligned}\boldsymbol{\varepsilon}_{(0)}^H &:= \mathbf{0}, \\ \boldsymbol{\varepsilon}_{(j)}^H &:= \mathbf{M}_H \boldsymbol{\varepsilon}_{(j-1)}^H + \mathbf{d}^H, \quad 1 \leq j \leq \ell.\end{aligned}$$

As this is a consistent method for solving $\mathbf{A}_H \boldsymbol{\varepsilon}^H = \mathbf{r}^H$, we have

$$\boldsymbol{\varepsilon}_{(\ell)}^H - \boldsymbol{\varepsilon}^H = \mathbf{M}_H (\boldsymbol{\varepsilon}_{(\ell-1)}^H - \boldsymbol{\varepsilon}^H) = \mathbf{M}_H^\ell (\boldsymbol{\varepsilon}_{(0)}^H - \boldsymbol{\varepsilon}^H) = -\mathbf{M}_H^\ell \boldsymbol{\varepsilon}^H.$$

Then, we have

$$\boldsymbol{\varepsilon}_{(\ell)}^H = (\mathbf{I} - \mathbf{M}_H^\ell) \boldsymbol{\varepsilon}^H = (\mathbf{I} - \mathbf{M}_H^\ell) \mathbf{A}_H^{-1} \mathbf{r}^H.$$

Replacing \mathbf{A}_H^{-1} by $(\mathbf{I} - \mathbf{M}_H^\ell)\mathbf{A}_H^{-1}$ yields

$$\begin{aligned}\mathbf{M}_h &= \mathbf{R}_h^{\ell_2} [\mathbf{I} - \mathbf{I}_H^h (\mathbf{I} - \mathbf{M}_H^\ell) \mathbf{A}_H^{-1} \mathbf{I}_h^H \mathbf{A}_h] \mathbf{R}_h^{\ell_1} \\ &= \mathbf{R}_h^{\ell_2} [\mathbf{I} - \mathbf{I}_H^h \mathbf{A}_H^{-1} \mathbf{I}_h^H \mathbf{A}_h + \mathbf{I}_H^h \mathbf{M}_H^\ell \mathbf{A}_H^{-1} \mathbf{I}_h^H \mathbf{A}_h] \mathbf{R}_h^{\ell_1} \\ &= \mathbf{T}_h + \mathbf{R}_h^{\ell_2} \mathbf{I}_H^h \mathbf{M}_H^\ell \mathbf{A}_H^{-1} \mathbf{I}_h^H \mathbf{A}_h \mathbf{R}_h^{\ell_1},\end{aligned}$$

which can be seen as a perturbation of \mathbf{T}_h .

- *Algebraic multigrid* (AMG): $\mathbf{A}_h \mathbf{u}^h = \mathbf{f}^h$ with $\mathbf{A}_h \in \mathbb{R}^{n_h \times n_h}$
 - define the coarse subset \mathbb{R}^{n_H} from the fine set \mathbb{R}^{n_h} ,
 - define the coarsening operator \mathbf{I}_h^H from \mathbb{R}^{n_h} to \mathbb{R}^{n_H}
 - use the abstract definitions

$$\mathbf{I}_H^h = (\mathbf{I}_h^H)^\top, \quad \mathbf{A}_H = \mathbf{I}_h^H \mathbf{A}_h \mathbf{I}_H^h, \quad \mathbf{f}^H = \mathbf{I}_h^H \mathbf{f}^h$$

to complete the set-up.

4. Further reading

- Holger Wendland

Numerical Linear Algebra An Introduction

Cambridge University Press, 2018

- William L. Briggs, Van E. Henson, and Steve F. McCormick

A Multigrid Tutorial

Second Edition, SIAM, 2000

- Pieter Wesseling

An Introduction to Multigrid Methods

John Wiley & Sons, 1992