Numerical Linear Algebra Assignment 10

Exercise 1. (Zhihao Cao, 10 points)

Let

$$\mathbf{A} = \begin{bmatrix} a_1 & b_1 & & & \\ c_1 & a_2 & b_2 & & \\ & c_2 & a_3 & \ddots & \\ & & \ddots & \ddots & b_{m-1} \\ & & & c_{m-1} & a_m \end{bmatrix} \in \mathbb{R}^{m\times m}, \qquad c_j b_j > 0.$$

Prove that there exists a diagonal matrix $\mathbf{D}$ satisfying that $\mathbf{D}^{-1}\mathbf{A}\mathbf{D}$ is symmetric.

Exercise 2. (10 points)

How many eigenvalues does $\mathbf{A} = \begin{bmatrix} -2 & 2 & & \\ 2 & 2 & 1 & \\ & 1 & 2 & -1 \\ & & -1 & 1 \end{bmatrix}$ have in the interval $[1, 2]$? Determine the answer via Sturm sequences.

Exercise 3. (10 points)

Prove Proposition 12 of Lecture 10.

Exercise 4. (TreBau Exercise 30.3, 10 points)

Show that if the largest off-diagonal entry is annihilated at each step of the Jacobi algorithm, then the sum of the squares of the off-diagonal entries decreases by at least the factor $1 - 2/(m^2 - m)$ at each step.

Exercise 5. (TreBau Exercise 31.3, 10 points)

Show that if the entries on both principal diagonals of a bidiagonal matrix are all nonzero, then the singular values of the matrix are distinct.

Exercise 6. (Programming, TreBau Exercise 30.5, 10 points)

Write a program to find the eigenvalues of an $m \times m$ real symmetric matrix by the Jacobi algorithm with the standard row-wise ordering, plotting the sum of the squares of the off-diagonal entries on a log scale as a function of the number of sweeps. Apply your program to random matrices of dimensions 20, 40, and 80.