

Lecture 3: Low-rank matrix approximation



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1. Low-rank matrix approximation problem via matvecs

- Suppose $\mathbf{B} \in \mathbb{R}^{m \times n}$ is accessible via matvecs $\mathbf{x} \mapsto \mathbf{B}\mathbf{x}$, $\mathbf{y} \mapsto \mathbf{B}^\top \mathbf{y}$. The task is to produce a low-rank approximation of \mathbf{B} that is competitive with a best approximation of similar rank.
- The best rank- k approximation is unique if and only if $\sigma_k > \sigma_{k+1}$:

$$\min_{\text{rank}(\mathbf{M}) \leq k} \|\mathbf{B} - \mathbf{M}\|_F^2 = \|\mathbf{B} - \mathbf{U}_k \mathbf{U}_k^\top \mathbf{B}\|_F^2 = \sum_{i > k} \sigma_i^2,$$

where \mathbf{U}_k is the matrix consisting of the leading k left singular vectors. Cost: $\mathcal{O}(mnp)$ where $p = \min(m, n)$.

- Let $s = k + l$ for a small natural number l . For a tolerance $\varepsilon > 0$, we seek a rank- s approximation $\hat{\mathbf{B}}_s$ that competes with the best rank- k approximation:

$$\|\mathbf{B} - \hat{\mathbf{B}}_s\|_F^2 \leq (1 + \varepsilon) \|\mathbf{B} - \mathbf{U}_k \mathbf{U}_k^\top \mathbf{B}\|_F^2 = (1 + \varepsilon) \sum_{i > k} \sigma_i^2.$$

1.1 Randomized SVD: Intuition

- Draw a standard normal test vector $\mathbf{w} \in \mathbb{R}^n$. we have

$$\mathbf{B}\mathbf{w} = \sum_{i=1}^p \sigma_i \mathbf{u}_i (\mathbf{v}_i^\top \mathbf{w}) := \sum_{i=1}^p \sigma_i \mathbf{u}_i \hat{w}_i.$$

The component $\hat{w}_i := \mathbf{v}_i^\top \mathbf{w}$ of the random vector along the i th right singular vector follows a standard normal distribution, and the components $(\hat{w}_i, i = 1, \dots, p)$ compose an independent family.

- On average, $\mathbb{E}(\hat{w}_i^2) = 1$. Therefore, the image $\mathbf{B}\mathbf{w}$ tends to align with the left singular vectors associated with large singular values.
- By repeating this process with a statistically independent family $(\mathbf{w}^{(j)} : j = 1, \dots, s)$ of random test vectors, we can obtain a family $(\mathbf{B}\mathbf{w}^{(j)} : j = 1, \dots, s)$ of vectors whose span contains most of $\text{range}(\mathbf{U}_k)$. The number $s = k + l$ of test vectors needs to be a bit larger than the target rank k to obtain coverage of the subspace with high probability.

1.2 Randomized SVD: Algorithm (cost $\mathcal{O}(smn)$)

- For a rank parameter s , we draw a random test matrix:

$$\mathbf{\Omega} = [\mathbf{w}^{(1)} \quad \dots \quad \mathbf{w}^{(s)}] \quad \text{where} \quad \mathbf{w}^{(j)} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_n) \quad \text{i.i.d.}$$

- We obtain $\mathbf{Y} := \mathbf{B}\mathbf{\Omega}$. The orthogonal projector $\mathbf{P}_{\mathbf{Y}}$ onto $\text{range}(\mathbf{Y})$ serves as a proxy for the ideal projector $\mathbf{U}_k \mathbf{U}_k^\top$. Computationally,

$$\mathbf{P}_{\mathbf{Y}} := \mathbf{Q}\mathbf{Q}^\top \quad \text{where} \quad \mathbf{Q} := \text{orth}(\mathbf{Y}).$$

The function `orth` returns an orthonormal basis and costs $\mathcal{O}(s^2m)$.

- Finally, we report the approximation $\hat{\mathbf{B}}_s$ in factored form:

$$\hat{\mathbf{B}}_s := \mathbf{P}_{\mathbf{Y}}\mathbf{B} = \mathbf{Q}(\mathbf{Q}^\top \mathbf{B}).$$

- If desired, we can report the SVD of the approximation after a small amount of additional work ($\mathcal{O}(s^2n)$):

$$\hat{\mathbf{B}}_s = (\mathbf{Q}\hat{\mathbf{U}}_0)\hat{\mathbf{\Sigma}}\hat{\mathbf{V}}^\top \quad \text{where} \quad (\hat{\mathbf{U}}_0, \hat{\mathbf{\Sigma}}, \hat{\mathbf{V}}) = \text{svd}(\mathbf{Q}^\top \mathbf{B}).$$

Algorithm: Randomized SVD.

$$\mathbf{\Omega} = \text{randn}(n, s)$$

$$\mathbf{Y} = \mathbf{B}\mathbf{\Omega}$$

$$\mathbf{Q} = \text{orth}(\mathbf{Y})$$

$$\mathbf{C} = \mathbf{Q}^\top \mathbf{B}$$

$$(\hat{\mathbf{U}}_0, \hat{\mathbf{\Sigma}}, \hat{\mathbf{V}}) = \text{svd}(\mathbf{C})$$

$$\hat{\mathbf{U}} = \mathbf{Q}\hat{\mathbf{U}}_0$$

Theorem 1

Consider a matrix $\mathbf{B} \in \mathbb{R}^{m \times n}$, and fix the target rank $k \leq p$. When $s \geq k + 2$, the randomized SVD method produces a random rank- s approximation $\hat{\mathbf{B}}_s$ that satisfies

$$\mathbb{E} \|\mathbf{B} - \hat{\mathbf{B}}_s\|_F^2 \leq \left(1 + \frac{k}{s - k - 1}\right) \sum_{i>k} \sigma_i^2(\mathbf{B}).$$

Proof. See A. Kireeva and J.A. Tropp, arXiv:2402.17873, 2024. □

1.3 Randomized subspace iteration

Algorithm: Randomized subspace iteration.

```
 $\mathbf{X}_0 = \text{randn}(n, s)$   
for  $t = 1, 2, \dots, T$   
     $\mathbf{Q}_t := \text{orth}(\mathbf{B}\mathbf{X}_{t-1})$   
     $\mathbf{X}_t := \mathbf{B}^\top \mathbf{Q}_t$   
end  
 $\hat{\mathbf{B}}_s := \mathbf{Q}_T \mathbf{X}_T^\top$ 
```

- Randomized SVD is the special case of this algorithm with $T = 1$.
- Randomized subspace iteration produces approximations

$$\hat{\mathbf{B}}_s = \mathbf{Q}_t (\mathbf{Q}_t^\top \mathbf{B}) \quad \text{where} \quad \mathbf{Q}_t = \text{orth}((\mathbf{B}\mathbf{B}^\top)^{t-1} \mathbf{B}\mathbf{\Omega}) \quad \text{for } t = 1, 2, \dots$$

- Much as the block power method drives its iterates toward the leading eigenspace, subspace iteration drives $\text{range}(\mathbf{Q}_t)$ so that it aligns with $\text{range}(\mathbf{U}_k)$, the leading left singular subspace of \mathbf{B} .

2. Low-rank spsd approximation from entries

Consider an spsd matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ that we access via entry evaluations: $(j, k) \mapsto a_{jk}$. The task is to produce a low-rank spsd approximation of \mathbf{A} using as few as entry evaluations.

2.1 Column Nyström approximation

- Given a list $S \subseteq \{1, 2, 3, \dots, n\}$ of column indices, the column Nyström approximation:

$$\mathbf{A}_{\langle S \rangle} := \mathbf{A}(:, S) \mathbf{A}(S, S)^\dagger \mathbf{A}(S, :).$$

- The column Nyström approximation has several remarkable properties: (1) $\text{range}(\mathbf{A}_{\langle S \rangle}) = \text{range}(\mathbf{A}(:, S))$; (2) $\mathbf{0} \preceq \mathbf{A}_{\langle S \rangle} \preceq \mathbf{A}$.
- Our goal is to find a set S of s columns that make the error

$$\|\mathbf{A} - \mathbf{A}_{\langle S \rangle}\|_* = \text{tr}(\mathbf{A} - \mathbf{A}_{\langle S \rangle})$$

as small as possible. ($\|\mathbf{A}\|_*$: the sum of the singular values of \mathbf{A})

2.2 Pivoted partial Cholesky

- Set $\hat{\mathbf{A}}_0 := \mathbf{0}$ and $\mathbf{A}_0 := \mathbf{A}$.

At each step $t = 1, 2, \dots, s$, select $i_t \in \{1, 2, \dots, n\}$, and update

$$\hat{\mathbf{A}}_t := \hat{\mathbf{A}}_{t-1} + \frac{\mathbf{A}_{t-1}(:, i_t) \mathbf{A}_{t-1}(i_t, :)}{\mathbf{A}_{t-1}(i_t, i_t)};$$

$$\mathbf{A}_t := \mathbf{A}_{t-1} - \frac{\mathbf{A}_{t-1}(:, i_t) \mathbf{A}_{t-1}(i_t, :)}{\mathbf{A}_{t-1}(i_t, i_t)}.$$

Exercise: Prove the following results: (i) $\hat{\mathbf{A}}_t + \mathbf{A}_t = \mathbf{A}$;

(ii) $\text{diag}(\mathbf{A}_t) = \text{diag}(\mathbf{A}_{t-1}) - \frac{1}{\mathbf{A}_{t-1}(i_t, i_t)} |\mathbf{A}_{t-1}(:, i_t)|^2.$

Proposition 2

Suppose that we apply the pivoted partial Cholesky algorithm to an spsd matrix \mathbf{A} , and we select i_t from S in any order. Then $\hat{\mathbf{A}}_{|S|} = \mathbf{A}_{\langle S \rangle}$, where $|S|$ denotes the number of elements of the set S .

2.3 Pivoted partial Cholesky: evaluating fewer entries

- Set $\mathbf{F}_0 := \mathbf{0}$. At step $t = 1, 2, \dots, s$, select $i_t \in \{1, 2, \dots, n\}$ and set

$$\mathbf{c}_t := \mathbf{A}(:, i_t) - \mathbf{F}_{t-1}(\mathbf{F}_{t-1}(i_t, :))^{\top}.$$

Update $\mathbf{F}_t := [\mathbf{F}_{t-1} \quad \mathbf{c}_t / \sqrt{\mathbf{c}_t(i_t)}]$.

Exercise: Prove that $\hat{\mathbf{A}}_t = \mathbf{F}_t \mathbf{F}_t^{\top}$ for $t = 0, 1, 2, \dots, s$.

2.4 Pivot selection rules

- Uniform random pivoting: $i_t \sim \text{uniform}\{1, 2, \dots, n\}$.

Assumption: data points represent an i.i.d. sample from a population, so one is just as good as another.

- Greedy pivoting: $i_t \in \operatorname{argmax}\{\mathbf{A}_{t-1}(i, i) : i = 1, 2, \dots, n\}$.

Note that $\mathbf{A}_t(i_t, i_t) = 0$.

- Importance sampling pivoting: $\mathbb{P}\{i_t = j\} = \mathbf{A}_{t-1}(j, j) / \operatorname{tr}(\mathbf{A}_{t-1})$.

Balance between uniform random and greedy.

2.5 Randomly pivoted partial Cholesky

Algorithm: Randomly pivoted partial Cholesky.

```
F = zeros( $n, s$ )                                     (Preallocation)
d = diag(A)
for  $t = 1, 2, \dots, s$ 
    Sample  $i_t \sim \mathbf{d} / \sum_{j=1}^n \mathbf{d}(j)$ 
     $\mathbf{c} = \mathbf{A}(:, i_t) - \mathbf{F}(:, 1:t-1)(\mathbf{F}(i_t, 1:t-1))^\top$ 
     $\mathbf{F}(:, t) = \mathbf{c} / \sqrt{\mathbf{c}(i_t)}$ 
     $\mathbf{d} = \mathbf{d} - |\mathbf{F}(:, t)|^2$ 
     $\mathbf{d} = \max\{\mathbf{d}, \mathbf{0}\}$                                (Improve numerical stability)
    Stop when  $\sum_{j=1}^n \mathbf{d}(j) < \eta \cdot \text{tr}(\mathbf{A})$       (Optional)
end
```

- To produce a rank- s approximation, the algorithm only requires $(s+1)n - s$ entries of **A**: its diagonal and the s pivot columns.

- Define the expected residual map: $\Phi(\mathbf{A}) := \mathbb{E}(\mathbf{A}_1)$. This function measures the average progress that we make after one step of the algorithm. A quick calculation yields a formula for the expected residual map:

$$\begin{aligned}\Phi(\mathbf{A}) &= \sum_{j=1}^n \left[\mathbf{A} - \frac{\mathbf{A}(:,j)\mathbf{A}(j,:)}{\mathbf{A}(j,j)} \right] \frac{\mathbf{A}(j,j)}{\text{tr}(\mathbf{A})} \\ &= \mathbf{A} - \frac{1}{\text{tr}(\mathbf{A})} \sum_{j=1}^n \mathbf{A}(:,j)\mathbf{A}(j,:) = \mathbf{A} - \frac{\mathbf{A}^2}{\text{tr}(\mathbf{A})}.\end{aligned}$$

As a result, we have

$$\mathbb{E}(\text{tr}(\mathbf{A}_1)) = \text{tr}(\mathbb{E}(\mathbf{A}_1)) = \left(1 - \frac{\text{tr}(\mathbf{A}^2)}{(\text{tr}(\mathbf{A}))^2} \right) \text{tr}(\mathbf{A}) \leq \frac{n-1}{n} \text{tr}(\mathbf{A}).$$

In each iteration, we decrease the expected trace of the residual on average.

- The best rank- k approximation in $\|\cdot\|_*$:

$$\min_{\text{rank}(\mathbf{M}) \leq k} \|\mathbf{A} - \mathbf{M}\|_* = \sum_{j>k} \sigma_j(\mathbf{A}).$$

- Fix a comparison rank k and a tolerance $\varepsilon > 0$. Randomly pivoted Cholesky produces an approximation $\hat{\mathbf{A}}_s$ that attains the error bound

$$\mathbb{E}(\|\mathbf{A} - \hat{\mathbf{A}}_s\|_*) \leq (1 + \varepsilon) \sum_{j>k} \sigma_j(\mathbf{A})$$

after selecting s columns where

$$s \geq \frac{k}{\varepsilon} + k \log \left(\frac{1}{\varepsilon \eta} \right) \quad \text{and} \quad \eta := \frac{1}{\text{tr}(\mathbf{A})} \sum_{j>k} \sigma_j(\mathbf{A}).$$

- Y. Chen, E.N. Epperly, J.A. Tropp, and R.J. Webber, **Randomly pivoted Cholesky: Practical approximation of a kernel matrix with few entry evaluations**, Comm. Pure Appl. Math. 78 (2025), no. 5.