Numerical Linear Algebra Assignment 4

Exercise 1. (10 points)

Let $\mathbf{x} \in \mathbb{C}^m$ and $x_1 = \mathbf{e}_1^\top \mathbf{x} \neq 0$. Show that the matrix

$$\mathbf{H} = \mathbf{I} - 2\frac{\mathbf{v}\mathbf{v}^*}{\mathbf{v}^*\mathbf{v}}, \qquad \mathbf{v} = \pm\frac{x_1}{|x_1|}\|\mathbf{x}\|_2\mathbf{e}_1 - \mathbf{x},$$

satisfies that

$$\mathbf{H}\mathbf{x} = \pm\frac{x_1}{|x_1|}\|\mathbf{x}\|_2\mathbf{e}_1.$$

Exercise 2. (10 points)

Prove that $\mathbf{I} - \dfrac{\mathbf{v}\mathbf{v}^*}{\mathbf{v}^*\mathbf{v}}$ is the orthogonal projector which projects $\mathbb{C}^m$ onto the *hyperplane* $\operatorname{span}\{\mathbf{v}\}^\perp$ along $\operatorname{span}\{\mathbf{v}\}$.

Exercise 3. (10 points)

Let $\mathbf{x} = \begin{bmatrix} 3 & 4 & 0 \end{bmatrix}^\top$ and $\mathbf{e}_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top$. Try to find all possible numbers $z \in \mathbb{C}$ and the corresponding Householder reflectors $\mathbf{H}$ such that $\mathbf{H}\mathbf{x} = z\mathbf{e}_3$.

Exercise 4. (10 points)

Let $m > n$, $\mathbf{A} \in \mathbb{C}^{m\times n}$ of rank n, $\mathbf{b} \in \mathbb{C}^m$, $\mathbf{b} \notin \operatorname{range}(\mathbf{A})$ and $\mathbf{QR} = \begin{bmatrix} \mathbf{A} & \mathbf{b} \end{bmatrix}$ (i.e., full QR factorization of $\begin{bmatrix} \mathbf{A} & \mathbf{b} \end{bmatrix}$). Show that (we use Matlab's notation for convenience)

$$\min_{\mathbf{x}\in\mathbb{C}^n} \|\mathbf{b} - \mathbf{A}\mathbf{x}\|_2 = |\mathbf{R}(n+1, n+1)|,$$

and the least squares solution is given by

$$\mathbf{x} = \mathbf{R}(1\colon n, 1\colon n)\backslash\mathbf{R}(1\colon n, n+1).$$

Exercise 5. (TreBau Exercise 19.1, 10 points)

Given $\mathbf{A} \in \mathbb{C}^{m\times n}$ of rank n and $\mathbf{b} \in \mathbb{C}^m$, consider the block 2×2 system of equations

$$\begin{bmatrix} \mathbf{I} & \mathbf{A} \\ \mathbf{A}^* & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix},$$

where $\mathbf{I}$ is the $m \times m$ identity. Show that this system has a unique solution $\begin{bmatrix} \mathbf{r} \\ \mathbf{x} \end{bmatrix}$, and that the vectors $\mathbf{r}$ and $\mathbf{x}$ are the residual and the least squares solution of the least squares problem: Given $\mathbf{A} \in \mathbb{C}^{m\times n}$ of full rank, $m \geq n$, $\mathbf{b} \in \mathbb{C}^m$, find $\mathbf{x} \in \mathbb{C}^n$ such that $\|\mathbf{b} - \mathbf{A}\mathbf{x}\|_2$ is minimized.

Exercise 6. (Demmel Question 3.11, 10 points)

Let $\mathbf{A} \in \mathbb{C}^{m\times n}$. Show that $\mathbf{X} = \mathbf{A}^\dagger$ (the Moore–Penrose pseudoinverse) minimizes $\|\mathbf{A}\mathbf{X} - \mathbf{I}\|_{\mathrm{F}}$ over all $n \times m$ matrices. What is the value of this minimum?

Exercise 7. (10 points)

Let $\mathbf{A} \in \mathbb{C}^{m\times n}$ and $\mathbf{b} \in \mathbb{C}^m$. Solve the *penalized* problem

$$\min_{\mathbf{x}\in\mathbb{C}^n} \left\{ \|\mathbf{b} - \mathbf{A}\mathbf{x}\|_2^2 + \lambda^2\|\mathbf{x}\|_2^2 \right\},$$

where $\lambda > 0$. Hint: consider the LSP

$$\min_{\mathbf{x}\in\mathbb{C}^n} \left\| \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{A} \\ \lambda\mathbf{I} \end{bmatrix} \mathbf{x} \right\|_2.$$

Exercise 8. (10 points)

Suppose a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ satisfies

$$\mathbf{A} = \mathbf{U}\mathbf{V}, \qquad \mathbf{U} \in \mathbb{R}^{m\times l}, \qquad \mathbf{V} \in \mathbb{R}^{l\times n}.$$

Prove the following statements:

(i) If $\operatorname{rank}(\mathbf{U}) = \operatorname{rank}(\mathbf{V}) = l$ (i.e., $\mathbf{A} = \mathbf{U}\mathbf{V}$ is a full-rank factorization), then $\mathbf{A}^\dagger = \mathbf{V}^\dagger\mathbf{U}^\dagger$.

(ii) For all $\mathbf{b} \in \operatorname{range}(\mathbf{A})$, if $\operatorname{rank}(\mathbf{U}) = l$ and $\operatorname{rank}(\mathbf{V}) = n$, then $\mathbf{A}^\dagger\mathbf{b} = \mathbf{V}^\dagger\mathbf{U}^\dagger\mathbf{b}$.

Exercise 9. (10 points)

For a given $\mathbf{b} \in \mathbb{C}^m$ and any $\mathbf{A} \in \mathbb{C}^{m\times n}$, let $\mathbf{y}$ be the closest point (we use $\|\cdot\|_2$) to $\mathbf{b}$ in $\operatorname{range}(\mathbf{A})$. Prove that $\mathbf{y}$ is located on the sphere of radius $\|\mathbf{b}\|_2/2$ centered at $\mathbf{b}/2$.

Exercise 10. (10 points)

Let $\mathbf{A} \in \mathbb{C}^{m\times n}$, $\mathbf{b} \in \mathbb{C}^n$, and $\mathbf{y} \in \mathbb{C}^m$ be given. Assume that $\operatorname{rank}(\mathbf{A}) < \min\{m, n\}$ and $\mathbf{y} \in \operatorname{range}(\mathbf{A})$. Solve the following problem

$$\min_{\mathbf{x}\in\mathbb{C}^n,\ \mathbf{A}\mathbf{x}=\mathbf{y}} \|\mathbf{b} - \mathbf{x}\|_2.$$

Compulsory requirement for programming: Use Matlab's `publish` to save all your code, comments, and results to a PDF file. You must use the programming format files: example_format.zip.

Programming 1. (TreBau Exercises 10.2, 10 points)

Programming 2. (TreBau Exercises 10.3, 10 points)

Programming 3. (TreBau Exercises 11.3, 10 points)

About 11.3 (g), you don't have to shade with red pen the digits that appear to be wrong.