Numerical Linear Algebra Assignment 5

Exercise 1. (10 points)

Let

$$\widehat{\mathbf{L}}_k = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & l_{kk} & & & \\ & & & l_{k+1,k} & 1 & & \\ & & & \vdots & & \ddots & \\ & & & l_{mk} & & & 1 \end{bmatrix}, \qquad \widehat{\mathbf{L}} = \begin{bmatrix} l_{11} & & & \\ l_{21} & l_{22} & & \\ \vdots & \vdots & \ddots & \\ l_{m1} & l_{m2} & \cdots & l_{mm} \end{bmatrix}$$

Prove that

$$\widehat{\mathbf{L}}_1 \widehat{\mathbf{L}}_2 \cdots \widehat{\mathbf{L}}_m = \widehat{\mathbf{L}}$$

by the same approach as we discussed in LU factorization for the lower triangular matrix $\mathbf{L}$.

Exercise 2. (TreBau Exercise 20.1, 10 points)

Let $\mathbf{A} \in \mathbb{C}^{m\times m}$ be nonsingular. Show that $\mathbf{A}$ has an LU factorization if and only if for each k with $1 \le k \le m$, the upper-left $k \times k$ block $\mathbf{A}_{1:k,1:k}$ is nonsingular. (Hints: The row operations of Gaussian elimination leave the determinants $\det(\mathbf{A}_{1:k,1:k})$ unchanged.) Prove that this LU factorization is unique.

Exercise 3. (TreBau Exercise 20.3, 10 points)

Suppose an $m\times m$ matrix $\mathbf{A}$ is written in the block form $\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$, where $\mathbf{A}_{11}$ is $n\times n$ and $\mathbf{A}_{22}$ is $(m-n)\times(m-n)$. Assume that $\mathbf{A}$ satisfies the condition of Exercise 2 (TreBau Exercise 20.1).

(a) Verify the formula

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12} \end{bmatrix}$$

for "elimination" of the block $\mathbf{A}_{21}$. The matrix $\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}$ is known as the *Schur complement* of $\mathbf{A}_{11}$ in $\mathbf{A}$.

(b) Suppose $\mathbf{A}_{21}$ is eliminated row by row by means of n steps of Gaussian elimination. Show that the bottom-right $(m-n)\times(m-n)$ block of the result is again $\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}$.

Exercise 4. (TreBau Exercise 21.3, 10 points)

Consider Gaussian elimination carried out with pivoting by columns instead of rows, leading to a factorization $\mathbf{AQ} = \mathbf{LU}$, where $\mathbf{Q}$ is a permutation matrix.

(a) Show that if $\mathbf{A}$ is nonsingular, such a factorization always exists.

(b) Show that if $\mathbf{A}$ is singular, such a factorization does not always exist.

Exercise 5. (TreBau Exercise 21.6, 10 points)

Suppose $\mathbf{A} \in \mathbb{C}^{m\times m}$ is *strictly column diagonally dominant*, which means that for each k,

$$|a_{kk}| > \sum_{j\neq k} |a_{jk}|.$$

Show that if Gaussian elimination with partial pivoting is applied to $\mathbf{A}$, no row interchanges take place.

Exercise 6. (TreBau Exercise 22.1, 10 points)

Show that for Gaussian elimination with partial pivoting applied to any matrix $\mathbf{A} \in \mathbb{C}^{m\times m}$, the growth factor satisfies $\rho \leq 2^{m-1}$.

Exercise 7. (TreBau Exercise 23.1, 10 points)

Let $\mathbf{A}$ be a nonsingular square matrix and let $\mathbf{A} = \mathbf{QR}$ and $\mathbf{A}^*\mathbf{A} = \mathbf{U}^*\mathbf{U}$ be QR and Cholesky factorizations, respectively, with the usual normalizations $r_{jj}, u_{jj} > 0$. Is it true or false that $\mathbf{R} = \mathbf{U}$? Explain your answer.

Exercise 8. (10 points)

Compute the Cholesky factorization of the matrix $\mathbf{A} = \begin{bmatrix} 2 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 3 & 1+\sqrt{2} \\ \sqrt{2} & 1+\sqrt{2} & 4 \end{bmatrix}$.

Exercise 9. (10 points)

Let

$$\mathcal{A} = \begin{bmatrix} \mathbf{A} & \mathbf{B}^* \\ \mathbf{C} & \mathbf{0} \end{bmatrix}, \quad \mathcal{P} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}\mathbf{A}^{-1}\mathbf{B}^* \end{bmatrix}$$

where $\mathbf{A} \in \mathbb{C}^{m\times m}$ is invertible, and $\mathbf{B}, \mathbf{C} \in \mathbb{C}^{n\times m}$ with $m \geq n$. Assume that the *Schur complement* $-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}^*$ is invertible. Prove that the matrix $\mathcal{T} = \mathcal{P}^{-1}\mathcal{A}$ is diagonalizable and has at most three distinct eigenvalues

$$1, \quad \frac{1}{2} \pm \frac{\sqrt{5}}{2}.$$

(Hint: consider the minimal polynomial of the matrix $\mathcal{T} = \mathcal{P}^{-1}\mathcal{A}$.)

Compulsory requirement for programming: Use Matlab's `publish` to save all your code, comments, and results to a PDF file. You must use the programming format files: example_format.zip.

Programming 1. (TreBau Exercises 20.2, 10 points)

Answer the question in Exercises 20.2 and write matlab codes to provide an example with $p = 3$ for a 20×20 matrix $\mathbf{A}$. Plot the sparisity patterns of $\mathbf{A}, \mathbf{L}$ and $\mathbf{U}$ by using matlab's `spy`.

Programming 2. (TreBau Exercises 20.4, 10 points)

Write two matlab functions, `[L,U]=gelu(A)` and `[L,U]=geoplu(A)`, to implement Algorithm 20.1 and the "outer product" form of Guassian elimination you have designed in Exercises 20.4, respectively. Compare the CPU times of `gelu` and `geoplu` for a 500×500 matrix $\mathbf{A}$.

Programming 3. (10 points)

Write a matlab function, `[L,U,P]=gepp(A)`, to implement Algorithm 21.1 of TreBau's book. Test the 4×4 complex matrix ($\mathrm{i} = \sqrt{-1}$)

$$\mathbf{A} = \begin{bmatrix} 1+1\mathrm{i} & -1\mathrm{i} & 0 & 1\mathrm{i} \\ 1 & 1+1\mathrm{i} & 1-1\mathrm{i} & 1+3\mathrm{i} \\ 0 & 1\mathrm{i} & -1\mathrm{i} & -1\mathrm{i} \\ 2\mathrm{i} & 1 & 0 & 0 \end{bmatrix}.$$

Programming 4. (10 points)

Write a matlab function, `R=mychol(A)`, to implement Algorithm 23.1 of TreBau's book. Test the 4×4 Hermitian positive definite matrix ($\mathrm{i} = \sqrt{-1}$)

$$\mathbf{A} = \begin{bmatrix} 7 & -2\mathrm{i} & 1-1\mathrm{i} & 2+4\mathrm{i} \\ 2\mathrm{i} & 5 & -1-2\mathrm{i} & 2+2\mathrm{i} \\ 1+1\mathrm{i} & -1+2\mathrm{i} & 3 & -1+4\mathrm{i} \\ 2-4\mathrm{i} & 2-2\mathrm{i} & -1-4\mathrm{i} & 12 \end{bmatrix}.$$

Programming 5. (10 points)

Write a matlab function, `[Q,R,P]=hqrp(A)`, via Householder reflectors, to compute the so-called QR factorization with column pivoting: `AP=QR`, where `Q` is unitary, `R` is upper triangular, `P` is a permutation matrix, and `abs(diag(R))` is decreasing. Test the 4×4 matrix in Programming 4.