Numerical Linear Algebra Assignment 3

Exercise 1. (10 points)

(1) Let $\mathbf{P}$ be a projector. Given an explicit expression for the inverse of $\lambda\mathbf{I}-\mathbf{P}$, where $\lambda \neq 0, 1$.

(2) Suppose $\mathbf{A} \in \mathbb{C}^{m\times n}$ has a full SVD $\mathbf{A} = \mathbf{U\Sigma V}^*$, where

$$\mathbf{U} = [\mathbf{U}_r \quad \mathbf{U}_c], \quad \mathbf{V} = [\mathbf{V}_r \quad \mathbf{V}_c], \quad r = \text{rank}(\mathbf{A}).$$

What are the orthogonal projections onto $\text{null}(\mathbf{A})^\perp$, $\text{null}(\mathbf{A})$, $\text{range}(\mathbf{A})$ and $\text{range}(\mathbf{A})^\perp$?

Exercise 2. (10 points)

Two subspaces $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathbb{C}^m$ are called *complementary subspaces* if they satisfy

$$\mathcal{S}_1 \cap \mathcal{S}_2 = \{\mathbf{0}\}, \qquad \mathcal{S}_1 + \mathcal{S}_2 = \mathbb{C}^m.$$

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be complementary subspaces. Prove that there exists a projector $\mathbf{P}$ with

$$\text{range}(\mathbf{P}) = \mathcal{S}_1, \qquad \text{null}(\mathbf{P}) = \mathcal{S}_2.$$

Exercise 3. (TreBau Exercise 6.1, 10 points)

If $\mathbf{P}$ is an orthogonal projector, then $\mathbf{I}-2\mathbf{P}$ is unitary. Prove this algebraically, and give a geometric interpretation.

Exercise 4. (TreBau Exercise 6.5, 10 points)

Let $\mathbf{P} \in \mathbb{C}^{m\times m}$ be a nonzero projector. Show that $\|\mathbf{P}\|_2 \geq 1$, with equality if and only if $\mathbf{P}$ is an orthogonal projector.

Exercise 5. (10 points)

Compute a QR factorization of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ \sqrt{2} & 1+\sqrt{2} & 1 \\ 1 & 2 & 1 \end{bmatrix}$.

Exercise 6. (Programming, TreBau Exercise 8.2, 10 points)

Additional Exercise 1.

Let $C[-1,1]$ denote the linear space of real-valued continuous functions on $[-1,1]$ with the inner product

$$\forall f, g \in C[-1,1], \qquad \langle f, g\rangle_w = \int_{-1}^{1} w(x)f(x)g(x)\mathrm{d}x,$$

where $w(x) \geq 0 (\not\equiv 0)$ is a weight function (continuous). For the case $w(x) = 1 + x^2$, complete the following:

(i) Write Matlab code to compute the first six orthogonal (with respect to the inner product $\langle\cdot,\cdot\rangle_w$) polynomials ($P_j(x), j = 0, 1, 2, 3, 4, 5$, which are conventionally normalized so that $P_j(1) = 1$). Hint: you can use Matlab's symbolic toolbox. For your reference, the polynomials are given by:

```
P =
 
                                          1
                                          x
                             (5*x^2)/3 - 2/3
                        (14*x^3)/5 - (9*x)/5
         (119*x^4)/24 - (161*x^2)/36 + 37/72
(1221*x^5)/136 - (705*x^3)/68 + (325*x)/136
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(ii) Modify the code we used for discrete Legendre polynomials to plot the discrete polynomials corresponding to those obtained in (i).

Additional Exercise 2.

Let Π_n denote the linear operator that maps $f \in C[a,b]$ to the polynomial p_n that interpolates f at the distinct points $x_0, x_1 \cdots, x_n \in [a,b]$. In other words,

$$\Pi_n f = p_n,$$

where p_n is the unique polynomial of degree n (or less) for which $f(x_i) = p_n(x_i)$ for $i = 0, 1, \cdots, n$. The infinity norm $\|\cdot\|_\infty$ induces the operator norm

$$\|\Pi_n\|_\infty = \max_{f \in C[a,b], f \neq 0} \frac{\|\Pi_n f\|_\infty}{\|f\|_\infty} = \max_{\|f\|_\infty = 1} \|\Pi_n f\|_\infty.$$

(a) Explain why Π_n is a projector: That is, for any $f \in C[a,b]$, show that

$$\Pi_n(\Pi_n f) = \Pi_n f.$$

(Hint: What does $\Pi_n p_n$ equal if p_n is a polynomial of degree n?)

(b) Show that if $x_0 = a$ and $x_1 = b$, then $\|\Pi_0\|_\infty = \|\Pi_1\|_\infty = 1$.

(c) Recall that we can write the polynomial $p_n = \Pi_n f$ in the Lagrange form

$$\Pi_n f = \sum_{i=0}^{n} f(x_i)\ell_i(x),$$

where ℓ_i denotes the ith Lagrange basis polynomial. Prove that

$$\|\Pi_n\|_\infty = \max_{x \in [a,b]} \sum_{i=0}^{n} |\ell_i(x)|.$$

(d) Let p_* denote any polynomial of degree n (e.g., p_* minimizes $\|f - p\|_\infty$ over all $p \in \mathbb{P}_n$). Prove that

$$\|f - p_n\|_\infty \leq (1 + \|\Pi_n\|_\infty)\|f - p_*\|_\infty.$$

(e) Computationally estimate $\|\Pi_n\|_\infty$ for $n = 1, 2, \cdots, 20$ with (i) uniformly spaced points $x_i = -1 + 2i/n$, $i = 0, 1, \cdots, n$ and (ii) Chebyshev points $x_i = \cos(i\pi/n)$ over $[-1, 1]$.