On TriMR for solving symmetric quasi-definite linear systems

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joint work with Jia-Jun Fan and Ya-Lan Zhang

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Outline

Symmetric quasi-definite (SQD) linear systems

- $\mathbf{M} \in \mathbb{R}^{m\times m}$ and $\mathbf{N} \in \mathbb{R}^{n\times n}$ are SPD, $\mathbf{A} \in \mathbb{R}^{m\times n}$ is nonzero, $\mathbf{b} \in \mathbb{R}^{m}$, and $\mathbf{c} \in \mathbb{R}^{n}$:

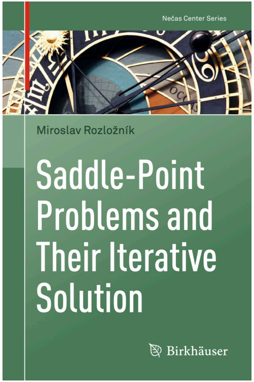
$$\begin{bmatrix} \mathbf{M} & \mathbf{A} \\ \mathbf{A}^\top & -\mathbf{N} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{c} \end{bmatrix}.$$

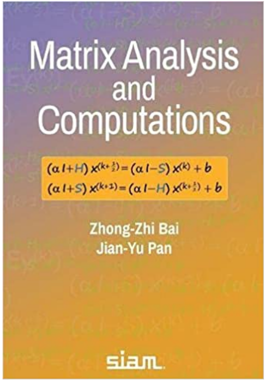
- Computational optimization and computational partial differential equations, etc.
- Symmetric, indefinite, nonsingular
- Monolithic methods: solving the system as a whole, for example, SYMMLQ, MINRES

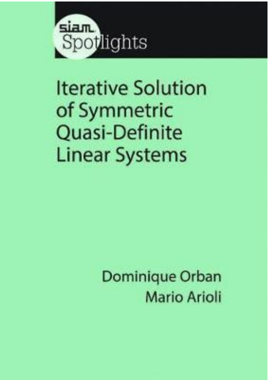
  Segregated methods: exploiting the block structure, excluding the preconditioning stage, for example, TriCG, TriMR

Review paper and books

- Michele Benzi, Gene H. Golub, and Jörg Liesen
  Numerical solution of saddle point problems. Acta Numerica (2005).
- Books







The generalized Saunders–Simon–Yip (gSSY) tridiagonalization

- The generalized Saunders–Simon–Yip tridiagonalization:

$$\begin{aligned}
\mathbf{A}\mathbf{V}_k &= \mathbf{M}\mathbf{U}_{k+1}\mathbf{T}_{k+1,k} = \mathbf{M}\mathbf{U}_k\mathbf{T}_k + \beta_{k+1}\mathbf{M}\mathbf{u}_{k+1}\mathbf{e}_k^\top, \\
\mathbf{A}^\top\mathbf{U}_k &= \mathbf{N}\mathbf{V}_{k+1}\mathbf{T}_{k,k+1}^\top = \mathbf{N}\mathbf{V}_k\mathbf{T}_k^\top + \gamma_{k+1}\mathbf{N}\mathbf{v}_{k+1}\mathbf{e}_k^\top, \\
\mathbf{U}_{k+1}^\top\mathbf{M}\mathbf{U}_{k+1} &= \mathbf{V}_{k+1}^\top\mathbf{N}\mathbf{V}_{k+1} = \mathbf{I}_{k+1},
\end{aligned}$$

where

$$\mathbf{U}_k = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{bmatrix}, \quad \mathbf{V}_k = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \end{bmatrix},$$

and

$$\mathbf{T}_k = \mathrm{tridiag}(\beta_i, \alpha_i, \gamma_{i+1}), \quad \mathbf{T}_{k+1,k} = \begin{bmatrix} \mathbf{T}_k \\ \beta_{k+1}\mathbf{e}_k^\top \end{bmatrix}, \quad \mathbf{T}_{k,k+1} = \begin{bmatrix} \mathbf{T}_k & \gamma_{k+1}\mathbf{e}_k \end{bmatrix}.$$

M. A. Saunders, H. D. Simon, and E. L. Yip. *Two conjugate-gradient-type methods for unsymmetric linear equations.* SINUM, Vol. 25, Iss. 4 (1988)

The generalized Saunders–Simon–Yip tridiagonalization

Algorithm 1 Generalized Saunders–Simon–Yip tridiagonalization

Require: $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{c} \in \mathbb{R}^n$ are nonzero, $\mathbf{M} \in \mathbb{R}^{m \times m}$ and $\mathbf{N} \in \mathbb{R}^{n \times n}$ are SPD, subroutines for performing $\mathbf{M}^{-1}\mathbf{u}$ and $\mathbf{N}^{-1}\mathbf{v}$

$\mathbf{u}_0 = \mathbf{0}$, $\mathbf{v}_0 = \mathbf{0}$, $\beta_1 \mathbf{M}\mathbf{u}_1 = \mathbf{b}$, $\gamma_1 \mathbf{N}\mathbf{v}_1 = \mathbf{c}$,

for $k = 1, 2, \ldots$ **do**

  $\mathbf{p} = \mathbf{A}\mathbf{v}_k - \gamma_k \mathbf{M}\mathbf{u}_{k-1}$, $\alpha_k = \mathbf{u}_k^\top \mathbf{p}$,

  $\beta_{k+1} \mathbf{M}\mathbf{u}_{k+1} = \mathbf{p} - \alpha_k \mathbf{M}\mathbf{u}_k$, $\gamma_{k+1} \mathbf{N}\mathbf{v}_{k+1} = \mathbf{A}^\top \mathbf{u}_k - \beta_k \mathbf{N}\mathbf{v}_{k-1} - \alpha_k \mathbf{N}\mathbf{v}_k$,

  if $\beta_{k+1} = 0$ or $\gamma_{k+1} = 0$, **terminate**

end for

TriMR: A minimal residual method for SQD linear systems

- The kth TriMR iterate is

$$\begin{bmatrix} \mathbf{x}_k \\ \mathbf{y}_k \end{bmatrix} = \underset{\mathbf{x} \in \text{range}(\mathbf{U}_k),\ \mathbf{y} \in \text{range}(\mathbf{V}_k)}{\text{argmin}} \left\| \begin{bmatrix} \mathbf{b} \\ \mathbf{c} \end{bmatrix} - \begin{bmatrix} \mathbf{M} & \mathbf{A} \\ \mathbf{A}^\top & -\mathbf{N} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \right\|_{\mathbf{H}^{-1}},$$

  where

$$\mathbf{H} = \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{N} \end{bmatrix}.$$

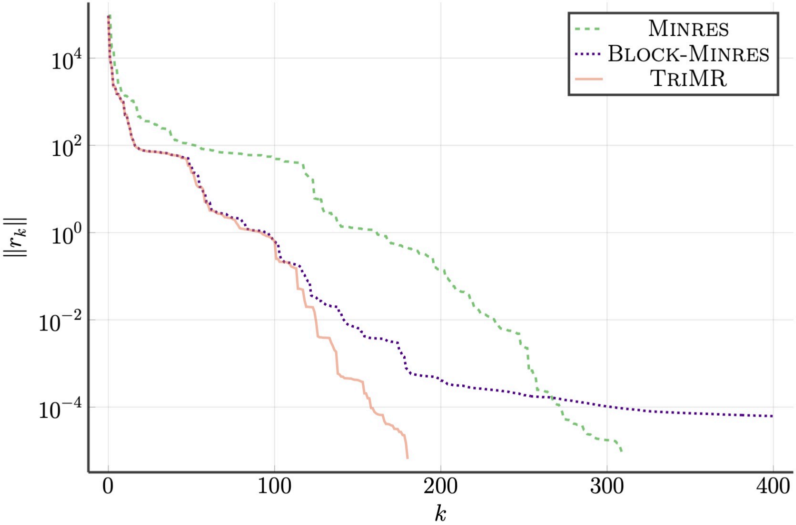
- Equivalent to preconditioned block-MINRES:

$$\begin{bmatrix} \mathbf{M} & \mathbf{A} \\ \mathbf{A}^\top & -\mathbf{N} \end{bmatrix} \begin{bmatrix} \mathbf{x}^1 & \mathbf{x}^2 \\ \mathbf{y}^1 & \mathbf{y}^2 \end{bmatrix} = \begin{bmatrix} \mathbf{b} & \mathbf{0} \\ \mathbf{0} & \mathbf{c} \end{bmatrix}.$$

- Faster than preconditioned MINRES.

A. Montoison and D. Orban. *TriCG and TriMR: Two iterative methods for symmetric quasi-definite systems.* SISC, Vol. 43, Iss. 4 (2021)

Example: $\mathbf{M} = \mathbf{I}$, $\mathbf{N} = \mathbf{I}$, $\mathbf{A}$ = `lp_osa_07`



The subproblem of TriMR

Let

$$\mathbf{W}_k = \begin{bmatrix} \mathbf{U}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_k \end{bmatrix} \Pi_{2k}, \quad \Pi_{2k} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_{k+1} & \cdots & \mathbf{e}_k & \mathbf{e}_{2k} \end{bmatrix},$$

and

$$\mathbf{S}_{k+1,k} = \Pi_{2k+2}^\top \begin{bmatrix} \begin{bmatrix} \mathbf{I}_k \\ \mathbf{0} \end{bmatrix} & \mathbf{T}_{k+1,k} \\ \mathbf{T}_{k,k+1}^\top & -\begin{bmatrix} \mathbf{I}_k \\ \mathbf{0} \end{bmatrix} \end{bmatrix} \Pi_{2k} = \begin{bmatrix} \Theta_1 & \Psi_2 & & \\ \Psi_2^\top & \Theta_2 & \ddots & \\ & \ddots & \ddots & \Psi_k \\ & & \ddots & \Theta_k \\ & & & \Psi_{k+1} \end{bmatrix},$$

where

$$\Theta_k = \begin{bmatrix} 1 & \alpha_k \\ \alpha_k & -1 \end{bmatrix} \quad \text{and} \quad \Psi_k = \begin{bmatrix} 0 & \gamma_k \\ \beta_k & 0 \end{bmatrix}.$$

The subproblem of TriMR

We have

$$\begin{bmatrix} \mathbf{M} & \mathbf{A} \\ \mathbf{A}^\top & -\mathbf{N} \end{bmatrix} \mathbf{W}_k = \mathbf{H}\mathbf{W}_{k+1}\mathbf{S}_{k+1,k}.$$

Then the kth TriMR iterate can be determined by

$$\begin{bmatrix} \mathbf{x}_k \\ \mathbf{y}_k \end{bmatrix} = \mathbf{W}_k \mathbf{z}_k$$

where $\mathbf{z}_k \in \mathbb{R}^{2k}$ solves

$$\min_{\mathbf{z} \in \mathbb{R}^{2k}} \|\mathbf{S}_{k+1,k}\mathbf{z} - (\beta_1 \mathbf{e}_1 + \gamma_1 \mathbf{e}_2)\|.$$

The vector $\mathbf{z}_k$ can be determined via the QR factorization

$$\mathbf{S}_{k+1,k} = \mathbf{Q}_k \begin{bmatrix} \mathbf{R}_k \\ \mathbf{0} \end{bmatrix},$$

The subproblem of TriMR

where

$$\mathbf{Q}_k \in \mathbb{R}^{(2k+2)\times(2k+2)}$$

is a product of reflections, and

$$\mathbf{R}_k = \begin{bmatrix} \delta_1 & \sigma_1 & \eta_1 & \lambda_1 & \mu_1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots & \mu_{2k-4} \\ & & & \ddots & \ddots & \ddots & \lambda_{2k-3} \\ & & & & \ddots & \ddots & \eta_{2k-2} \\ & & & & & \ddots & \sigma_{2k-1} \\ & & & & & & \delta_{2k} \end{bmatrix} \in \mathbb{R}^{(2k)\times(2k)}.$$

The subproblem of TriMR

Theorem ($\mathbf{R}_k$ has only three nonzero diagonals)

The upper triangular matrix $\mathbf{R}_k$ of the QR factorization has the following form:

$$\mathbf{R}_k = \begin{bmatrix} \delta_1 & 0 & \eta_1 & 0 & \mu_1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots & \mu_{2k-4} \\ & & & \ddots & \ddots & \ddots & 0 \\ & & & & \ddots & \ddots & \eta_{2k-2} \\ & & & & & \ddots & 0 \\ & & & & & & \delta_{2k} \end{bmatrix}.$$

Rapoport's method for nonsymmetric positive definite systems

- The equivalent nonsymmetric positive definite system

$$\begin{bmatrix} \mathbf{M} & \mathbf{A} \\ -\mathbf{A}^\top & \mathbf{N} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ -\mathbf{c} \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} \mathbf{M} & \\ & \mathbf{N} \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} & \mathbf{A} \\ -\mathbf{A}^\top & \end{bmatrix}.$$

- The kth Rapoport iterate is

$$\begin{bmatrix} \mathbf{x}_k \\ \mathbf{y}_k \end{bmatrix} = \operatorname*{argmin}_{\mathbf{z} \in \mathcal{K}_k} \left\| \begin{bmatrix} \mathbf{b} \\ -\mathbf{c} \end{bmatrix} - \begin{bmatrix} \mathbf{M} & \mathbf{A} \\ -\mathbf{A}^\top & \mathbf{N} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \right\|_{\mathbf{H}^{-1}},$$

  where the Krylov subspace

$$\mathcal{K}_k = \operatorname{span}\left\{ \mathbf{H}^{-1} \begin{bmatrix} \mathbf{b} \\ -\mathbf{c} \end{bmatrix}, \; \mathbf{H}^{-1}\mathbf{S} \begin{bmatrix} \mathbf{b} \\ -\mathbf{c} \end{bmatrix}, \; \ldots, \; (\mathbf{H}^{-1}\mathbf{S})^{k-1} \begin{bmatrix} \mathbf{b} \\ -\mathbf{c} \end{bmatrix} \right\}.$$

- The skew-symmetric Lanczos: $\mathbf{S}\mathbf{Q}_k = \mathbf{H}\mathbf{Q}_{k+1}\mathbf{T}_{k+1,k}$.

TriMR is faster than Rapoport's method

- The kth TriMR iterate is

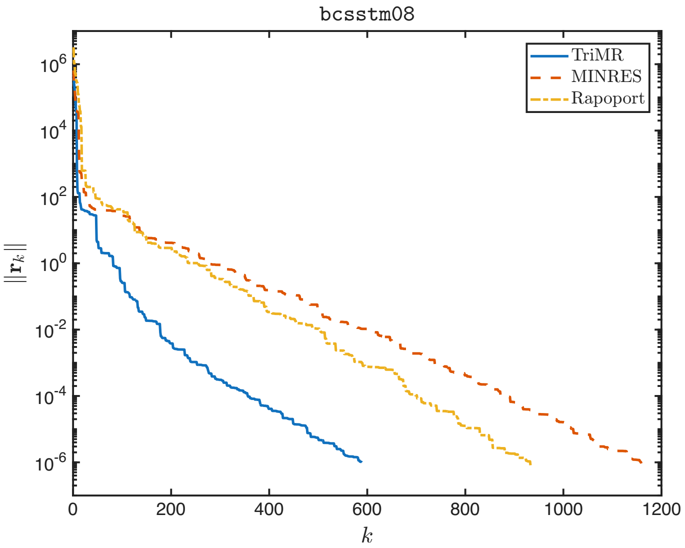
$$\begin{bmatrix} \mathbf{x}_k \\ \mathbf{y}_k \end{bmatrix} = \underset{\mathbf{x} \in \operatorname{range}(\mathbf{U}_k),\ \mathbf{y} \in \operatorname{range}(\mathbf{V}_k)}{\operatorname{argmin}} \left\| \begin{bmatrix} \mathbf{b} \\ \mathbf{c} \end{bmatrix} - \begin{bmatrix} \mathbf{M} & \mathbf{A} \\ \mathbf{A}^\top & -\mathbf{N} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \right\|_{\mathbf{H}^{-1}}.$$

- The kth Rapoport iterate is

$$\begin{aligned} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{y}_k \end{bmatrix} &= \underset{\mathbf{z} \in \mathcal{K}_k}{\operatorname{argmin}} \left\| \begin{bmatrix} \mathbf{b} \\ -\mathbf{c} \end{bmatrix} - \begin{bmatrix} \mathbf{M} & \mathbf{A} \\ -\mathbf{A}^\top & \mathbf{N} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \right\|_{\mathbf{H}^{-1}} \\ &= \underset{\mathbf{z} \in \mathcal{K}_k}{\operatorname{argmin}} \left\| \begin{bmatrix} \mathbf{b} \\ \mathbf{c} \end{bmatrix} - \begin{bmatrix} \mathbf{M} & \mathbf{A} \\ \mathbf{A}^\top & -\mathbf{N} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \right\|_{\mathbf{H}^{-1}}. \end{aligned}$$

- TriMR is faster than Rapoport's method since $\mathcal{K}_k \subset \operatorname{range}\left(\begin{bmatrix} \mathbf{U}_k & \\ & \mathbf{V}_k \end{bmatrix} \right)$.

Example: $\mathrm{M} = \mathrm{I}$, $\mathrm{N} = \mathrm{I}$, A = `bcsstm08`



Terminations of gSSY

Algorithm 1 Generalized Saunders–Simon–Yip tridiagonalization

Require: $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{c} \in \mathbb{R}^n$ are nonzero, $\mathbf{M} \in \mathbb{R}^{m\times m}$ and $\mathbf{N} \in \mathbb{R}^{n\times n}$ are SPD, subroutines for performing $\mathbf{M}^{-1}\mathbf{u}$ and $\mathbf{N}^{-1}\mathbf{v}$

$\mathbf{u}_0 = \mathbf{0}$, $\mathbf{v}_0 = \mathbf{0}$, $\beta_1 \mathbf{M}\mathbf{u}_1 = \mathbf{b}$, $\gamma_1 \mathbf{N}\mathbf{v}_1 = \mathbf{c}$,
for $k = 1, 2, \ldots$ **do**
    $\mathbf{p} = \mathbf{A}\mathbf{v}_k - \gamma_k \mathbf{M}\mathbf{u}_{k-1}$, $\alpha_k = \mathbf{u}_k^\top \mathbf{p}$,
    $\beta_{k+1}\mathbf{M}\mathbf{u}_{k+1} = \mathbf{p} - \alpha_k \mathbf{M}\mathbf{u}_k$, $\gamma_{k+1}\mathbf{N}\mathbf{v}_{k+1} = \mathbf{A}^\top \mathbf{u}_k - \beta_k \mathbf{N}\mathbf{v}_{k-1} - \alpha_k \mathbf{N}\mathbf{v}_k$,
    if $\beta_{k+1} = 0$ or $\gamma_{k+1} = 0$, **terminate**
end for

- gSSY must terminate in $\ell \le \min(m, n)$ steps in exact arithmetic, and either $\beta_{\ell+1} = 0$ or $\beta_{\ell+1} \neq 0$ and $\gamma_{\ell+1} = 0$. $\beta_{\ell+1} = \gamma_{\ell+1} = 0$ ensures a lucky termination (i.e., the exact solution is in the final subspace generated by gSSY). When $\beta_{\ell+1}$ and $\gamma_{\ell+1}$ are not simultaneous zero, unlucky terminations.

Unlucky terminations of gSSY: case I

Example (The case that $\beta_{\ell+1} = 0$ and $\gamma_{\ell+1} \neq 0$)

The solution to the SQD linear system with

$$\mathbf{M} = \mathbf{N} = \mathbf{I}_3, \quad \mathbf{A} = \begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \quad \mathbf{b} = \mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

is $\begin{bmatrix} 1 & 2 & 1 & -3 & 0 & 1 \end{bmatrix}^\top / 4$. gSSY terminates at step $\ell = 2$ with $\beta_{\ell+1} = 0$, and we have $\gamma_{\ell+1} = 1 \neq 0$ and $\mathbf{U}_\ell = \mathbf{V}_\ell = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{bmatrix}$. Obviously,

$$\begin{bmatrix} 1 & 2 & 1 & -3 & 0 & 1 \end{bmatrix}^\top \notin \mathrm{range}\left(\begin{bmatrix} \mathbf{U}_\ell & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_\ell \end{bmatrix} \right).$$

Unlucky terminations of gSSY: case II

Example (The case that $\beta_{\ell+1} \neq 0$ and $\gamma_{\ell+1} = 0$)

The solution to the SQD linear system with

$$\mathbf{M} = \mathbf{N} = \mathbf{I}_3, \quad \mathbf{A} = \begin{bmatrix} -1 & 1 & 0 \\ 3 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \quad \mathbf{b} = \mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

is $\begin{bmatrix} 11 & 8 & -1 & -2 & 2 & 1 \end{bmatrix}^\top / 15$. gSSY terminates at step $\ell = 2$ with $\beta_{\ell+1} = 1 \neq 0$ and $\gamma_{\ell+1} = 0$, and we have $\mathbf{U}_{\ell+1} = \mathbf{I}_3$, $\mathbf{V}_\ell = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{bmatrix}$. Obviously,

$$\begin{bmatrix} 11 & 8 & -1 & -2 & 2 & 1 \end{bmatrix}^\top \notin \operatorname{range}\left(\begin{bmatrix} \mathbf{U}_{\ell+1} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_\ell \end{bmatrix} \right).$$

Improved generalized SSY tridiagonalization

Algorithm 2 Improved generalized Saunders–Simon–Yip tridiagonalization: igSSY$(\mathbf{M}, \mathbf{N}, \mathbf{A}, \mathbf{b}, \mathbf{c})$

Require: $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{c} \in \mathbb{R}^n$ are not simultaneously zero, $\mathbf{M} \in \mathbb{R}^{m\times m}$ and $\mathbf{N} \in \mathbb{R}^{n\times n}$ are SPD, subroutines for performing $\mathbf{M}^{-1}\mathbf{u}$ and $\mathbf{N}^{-1}\mathbf{v}$

$\mathbf{u}_0 = \mathbf{0}$, $\mathbf{v}_0 = \mathbf{0}$
$\mathbf{u} = \mathbf{M}^{-1}\mathbf{b}$, $\beta_1 = \sqrt{\mathbf{b}^\top \mathbf{u}}$; **if** $\beta_1 \neq 0$, **then** $\mathbf{u}_1 = \mathbf{u}/\beta_1$ **end if**
$\mathbf{v} = \mathbf{N}^{-1}\mathbf{c}$, $\gamma_1 = \sqrt{\mathbf{c}^\top \mathbf{v}}$; **if** $\gamma_1 \neq 0$, **then** $\mathbf{v}_1 = \mathbf{v}/\gamma_1$ **end if**
$k = 1$
while $\beta_k\gamma_k \neq 0$ **do**
    $\mathbf{p} = \mathbf{A}\mathbf{v}_k - \gamma_k \mathbf{M}\mathbf{u}_{k-1}$
    $\alpha_k = \mathbf{u}_k^\top \mathbf{p}$, $\mathbf{p} = \mathbf{p} - \alpha_k \mathbf{M}\mathbf{u}_k$
    $\mathbf{u} = \mathbf{M}^{-1}\mathbf{p}$, $\beta_{k+1} = \sqrt{\mathbf{p}^\top \mathbf{u}}$; **if** $\beta_{k+1} \neq 0$, **then** $\mathbf{u}_{k+1} = \mathbf{u}/\beta_{k+1}$ **end if**
    $\mathbf{q} = \mathbf{A}^\top \mathbf{u}_k - \beta_k \mathbf{N}\mathbf{v}_{k-1} - \alpha_k \mathbf{N}\mathbf{v}_k$
    $\mathbf{v} = \mathbf{N}^{-1}\mathbf{q}$, $\gamma_{k+1} = \sqrt{\mathbf{q}^\top \mathbf{v}}$; **if** $\gamma_{k+1} \neq 0$, **then** $\mathbf{v}_{k+1} = \mathbf{v}/\gamma_{k+1}$ **end if**
    $k = k + 1$
end while
$\ell = k - 1$
if $\beta_{\ell+1} = 0$ and $\gamma_{\ell+1} = 0$, **terminate**
if $\beta_{\ell+1} = 0$ **then**
    for $k = \ell+1, \ell+2, \dots$ **do**
        $\alpha_k \mathbf{M}\mathbf{u}_k = \mathbf{A}\mathbf{v}_k - \gamma_k \mathbf{M}\mathbf{u}_{k-1}$
        $\gamma_{k+1}\mathbf{N}\mathbf{v}_{k+1} = \mathbf{A}^\top \mathbf{u}_k - \alpha_k \mathbf{N}\mathbf{v}_k$
        if $\alpha_k = 0$ or $\gamma_{k+1} = 0$, **terminate**
    end for
else if $\gamma_{\ell+1} = 0$ **then**
    for $k = \ell+1, \ell+2, \dots$ **do**
        $\alpha_k \mathbf{N}\mathbf{v}_k = \mathbf{A}^\top \mathbf{u}_k - \beta_k \mathbf{N}\mathbf{v}_{k-1}$
        $\beta_{k+1}\mathbf{M}\mathbf{u}_{k+1} = \mathbf{A}\mathbf{v}_k - \alpha_k \mathbf{M}\mathbf{u}_k$
        if $\alpha_k = 0$ or $\beta_{k+1} = 0$, **terminate**
    end for
end if

Terminations of igSSY

- Assume that gSSY terminates at step ℓ, i.e., $\beta_{\ell+1} = 0$ or $\gamma_{\ell+1} = 0$.
- Assume that igSSY terminates at step $L \geq \ell$. Five cases occur (see lines 14, 19, and 25): for $k = \ell, \ldots, L$,

  Case I: $\beta_{\ell+1} = \gamma_{\ell+1} = 0$;

  Case II: $\alpha_{L+1} = 0$, $\beta_{k+1} = 0$, $\gamma_{k+1} \neq 0$;

  Case III: $\alpha_{L+1} \neq 0$, $\beta_{k+1} = 0$, $\gamma_{k+1} \neq 0$, $\gamma_{L+2} = 0$;

  Case IV: $\alpha_{L+1} = 0$, $\beta_{k+1} \neq 0$, $\gamma_{k+1} = 0$;

  Case V: $\alpha_{L+1} \neq 0$, $\beta_{k+1} \neq 0$, $\gamma_{k+1} = 0$, $\beta_{L+2} = 0$.

  All are lucky terminations. The solution of the SQD linear system belongs to the final subspace generated by igSSY.

Elliptic singular value decomposition (ESVD)

- Given SPD $\mathbf{M}$ and $\mathbf{N}$, ESVD of $\mathbf{A}$ is

$$\mathbf{A} = \mathbf{M}\mathbf{P}\boldsymbol{\Sigma}\mathbf{Q}^\top\mathbf{N},$$

where $\boldsymbol{\Sigma} = \mathrm{diag}(\sigma_1, \ldots, \sigma_p)$, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0$, $p = \min(m, n)$, and $\mathbf{P}$ and $\mathbf{Q}$ satisfy

$$\mathbf{P}^\top\mathbf{M}\mathbf{P} = \mathbf{I}_m, \quad \mathbf{Q}^\top\mathbf{N}\mathbf{Q} = \mathbf{I}_n.$$

Theorem

Assume that igSSY terminates at step L. If d is the number of distinct elliptic singular values of $\mathbf{A}$ and r is the rank of $\mathbf{A}$, then we have $L \leq \min(2d, r)$.

M. Arioli. *Generalized Golub–Kahan bidiagonalization and stopping criteria*. SIMAX, Vol. 34, Iss. 2 (2013)

Summary

- We proved that the upper triangular factor of the QR factorization used in TriMR only has three nonzero diagonals, and based on this fact we provided simplified short recurrences for TriMR, which reduce the work per iteration.
- We proved that the TriMR method for SQD linear systems is faster than Rapoport's method for the equivalent nonsymmetric positive definite linear systems.
- We proposed igSSY, which avoids unlucky terminations of gSSY. Improved TriMR can be defined in the same fashion as TriMR, but is based on igSSY instead of gSSY.
- Algorithms that fully leverage the underlying structure of a problem can be significantly more efficient than general-purpose algorithms!

The related work

- Kui Du, Jia-Jun Fan, Xiao-Hui Sun, Fang Wang, and Ya-Lan Zhang

  On Krylov subspace methods for skew-symmetric and shifted skew-symmetric linear systems

  Advances in Computational Mathematics (2024) 50:78

- Kui Du, Jia-Jun Fan, and Ya-Lan Zhang

  Improved TriCG and TriMR methods for symmetric quasi-definite linear systems

  Numerical Linear Algebra with Applications, 2025, 32:e70026

Thanks!