Numerical Linear Algebra Assignment 13

Exercise 1. (TreBau Exercise 39.1, 10 points)

Consider a problem $\mathbf{Ax} = \mathbf{b}$ for the matrix $\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_{m-1} \\ 1 & \mathbf{0} \end{bmatrix}$.

(a) Show that the singular values are all 1 and that this implies that CGN converges in one step.

(b) Show that the eigenvalues are the mth roots of unity and that this implies that GMRES requires m steps to converge for general $\mathbf{b}$.

(c) This matrix $\mathbf{A}$ has so much structure that one does not need to consider eigenvalues or singular values to understand its convergence behavior. In particular, explain by elementary argument why GMRES takes m steps to converge for the right-hand side $\mathbf{b} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^\top$.

Exercise 2. (TreBau Exercise 39.2, 10 points)

As a converse to Exercise 1, devise an example of a matrix of arbitrary dimension m with almost the opposite property: GMRES converges in two steps, but CGN requires m steps.

Exercise 3. (Programming, 10 points)

(1) Write MATLAB code to generate matrices for the cases (a)–(g) in TreBau Exercise 39.5.

(2) Compare CG, GMRES, CGN, and Bi-CG for linear systems with these matrices and right-hand-side $\mathbf{b} = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^\top$.

(3) Explain your numerical results.