Some new developments in iterative solvers for some structured linear systems

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joint work with Jia-Jun Fan and Fang Wang

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Outline

1. Linear systems and Krylov subspace methods
2. Range-symmetric linear systems
3. Symmetric quasi-definite linear systems
4. Block two-by-two nonsymmetric linear systems
5. Summary

Linear systems and Krylov subspace methods

- Linear systems of equations

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \qquad \mathbf{A} \in \mathbb{R}^{m \times m}, \qquad \mathbf{b} \in \mathbb{R}^m$$

- Krylov subspace methods

CG, MINRES; GMRES, CMRH, Bi-CG, QMR, Bi-CGSTAB ...

- Structured linear systems

(1) Range-symmetric linear systems $\quad (\mathrm{range}(\mathbf{A}) = \mathrm{range}(\mathbf{A}^\top))$

(2) Symmetric quasi-definite linear systems

(3) Block two-by-two nonsymmetric linear systems

The pseudoinverse solution

- $\mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{A} \in \mathbb{R}^{m \times n}, \quad \mathbf{b} \in \mathbb{R}^m.$
  Consistent if $\mathbf{b} \in \mathrm{range}(\mathbf{A})$, otherwise, inconsistent.
- $\mathbf{A}^\dagger$: the Moore–Penrose inverse of $\mathbf{A}$
- $\mathbf{A}^\dagger \mathbf{b}$: the pseudoinverse solution

$\mathbf{A}\mathbf{x} = \mathbf{b}$	$\mathrm{rank}(\mathbf{A})$	$\mathbf{A}^\dagger \mathbf{b}$
consistent	$= n$	unique solution
consistent	$< n$	unique minimum 2-norm solution
inconsistent	$= n$	unique least-squares (LS) solution
inconsistent	$< n$	unique minimum 2-norm LS solution

Krylov subspaces and (least squares) solutions

- $\mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{A} \in \mathbb{R}^{n\times n}, \qquad \mathbf{b},\ \mathbf{x}_0 \in \mathbb{R}^n, \qquad \mathbf{r}_0 := \mathbf{b} - \mathbf{A}\mathbf{x}_0,$

$$\mathcal{K}_k(\mathbf{A}, \mathbf{r}_0) := \operatorname{span}\{\mathbf{r}_0, \mathbf{A}\mathbf{r}_0, \dots, \mathbf{A}^{k-1}\mathbf{r}_0\}.$$

- ℓ: the grade of $\mathbf{r}_0$ with respect to $\mathbf{A}$, i.e., ℓ satisfies

$$\dim \mathcal{K}_k(\mathbf{A}, \mathbf{r}_0) = \begin{cases} k, & \text{if } k \le \ell, \\ \ell, & \text{if } k \ge \ell + 1. \end{cases}$$

- For any $\mathbf{A} \in \mathbb{R}^{n\times n}$,
  (i) $\mathbf{b} \notin \operatorname{range}(\mathbf{A})$: # LS solution in $\mathbf{x}_0 + \mathcal{K}_{\ell-1}(\mathbf{A}, \mathbf{r}_0) \le 1$;
  (ii) $\mathbf{b} \in \operatorname{range}(\mathbf{A})$: # solution in $\mathbf{x}_0 + \mathcal{K}_{\ell}(\mathbf{A}, \mathbf{r}_0) \le 1$.

GMRES for singular range-symmetric systems

- GMRES: $\mathbf{x}_k := \underset{\mathbf{x} \in \mathbf{x}_0 + \mathcal{K}_k(\mathbf{A}, \mathbf{r}_0)}{\operatorname{argmin}} \|\mathbf{b} - \mathbf{A}\mathbf{x}\|.$
- For singular range-symmetric $\mathbf{A}$ [BW97]:

  (i) $\mathbf{b} \in \operatorname{range}(\mathbf{A})$: $\mathbf{x}_\ell$ is a solution. More precisely,

$$\mathbf{x}_\ell = \mathbf{A}^\dagger \mathbf{b} + (\mathbf{I} - \mathbf{A}^\dagger \mathbf{A})\mathbf{x}_0,$$

  the orthogonal projection of $\mathbf{x}_0$ onto the solution set

$$\{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b}\} = \mathbf{A}^\dagger \mathbf{b} + \operatorname{null}(\mathbf{A}).$$

  (ii) $\mathbf{b} \notin \operatorname{range}(\mathbf{A})$: $\mathbf{x}_{\ell-1}$ is a least-squares solution. Which one?

[BW97] P. N. Brown and H. F. Walker. *GMRES on (nearly) singular systems*. SIMAX, 1997.

GMRES for singular (skew-)symmetric systems

- "(skew-)symmetric" $\in$ "range-symmetric"
- For skew-symmetric $\mathbf{A}$, i.e., $\mathbf{A}^\top = -\mathbf{A}$, if $\mathbf{b} \notin \mathrm{range}(\mathbf{A})$, then

$$\mathbf{r}_{\ell-1}^\top(\mathbf{x}_{\ell-1} - \mathbf{x}_0) = 0,$$

  which implies

$$\mathbf{x}_{\ell-1} = \mathbf{A}^\dagger\mathbf{b} + (\mathbf{I} - \mathbf{A}^\dagger\mathbf{A})\mathbf{x}_0.$$

- For symmetric $\mathbf{A}$, if $\mathbf{b} \notin \mathrm{range}(\mathbf{A})$, then $\mathbf{x}_{\ell-1}$ is a least-squares solution, but not necessarily the pseudoinverse solution $\mathbf{A}^\dagger\mathbf{b}$.
  MINRES-QLP or MINRES with a minimum-norm (MN) refinement

[CPS11] S.-C. T. Choi, C. C. Paige, and M. A. Saunders. *MINRES-QLP: A Krylov subspace method for indefinite or singular symmetric systems*. SISC, 2011.

[LMR25] Y. Liu, A. Milzarek, and F. Roosta. *Obtaining pseudoinverse solutions with MINRES*. SIMAX, 2025.

A minimum-norm (MN) refinement for GMRES iterates

- If $\mathrm{range}(\mathbf{A}) = \mathrm{range}(\mathbf{A}^\top)$ and $\mathbf{b} \notin \mathrm{range}(\mathbf{A})$, then the MN refinement vector,

$$\widetilde{\mathbf{x}}_{\ell-1} := \mathbf{x}_{\ell-1} - \frac{\mathbf{r}_{\ell-1}^\top(\mathbf{x}_{\ell-1} - \mathbf{x}_0)}{\mathbf{r}_{\ell-1}^\top \mathbf{r}_{\ell-1}} \mathbf{r}_{\ell-1},$$

  is the orthogonal projection of $\mathbf{x}_0$ onto the least squares solution set $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}^\top \mathbf{A}\mathbf{x} = \mathbf{A}^\top \mathbf{b}\}$, i.e.,

$$\widetilde{\mathbf{x}}_{\ell-1} = \mathbf{A}^\dagger \mathbf{b} + (\mathbf{I} - \mathbf{A}^\dagger \mathbf{A})\mathbf{x}_0.$$

- $\mathbf{x}_0 = \mathbf{0} \;\Rightarrow\; \widetilde{\mathbf{x}}_{\ell-1} = \mathbf{A}^\dagger \mathbf{b}.$

[DFW26] D., Fan, and Wang. *RSMAR: An iterative method for range-symmetric linear systems*. LAA, 2026.

RSMAR for range-symmetric systems

- RSMAR: $\mathbf{x}_k^{\mathrm{A}} := \underset{\mathbf{x}\in\mathcal{K}_k(\mathbf{A},\mathbf{b})}{\operatorname{argmin}} \|\mathbf{A}(\mathbf{b}-\mathbf{A}\mathbf{x})\|,$ (well-defined?)

- For range-symmetric $\mathbf{A}$, if $\mathbf{b} \in \mathrm{range}(\mathbf{A})$, then $\mathbf{x}_\ell^{\mathrm{A}} = \mathbf{x}_\ell$, and if $\mathbf{b} \notin \mathrm{range}(\mathbf{A})$, then $\mathbf{x}_{\ell-1}^{\mathrm{A}} = \mathbf{x}_{\ell-1}$. In other words, the final iterates of GMRES and RSMAR are the same.

- RSMAR for $\mathbf{A}\mathbf{x} = \mathbf{b}$ and GMRES for $\mathbf{A}\mathbf{y} = \mathbf{A}\mathbf{b}$:

$$\min_{\mathbf{x}\in\mathcal{K}_k(\mathbf{A},\mathbf{b})} \|\mathbf{A}(\mathbf{b}-\mathbf{A}\mathbf{x})\| = \min_{\mathbf{y}\in\mathcal{K}_k(\mathbf{A},\mathbf{A}\mathbf{b})} \|\mathbf{A}\mathbf{b}-\mathbf{A}\mathbf{y}\|,$$

$$\mathbf{y}_k = \mathbf{A}\mathbf{x}_k^{\mathrm{A}} = \underset{\mathbf{y}\in\mathcal{K}_k(\mathbf{A},\mathbf{A}\mathbf{b})}{\operatorname{argmin}} \|\mathbf{A}\mathbf{b}-\mathbf{A}\mathbf{y}\|.$$

[MOS25] A. Montoison, D. Orban, and M. A. Saunders. *MINARES: An iterative solver for symmetric linear systems.* SIMAX, 2025.

Implementation I (inspired by simpler GMRES)

- Arnoldi process yields $\operatorname{span}\{\widehat{\mathbf{v}}_1, \widehat{\mathbf{v}}_2, \ldots, \widehat{\mathbf{v}}_k\} = \mathcal{K}_k(\mathbf{A}, \mathbf{A}\mathbf{b})$,
$$\widehat{\beta}_1 \widehat{\mathbf{v}}_1 = \mathbf{A}\mathbf{b}, \quad \mathbf{A}\widehat{\mathbf{V}}_k = \widehat{\mathbf{V}}_{k+1}\widehat{\mathbf{H}}_{k+1,k}, \quad \widehat{\mathbf{V}}_k^\top \widehat{\mathbf{V}}_k = \mathbf{I}_k.$$
- $\min_{\mathbf{y} \in \mathcal{K}_k(\mathbf{A},\mathbf{A}\mathbf{b})} \|\mathbf{A}\mathbf{b} - \mathbf{A}\mathbf{y}\| = \min_{\widehat{\mathbf{z}} \in \mathbb{R}^k} \|\widehat{\beta}_1 \mathbf{e}_1 - \widehat{\mathbf{H}}_{k+1,k}\widehat{\mathbf{z}}\| \;\Rightarrow\; \mathbf{y}_k = \widehat{\mathbf{V}}_k \widehat{\mathbf{z}}_k$ with
$\widehat{\mathbf{z}}_k = \operatorname{argmin}_{\widehat{\mathbf{z}} \in \mathbb{R}^k} \|\widehat{\beta}_1 \mathbf{e}_1 - \widehat{\mathbf{H}}_{k+1,k}\widehat{\mathbf{z}}\|.$
- $\mathcal{K}_k(\mathbf{A}, \mathbf{b}) = \operatorname{span}\{\mathbf{b}, \widehat{\mathbf{v}}_1, \ldots, \widehat{\mathbf{v}}_{k-1}\}$ and $\mathbf{y}_k = \mathbf{A}\mathbf{x}_k^{\mathrm{A}} \;\Rightarrow$
$$\mathbf{x}_k^{\mathrm{A}} = \begin{bmatrix} \mathbf{b} & \widehat{\mathbf{V}}_{k-1} \end{bmatrix} \mathbf{z}_k,$$
where $\mathbf{z}_k$ solves
$$\mathbf{A} \begin{bmatrix} \mathbf{b} & \widehat{\mathbf{V}}_{k-1} \end{bmatrix} \mathbf{z} = \widehat{\mathbf{V}}_k \begin{bmatrix} \widehat{\beta}_1 \mathbf{e}_1 & \widehat{\mathbf{H}}_{k,k-1} \end{bmatrix} \mathbf{z} = \widehat{\mathbf{V}}_k \widehat{\mathbf{z}}_k.$$

Implementation II (inspired by RRGMRES)

- Arnoldi process yields $\mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} = \mathcal{K}_k(\mathbf{A}, \mathbf{b})$,

$$\beta_1 \mathbf{v}_1 = \mathbf{b}, \quad \mathbf{A}\mathbf{V}_k = \mathbf{V}_{k+1}\mathbf{H}_{k+1,k}, \quad \mathbf{V}_k^\top \mathbf{V}_k = \mathbf{I}_k.$$

- The subproblem:

$$\min_{\mathbf{x} \in \mathcal{K}_k(\mathbf{A},\mathbf{b})} \|\mathbf{A}(\mathbf{b} - \mathbf{A}\mathbf{x})\| = \min_{\mathbf{z} \in \mathbb{R}^k} \|\beta_1 \mathbf{H}_{k+2,k+1}\mathbf{e}_1 - \mathbf{H}_{k+2,k+1}\mathbf{H}_{k+1,k}\mathbf{z}\|.$$

- Two QR factorizations are required:

$$\mathbf{H}_{k+1,k} = \mathbf{Q}_{k+1} \begin{bmatrix} \mathbf{R}_k \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{H}_{k+2,k+1}\mathbf{Q}_{k+1} \begin{bmatrix} \mathbf{I}_k \\ \mathbf{0} \end{bmatrix} = \widetilde{\mathbf{Q}}_{k+2} \begin{bmatrix} \widetilde{\mathbf{R}}_k \\ \mathbf{0} \end{bmatrix}.$$

- $\mathbf{x}_k^{\mathrm{A}} = \mathbf{V}_k \mathbf{R}_k^{-1} \widetilde{\mathbf{R}}_k^{-1} \begin{bmatrix} \mathbf{I}_k & \mathbf{0} \end{bmatrix} \widetilde{\mathbf{Q}}_{k+2}^\top \beta_1 (h_{11}\mathbf{e}_1 + h_{21}\mathbf{e}_2)$.

Numerical experiments

- A boundary value problem (d is a constant and f is a given function)

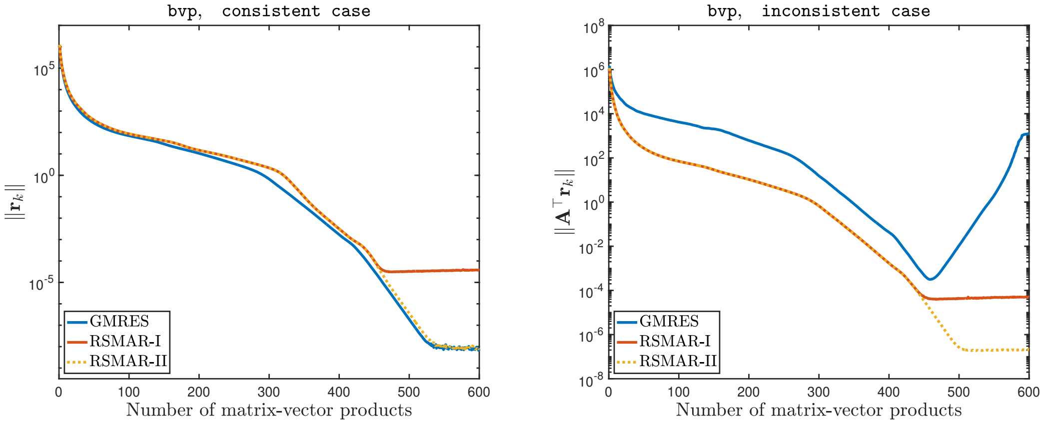
$$\begin{cases} \Delta u + d\dfrac{\partial u}{\partial x} = f, & \text{in} \quad \Omega := [0,1]\times[0,1], \\ u(x,0) = u(x,1), & \text{for} \quad 0 \le x \le 1, \\ u(0,y) = u(1,y), & \text{for} \quad 0 \le y \le 1. \end{cases}$$

- FD discretization yields a singular range-symmetric $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} \mathbf{T}_m & \mathbf{I}_m & & \mathbf{I}_m \\ \mathbf{I}_m & \ddots & \ddots & \\ & \ddots & \ddots & \mathbf{I}_m \\ \mathbf{I}_m & & \mathbf{I}_m & \mathbf{T}_m \end{bmatrix}, \quad \mathbf{T}_m = \begin{bmatrix} -4 & \alpha_+ & & \alpha_- \\ \alpha_- & \ddots & \ddots & \\ & \ddots & \ddots & \alpha_+ \\ \alpha_+ & & \alpha_- & -4 \end{bmatrix},$$

where $m = 100$, $h = 1/m$, $\alpha_\pm = 1 \pm dh/2$, and $d = 10$.

Numerical experiments



Symmetric quasi-definite linear systems

- $\mathbf{M} \in \mathbb{R}^{m\times m}$ and $\mathbf{N} \in \mathbb{R}^{n\times n}$ are SPD, $\mathbf{A} \in \mathbb{R}^{m\times n}$ is nonzero, $\mathbf{b} \in \mathbb{R}^{m}$, and $\mathbf{c} \in \mathbb{R}^{n}$:

$$\begin{bmatrix} \mathbf{M} & \mathbf{A} \\ \mathbf{A}^\top & -\mathbf{N} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{c} \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} \mathbf{M} & \\ & \mathbf{N} \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} \mathbf{M} & \mathbf{A} \\ \mathbf{A}^\top & -\mathbf{N} \end{bmatrix}.$$

- Computational optimization and computational partial differential equations, etc.
- Symmetric, indefinite, nonsingular
- Monolithic methods: solving the system as a whole, for example, SYMMLQ, MINRES

  Segregated methods: tailored specifically to the block structure, for example, TriCG and TriMR

The generalized SSY tridiagonalization

- Let $\beta_1 \mathbf{M}\mathbf{u}_1 = \mathbf{b}$ and $\gamma_1 \mathbf{N}\mathbf{v}_1 = \mathbf{v}$. After j steps of gSSY, we have

$$\mathbf{A}\mathbf{V}_j = \mathbf{M}\mathbf{U}_{j+1}\mathbf{T}_{j+1,j}, \quad \mathbf{A}^\top \mathbf{U}_j = \mathbf{N}\mathbf{V}_{j+1}\mathbf{T}_{j,j+1}^\top,$$
$$\mathbf{U}_{j+1}^\top \mathbf{M}\mathbf{U}_{j+1} = \mathbf{V}_{j+1}^\top \mathbf{N}\mathbf{V}_{j+1} = \mathbf{I}_{j+1}.$$

with

$$\mathbf{T}_{j+1,j} = \begin{bmatrix} \alpha_1 & \gamma_2 & & \\ \beta_2 & \alpha_2 & \ddots & \\ & \ddots & \ddots & \gamma_j \\ & & \beta_j & \alpha_j \\ & & & \beta_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{T}_j \\ \beta_{j+1}\mathbf{e}_j^\top \end{bmatrix}.$$

M. A. Saunders, H. D. Simon, and E. L. Yip. *Two conjugate-gradient-type methods for unsymmetric linear equations.* SINUM, Vol. 25, Iss. 4 (1988)

TriCG

- Assume that $\mathbf{U}_j$, $\mathbf{V}_j$, and $\mathbf{T}_j$ are well defined. The jth TriCG iterate is

$$\begin{bmatrix} \mathbf{x}_j \\ \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{U}_j & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_j \end{bmatrix} \begin{bmatrix} \mathbf{I}_j & \mathbf{T}_j \\ \mathbf{T}_j^\top & -\mathbf{I}_j \end{bmatrix}^{-1} \begin{bmatrix} \beta_1 \mathbf{e}_1 \\ \gamma_1 \mathbf{e}_1 \end{bmatrix},$$

  which satisfies the Galerkin condition

$$\begin{bmatrix} \mathbf{U}_j & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_j \end{bmatrix}^\top \left(\begin{bmatrix} \mathbf{b} \\ \mathbf{c} \end{bmatrix} - \begin{bmatrix} \mathbf{M} & \mathbf{A} \\ \mathbf{A}^\top & -\mathbf{N} \end{bmatrix} \begin{bmatrix} \mathbf{x}_j \\ \mathbf{y}_j \end{bmatrix} \right) = \mathbf{0}.$$

- Equivalent to preconditioned block-CG:

$$\begin{bmatrix} \mathbf{M} & \mathbf{A} \\ \mathbf{A}^\top & -\mathbf{N} \end{bmatrix} \begin{bmatrix} \mathbf{x}^1 & \mathbf{x}^2 \\ \mathbf{y}^1 & \mathbf{y}^2 \end{bmatrix} = \begin{bmatrix} \mathbf{b} & \mathbf{0} \\ \mathbf{0} & \mathbf{c} \end{bmatrix}.$$

A. Montoison and D. Orban. *TriCG and TriMR: Two iterative methods for symmetric quasi-definite systems.* SISC, Vol. 43, Iss. 4 (2021)

Elliptic singular value decomposition (ESVD)

- Given SPD $\mathbf{M}$ and $\mathbf{N}$, ESVD of $\mathbf{A}$ is
$$\mathbf{A} = \mathbf{M}\mathbf{P}\boldsymbol{\Sigma}\mathbf{Q}^\top\mathbf{N},$$
where $\boldsymbol{\Sigma} = \mathrm{diag}(\sigma_1, \ldots, \sigma_p)$, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0$, $p = \min(m, n)$, and $\mathbf{P}$ and $\mathbf{Q}$ satisfy
$$\mathbf{P}^\top\mathbf{M}\mathbf{P} = \mathbf{I}_m, \quad \mathbf{Q}^\top\mathbf{N}\mathbf{Q} = \mathbf{I}_n.$$
- Eigenvalues of a two-sided preconditioned matrix (let $r = \mathrm{rank}(\mathbf{A})$):
$$\lambda\left(\mathbf{H}^{-\frac{1}{2}}\mathbf{K}\mathbf{H}^{-\frac{1}{2}}\right) = \begin{cases} \pm\sqrt{\sigma_i^2 + 1}, & i = 1, \ldots, r, \\ 1, & (m - r) \text{ times}, \\ -1, & (n - r) \text{ times}. \end{cases}$$

M. Arioli. *Generalized Golub–Kahan bidiagonalization and stopping criteria*. SIMAX, Vol. 34, Iss. 2 (2013)

A gSSY process with deflated restarting

- gSSY-DR(p, k):

$$
\begin{aligned}
\mathbf{A}\mathbf{V}_p^{(i)} &= \mathbf{M}\mathbf{U}_p^{(i)}\mathbf{T}_p^{(i)} + \beta_{p+1}^{(i)}\mathbf{M}\mathbf{u}_{p+1}^{(i)}\mathbf{e}_p^\top, \\
\mathbf{A}^\top\mathbf{U}_p^{(i)} &= \mathbf{N}\mathbf{V}_p^{(i)}(\mathbf{T}_p^{(i)})^\top + \gamma_{p+1}^{(i)}\mathbf{N}\mathbf{v}_{p+1}^{(i)}\mathbf{e}_p^\top.
\end{aligned}
$$

For $i = 2, 3, \ldots,$

$$
\mathbf{T}_p^{(i)} = \begin{bmatrix}
\alpha_1^{(i)} & & & \gamma_2^{(i)} & & & \\
& \ddots & & \vdots & & & \\
& & \ddots & \gamma_{k+1}^{(i)} & & & \\
\beta_2^{(i)} & \ldots & \beta_{k+1}^{(i)} & \alpha_{k+1}^{(i)} & \gamma_{k+2}^{(i)} & & \\
& & & \beta_{k+2}^{(i)} & \alpha_{k+2}^{(i)} & \ddots & \\
& & & & \ddots & \ddots & \gamma_p^{(i)} \\
& & & & & \beta_p^{(i)} & \alpha_p^{(i)}
\end{bmatrix}.
$$

TriCG with deflated restarting

- The recurrences in the first cycle are the same as that of TriCG. Now consider cycle $i \geq 2$. The jth ($k+1 \leq j \leq p$) TriCG-DR(p,k) iterate is

$$\begin{bmatrix} \mathbf{x}_j^{(i)} \\ \mathbf{y}_j^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{U}_j^{(i)} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_j^{(i)} \end{bmatrix} \begin{bmatrix} \mathbf{I}_j & \mathbf{T}_j^{(i)} \\ (\mathbf{T}_j^{(i)})^\top & -\mathbf{I}_j \end{bmatrix}^{-1} \begin{bmatrix} \beta_1^{(i)} \mathbf{e}_{k+1} \\ \gamma_1^{(i)} \mathbf{e}_{k+1} \end{bmatrix},$$

  which satisfies the Galerkin condition.
- Using an LDL$^\top$ decomposition and the same strategy in TriCG, short recurrences can be obtained to compute $\mathbf{x}_j^{(i)}$ and $\mathbf{y}_j^{(i)}$ for $k+1 \leq j \leq p$.
- If the desired k approximate elliptic singular triplets are sufficiently accurate, we stop restarting. In other words, the last cycle is implemented completely until a sufficiently accurate approximate solution is found or the maximum number of iterations is reached. Some reorthogonalization is necessary.

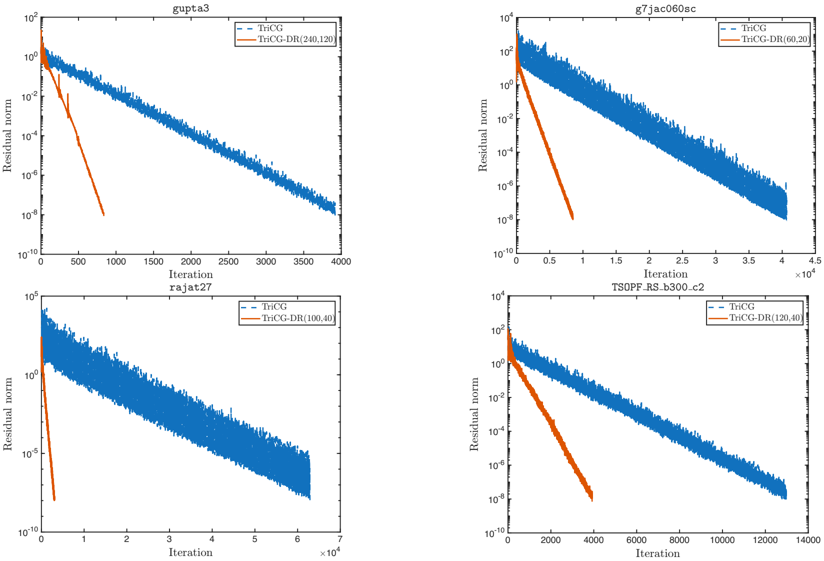
TriCG vs. TriCG-DR(p, k)

$\mathbf{M} = \mathbf{I}$, $\mathbf{N} = \mathbf{I}$, $\mathbf{A}$ is from the SuiteSparse Matrix Collection.

Table: The information of square matrices from the SuiteSparse Matrix Collection, runtime of TriCG and TriCG-DR, and values of parameters p and k of TriCG-DR.

Matrix	Size	Nnz	TriCG	TriCG-DR		
			Time(s)	Time(s)	p	k
gupta3	16783	9323427	17.55	7.61	240	120
g7jac060sc	17730	183325	16.82	10.10	60	20
rajat27	20640	97353	24.70	6.42	100	40
TSOPF_RS_b300_c2	28338	2943887	30.48	17.64	120	40

Numerical experiments



Block two-by-two nonsymmetric linear systems

- Block two-by-two nonsymmetric linear systems of the form

$$\begin{bmatrix} \mathbf{M} & \mathbf{A} \\ \mathbf{B} & \mathbf{N} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{c} \end{bmatrix}, \quad \mathbf{M} \in \mathbb{R}^{m \times m}, \quad \mathbf{N} \in \mathbb{R}^{n \times n}.$$

- Monolithic methods: solving the system as a whole.
  For example: GMRES, Bi-CG, QMR, Bi-CGSTAB, CMRH ...
  Segregated methods: tailored specifically to the block structure. For example: GPMR, GPBiLQ, GPQMR ...
- We consider a simple case: $\mathbf{A} \neq \mathbf{B}^\top$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$,

$$\begin{bmatrix} \lambda \mathbf{I} & \mathbf{A} \\ \mathbf{B} & \mu \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{c} \end{bmatrix}.$$

A. Montoison and D. Orban. *GPMR: An iterative method for unsymmetric partitioned linear systems*. SIMAX, Vol. 44, No. 1 (2023)

Simultaneous orthogonal Hessenberg reduction for $(\mathbf{A}, \mathbf{B})$

- Simultaneous orthogonal Hessenberg reduction

$$\mathbf{A}\mathbf{U}_k = \mathbf{V}_{k+1}\mathbf{H}_{k+1,k}, \quad \mathbf{B}\mathbf{V}_k = \mathbf{U}_{k+1}\mathbf{F}_{k+1,k},$$

$$\mathbf{V}_{k+1}^\top \mathbf{V}_{k+1} = \mathbf{U}_{k+1}^\top \mathbf{U}_{k+1} = \mathbf{I}_{k+1},$$

where

$$\mathbf{H}_{k+1,k} = \begin{bmatrix} h_{11} & \cdots & h_{1k} \\ h_{21} & \ddots & \vdots \\ & \ddots & h_{kk} \\ & & h_{k+1,k} \end{bmatrix}, \qquad \mathbf{F}_{k+1,k} = \begin{bmatrix} f_{11} & \cdots & f_{1k} \\ f_{21} & \ddots & \vdots \\ & \ddots & f_{kk} \\ & & f_{k+1,k} \end{bmatrix}.$$

A. Montoison and D. Orban. *GPMR: An iterative method for unsymmetric partitioned linear systems*. SIMAX, Vol. 44, No. 1 (2023)

GPMR

- The kth GPMR iterate is

$$\begin{bmatrix} \mathbf{x}_k \\ \mathbf{y}_k \end{bmatrix} = \underset{\mathbf{x} \in \mathrm{range}(\mathbf{V}_k),\ \mathbf{y} \in \mathrm{range}(\mathbf{U}_k)}{\mathrm{argmin}} \left\| \begin{bmatrix} \mathbf{b} \\ \mathbf{c} \end{bmatrix} - \begin{bmatrix} \lambda \mathbf{I} & \mathbf{A} \\ \mathbf{B} & \mu \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \right\|.$$

- Equivalent to Block-GMRES:

$$\begin{bmatrix} \lambda \mathbf{I} & \mathbf{A} \\ \mathbf{B} & \mu \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x}^1 & \mathbf{x}^2 \\ \mathbf{y}^1 & \mathbf{y}^2 \end{bmatrix} = \begin{bmatrix} \mathbf{b} & \mathbf{0} \\ \mathbf{0} & \mathbf{c} \end{bmatrix}.$$

- GPMR terminates significantly earlier than GMRES on a residual-based stopping criterion with an improvement up to 50% in terms of number of iterations.

A. Montoison and D. Orban. *GPMR: An iterative method for unsymmetric partitioned linear systems*. SIMAX, Vol. 44, No. 1 (2023)

Simultaneous Hessenberg reduction with pivoting for (A, B)

- Simultaneous Hessenberg reduction with pivoting

$$\mathbf{A}\mathbf{L}_k = \mathbf{D}_{k+1}\widetilde{\mathbf{H}}_{k+1,k}, \quad \mathbf{B}\mathbf{D}_k = \mathbf{L}_{k+1}\widetilde{\mathbf{F}}_{k+1,k},$$

where

$$\widetilde{\mathbf{H}}_{k+1,k} = \begin{bmatrix} \widetilde{h}_{11} & \cdots & \widetilde{h}_{1k} \\ \widetilde{h}_{21} & \ddots & \vdots \\ & \ddots & \widetilde{h}_{kk} \\ & & \widetilde{h}_{k+1,k} \end{bmatrix}, \qquad \widetilde{\mathbf{F}}_{k+1,k} = \begin{bmatrix} \widetilde{f}_{11} & \cdots & \widetilde{f}_{1k} \\ \widetilde{f}_{21} & \ddots & \vdots \\ & \ddots & \widetilde{f}_{kk} \\ & & \widetilde{f}_{k+1,k} \end{bmatrix}.$$

We have

$$\mathrm{range}(\mathbf{D}_k) = \mathrm{range}(\mathbf{V}_k), \quad \mathrm{range}(\mathbf{L}_k) = \mathrm{range}(\mathbf{U}_k).$$

GP-CMRH

- The kth GP-CMRH iterate is

$$\begin{bmatrix} \mathbf{x}_k \\ \mathbf{y}_k \end{bmatrix} = \underset{\mathbf{x}\in\mathrm{range}(\mathbf{D}_k),\ \mathbf{y}\in\mathrm{range}(\mathbf{L}_k)}{\mathrm{argmin}} \left\| \begin{bmatrix} \mathbf{D}_{k+1} & \\ & \mathbf{L}_{k+1} \end{bmatrix}^{\dagger} \left(\begin{bmatrix} \mathbf{b} \\ \mathbf{c} \end{bmatrix} - \begin{bmatrix} \lambda \mathbf{I} & \mathbf{A} \\ \mathbf{B} & \mu \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \right) \right\|.$$

Theorem

Let $\mathbf{r}_k^{\text{GP-CMRH}}$ *and* $\mathbf{r}_k^{\text{GPMR}}$ *be the kth residuals of* GP-CMRH *and* GPMR, *respectively. Let* $\mathbf{W}_{k+1} = \begin{bmatrix} \mathbf{D}_{k+1} & \\ & \mathbf{L}_{k+1} \end{bmatrix}$. *Then,*

$$\|\mathbf{r}_k^{\text{GPMR}}\| \leq \|\mathbf{r}_k^{\text{GP-CMRH}}\| \leq \kappa(\mathbf{W}_{k+1}) \|\mathbf{r}_k^{\text{GPMR}}\|,$$

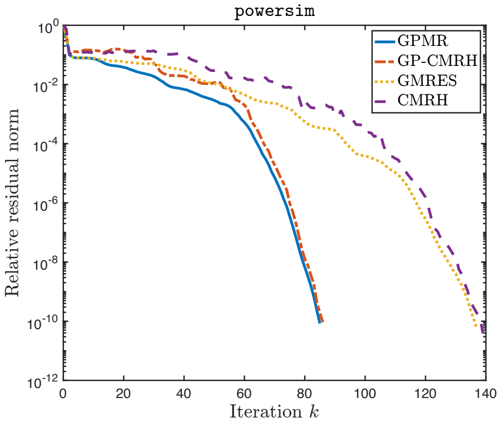
where $\kappa(\mathbf{W}_{k+1}) = \|\mathbf{W}_{k+1}\| \|\mathbf{W}_{k+1}^{\dagger}\|$ *is the condition number of* $\mathbf{W}_{k+1}$.

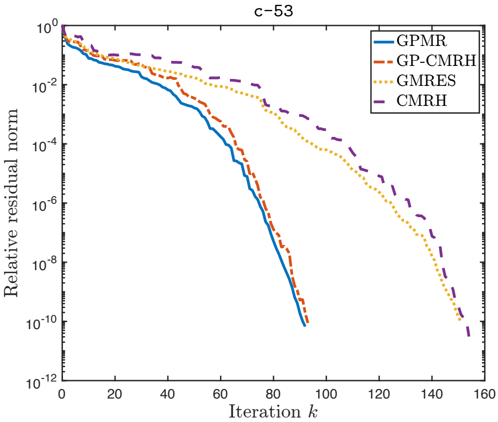
Numerical experiments

Table 1: Numbers of iterations (Iter), runtimes (Time), and relative residual norms (Rel) of GPMR, GP-CMRH, GMRES, and CMRH on twenty-two matrices from the SuitSparse Matrix Collection. "Nnz" denotes the number of nonzero elements in each sparse matrix. Bold-faced values in the runtime column highlight the shortest time taken among the four methods.

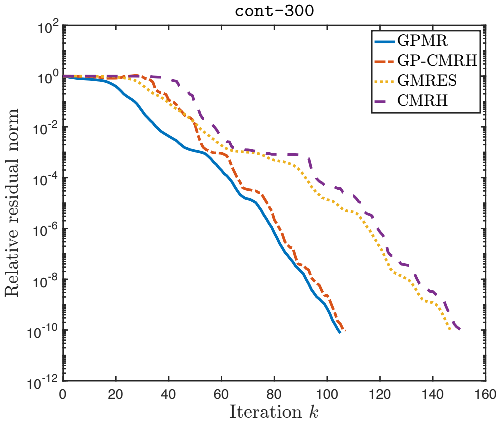
Name	Size	Nnz	GPMR			GP-CMRH			GMRES			CMRH		
			Iter	Time	Rel	Iter	Time	Rel	Iter	Time	Rel	Iter	Time	Rel
bcsstk17	10974	428650	121	0.39	3.66e-11	121	**0.28**	8.51e-11	213	1.57	8.17e-11	216	0.72	8.64e-11
bcsstk25	15439	252241	62	0.18	5.54e-11	63	**0.15**	8.63e-11	100	0.47	8.13e-11	116	0.37	8.41e-11
powersim	15838	64424	85	0.26	8.40e-11	86	**0.20**	9.28e-11	137	1.17	5.64e-11	139	0.42	3.91e-11
raefsky3	21200	1488768	37	**0.68**	9.56e-11	40	0.69	8.86e-11	63	1.24	8.59e-11	67	1.15	9.28e-11
sme3Db	29067	2081063	65	2.42	6.60e-11	66	**2.23**	7.30e-11	97	4.65	6.30e-11	98	3.27	9.40e-11
c-53	30235	355139	92	1.38	6.73e-11	93	**1.21**	8.85e-11	151	3.29	9.34e-11	154	2.17	3.08e-11
sme3Dc	42930	3148656	97	5.72	6.12e-11	98	**5.36**	7.67e-11	161	11.05	7.39e-11	163	8.75	6.85e-11
bcsstk39	46772	2060662	205	5.72	7.54e-11	209	**3.25**	7.33e-11	381	23.32	9.73e-11	392	5.39	9.50e-11
rma10	46835	2329092	41	1.43	6.39e-11	42	**1.33**	6.34e-11	49	1.69	7.02e-11	51	1.53	5.08e-11
copter2	55476	759952	211	19.86	7.38e-11	214	**16.11**	9.18e-11	367	50.06	7.04e-11	371	27.06	6.81e-11
Goodwin_071	56021	1797934	70	2.77	8.93e-11	72	**2.39**	7.56e-11	88	4.24	8.20e-11	91	2.94	9.34e-11
water_tank	60740	2035281	324	46.00	8.07e-11	338	**35.09**	7.20e-11	430	75.59	9.73e-11	464	51.15	8.34e-11
venkat50	62424	1717777	34	1.06	5.23e-11	35	**0.97**	6.24e-11	46	1.66	7.99e-11	48	1.29	4.17e-11
poisson3Db	85623	2374949	50	**7.57**	6.94e-11	51	7.64	8.84e-11	57	8.76	6.66e-11	59	9.00	5.97e-11
ifiss_mat	96307	3599932	33	**2.27**	8.76e-11	35	2.32	3.38e-11	42	3.03	7.08e-11	43	2.73	9.70e-11
hcircuit	105676	513072	46	0.80	9.69e-11	46	**0.38**	7.84e-11	58	1.44	4.99e-11	58	0.49	8.66e-11
PR02R	161070	8185136	61	**25.13**	8.31e-11	64	26.39	5.15e-11	100	42.66	8.91e-11	105	46.23	4.92e-11
cont-300	180895	988195	105	24.34	7.55e-11	107	**21.57**	9.34e-11	147	39.66	8.68e-11	151	32.55	9.49e-11
thermomech_dK	204316	2846228	108	14.06	9.26e-11	110	**8.59**	9.20e-11	164	27.30	8.81e-11	167	13.53	4.48e-11
pwtk	217918	11524432	190	28.61	8.36e-11	197	**13.83**	8.64e-11	283	56.32	9.94e-11	292	21.83	7.55e-11

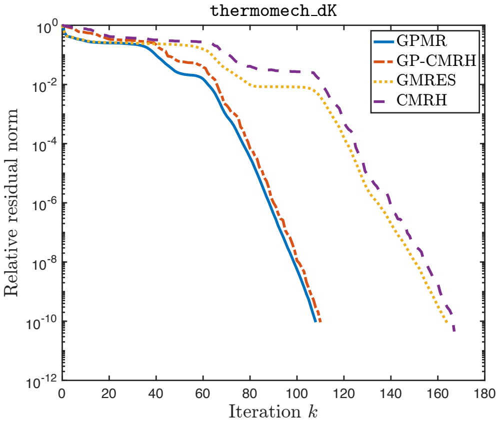
Numerical experiments





Numerical experiments





Summary

- We have proposed RSMAR for solving range-symmetric linear systems. On singular inconsistent range-symmetric systems, RSMAR outperforms GMRES, and thus should be the preferred method in finite precision arithmetic.
- We have proposed TriCG with deflated restarting for solving symmetric quasi-definite linear systems. TriCG-DR significantly outperforms TriCG when the off-diagonal block has a significant number of outlying elliptic singular values.
- We have proposed an inner product free iterative method called GP-CMRH for solving block two-by-two nonsymmetric linear systems. Our numerical experiments demonstrate that GP-CMRH and GPMR exhibit comparable convergence behavior (with GP-CMRH requiring slightly more iterations), yet GP-CMRH consumes less computational time in most cases.

Our recent related work

- Kui Du, Jia-Jun Fan, and Fang Wang
  RSMAR: An iterative method for range-symmetric linear systems
  Linear Algebra and its Applications, 729 (2026), 49–66.

- Kui Du and Jia-Jun Fan
  TriCG with deflated restarting for symmetric quasi-definite linear systems.
  In preparation, 2025.

- Kui Du and Jia-Jun Fan
  GP-CMRH: An inner product free iterative method for block two-by-two nonsymmetric linear systems.
  arXiv:2509.11272, 2025.

Thank you for your attention!