Lecture 9: QR algorithm

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1. Simultaneous iteration (SI)

- Sometimes also called *subspace iteration* or *orthogonal iteration* or *block power iteration*

Algorithm 1: Simultaneous iteration

    Pick $\mathbf{Q}_n^{(0)} \in \mathbb{C}^{m \times n}$ with orthonormal columns
    for $k = 1, 2, 3, \ldots,$
        $\mathbf{Z}_n^{(k)} = \mathbf{A}\mathbf{Q}_n^{(k-1)}$
        $\mathbf{Q}_n^{(k)}\mathbf{R}_n^{(k)} = \mathbf{Z}_n^{(k)}$ (QR factorization)
    end

- Here is an informal analysis of this algorithm.

  Assume $\mathbf{A} = \mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}$ with $\mathbf{\Lambda} = \text{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_m\}$ and

$$|\lambda_1| \geq \cdots \geq |\lambda_n| > |\lambda_{n+1}| \geq \cdots \geq |\lambda_m|.$$

If $n = 1$, then simultaneous iteration reduces to power iteration; see Lecture 8 for details.

Now we consider the case $n > 1$. Let

$$\mathbf{X}_n := \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \end{bmatrix} \mathbf{S}^{-1} \mathbf{Q}_n^{(0)},$$

and

$$\mathbf{X}_c := \begin{bmatrix} \mathbf{0} & \mathbf{I}_{m-n} \end{bmatrix} \mathbf{S}^{-1} \mathbf{Q}_n^{(0)}.$$

Assume that $\mathbf{X}_n$ has full rank (a generalization of the assumption $\alpha_1 \neq 0$ in power iteration). We have (the proof is left as an exercise)

$$\mathbf{Q}_n^{(k)} = \mathbf{A}^k \mathbf{Q}_n^{(0)} (\mathbf{R}_n^{(1)})^{-1} (\mathbf{R}_n^{(2)})^{-1} \cdots (\mathbf{R}_n^{(k)})^{-1}.$$

By $\mathbf{A} = \mathbf{S} \mathbf{\Lambda} \mathbf{S}^{-1}$, we have

$$\mathbf{Q}_n^{(k)} = \mathbf{S} \mathbf{\Lambda}^k \mathbf{S}^{-1} \mathbf{Q}_n^{(0)} (\mathbf{R}_n^{(1)})^{-1} (\mathbf{R}_n^{(2)})^{-1} \cdots (\mathbf{R}_n^{(k)})^{-1}.$$

Write $\mathbf{S} = \begin{bmatrix} \mathbf{S}_n & \mathbf{S}_c \end{bmatrix}$ and $\boldsymbol{\Lambda} = \mathrm{diag}\{\boldsymbol{\Lambda}_n, \boldsymbol{\Lambda}_c\}$. By $\mathbf{S}^{-1}\mathbf{Q}_n^{(0)} = \begin{bmatrix} \mathbf{X}_n \\ \mathbf{X}_c \end{bmatrix}$, we have

$$
\begin{aligned}
\mathbf{Q}_n^{(k)} &= \begin{bmatrix} \mathbf{S}_n & \mathbf{S}_c \end{bmatrix} \begin{bmatrix} \boldsymbol{\Lambda}_n^k & \\ & \boldsymbol{\Lambda}_c^k \end{bmatrix} \begin{bmatrix} \mathbf{X}_n \\ \mathbf{X}_c \end{bmatrix} (\mathbf{R}_n^{(1)})^{-1}(\mathbf{R}_n^{(2)})^{-1}\cdots(\mathbf{R}_n^{(k)})^{-1} \\
&= \begin{bmatrix} \mathbf{S}_n & \mathbf{S}_c \end{bmatrix} \begin{bmatrix} \boldsymbol{\Lambda}_n^k\mathbf{X}_n(\mathbf{R}_n^{(1)})^{-1}(\mathbf{R}_n^{(2)})^{-1}\cdots(\mathbf{R}_n^{(k)})^{-1} \\ \boldsymbol{\Lambda}_c^k\mathbf{X}_c(\mathbf{R}_n^{(1)})^{-1}(\mathbf{R}_n^{(2)})^{-1}\cdots(\mathbf{R}_n^{(k)})^{-1} \end{bmatrix}.
\end{aligned}
$$

Exercise: Prove that

$$
\mathbf{S}_c\boldsymbol{\Lambda}_c^k\mathbf{X}_c(\mathbf{R}_n^{(1)})^{-1}(\mathbf{R}_n^{(2)})^{-1}\cdots(\mathbf{R}_n^{(k)})^{-1} \to \mathbf{0}
$$

like $\left|\dfrac{\lambda_{n+1}}{\lambda_n}\right|^k$. Equivalently, we have the convergence result:

$$
\mathbf{Q}_n^{(k)} \to \mathbf{S}_n\boldsymbol{\Lambda}_n^k\mathbf{X}_n(\mathbf{R}_n^{(1)})^{-1}(\mathbf{R}_n^{(2)})^{-1}\cdots(\mathbf{R}_n^{(k)})^{-1},
$$

which means that $\mathrm{span}\{\mathbf{Q}_n^{(k)}\}$ converges to $\mathrm{span}\{\mathbf{S}_n\}$.

- Note that if we follow only the first $1 \leq j \leq n$ columns of $\mathbf{Q}_n^{(k)}$ through the iterations of the algorithm, they are *identical* to the iterates that we would compute if we had started with only the first j columns of $\mathbf{Q}_n^{(0)}$ instead of n columns.

In other words, simultaneous iteration is effectively running the algorithm for $j = 1, 2, \cdots, n$ all at the same time.

So if *all* the first $n + 1$ eigenvalues have distinct absolute values, i.e.,

$$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n| > |\lambda_{n+1}|,$$

and if *all* the leading principal submatrices of

$$\mathbf{X}_n = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \end{bmatrix} \mathbf{S}^{-1} \mathbf{Q}_n^{(0)}$$

have full rank, the same convergence analysis as before implies that $\text{span}\{\mathbf{Q}_j^{(k)}\}$ with $\mathbf{Q}_j^{(k)} := \mathbf{Q}_n^{(k)} \begin{bmatrix} \mathbf{I}_j \\ \mathbf{0} \end{bmatrix}$ converges to $\text{span}\{\mathbf{S}_j\}$ for each $j = 1, 2, \cdots, n$.

Theorem 1

Consider running simultaneous iteration on matrix $\mathbf{A} \in \mathbb{C}^{m \times m}$ with $n = m$ and $\mathbf{Q}_n^{(0)} = \mathbf{I}$. If $\mathbf{A} = \mathbf{S} \mathbf{\Lambda} \mathbf{S}^{-1}$ with

$$\mathbf{\Lambda} = \mathrm{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_m\}, \quad |\lambda_1| > |\lambda_2| > \cdots > |\lambda_m|,$$

and if all the leading principal submatrices of $\mathbf{S}^{-1}$ have full rank, then $\mathbf{A}^{(k)} := (\mathbf{Q}_n^{(k)})^* \mathbf{A} \mathbf{Q}_n^{(k)}$ converges to a Schur form of $\mathbf{A}$. The eigenvalues will appear in decreasing order of absolute value.

Proof: See Demmel's book: Theorem 4.8, Page 158, Applied numerical linear algebra.

- The entry $\mathbf{A}_{j,j}^{(k)}$ converges to λ_j like $\max\left(\left|\dfrac{\lambda_{j+1}}{\lambda_j}\right|^k, \left|\dfrac{\lambda_j}{\lambda_{j-1}}\right|^k \right)$.

- The block $\mathbf{A}^{(k)}(j+1:m, 1:j)$ converges to zero like $\left|\dfrac{\lambda_{j+1}}{\lambda_j}\right|^k$.

2. QR algorithm without shifts

Algorithm 2: "Pure" QR algorithm

$\mathbf{A}^{(0)} = \mathbf{A}$
for $k = 1, 2, 3, \ldots,$
$\qquad \mathbf{Q}^{(k)}\mathbf{R}^{(k)} = \mathbf{A}^{(k-1)}$ $\qquad$ (QR factorization)
$\qquad \mathbf{A}^{(k)} = \mathbf{R}^{(k)}\mathbf{Q}^{(k)}$
end

Proposition 2

We have $\mathbf{A}^{(k)} = (\underline{\mathbf{Q}}^{(k)})^*\mathbf{A}\underline{\mathbf{Q}}^{(k)}$, where $\underline{\mathbf{Q}}^{(k)} := \mathbf{Q}^{(1)}\mathbf{Q}^{(2)}\cdots\mathbf{Q}^{(k)}$.

Proof.

We use induction. Note that $\mathbf{A}^{(k)} = (\mathbf{Q}^{(k)})^*\mathbf{A}^{(k-1)}\mathbf{Q}^{(k)}$. The proof is easy. $\qquad\square$

Proposition 3

The QR *factorization of the kth power of* $\mathbf{A}$ *is given by*

$$\mathbf{A}^k = \underline{\mathbf{Q}}^{(k)}\underline{\mathbf{R}}^{(k)},$$

where $\underline{\mathbf{Q}}^{(k)} := \mathbf{Q}^{(1)}\mathbf{Q}^{(2)}\cdots\mathbf{Q}^{(k)}$, *and* $\underline{\mathbf{R}}^{(k)} := \mathbf{R}^{(k)}\mathbf{R}^{(k-1)}\cdots\mathbf{R}^{(1)}$.

Proof.

We use induction. For $k = 1$, $\mathbf{A} = \mathbf{A}^{(0)} = \mathbf{Q}^{(1)}\mathbf{R}^{(1)} = \underline{\mathbf{Q}}^{(1)}\underline{\mathbf{R}}^{(1)}$. Now we prove the case $k > 1$ with the assumption $\mathbf{A}^{k-1} = \underline{\mathbf{Q}}^{(k-1)}\underline{\mathbf{R}}^{(k-1)}$. By Proposition 2, we have $\mathbf{A}^{(k-1)} = (\underline{\mathbf{Q}}^{(k-1)})^*\mathbf{A}\underline{\mathbf{Q}}^{(k-1)}$, which implies $\mathbf{A}\underline{\mathbf{Q}}^{(k-1)} = \underline{\mathbf{Q}}^{(k-1)}\mathbf{A}^{(k-1)}$. Then we have

$$\begin{aligned}
\mathbf{A}^k &= \mathbf{A}\cdot\mathbf{A}^{k-1} = \mathbf{A}\underline{\mathbf{Q}}^{(k-1)}\underline{\mathbf{R}}^{(k-1)} = \underline{\mathbf{Q}}^{(k-1)}\mathbf{A}^{(k-1)}\underline{\mathbf{R}}^{(k-1)} \\
&= \underline{\mathbf{Q}}^{(k-1)}\mathbf{Q}^{(k)}\mathbf{R}^{(k)}\underline{\mathbf{R}}^{(k-1)} = \underline{\mathbf{Q}}^{(k)}\underline{\mathbf{R}}^{(k)}.
\end{aligned}$$

This completes the proof. $\qquad\square$

- Connection with power iteration: By $\mathbf{A}^k = \underline{\mathbf{Q}}^{(k)}\underline{\mathbf{R}}^{(k)}$, the first column of $\underline{\mathbf{Q}}^{(k)}$ is the result of applying k steps of power iteration on $\mathbf{A}$ to the vector $\mathbf{e}_1$.

- Connection with inverse iteration: By $\underline{\mathbf{Q}}^{(k)} = (\mathbf{A}^*)^{-k}(\underline{\mathbf{R}}^{(k)})^*$, the last column of $\underline{\mathbf{Q}}^{(k)}$ is the result of applying k steps of inverse iteration on $\mathbf{A}^*$ to the vector $\mathbf{e}_m$.

Theorem 4

If $\mathbf{A} = \mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}$ is diagonalizable with

$$\mathbf{\Lambda} = \mathrm{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_m\}, \quad |\lambda_1| > |\lambda_2| > \cdots > |\lambda_m|,$$

and if all the leading principal submatrices of $\mathbf{S}^{-1}$ have full rank, then $\mathbf{A}^{(k)}$ computed by "pure" QR algorithm converges to a Schur form of $\mathbf{A}$. The eigenvalues will appear in decreasing order of absolute value.

This theorem is a direct result of the following lemma.

Lemma 5

The $\mathbf{A}^{(k)}$ computed by "pure" QR algorithm is identical to the matrix $(\mathbf{Q}_n^{(k)})^*\mathbf{A}\mathbf{Q}_n^{(k)}$ implicitly computed by running simultaneous iteration on matrix $\mathbf{A} \in \mathbb{C}^{m \times m}$ with $n = m$ and $\mathbf{Q}_n^{(0)} = \mathbf{I}$. (We need an assumption about QR factorizations used in simultaneous iteration.)

Proof. We use induction. For $k = 1$, let $\mathbf{Q}_n^{(1)} = \mathbf{Q}^{(1)}$ and $\mathbf{R}_n^{(1)} = \mathbf{R}^{(1)}$. We have $\mathbf{A}^{(1)} = (\mathbf{Q}_n^{(1)})^*\mathbf{A}\mathbf{Q}_n^{(1)}$. Assume $\mathbf{A}^{(k-1)} = (\mathbf{Q}_n^{(k-1)})^*\mathbf{A}\mathbf{Q}_n^{(k-1)}$. Then from the "pure" QR algorithm and the induction hypothesis,

$$\mathbf{Q}^{(k)}\mathbf{R}^{(k)} = \mathbf{A}^{(k-1)} = (\mathbf{Q}_n^{(k-1)})^*\mathbf{A}\mathbf{Q}_n^{(k-1)}.$$

Let $\mathbf{Q}_n^{(k)} = \mathbf{Q}_n^{(k-1)}\mathbf{Q}^{(k)}$ and $\mathbf{R}_n^{(k)} = \mathbf{R}^{(k)}$ be the QR factorization of $\mathbf{A}\mathbf{Q}_n^{(k-1)}$ used in simultaneous iteration. By $\mathbf{R}_n^{(k)} = (\mathbf{Q}_n^{(k)})^*\mathbf{A}\mathbf{Q}_n^{(k-1)}$, $\mathbf{R}^{(k)} = \mathbf{R}_n^{(k)}$, and $\mathbf{Q}^{(k)} = (\mathbf{Q}_n^{(k-1)})^*\mathbf{Q}_n^{(k)}$, we have

$$\mathbf{A}^{(k)} = \mathbf{R}^{(k)}\mathbf{Q}^{(k)} = (\mathbf{Q}_n^{(k)})^*\mathbf{A}\mathbf{Q}_n^{(k-1)}(\mathbf{Q}_n^{(k-1)})^*\mathbf{Q}_n^{(k)} = (\mathbf{Q}_n^{(k)})^*\mathbf{A}\mathbf{Q}_n^{(k)}.$$

This completes the proof. $\qquad\square$

- From earlier analysis, we know that the convergence rate of "pure" QR algorithm depends on the absolute values of the ratios of eigenvalues. To speed convergence, we can use shift and invert techniques.

3. QR algorithm with shifts

Algorithm 3: QR algorithm with shifts

$\mathbf{A}^{(0)} = \mathbf{A}$

for $k = 1, 2, 3, \ldots,$

     Pick a shift $\mu^{(k)}$ near an eigenvalue of $\mathbf{A}$

     $\mathbf{Q}^{(k)}\mathbf{R}^{(k)} = \mathbf{A}^{(k-1)} - \mu^{(k)}\mathbf{I}$ (QR factorization)

     $\mathbf{A}^{(k)} = \mathbf{R}^{(k)}\mathbf{Q}^{(k)} + \mu^{(k)}\mathbf{I}$

end

Proposition 6

We have $\mathbf{A}^{(k)} = (\underline{\mathbf{Q}}^{(k)})^* \mathbf{A} \underline{\mathbf{Q}}^{(k)}$*, where* $\underline{\mathbf{Q}}^{(k)} := \mathbf{Q}^{(1)}\mathbf{Q}^{(2)} \cdots \mathbf{Q}^{(k)}$*.*

We have the factorization (for $k \geq 1$)

$$(\mathbf{A} - \mu^{(k)}\mathbf{I})(\mathbf{A} - \mu^{(k-1)}\mathbf{I}) \cdots (\mathbf{A} - \mu^{(1)}\mathbf{I}) = \underline{\mathbf{Q}}^{(k)}\underline{\mathbf{R}}^{(k)},$$

where $\underline{\mathbf{Q}}^{(k)} := \mathbf{Q}^{(1)}\mathbf{Q}^{(2)} \cdots \mathbf{Q}^{(k)}$, and $\underline{\mathbf{R}}^{(k)} := \mathbf{R}^{(k)}\mathbf{R}^{(k-1)} \cdots \mathbf{R}^{(1)}$.

Proof. We use induction. For $k = 1$, $\mathbf{A} - \mu^{(1)}\mathbf{I} = \mathbf{Q}^{(1)}\mathbf{R}^{(1)} = \underline{\mathbf{Q}}^{(1)}\underline{\mathbf{R}}^{(1)}$. Assume $(\mathbf{A} - \mu^{(k-1)}\mathbf{I})(\mathbf{A} - \mu^{(k-2)}\mathbf{I}) \cdots (\mathbf{A} - \mu^{(1)}\mathbf{I}) = \underline{\mathbf{Q}}^{(k-1)}\underline{\mathbf{R}}^{(k-1)}$. By Proposition 6, we have $\mathbf{A}^{(k-1)} = (\underline{\mathbf{Q}}^{(k-1)})^*\mathbf{A}\underline{\mathbf{Q}}^{(k-1)}$. Then

$$
\begin{aligned}
&(\mathbf{A} - \mu^{(k)}\mathbf{I})(\mathbf{A} - \mu^{(k-1)}\mathbf{I}) \cdots (\mathbf{A} - \mu^{(1)}\mathbf{I}) = (\mathbf{A} - \mu^{(k)}\mathbf{I})\underline{\mathbf{Q}}^{(k-1)}\underline{\mathbf{R}}^{(k-1)} \\
&= (\mathbf{A}\underline{\mathbf{Q}}^{(k-1)} - \mu^{(k)}\underline{\mathbf{Q}}^{(k-1)})\underline{\mathbf{R}}^{(k-1)} \\
&= (\underline{\mathbf{Q}}^{(k-1)}\mathbf{A}^{(k-1)} - \mu^{(k)}\underline{\mathbf{Q}}^{(k-1)})\underline{\mathbf{R}}^{(k-1)} \\
&= \underline{\mathbf{Q}}^{(k-1)}(\mathbf{A}^{(k-1)} - \mu^{(k)}\mathbf{I})\underline{\mathbf{R}}^{(k-1)} = \underline{\mathbf{Q}}^{(k-1)}\mathbf{Q}^{(k)}\mathbf{R}^{(k)}\underline{\mathbf{R}}^{(k-1)} = \underline{\mathbf{Q}}^{(k)}\underline{\mathbf{R}}^{(k)}.
\end{aligned}
$$

This completes the proof. $\qquad\square$

- Connection with shifted power iteration: By

$$(\mathbf{A} - \mu^{(k)}\mathbf{I})(\mathbf{A} - \mu^{(k-1)}\mathbf{I}) \cdots (\mathbf{A} - \mu^{(1)}\mathbf{I}) = \underline{\mathbf{Q}}^{(k)}\underline{\mathbf{R}}^{(k)},$$

the first column of $\underline{\mathbf{Q}}^{(k)}$ is the result of applying k steps of shifted power iteration on the matrix $\mathbf{A}$ using the starting vector $\mathbf{e}_1$ and the shifts $\mu^{(j)}$, $j = 1 : k$.

- Connection with shifted inverse iteration: By

$$\underline{\mathbf{Q}}^{(k)} = (\mathbf{A} - \mu^{(k)}\mathbf{I})^{-*}(\mathbf{A} - \mu^{(k-1)}\mathbf{I})^{-*} \cdots (\mathbf{A} - \mu^{(1)}\mathbf{I})^{-*}(\underline{\mathbf{R}}^{(k)})^{*},$$

the last column of $\underline{\mathbf{Q}}^{(k)}$ is the result of applying k steps of shifted inverse iteration on the matrix $\mathbf{A}^{*}$ using the starting vector $\mathbf{e}_m$ and the shifts $\overline{\mu^{(j)}}$, $j = 1 : k$.

If the shifts are good eigenvalue estimates, the last column of $\underline{\mathbf{Q}}^{(k)}$, i.e., $\underline{\mathbf{Q}}^{(k)}\mathbf{e}_m$, converges quickly to an eigenvector of $\mathbf{A}^{*}$.

- Connection with Rayleigh quotient iteration: Choose

$$\mu^{(1)} = r(\mathbf{e}_m) = \mathbf{e}_m^* \mathbf{A} \mathbf{e}_m,$$
$$\mu^{(k+1)} = r(\underline{\mathbf{Q}}^{(k)} \mathbf{e}_m) = (\underline{\mathbf{Q}}^{(k)} \mathbf{e}_m)^* \mathbf{A} (\underline{\mathbf{Q}}^{(k)} \mathbf{e}_m), \quad k \geq 1,$$

as the shifts. Then $\overline{\mu^{(k+1)}}$ and $\underline{\mathbf{Q}}^{(k)} \mathbf{e}_m$ are identical to those computed by the Rayleigh quotient iteration on $\mathbf{A}^*$ starting with $\mathbf{e}_m$. Assume the algorithm converges. Then $\underline{\mathbf{Q}}^{(k)} \mathbf{e}_m$ converges quadratically or cubically to an eigenvector of $\overline{\mathbf{A}^*}$.

- *Rayleigh quotient shift* $\mu^{(k+1)} = \mathbf{A}_{mm}^{(k)}$: In the QR algorithm, we have

$$\mathbf{A}_{mm}^{(k)} = \mathbf{e}_m^* \mathbf{A}^{(k)} \mathbf{e}_m = \mathbf{e}_m^* (\underline{\mathbf{Q}}^{(k)})^* \mathbf{A} \underline{\mathbf{Q}}^{(k)} \mathbf{e}_m = r(\underline{\mathbf{Q}}^{(k)} \mathbf{e}_m),$$

which means that the Rayleigh quotient $r(\underline{\mathbf{Q}}^{(k)} \mathbf{e}_m)$ appears as the (m, m) entry of $\mathbf{A}^{(k)}$. So it comes for free!

- Other issues: *Wilkinson shift ...*

4. Upper Hessenberg form in QR algorithm

Proposition 8

Upper Hessenberg form is preserved by QR *algorithm.*

Proof.

For the upper Hessenberg matrix $\mathbf{H}^{(k-1)} - \mu^{(k)}\mathbf{I}$, it is easy to show that there exists a QR factorization $\mathbf{Q}^{(k)}\mathbf{R}^{(k)} = \mathbf{H}^{(k-1)} - \mu^{(k)}\mathbf{I}$ such that $\mathbf{Q}^{(k)}$ is upper Hessenberg. Then it is easy to confirm that $\mathbf{R}^{(k)}\mathbf{Q}^{(k)}$ remains upper Hessenberg and adding $\mu^{(k)}\mathbf{I}$ does not change this. □

Proposition 9

Hermitian tridiagonal form is preserved by QR *algorithm* (*real shifts*).

Proof.

Hermitian + tridiagonal = Hermitian + upper Hessenberg. □

For simplicity, in subsections 4.1 – 4.3, we only consider the real case.

4.1. Implicit Q theorem

Definition 10

An upper Hessenberg matrix $\mathbf{H}$ is unreduced if all $(j+1, j)$ entries of $\mathbf{H}$ are nonzero.

Theorem 11 (Implicit Q theorem)

Let $\mathbf{A} \in \mathbb{R}^{m \times m}$. Suppose that $\mathbf{Q}^{\top} \mathbf{A} \mathbf{Q} = \mathbf{H}$ is unreduced upper Hessenberg and $\mathbf{Q}$ is orthogonal. Then columns 2 to m of $\mathbf{Q}$ are determined uniquely (up to signs) by the first column of $\mathbf{Q}$.

Remark 12

Implicit Q theorem implies that QR algorithm can be implemented cheaply on an upper Hessenberg matrix. The implementation will be implicit in the sense that we do not explicitly compute the QR factorization of an upper Hessenberg matrix each iteration but rather construct $\mathbf{Q}$ implicitly as a product of Givens rotations and other simple orthogonal/unitary matrices. See subsections 4.2 – 4.3.

Proof of implicit Q theorem.

Suppose that $\mathbf{Q}^\top \mathbf{A}\mathbf{Q} = \mathbf{H}$ and $\mathbf{V}^\top \mathbf{A}\mathbf{V} = \mathbf{G}$ are both unreduced upper Hessenberg, $\mathbf{Q}$ and $\mathbf{V}$ are orthogonal, and the first columns of $\mathbf{Q}$ and $\mathbf{V}$ are equal. Let $(\mathbf{X})_i$ denote the ith column of $\mathbf{X}$. Let $\mathbf{W} := \mathbf{V}^\top \mathbf{Q}$. By

$$\mathbf{GW} = \mathbf{GV}^\top \mathbf{Q} = \mathbf{V}^\top \mathbf{A}\mathbf{Q} = \mathbf{V}^\top \mathbf{Q}\mathbf{H} = \mathbf{WH},$$

we have

$$\mathbf{G}(\mathbf{W})_i = \mathbf{W}(\mathbf{H})_i = \sum\nolimits_{j=1}^{i+1} h_{ji}(\mathbf{W})_j.$$

Thus,

$$h_{i+1,i}(\mathbf{W})_{i+1} = \mathbf{G}(\mathbf{W})_i - \sum\nolimits_{j=1}^{i} h_{ji}(\mathbf{W})_j.$$

Since $(\mathbf{W})_1 = \mathbf{e}_1$ and $\mathbf{G}$ is upper Hessenberg, we can use induction on i to show that $(\mathbf{W})_i$ is nonzero in entries 1 to i only; i.e., $\mathbf{W}$ is upper triangular. Since $\mathbf{W}$ is also orthogonal, then $\mathbf{W}$ is diagonal and

$$\mathbf{W} = \mathrm{diag}\{1, \pm 1, \cdots, \pm 1\},$$

which implies

$$\mathbf{V}\mathrm{diag}\{1, \pm 1, \cdots, \pm 1\} = \mathbf{Q}. \quad \square$$

4.2. Implicit single shift QR algorithm ($\mu^{(k)} \in \mathbb{R}$)

- To compute $\mathbf{H}^{(k)} = (\mathbf{Q}^{(k)})^\top \mathbf{H}^{(k-1)} \mathbf{Q}^{(k)}$ from $\mathbf{H}^{(k-1)}$ in the QR algorithm (assume that $\mathbf{H}^{(k)}$ is unreduced), we will need only to

  (1) compute the first column of $\mathbf{Q}^{(k)}$ (which is parallel to the first column of $\mathbf{H}^{(k-1)} - \mu^{(k)}\mathbf{I}$ and so can be gotten just by normalizing this column vector).

  (2) choose other columns of $\mathbf{Q}^{(k)}$ such that $\mathbf{Q}^{(k)}$ is orthogonal and $(\mathbf{Q}^{(k)})^\top \mathbf{H}^{(k-1)} \mathbf{Q}^{(k)}$ is unreduced upper Hessenberg.

- By the implicit Q theorem, we know that we will have computed $\mathbf{H}^{(k)}$ correctly because $\mathbf{Q}^{(k)}$ is unique up to signs, which do not matter. (Signs do not matter because changing the signs of the columns of $\mathbf{Q}^{(k)}$ is the same as changing $\mathbf{H}^{(k-1)} - \mu^{(k)}\mathbf{I} = \mathbf{Q}^{(k)}\mathbf{R}^{(k)}$ to $(\mathbf{Q}^{(k)}\mathbf{S}^{(k)})(\mathbf{S}^{(k)}\mathbf{R}^{(k)})$, where $\mathbf{S}^{(k)} = \operatorname{diag}\{\pm 1, \pm 1, \cdots, \pm 1\}$. Then

  $$\mathbf{H}^{(k)} = (\mathbf{S}^{(k)}\mathbf{R}^{(k)})(\mathbf{Q}^{(k)}\mathbf{S}^{(k)}) + \mu^{(k)}\mathbf{I} = \mathbf{S}^{(k)}(\mathbf{R}^{(k)}\mathbf{Q}^{(k)} + \mu^{(k)}\mathbf{I})\mathbf{S}^{(k)},$$

  which is an orthogonal similarity that just changes the signs of some columns and rows of $\mathbf{H}^{(k)}$.)

- To see how to use the implicit Q theorem to compute $\mathbf{H}^{(1)}$ from $\mathbf{H}^{(0)} = \mathbf{H}$, we use a 5×5 example.

1. $\mathbf{Q}_1^{\top} = \begin{bmatrix} c_1 & s_1 & & & \\ -s_1 & c_1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$, $\mathbf{H}_1 = \mathbf{Q}_1^{\top}\mathbf{H}\mathbf{Q}_1 = \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ + & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix}$

2. $\mathbf{Q}_2^{\top} = \begin{bmatrix} 1 & & & & \\ & c_2 & s_2 & & \\ & -s_2 & c_2 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$, $\mathbf{Q}_2^{\top}\mathbf{H}_1 = \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix}$

$\mathbf{H}_2 = \mathbf{Q}_2^{\top}\mathbf{H}_1\mathbf{Q}_2 = \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & + & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix}$

3. $\mathbf{Q}_3^\top = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & c_3 & s_3 & \\ & & -s_3 & c_3 & \\ & & & & 1 \end{bmatrix}$, $\quad \mathbf{Q}_3^\top \mathbf{H}_2 = \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix}$

$$\mathbf{H}_3 = \mathbf{Q}_3^\top \mathbf{H}_2 \mathbf{Q}_3 = \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & + & \times & \times \end{bmatrix}$$

4. $\mathbf{Q}_4^\top = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & c_4 & s_4 \\ & & & -s_4 & c_4 \end{bmatrix}$, $\quad \mathbf{Q}_4^\top \mathbf{H}_3 = \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix}$

$$\mathbf{H}_4 = \mathbf{Q}_4^\top \mathbf{H}_3 \mathbf{Q}_4 = \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix}$$

Altogether $\mathbf{Q}^\top \mathbf{H} \mathbf{Q} = \mathbf{H}_4$ is upper Hessenberg, where

$$\mathbf{Q} = \mathbf{Q}_1 \mathbf{Q}_2 \mathbf{Q}_3 \mathbf{Q}_4 = \begin{bmatrix} c_1 & \times & \times & \times & \times \\ s_1 & \times & \times & \times & \times \\ & s_2 & \times & \times & \times \\ & & s_3 & \times & \times \\ & & & s_4 & c_4 \end{bmatrix},$$

so the first column of $\mathbf{Q}$ is $\begin{bmatrix} c_1 & s_1 & 0 & \cdots & 0 \end{bmatrix}^\top$, which by the implicit Q theorem has uniquely determined the other columns of $\mathbf{Q}$ (up to signs). We now choose the first column of $\mathbf{Q}$ to be proportional to the first column of $\mathbf{H}^{(0)} - \mu^{(1)}\mathbf{I}$. This means $\mathbf{Q}$ is the same (up to signs) as in the QR factorization of $\mathbf{H}^{(0)} - \mu^{(1)}\mathbf{I}$.

4.3. Implicit double shift QR algorithm ($\mu^{(k)} \in \mathbb{C}$)

- We describe how to maintain real arithmetic by shifting $\mu^{(k)}$ and $\overline{\mu^{(k)}}$ in succession:

$$
\begin{aligned}
\mathbf{Q}^{(k-1/2)}\mathbf{R}^{(k-1/2)} &= \mathbf{H}^{(k-1)} - \mu^{(k)}\mathbf{I} \\
\mathbf{H}^{(k-1/2)} &= \mathbf{R}^{(k-1/2)}\mathbf{Q}^{(k-1/2)} + \mu^{(k)}\mathbf{I} \\
&= (\mathbf{Q}^{(k-1/2)})^*\mathbf{H}^{(k-1)}\mathbf{Q}^{(k-1/2)}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{Q}^{(k)}\mathbf{R}^{(k)} &= \mathbf{H}^{(k-1/2)} - \overline{\mu^{(k)}}\mathbf{I} \\
\mathbf{H}^{(k)} &= \mathbf{R}^{(k)}\mathbf{Q}^{(k)} + \overline{\mu^{(k)}}\mathbf{I} \\
&= (\mathbf{Q}^{(k)})^*\mathbf{H}^{(k-1/2)}\mathbf{Q}^{(k)} \\
&= (\mathbf{Q}^{(k-1/2)}\mathbf{Q}^{(k)})^*\mathbf{H}^{(k-1)}\mathbf{Q}^{(k-1/2)}\mathbf{Q}^{(k)}
\end{aligned}
$$

> ### Lemma 13
>
> Assume $\mathbf{H}^{(0)} = \mathbf{H}$ is real. We can choose $\mathbf{Q}^{(k-1/2)}$ and $\mathbf{Q}^{(k)}$ such that
> (1) $\mathbf{Q}^{(k-1/2)}\mathbf{Q}^{(k)}$ is real,
> (2) $\mathbf{H}^{(k)}$ is therefore real,
> (3) the first column of $\mathbf{Q}^{(k-1/2)}\mathbf{Q}^{(k)}$ is easy to compute.

Proof. Since

$$\mathbf{Q}^{(k)}\mathbf{R}^{(k)} = \mathbf{H}^{(k-1/2)} - \overline{\mu^{(k)}}\mathbf{I} = \mathbf{R}^{(k-1/2)}\mathbf{Q}^{(k-1/2)} + (\mu^{(k)} - \overline{\mu^{(k)}})\mathbf{I},$$

we get

$$\begin{aligned}
&\mathbf{Q}^{(k-1/2)}\mathbf{Q}^{(k)}\mathbf{R}^{(k)}\mathbf{R}^{(k-1/2)}\\
&= \mathbf{Q}^{(k-1/2)}(\mathbf{R}^{(k-1/2)}\mathbf{Q}^{(k-1/2)} + (\mu^{(k)} - \overline{\mu^{(k)}})\mathbf{I})\mathbf{R}^{(k-1/2)}\\
&= \mathbf{Q}^{(k-1/2)}\mathbf{R}^{(k-1/2)}\mathbf{Q}^{(k-1/2)}\mathbf{R}^{(k-1/2)} + (\mu^{(k)} - \overline{\mu^{(k)}})\mathbf{Q}^{(k-1/2)}\mathbf{R}^{(k-1/2)}\\
&= (\mathbf{H}^{(k-1)} - \mu^{(k)}\mathbf{I})^2 + (\mu^{(k)} - \overline{\mu^{(k)}})(\mathbf{H}^{(k-1)} - \mu^{(k)}\mathbf{I})\\
&= (\mathbf{H}^{(k-1)})^2 - 2\mathrm{Re}(\mu^{(k)})\mathbf{H}^{(k-1)} + |\mu^{(k)}|^2\mathbf{I} =: \mathbf{M}.
\end{aligned}$$

Note that
$$\mathbf{Q}^{(k-1/2)}\mathbf{Q}^{(k)}\mathbf{R}^{(k)}\mathbf{R}^{(k-1/2)} = \mathbf{M}$$

is a QR factorization of the real matrix $\mathbf{M}$. Therefore, $\mathbf{Q}^{(k-1/2)}\mathbf{Q}^{(k)}$ and $\mathbf{R}^{(k)}\mathbf{R}^{(k-1/2)}$ can be chosen real. This means that

$$\mathbf{H}^{(k)} = (\mathbf{Q}^{(k-1/2)}\mathbf{Q}^{(k)})^*\mathbf{H}^{(k-1)}\mathbf{Q}^{(k-1/2)}\mathbf{Q}^{(k)}$$

also is real if $\mathbf{H}^{(k-1)}$ is real. The first column of $\mathbf{Q}^{(k-1/2)}\mathbf{Q}^{(k)}$ is proportional to the first column of

$$(\mathbf{H}^{(k-1)})^2 - 2\mathrm{Re}(\mu^{(k)})\mathbf{H}^{(k-1)} + |\mu^{(k)}|^2\mathbf{I},$$

whose sparsity pattern is $\begin{bmatrix} \times & \times & \times & 0 & \cdots & 0 \end{bmatrix}^\top$. Obviously, the first column of $\mathbf{Q}^{(k-1/2)}\mathbf{Q}^{(k)}$ is easy to compute since $\mathbf{H}^{(k-1)}$ is upper Hessenberg. $\qquad\square$

The implicit Q theorem and the last lemma can be used to compute $\mathbf{H}^{(k)}$ from $\mathbf{H}^{(k-1)}$. We provide a 6×6 example.

- Assume $\mathbf{H}$ is upper Hessenberg and the shifts are μ and $\overline{\mu}$.

  1. Choose an orthogonal matrix

  $$\mathbf{Q}_1^\top = \begin{bmatrix} \widetilde{\mathbf{Q}}^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad \widetilde{\mathbf{Q}}^\top \widetilde{\mathbf{Q}} = \mathbf{I}_3,$$

  where the first column of $\mathbf{Q}_1$ is proportional to the first column of

  $$\mathbf{H}^2 - 2\mathrm{Re}(\mu)\mathbf{H} + |\mu|^2 \mathbf{I},$$

  so

  $$\mathbf{Q}_1^\top \mathbf{H} = \begin{bmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ + & \times & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times & \times \\ 0 & 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & 0 & \times & \times \end{bmatrix}, \mathbf{Q}_1^\top \mathbf{H} \mathbf{Q}_1 = \begin{bmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ + & \times & \times & \times & \times & \times \\ + & + & \times & \times & \times & \times \\ 0 & 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & 0 & \times & \times \end{bmatrix}$$

2. Choose a Householder reflector $\mathbf{Q}_2^\top$, which affects only rows 2,3, and 4 of $\mathbf{Q}_2^\top \mathbf{H}_1$, zeroing out entries (3,1) and (4,1) of $\mathbf{H}_1 = \mathbf{Q}_1^\top \mathbf{H} \mathbf{Q}_1$ (this means that $\mathbf{Q}_2^\top$ is the identity matrix outside rows and columns 2 through 4):

$$\mathbf{Q}_2^\top \mathbf{H}_1 = \begin{bmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times & \times \\ 0 & + & \times & \times & \times & \times \\ 0 & 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & 0 & \times & \times \end{bmatrix},$$

$$\mathbf{H}_2 = \mathbf{Q}_2^\top \mathbf{H}_1 \mathbf{Q}_2 = \begin{bmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times & \times \\ 0 & + & \times & \times & \times & \times \\ 0 & + & + & \times & \times & \times \\ 0 & 0 & 0 & 0 & \times & \times \end{bmatrix}$$

3. Choose a Householder reflector $\mathbf{Q}_3^\top$, which affects only rows 3,4, and 5 of $\mathbf{Q}_3^\top \mathbf{H}_2$, zeroing out entries (4,2) and (5,2) of $\mathbf{H}_2$ (this means that $\mathbf{Q}_3^\top$ is the identity matrix outside rows and columns 3 through 5):

$$\mathbf{Q}_3^\top \mathbf{H}_2 = \begin{bmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times & \times \\ 0 & 0 & + & \times & \times & \times \\ 0 & 0 & 0 & 0 & \times & \times \end{bmatrix}$$

$$\mathbf{H}_3 = \mathbf{Q}_3^\top \mathbf{H}_2 \mathbf{Q}_3 = \begin{bmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times & \times \\ 0 & 0 & + & \times & \times & \times \\ 0 & 0 & + & + & \times & \times \end{bmatrix}$$

4. Choose a Householder reflector $\mathbf{Q}_4^\top$, which affects only rows 4,5, and 6 of $\mathbf{Q}_4^\top \mathbf{H}_3$, zeroing out entries (5,3) and (6,3) of $\mathbf{H}_2$ (this means that $\mathbf{Q}_4^\top$ is the identity matrix outside rows and columns 4 through 6):

$$\mathbf{Q}_4^\top \mathbf{H}_3 = \begin{bmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times & \times \\ 0 & 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & + & \times & \times \end{bmatrix}$$

$$\mathbf{H}_4 = \mathbf{Q}_4^\top \mathbf{H}_3 \mathbf{Q}_4 = \begin{bmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times & \times \\ 0 & 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & + & \times & \times \end{bmatrix}$$

5. Choose a Givens rotation $\mathbf{Q}_5^\top$

$$\mathbf{Q}_5^\top = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & c & s \\ & & & & -s & c \end{bmatrix}, \quad \mathbf{Q}_5^\top \mathbf{H}_4 = \begin{bmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times & \times \\ 0 & 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & 0 & \times & \times \end{bmatrix}$$

$$\mathbf{H}_5 = \mathbf{Q}_5^\top \mathbf{H}_4 \mathbf{Q}_5 = \begin{bmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times & \times \\ 0 & 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & 0 & \times & \times \end{bmatrix}.$$

Altogether $\mathbf{Q}^\top \mathbf{H} \mathbf{Q}$ is upper Hessenberg, where

$$\mathbf{Q} = \mathbf{Q}_1 \mathbf{Q}_2 \mathbf{Q}_3 \mathbf{Q}_4 \mathbf{Q}_5 \quad \text{with} \quad \mathbf{Q}\mathbf{e}_1 = \mathbf{Q}_1 \mathbf{e}_1.$$

4.4. Two phases of QR algorithm

- First phase: Reduction to Hessenberg or tridiagonal form

$$
\begin{bmatrix}
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times
\end{bmatrix}
\xrightarrow{\mathbf{Q}_1^*:}
\begin{bmatrix}
\times & \times & \times & \times & \times \\
\mathbf{\times} & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \\
\mathbf{0} & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \\
\mathbf{0} & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \\
\mathbf{0} & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} & \mathbf{\times}
\end{bmatrix}
\xrightarrow{\cdot\,\mathbf{Q}_1}
\begin{bmatrix}
\times & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \\
\times & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \\
 & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \\
 & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \\
 & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} & \mathbf{\times}
\end{bmatrix}
$$

$$
\mathbf{A} \qquad\qquad \mathbf{Q}_1^*\mathbf{A} \qquad\qquad \mathbf{Q}_1^*\mathbf{A}\mathbf{Q}_1
$$

$$
\begin{bmatrix}
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
 & \times & \times & \times & \times \\
 & \times & \times & \times & \times \\
 & \times & \times & \times & \times
\end{bmatrix}
\xrightarrow{\mathbf{Q}_2^*:}
\begin{bmatrix}
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
 & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \\
 & \mathbf{0} & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \\
 & \mathbf{0} & \mathbf{\times} & \mathbf{\times} & \mathbf{\times}
\end{bmatrix}
\xrightarrow{\cdot\,\mathbf{Q}_2}
\begin{bmatrix}
\times & \times & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \\
\times & \times & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \\
 & \times & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \\
 & & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \\
 & & \mathbf{\times} & \mathbf{\times} & \mathbf{\times}
\end{bmatrix}
$$

$$
\mathbf{Q}_1^*\mathbf{A}\mathbf{Q}_1 \qquad\qquad \mathbf{Q}_2^*\mathbf{Q}_1^*\mathbf{A}\mathbf{Q}_1 \qquad\qquad \mathbf{Q}_2^*\mathbf{Q}_1^*\mathbf{A}\mathbf{Q}_1\mathbf{Q}_2
$$

$$
\begin{bmatrix}
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
 & \times & \times & \times & \times \\
 & & \times & \times & \times \\
 & & & \times & \times
\end{bmatrix}
\qquad
\underbrace{\mathbf{Q}_{m-2}^*\cdots\mathbf{Q}_2^*\mathbf{Q}_1^*}_{\mathbf{Q}^*}\,\mathbf{A}\,\underbrace{\mathbf{Q}_1\mathbf{Q}_2\cdots\mathbf{Q}_{m-2}}_{\mathbf{Q}} = \mathbf{H}.
$$

- Second phase: generate a sequence of Hessenberg (or tridiagonal) matrices that converge to a triangular (or diagonal) form.

$$
\begin{bmatrix}
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times
\end{bmatrix}
\xrightarrow{\text{Phase 1}}
\begin{bmatrix}
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
& \times & \times & \times & \times \\
& & \times & \times & \times \\
& & & \times & \times
\end{bmatrix}
\xrightarrow{\text{Phase 2}}
\begin{bmatrix}
\times & \times & \times & \times & \times \\
& \times & \times & \times & \times \\
& & \times & \times & \times \\
& & & \times & \times \\
& & & & \times
\end{bmatrix}
$$
$$\mathbf{A} \neq \mathbf{A}^* \qquad\qquad\qquad \mathbf{H} \qquad\qquad\qquad \mathbf{T}$$

$$
\begin{bmatrix}
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times
\end{bmatrix}
\xrightarrow{\text{Phase 1}}
\begin{bmatrix}
\times & \times & & & \\
\times & \times & \times & & \\
& \times & \times & \times & \\
& & \times & \times & \times \\
& & & \times & \times
\end{bmatrix}
\xrightarrow{\text{Phase 2}}
\begin{bmatrix}
\times & & & & \\
& \times & & & \\
& & \times & & \\
& & & \times & \\
& & & & \times
\end{bmatrix}
$$
$$\mathbf{A} = \mathbf{A}^* \qquad\qquad\qquad \mathbf{H} \qquad\qquad\qquad \mathbf{D}$$

5. Further reading

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