Lecture 1: Inner product, Orthogonality, Vector/Matrix norms

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1. **Inner product** on a linear space $\mathbb{V}$ over a number field $\mathbb{F}$ ($\mathbb{C}$ or $\mathbb{R}$)

- Definition: A function $\langle\cdot,\cdot\rangle : \mathbb{V}\times\mathbb{V}\to\mathbb{F}$ is called an *inner product*, if it satisfies the following three conditions ($\forall\ \mathbf{x},\mathbf{y},\mathbf{z}\in\mathbb{V}$, $\forall\ \alpha\in\mathbb{F}$):

  (1) Conjugate symmetry:

$$\langle\mathbf{x},\mathbf{y}\rangle = \overline{\langle\mathbf{y},\mathbf{x}\rangle}$$

  (2) Positive definiteness:

$$\langle\mathbf{x},\mathbf{x}\rangle \geq 0, \quad \langle\mathbf{x},\mathbf{x}\rangle = 0 \Leftrightarrow \mathbf{x}=\mathbf{0}$$

  (3) Linearity in the first variable:

$$\langle\mathbf{x}+\mathbf{y},\mathbf{z}\rangle = \langle\mathbf{x},\mathbf{z}\rangle + \langle\mathbf{y},\mathbf{z}\rangle, \quad \langle\alpha\mathbf{x},\mathbf{y}\rangle = \alpha\langle\mathbf{x},\mathbf{y}\rangle$$

  Example: the standard inner product on the space $\mathbb{V}=\mathbb{C}^m$:

$$\forall\ \mathbf{x},\mathbf{y}\in\mathbb{C}^m, \quad \langle\mathbf{x},\mathbf{y}\rangle = \mathbf{y}^*\mathbf{x} = \sum\nolimits_{i=1}^{m} x_i\overline{y}_i.$$

Example: the $\mathbf{A}$-inner product on the space $\mathbb{V} = \mathbb{C}^m$:

$$\forall\ \mathbf{x}, \mathbf{y} \in \mathbb{C}^m, \quad \langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^* \mathbf{A} \mathbf{x},$$

where $\mathbf{A}$ is a given Hermitian positive definite matrix.

2. Orthogonality

- Orthogonality is a mathematical concept with respect to a given inner product $\langle \cdot, \cdot \rangle$.
  - (1) Two vectors $\mathbf{x}$ and $\mathbf{y}$ are called *orthogonal* if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.
  - (2) Two sets of vectors $\mathcal{X}$ and $\mathcal{Y}$ are called orthogonal if $\forall\ \mathbf{x} \in \mathcal{X}$ and $\mathbf{y} \in \mathcal{Y}$, $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.
  - (3) A set of nonzero vectors $\mathcal{S}$ is orthogonal if $\forall\ \mathbf{x}, \mathbf{y} \in \mathcal{S}$ and $\mathbf{x} \neq \mathbf{y}$, $\langle \mathbf{x}, \mathbf{y} \rangle = 0$; if further $\forall\ \mathbf{x} \in \mathcal{S}$, $\langle \mathbf{x}, \mathbf{x} \rangle =1$, $\mathcal{S}$ is called *orthonormal*.

Proposition 1

The vectors in an orthogonal set $\mathcal{S}$ are linearly independent.

2.1. Orthogonal components of a vector

- Inner products can be used to decompose arbitrary vectors into orthogonal components. Given an *orthonormal* set $\{\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n\}$ and an arbitrary vector $\mathbf{v}$, let

$$\mathbf{r} = \mathbf{v} - \langle \mathbf{v}, \mathbf{q}_1 \rangle \mathbf{q}_1 - \langle \mathbf{v}, \mathbf{q}_2 \rangle \mathbf{q}_2 - \cdots - \langle \mathbf{v}, \mathbf{q}_n \rangle \mathbf{q}_n.$$

  Obviously,

$$\mathbf{r} \in \operatorname{span}\{\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n\}^{\perp}.$$

  Thus we see that $\mathbf{v}$ can be decomposed into $n+1$ orthogonal components:

$$\mathbf{v} = \mathbf{r} + \langle \mathbf{v}, \mathbf{q}_1 \rangle \mathbf{q}_1 + \langle \mathbf{v}, \mathbf{q}_2 \rangle \mathbf{q}_2 + \cdots + \langle \mathbf{v}, \mathbf{q}_n \rangle \mathbf{q}_n.$$

  We call $\langle \mathbf{v}, \mathbf{q}_i \rangle \mathbf{q}_i$ the part of $\mathbf{v}$ in the direction of $\mathbf{q}_i$, and $\mathbf{r}$ the part of $\mathbf{v}$ orthogonal to the subspace $\operatorname{span}\{\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n\}$.

Exercise: Write the expression for $\mathbf{v}$ when the set $\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n\}$ is only orthogonal.

- Cauchy–Schwarz inequality: For any given inner product $\langle\cdot,\cdot\rangle$,

$$|\langle \mathbf{x}, \mathbf{y}\rangle| \le \sqrt{\langle \mathbf{x}, \mathbf{x}\rangle}\sqrt{\langle \mathbf{y}, \mathbf{y}\rangle}.$$

The equality holds if and only if $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent.

Exercise: Prove the inequality. Hint: write

$$\mathbf{x} = \frac{\langle \mathbf{x}, \mathbf{y}\rangle}{\langle \mathbf{y}, \mathbf{y}\rangle}\mathbf{y} + \mathbf{z}.$$

Then $\langle \mathbf{z}, \mathbf{y}\rangle = 0$. Consider $\langle \mathbf{x}, \mathbf{x}\rangle$.

Application: For any Hermitian positive definite matrix $\mathbf{A}$,

$$|\mathbf{y}^*\mathbf{A}\mathbf{x}|^2 \le (\mathbf{x}^*\mathbf{A}\mathbf{x})(\mathbf{y}^*\mathbf{A}\mathbf{y}).$$

3. Norm on a linear space $\mathbb{V}$ over a number field $\mathbb{F}$ ($\mathbb{C}$ or $\mathbb{R}$)

- Definition: A function $\|\cdot\| : \mathbb{V} \to \mathbb{R}$ is called a *norm* if it satisfies the following three conditions ($\forall\ \mathbf{x}, \mathbf{y} \in \mathbb{V}$ and $\forall\ \alpha \in \mathbb{F}$):

  (1) Positive definiteness:

  $$\|\mathbf{x}\| \geq 0, \quad \|\mathbf{x}\| = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$$

  (2) Absolute homogeneity:

  $$\|\alpha \mathbf{x}\| = |\alpha| \|\mathbf{x}\|$$

  (3) Triangle inequality:

  $$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$$

  Exercise: Show that any norm is continuous.

- More on metric, norm, and inner product (click!)

Exercise: For any given inner product $\langle \cdot, \cdot \rangle$, let $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$.

(1) Prove that the function $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$ is a norm.

(2) Prove the parallelogram law

$$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2.$$

(3) For a set of n orthogonal (with respect to the inner product $\langle \cdot, \cdot \rangle$) vectors $\{\mathbf{x}_i\}$, prove that

$$\left\| \sum_{i=1}^{n} \mathbf{x}_i \right\|^2 = \sum_{i=1}^{n} \|\mathbf{x}_i\|^2.$$

The function $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$ is called the norm *induced* by the inner product $\langle \cdot, \cdot \rangle$. Using this norm, we can write the Cauchy–Schwarz inequality as

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \|\mathbf{y}\|.$$

Theorem 2 (Equivalence of norms)

For each pair of norms $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ on a finite-dimensional linear space $\mathbb{V}$, there exist positive constants $a > 0$ and $b > 0$ (depending only on the norms) such that

$$a\|\mathbf{x}\|_\beta \le \|\mathbf{x}\|_\alpha \le b\|\mathbf{x}\|_\beta, \qquad \forall\ \mathbf{x} \in \mathbb{V}.$$

Sketch of the proof.

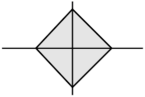
For all $\mathbf{x} = \sum_i x_i \mathbf{v}_i$, where $\{\mathbf{v}_i\}$ is a basis of $\mathbb{V}$, $\|\mathbf{x}\| = \sum_i |x_i|$ is a norm on $\mathbb{V}$. By $\|\mathbf{x}\|_\alpha = \|\sum_i x_i \mathbf{v}_i\|_\alpha \le \sum_i |x_i| \|\mathbf{v}_i\|_\alpha \le \|\mathbf{x}\| \cdot \max_i \|\mathbf{v}_i\|_\alpha$, we know $\|\cdot\|_\alpha$ is a continuous function with respect to $\|\cdot\|$, which attains its minimum c and maximum C on the unit sphere $\{\mathbf{x} \in \mathbb{V},\ \|\mathbf{x}\| = 1\}$ (because it is a compact set). Then, $\forall\ \mathbf{x} \in \mathbb{V}$, $c\|\mathbf{x}\| \le \|\mathbf{x}\|_\alpha \le C\|\mathbf{x}\|$. □

- Convergence of a sequence $\{\mathbf{x}_k\} \subset \mathbb{V}$: $\mathbf{x}_k \to \mathbf{x}$

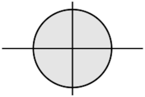
  We say $\mathbf{x}_k$ converges to $\mathbf{x}$ if $\lim_{k\to\infty} \|\mathbf{x}_k - \mathbf{x}\| = 0$.

3.1. Vector norms on $\mathbb{C}^m$

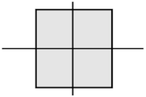
- ℓ_p-norm: $\|\mathbf{x}\|_1 = \sum_{i=1}^{m} |x_i|,$



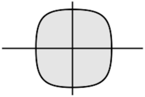
$$\|\mathbf{x}\|_2 = \left(\sum_{i=1}^{m} |x_i|^2\right)^{1/2} = \sqrt{\mathbf{x}^*\mathbf{x}},$$



$$\|\mathbf{x}\|_\infty = \max_{1\le i\le m} |x_i|,$$



$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{m} |x_i|^p\right)^{1/p}, \quad (1 \le p < \infty)$$



Minkowski's inequality: $\|\mathbf{x}+\mathbf{y}\|_p \le \|\mathbf{x}\|_p + \|\mathbf{y}\|_p$.

Equivalence of ℓ_1, ℓ_2, and ℓ_∞ norms: $\|\mathbf{x}\|_2 \le \|\mathbf{x}\|_1 \le \sqrt{m}\|\mathbf{x}\|_2$,

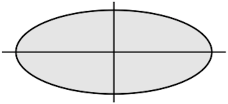
$$\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_2 \le \sqrt{m}\|\mathbf{x}\|_\infty, \quad \|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_1 \le m\|\mathbf{x}\|_\infty.$$

- Weighted norm: Let $\|\cdot\|$ denote any norm on $\mathbb{C}^m$. Suppose a diagonal matrix $\mathbf{W} = \mathrm{diag}\{w_1, \ldots, w_m\}$, $w_i \neq 0$. Then

$$\|\mathbf{x}\|_{\mathbf{W}} = \|\mathbf{W}\mathbf{x}\|$$

is a norm, called *weighted norm*. For example, weighted 2-norm

$$\|\mathbf{x}\|_{\mathbf{W}} = \|\mathbf{W}\mathbf{x}\|_2 = \left(\sum_{i=1}^{m} |w_i x_i|^2\right)^{1/2}.$$



- Dual norm: Let $\|\cdot\|$ denote any norm on $\mathbb{C}^m$. The corresponding *dual norm* $\|\cdot\|'$ (with respect to an inner product $\langle\cdot,\cdot\rangle$) is defined by

$$\|\mathbf{x}\|' = \sup_{\mathbf{y}\in\mathbb{C}^m,\ \|\mathbf{y}\|=1} |\langle \mathbf{x}, \mathbf{y}\rangle|.$$

Exercise: Let $\langle\cdot,\cdot\rangle$ denote the standard inner product on $\mathbb{C}^m$. If $p, q \in [1, \infty]$ with $1/p + 1/q = 1$, then $\|\cdot\|_p' = \|\cdot\|_q$. In particular, we have Hölder inequality: $|\mathbf{y}^*\mathbf{x}| \leq \|\mathbf{x}\|_p \|\mathbf{y}\|_q$.

3.2. Matrix norms on $\mathbb{C}^{m\times n}$

- Frobenius norm: $\forall\ \mathbf{A} = \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n\end{bmatrix} \in \mathbb{C}^{m\times n}$, define

$$\|\mathbf{A}\|_{\mathrm{F}} := \left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\right)^{1/2} = \left(\sum_{j=1}^{n}\|\mathbf{a}_j\|_2^2\right)^{1/2}$$

  or

$$\|\mathbf{A}\|_{\mathrm{F}} = \sqrt{\mathrm{tr}(\mathbf{A}^*\mathbf{A})} = \sqrt{\mathrm{tr}(\mathbf{A}\mathbf{A}^*)}.$$

- Max norm:

$$\|\mathbf{A}\|_{\max} := \max_{i,j}|a_{ij}|.$$

- Induced matrix norm (operator norm): $\forall\ \mathbf{A} \in \mathbb{C}^{m\times n}$, define

$$\|\mathbf{A}\|_{\alpha,\beta} := \sup_{\substack{\mathbf{x}\in\mathbb{C}^n\\ \mathbf{x}\neq\mathbf{0}}} \frac{\|\mathbf{A}\mathbf{x}\|_\alpha}{\|\mathbf{x}\|_\beta} = \sup_{\substack{\mathbf{x}\in\mathbb{C}^n\\ \|\mathbf{x}\|_\beta=1}} \|\mathbf{A}\mathbf{x}\|_\alpha = \sup_{\substack{\mathbf{x}\in\mathbb{C}^n\\ \|\mathbf{x}\|_\beta\leq 1}} \|\mathbf{A}\mathbf{x}\|_\alpha,$$

  where $\|\cdot\|_\alpha$ is a norm on $\mathbb{C}^m$ and $\|\cdot\|_\beta$ is a norm on $\mathbb{C}^n$. We say that $\|\cdot\|_{\alpha,\beta}$ is the matrix norm induced by $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$.

Exercise: $\forall\ \mathbf{x} \in \mathbb{C}^n$, prove that

$$\|\mathbf{Ax}\|_\alpha \leq \|\mathbf{A}\|_{\alpha,\beta}\|\mathbf{x}\|_\beta.$$

Exercise: Let $\mathbf{A} \in \mathbb{C}^{m\times n}$, $\mathbf{B} \in \mathbb{C}^{n\times r}$ and let $\|\cdot\|_\alpha$, $\|\cdot\|_\beta$, and $\|\cdot\|_\gamma$ be norms on $\mathbb{C}^m, \mathbb{C}^n$, and $\mathbb{C}^r$, respectively. Prove the induced matrix norms $\|\cdot\|_{\alpha,\gamma}$, $\|\cdot\|_{\alpha,\beta}$, and $\|\cdot\|_{\beta,\gamma}$ satisfy

$$\|\mathbf{AB}\|_{\alpha,\gamma} \leq \|\mathbf{A}\|_{\alpha,\beta}\|\mathbf{B}\|_{\beta,\gamma}.$$

Exercise: Prove that

$$\|\mathbf{A}\|_{\infty,1} = \max_{i,j} |a_{ij}|,$$

i.e., $\|\mathbf{A}\|_{\max}$ is the matrix norm induced by $\|\cdot\|_\infty$ and $\|\cdot\|_1$.

- The Frobenius norm $\|\cdot\|_{\mathrm{F}}$ on $\mathbb{C}^{m\times n}$ is not induced by norms on $\mathbb{C}^m$ and $\mathbb{C}^n$. (See Ref. 1 and Ref. 2)

- Induced matrix p-norm of $\mathbf{A} \in \mathbb{C}^{m \times n}$: For $p \in [1, +\infty]$,

$$\|\mathbf{A}\|_p := \|\mathbf{A}\|_{p,p} = \sup_{\mathbf{x} \in \mathbb{C}^n, \|\mathbf{x}\|_p = 1} \|\mathbf{A}\mathbf{x}\|_p.$$

  Example: For any diagonal matrix $\mathbf{D} = \mathrm{diag}\{d_1, \ldots, d_m\}$, we have

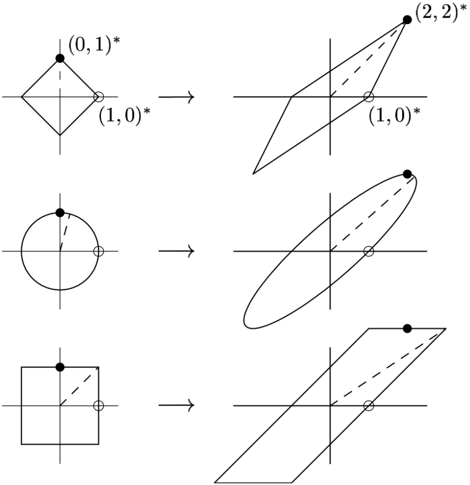
$$\|\mathbf{D}\|_p = \max_{1 \le i \le m} |d_i|.$$

  Example: $1, 2, \infty$-norm

$$\begin{aligned} \|\mathbf{A}\|_1 &= \max_j \sum_i |a_{ij}|, \quad \|\mathbf{A}\|_\infty = \max_i \sum_j |a_{ij}|, \\ \|\mathbf{A}\|_2 &= \sqrt{\lambda_{\max}(\mathbf{A}^*\mathbf{A})} = \sqrt{\lambda_{\max}(\mathbf{A}\mathbf{A}^*)} \le \|\mathbf{A}\|_{\mathrm{F}}. \end{aligned}$$

  The norm $\|\cdot\|_2$ on $\mathbb{C}^{m \times n}$ is also called the spectral norm.
  Inequalities: $\|\mathbf{A}\|_\infty \le \sqrt{n}\|\mathbf{A}\|_2, \quad \|\mathbf{A}\|_2 \le \sqrt{m}\|\mathbf{A}\|_\infty$.
- Matlab: `norm` for $1, 2, \infty$-norm

Example: $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$



1-norm: $\longrightarrow$ $\|\mathbf{A}\|_1 = 4$

2-norm: $\longrightarrow$ $\|\mathbf{A}\|_2 \approx 2.9208$

∞-norm: $\longrightarrow$ $\|\mathbf{A}\|_\infty = 3$

3.3. Unitary invariance of $\|\cdot\|_2$ and $\|\cdot\|_{\mathrm{F}}$: $\forall\ \mathbf{A} \in \mathbb{C}^{m\times n}$

- If $\mathbf{P}$ has orthonormal columns, i.e.,

$$\mathbf{P} \in \mathbb{C}^{p\times m}, \quad p \geq m, \quad \mathbf{P}^*\mathbf{P} = \mathbf{I}_m,$$

  then

$$\|\mathbf{PA}\|_2 = \|\mathbf{A}\|_2, \quad \|\mathbf{PA}\|_{\mathrm{F}} = \|\mathbf{A}\|_{\mathrm{F}}.$$

- If $\mathbf{Q}$ has orthonormal rows, i.e.,

$$\mathbf{Q} \in \mathbb{C}^{n\times q}, \quad n \leq q, \quad \mathbf{Q}\mathbf{Q}^* = \mathbf{I}_n,$$

  then

$$\|\mathbf{AQ}\|_2 = \|\mathbf{A}\|_2, \quad \|\mathbf{AQ}\|_{\mathrm{F}} = \|\mathbf{A}\|_{\mathrm{F}}.$$

4. Unitary matrix

- For $\mathbf{Q} \in \mathbb{C}^{m\times m}$, if $\mathbf{Q}^* = \mathbf{Q}^{-1}$, i.e., $\mathbf{Q}^*\mathbf{Q} = \mathbf{I}$, $\mathbf{Q}$ is called *unitary* (or *orthogonal* in the real case).

$$\left[\begin{array}{c} \mathbf{q}_1^* \\ \hline \mathbf{q}_2^* \\ \hline \vdots \\ \hline \mathbf{q}_m^* \end{array}\right] \left[\begin{array}{c|c|c|c} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_m \end{array}\right] = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

Exercise: Let $\mathbf{Q} \in \mathbb{C}^{m\times m}$ be a unitary matrix. Prove

$$\|\mathbf{Q}\|_2 = 1, \quad \|\mathbf{Q}\|_{\mathrm{F}} = \sqrt{m}.$$

- A unitary matrix has both orthonormal rows and orthonormal columns.

- The columns of a unitary matrix form an orthonormal basis of $\mathbb{C}^m$ and vice versa.