

# Numerical Linear Algebra Assignment 12

## Exercise 1. (10 points)

Let  $\mathbf{A} \in \mathbb{R}^{m \times m}$ ,  $\mathbf{b} \in \mathbb{R}^m$ , and  $\phi(\mathbf{x}) = \frac{1}{2} \mathbf{x}^\top \mathbf{A} \mathbf{x} - \mathbf{x}^\top \mathbf{b}$ . Compute the gradient of  $\phi(\mathbf{x})$ .

## Exercise 2. (TreBau Exercise 38.5, 10 points)

We have described CG as an iterative minimization of the function  $\varphi(\mathbf{x})$  of (38.7). Another way to minimize the same function—far slower, in general—is by the method of *steepest descent*.

- (a) Derive the formula  $\nabla \varphi(\mathbf{x}) = -\mathbf{r}$  for the gradient of  $\varphi(\mathbf{x})$ . Thus the steepest descent iteration corresponds to the choice  $\mathbf{p}_k = \mathbf{r}_k$  instead of  $\mathbf{p}_k = \mathbf{r}_k + \beta_k \mathbf{p}_{k-1}$  in Algorithm 38.1.
- (b) Determine the formula for the optimal step length  $\alpha_k$  of the steepest descent iteration.
- (c) Write down the full steepest descent iteration. There are three operations inside the main loop.

## Exercise 3. (Shufang Xu, 10 points)

Suppose that the steepest descent iteration finds the exact solution of a symmetric positive definite linear system  $\mathbf{A} \mathbf{x} = \mathbf{b}$  within finite steps. Prove that the last search direction is an eigenvector of  $\mathbf{A}$ .

## Exercise 4. (10 points)

Prove the following: if the vector  $\mathbf{b}$  is orthogonal to  $n$  linearly independent eigenvectors of the HPD matrix  $\mathbf{A} \in \mathbb{C}^{m \times m}$ , then the CG iteration for  $\mathbf{A} \mathbf{x} = \mathbf{b}$  with  $\mathbf{x}_0 = \mathbf{0}$  converges in at most  $m - n$  steps.

## Exercise 5. (10 points)

Assume  $\mathbf{A}$  has  $n$  distinct eigenvalues  $\{\lambda_i\}_{i=1}^n$  and the vector  $\mathbf{r}_0 = \mathbf{b} - \mathbf{A} \mathbf{x}_0$  is non-degenerate (i.e., the projections onto each of the  $n$  eigenspaces,  $\text{null}(\mathbf{A} - \lambda_i \mathbf{I})$ , are nonzero). Prove the following: CG converges at step  $n$  and the polynomial  $p(\lambda)$  in  $\mathbf{x}_n - \mathbf{x}_0 = p(\mathbf{A}) \mathbf{r}_0$  satisfies  $p(\lambda_i) = 1/\lambda_i$  and  $p(\mathbf{A}) = \mathbf{A}^{-1}$ .

## Exercise 6. (10 points)

Prove that PCG for  $\mathbf{A} \mathbf{M}^{-1} \mathbf{z} = \mathbf{b}$ ,  $\mathbf{x} = \mathbf{M}^{-1} \mathbf{z}$  has the convergence estimate

$$\frac{\|\varepsilon_j\|_{\mathbf{A}}}{\|\varepsilon_0\|_{\mathbf{A}}} \leq 2 \left( \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^j.$$

Here  $\varepsilon_j = \mathbf{A}^{-1} \mathbf{b} - \mathbf{x}_j$  and  $\kappa = \lambda_{\max}(\mathbf{A} \mathbf{M}^{-1}) / \lambda_{\min}(\mathbf{A} \mathbf{M}^{-1})$ .

## Exercise 7. (Programming, 10 points)

TreBau Exercise 38.6. Also plot the curve:  $(\kappa - 1)^n / (\kappa + 1)^n$ .

## Exercise 8. (Programming, 10 points)

Write matlab code to plot Figure 40.1 of TreBau's book.

## Additional Exercise 1.

How to recover the tridiagonal matrix  $\mathbf{T}_j$  generated in the Lanczos process via  $\alpha_i, \beta_i$  in CG, and  $\|\mathbf{r}_i\|_2, i = 1, 2, \dots, j$ .