Numerical Linear Algebra Assignment 9

Exercise 1. (10 points)

Consider the subspace iteration:

Algorithm 1: Subspace iteration

Pick $\mathbf{Q}_n^{(0)} \in \mathbb{C}^{m\times n}$ with orthonormal columns
for $k = 1, 2, 3, \ldots,$
$\quad \mathbf{Q}_n^{(k)}\mathbf{R}_n^{(k)} = \mathbf{A}\mathbf{Q}_n^{(k-1)} \qquad$ (QR factorization)
end

Assume $\mathbf{A}$ is diagonalizable with $\mathbf{A} = \mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}$, $\mathbf{\Lambda} = \operatorname{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_m\}$, and

$$|\lambda_1| \geq \cdots \geq |\lambda_n| > |\lambda_{n+1}| \geq \cdots \geq |\lambda_m|.$$

Assume $\mathbf{X}_n := \begin{bmatrix}\mathbf{I}_n & \mathbf{0}\end{bmatrix}\mathbf{S}^{-1}\mathbf{Q}_n^{(0)}$ has full rank. Prove that

$$\mathbf{Q}_n^{(k)} = \mathbf{A}^k\mathbf{Q}_n^{(0)}(\mathbf{R}_n^{(1)})^{-1}(\mathbf{R}_n^{(2)})^{-1}\cdots(\mathbf{R}_n^{(k)})^{-1}.$$

Exercise 2. (Zhihao Cao, 10 points)

Let $\mathbf{A}_1 = \begin{bmatrix} a & b \\ \varepsilon & c \end{bmatrix} \in \mathbb{R}^{2\times 2}$. Let $\mathbf{A}_2 = \mathbf{R}_1\mathbf{Q}_1 + c\mathbf{I} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, where $\mathbf{Q}_1\mathbf{R}_1 = \mathbf{A}_1 - c\mathbf{I}$ is a QR factorization of $\mathbf{A}_1 - c\mathbf{I}$. Prove:

(a) if a and c are not close (i.e., $\exists\delta > 0$, $|a - c| > \delta$), then $a_{21} = \mathcal{O}(\varepsilon^2)$;

(b) if further $b = \varepsilon$ (i.e., $\mathbf{A}_1$ is symmetric), then $a_{21} = \mathcal{O}(\varepsilon^3)$.

Exercise 3. (Programming, 10 points)

Write a function `[Q,H] = myhess(A)` that reduces a complex square matrix to upper Hessenberg form by unitary similarity transformations, i.e., $\mathbf{H} = \mathbf{Q}^*\mathbf{A}\mathbf{Q}$. Your program should use only elementary Matlab operations – not the function `hess`, for example. Apply your program to `A = randn(5) + 1i*randn(5)`.

Exercise 4. (Programming, TreBau Exercise 29.1, 10 points)