

# Numerical Linear Algebra Assignment 9

## Exercise 1. (10 points)

Consider the simultaneous iteration:

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### Algorithm 1: Simultaneous iteration

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Pick  $\mathbf{Q}_n^{(0)} \in \mathbb{C}^{m \times n}$  with orthonormal columns
for  $k = 1, 2, 3, \dots$ ,
     $\mathbf{Z}_n^{(k)} = \mathbf{A}\mathbf{Q}_n^{(k-1)}$ 
     $\mathbf{Q}_n^{(k)}\mathbf{R}_n^{(k)} = \mathbf{Z}_n^{(k)}$  (QR factorization)
end
```

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Assume  $\mathbf{A} = \mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}$  is diagonalizable with  $\mathbf{\Lambda} = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_m\}$  and

$$|\lambda_1| \geq \dots \geq |\lambda_n| > |\lambda_{n+1}| \geq \dots \geq |\lambda_m|.$$

Assume  $\mathbf{X}_n^{(0)} := [\mathbf{I}_n \quad \mathbf{0}] \mathbf{S}^{-1} \mathbf{Q}_n^{(0)}$  has full rank. Prove that

$$\mathbf{Q}_n^{(k)} = \mathbf{A}^k \mathbf{Q}_n^{(0)} (\mathbf{R}_n^{(1)})^{-1} (\mathbf{R}_n^{(2)})^{-1} \dots (\mathbf{R}_n^{(k)})^{-1}.$$

## Exercise 2. (Zhihao Cao, 10 points)

Let  $\mathbf{A}_1 = \begin{bmatrix} a & b \\ \varepsilon & c \end{bmatrix} \in \mathbb{R}^{2 \times 2}$ . Let  $\mathbf{A}_2 = \mathbf{R}_1 \mathbf{Q}_1 + c\mathbf{I} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ , where  $\mathbf{Q}_1 \mathbf{R}_1 = \mathbf{A}_1 - c\mathbf{I}$  is a QR factorization of  $\mathbf{A}_1 - c\mathbf{I}$ . Prove:

- (a) if  $a$  and  $c$  are not close (i.e.,  $\exists \delta > 0, |a - c| > \delta$ ), then  $a_{21} = \mathcal{O}(\varepsilon^2)$ ;
- (b) if further  $b = \varepsilon$  (i.e.,  $\mathbf{A}_1$  is symmetric), then  $a_{21} = \mathcal{O}(\varepsilon^3)$ .

## Exercise 3. (Programming, 10 points)

Write a function `[Q,H] = myhess(A)` that reduces a complex square matrix to upper Hessenberg form by unitary similarity transformations, i.e.,  $\mathbf{H} = \mathbf{Q}^* \mathbf{A} \mathbf{Q}$ . Your program should use only elementary Matlab operations – not the function `hess`, for example. Apply your program to  $\mathbf{A} = \text{randn}(5) + 1i * \text{randn}(5)$ .

## Exercise 4. (Programming, TreBau Exercise 29.1, 10 points)