Lecture 14: Matrix deviation inequality

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1. Gaussian width and Gaussian complexity

- Assume that $\boldsymbol{A}$ is an $m \times n$ random matrix whose rows are independent, mean zero, isotropic and sub-gaussian random vectors in $\mathbb{R}^n$. For example,

$$\texttt{A} = \texttt{randn}(\texttt{m}, \texttt{n}).$$

- For a fixed vector $\mathbf{x} \in \mathbb{R}^n$, we have

$$\begin{aligned}
\mathbb{E}\|\boldsymbol{A}\mathbf{x}\|_2^2 &= \mathbb{E}\sum_{i=1}^{m}(\boldsymbol{A}_{i,:}\mathbf{x})^2 = \sum_{i=1}^{m}\mathbb{E}(\boldsymbol{A}_{i,:}\mathbf{x})^2 \\
&= \sum_{i=1}^{m}\mathbf{x}^\top\mathbb{E}(\boldsymbol{A}_{i,:}^\top\boldsymbol{A}_{i,:})\mathbf{x} \\
&= m\|\mathbf{x}\|_2^2.
\end{aligned}$$

- If we assume that concentration about the mean holds here (and in fact, it does), we should expect that

$$\|\boldsymbol{A}\mathbf{x}\|_2 \approx \sqrt{m}\|\mathbf{x}\|_2$$

  with high probability. (Recall Johnson–Lindenstrauss Lemma)
- A natural problem is, for all $\mathbf{x}$ in some fixed set $\mathcal{T} \subset \mathbb{R}^n$, how large is the average uniform deviation:

$$\mathbb{E} \sup_{\mathbf{x} \in \mathcal{T}} \left| \|\boldsymbol{A}\mathbf{x}\|_2 - \sqrt{m}\|\mathbf{x}\|_2 \right| ?$$

- This quantity should clearly depend on some notion of the size of $\mathcal{T}$: the larger $\mathcal{T}$, the larger should the uniform deviation be.
- So, how can we quantify the size of $\mathcal{T}$ for this problem? Gaussian width: a geometric measure of the sizes of sets in $\mathbb{R}^n$.

- Definition. Let $\mathcal{T} \subset \mathbb{R}^n$ be a bounded set, and $\boldsymbol{g}$ be a standard normal random vector in $\mathbb{R}^n$, i.e. $\boldsymbol{g} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_n)$. Then the quantities

$$\omega(\mathcal{T}) := \mathbb{E} \sup_{\mathbf{x} \in \mathcal{T}} \langle \boldsymbol{g}, \mathbf{x} \rangle \quad \text{and} \quad \gamma(\mathcal{T}) := \mathbb{E} \sup_{\mathbf{x} \in \mathcal{T}} |\langle \boldsymbol{g}, \mathbf{x} \rangle|$$

  are called the *Gaussian width* of $\mathcal{T}$ and the *Gaussian complexity* of $\mathcal{T}$, respectively.

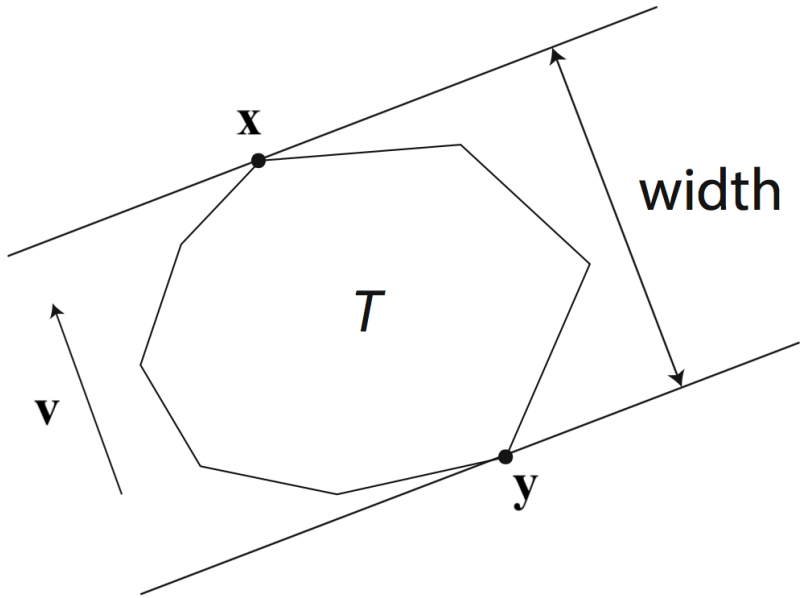
- Gaussian width and Gaussian complexity are closely related. Indeed, we have (see Vershynin's HDP book)

$$\begin{aligned} \omega(\mathcal{T} - \mathcal{T}) &= \mathbb{E} \sup_{\mathbf{x}, \mathbf{y} \in \mathcal{T}} \langle \boldsymbol{g}, \mathbf{x} - \mathbf{y} \rangle = 2\omega(\mathcal{T}) \\ &= \mathbb{E} \sup_{\mathbf{x}, \mathbf{y} \in \mathcal{T}} |\langle \boldsymbol{g}, \mathbf{x} - \mathbf{y} \rangle| \\ &= \gamma(\mathcal{T} - \mathcal{T}), \end{aligned}$$

  and if $\mathcal{T}$ contains the origin, then $\omega(\mathcal{T}) \leq \gamma(\mathcal{T}) \leq 2\omega(\mathcal{T})$.

- Geometric interpretation.

  Suppose $\mathbf{v}$ is a unit vector in $\mathbb{R}^n$. Then $\sup_{\mathbf{x},\mathbf{y}\in\mathcal{T}} \langle \mathbf{v}, \mathbf{x}-\mathbf{y}\rangle$ is simply the width of $\mathcal{T}$ in the direction of $\mathbf{v}$, i.e. the distance between the two hyperplanes with normal $\mathbf{v}$ that touch $\mathcal{T}$ on both sides as shown in the figure.



  For $\boldsymbol{g} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_n)$, we have $\mathbb{E}\|\boldsymbol{g}\|_2^2 = n$ and $\|\boldsymbol{g}\|_2 \approx \sqrt{n}$ with high probability. Then, $\omega(\mathcal{T})$ is approximately $\sqrt{n}/2$ larger than the usual, geometric width of $\mathcal{T}$ averaged over all directions.

- Gaussian complexity for the unit balls $\mathcal{B}_p^n = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_p \le 1\}$:

$$\gamma(\mathcal{B}_2^n) \sim \sqrt{n}, \quad \gamma(\mathcal{B}_1^n) \sim \sqrt{\log n}.$$

For any finite set $\mathcal{T} \subset \mathcal{B}_2^n$, we have $\gamma(\mathcal{T}) \lesssim \sqrt{\log |\mathcal{T}|}$.

2. Matrix deviation inequality

Theorem 1 (Matrix deviation inequality)

Let $\boldsymbol{A}$ be an $m \times n$ matrix whose rows $\boldsymbol{A}_{i,:}$ are independent, isotropic and sub-gaussian random vectors in $\mathbb{R}^n$. Let $\mathcal{T} \subset \mathbb{R}^n$ be a fixed bounded set. Then

$$\mathbb{E} \sup_{\mathbf{x} \in \mathcal{T}} \left| \|\boldsymbol{A}\mathbf{x}\|_2 - \sqrt{m}\|\mathbf{x}\|_2 \right| \le CK^2 \gamma(\mathcal{T})$$

where

$$K = \max_i \|\boldsymbol{A}_{i,:}\|_{\psi_2}$$

is the maximal sub-gaussian norm of the rows of $\boldsymbol{A}$.

2.1 Deriving Johnson–Lindenstrauss Lemma

- $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{T} = \{(\mathbf{x}-\mathbf{y})/\|\mathbf{x}-\mathbf{y}\|_2 : \ \mathbf{x}, \mathbf{y} \in \mathcal{X}\}$. Then we have

$$\gamma(\mathcal{T}) \lesssim \sqrt{\log|\mathcal{T}|} \leq \sqrt{\log|\mathcal{X}|^2} \lesssim \sqrt{\log|\mathcal{X}|}.$$

Matrix deviation inequality and $m \geq C\varepsilon^{-2}\log N$ then yield

$$\sup_{\mathbf{x},\mathbf{y}\in\mathcal{X}} \left| \frac{\|\boldsymbol{A}(\mathbf{x}-\mathbf{y})\|_2}{\|\mathbf{x}-\mathbf{y}\|_2} - \sqrt{m} \right| \lesssim \sqrt{\log N} \lesssim \varepsilon\sqrt{m}$$

with high probability, say 0.99. Multiplying both sides by $\|\mathbf{x}-\mathbf{y}\|_2/\sqrt{m}$, we can write the last bound as follows. With probability at least 0.99, we have

$$(1-\varepsilon)\|\mathbf{x}-\mathbf{y}\|_2 \leqslant \frac{1}{\sqrt{m}}\|\boldsymbol{A}\mathbf{x}-\boldsymbol{A}\mathbf{y}\|_2 \leqslant (1+\varepsilon)\|\mathbf{x}-\mathbf{y}\|_2$$

for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$.

3. Covariance estimation

- We already showed that $N \sim n \log n$ samples are enough to estimate the covariance matrix of a general distribution in $\mathbb{R}^n$.
- We can do better if the distribution is sub-gaussian: we can get rid of the logarithmic oversampling and the boundedness condition.

Theorem 2 (Covariance estimation for sub-gaussian distributions)

Let $\boldsymbol{X}$ be a random vector in $\mathbb{R}^n$ with covariance matrix Σ. Suppose $\boldsymbol{X}$ is sub-gaussian, mean zero, and more specifically for any $\mathbf{x} \in \mathbb{R}^n$

$$\|\langle \boldsymbol{X}, \mathbf{x} \rangle\|_{\psi_2} \lesssim \|\langle \boldsymbol{X}, \mathbf{x} \rangle\|_{L^2} = \|\Sigma^{1/2}\mathbf{x}\|_2.$$

Then, for every $N \geq 1$, we have

$$\mathbb{E}\|\Sigma_N - \Sigma\| \lesssim \|\Sigma\| \left(\sqrt{\frac{n}{N}} + \frac{n}{N} \right).$$

- This result shows $N \sim \varepsilon^{-2} n$ gives $\mathbb{E}\|\Sigma_N - \Sigma\| \lesssim \varepsilon \|\Sigma\|$.

3.1 Low-dimensional distributions

- We can show that much fewer samples are needed for covariance estimation of low-dimensional sub-gaussian distributions. We have

$$\mathbb{E}\|\Sigma_N - \Sigma\| \lesssim \|\Sigma\| \left(\sqrt{\frac{r}{N}} + \frac{r}{N} \right)$$

where

$$r = r(\Sigma^{1/2}) = \frac{\mathrm{tr}\Sigma}{\|\Sigma\|}$$

is the stable rank of $\Sigma^{1/2}$. This means that covariance estimation is possible with

$$N \sim r$$

samples.

4. Underdetermined linear systems

- Suppose we need to solve a severely underdetermined system of linear equations: say, we have m equations in $n \gg m$ variables

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{A} \in \mathbb{R}^{m \times n}, \quad \mathbf{b} \in \mathbb{R}^m.$$

- When the linear system is underdetermined, we can not find $\mathbf{x}$ with any accuracy, unless we know something extra about $\mathbf{x}$. So, let us assume that we do have some a-priori information. We can describe this situation mathematically by assuming that

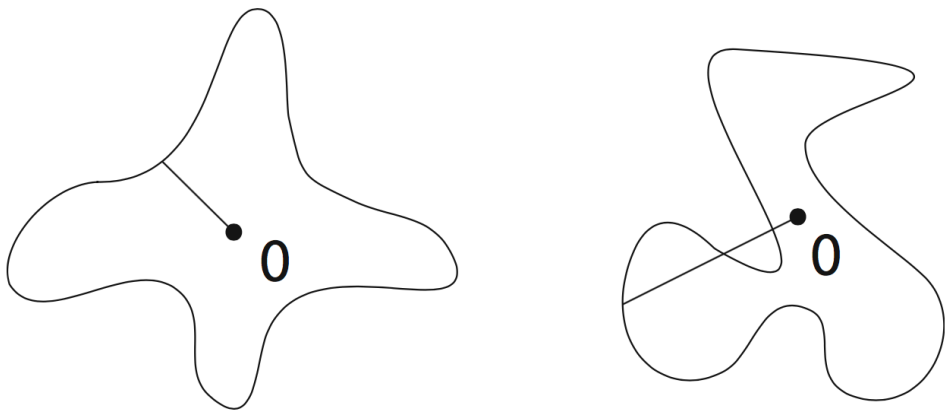
$$\mathbf{x} \in \mathcal{K}$$

  where $\mathcal{K} \subset \mathbb{R}^n$ is some known set in $\mathbb{R}^n$ that describes anything that we know about $\mathbf{x}$ a-priori.

- Summarizing, here is the problem we are trying to solve. Determine a solution $\mathbf{x} = \mathbf{x}(\mathbf{A}, \mathbf{b}, \mathcal{K})$ to the underdetermined linear equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ as accurately as possible, assuming that $\mathbf{x} \in \mathcal{K}$.

4.1 An optimization approach

- We convert the set $\mathcal{K}$ into a function on $\mathbb{R}^n$ which is called the Minkowski functional of $\mathcal{K}$. This is basically a function whose level sets are multiples of $\mathcal{K}$.
- To define it formally, assume that $\mathcal{K}$ is star-shaped, which means that together with any point $\mathbf{x}$, the set $\mathcal{K}$ must contain the entire interval that connects $\mathbf{x}$ with the origin; see the figure for illustration.



The set on the left (whose boundary is shown) is star-shaped, the set on the right is not.

- The Minkowski functional of $\mathcal{K}$ is defined as

$$\|\mathbf{x}\|_{\mathcal{K}} := \inf\{t > 0 : \mathbf{x}/t \in \mathcal{K}\}, \quad \mathbf{x} \in \mathbb{R}^n.$$

  If the set $\mathcal{K}$ is convex and symmetric about the origin, $\|\mathbf{x}\|_{\mathcal{K}}$ is actually a norm on $\mathbb{R}^n$. (Exercise)

- Now we propose the following way to solve the recovery problem: solve the optimization problem

$$\min \|\mathbf{x}\|_{\mathcal{K}} \quad \text{subject to} \quad \mathbf{b} = \mathbf{A}\mathbf{x}.$$

  It looks at all solutions to the equation $\mathbf{b} = \mathbf{A}\mathbf{x}$ and tries to "shrink" the solution $\mathbf{x}$ toward $\mathcal{K}$. (This is what minimization of Minkowski functional is about.)

- Also note that if $\mathcal{K}$ is convex, this is a convex optimization problem, and thus can be solved effectively by one of the many available numeric algorithms.

- The main question we should now be asking is – would the solution to this problem approximate the original vector $\mathbf{x}$? The following result bounds the approximation error for a probabilistic model of linear equations.

Theorem 3 (Recovery by optimization)

Assume that $\boldsymbol{A}$ is an $m \times n$ random matrix whose rows $\boldsymbol{A}_{i,:}$ are independent, mean zero, isotropic and sub-gaussian random vectors in $\mathbb{R}^n$. The solution $\widehat{\mathbf{x}}$ of the optimization problem satisfies

$$\mathbb{E}\|\widehat{\mathbf{x}} - \mathbf{x}\|_2 \lesssim \frac{\omega(\mathcal{K})}{\sqrt{m}},$$

where $\omega(\mathcal{K})$ is the Gaussian width of $\mathcal{K}$.

Proof. By $\mathbf{x} \in \mathcal{K}$ and the optimality, we have $\|\widehat{\mathbf{x}}\|_{\mathcal{K}} \leq \|\mathbf{x}\|_{\mathcal{K}} \leq 1$, which implies $\widehat{\mathbf{x}} \in \mathcal{K}$. By $\boldsymbol{A}\widehat{\mathbf{x}} = \boldsymbol{A}\mathbf{x} = \mathbf{b}$, we have $\boldsymbol{A}(\widehat{\mathbf{x}} - \mathbf{x}) = \mathbf{0}$.

Let us apply matrix deviation inequality for $\mathcal{T} := \mathcal{K} - \mathcal{K}$. It gives

$$\mathbb{E} \sup_{\mathbf{u},\mathbf{v} \in \mathcal{K}} \left| \|\boldsymbol{A}(\mathbf{u} - \mathbf{v})\|_2 - \sqrt{m}\|\mathbf{u} - \mathbf{v}\|_2 \right| \lesssim \gamma(\mathcal{T}) = 2\omega(\mathcal{K}).$$

Substitute $\mathbf{u} = \widehat{\mathbf{x}}$ and $\mathbf{v} = \mathbf{x}$ here. By $\boldsymbol{A}(\widehat{\mathbf{x}} - \mathbf{x}) = \mathbf{0}$, we get

$$\mathbb{E}\sqrt{m}\|\widehat{\mathbf{x}} - \mathbf{x}\|_2 \lesssim \omega(\mathcal{K}).$$

Dividing both sides by $\sqrt{m}$ we complete the proof. □

This theorem says that a solution vector $\mathbf{x} \in \mathcal{K}$ can be efficiently recovered from

$$m \sim \omega(\mathcal{K})^2$$

random linear measurements.

4.2 Sparse recovery

- Suppose we know that the signal $\mathbf{x}$ is sparse, which means that only a few coordinates of $\mathbf{x}$ are nonzero. As before, our task is to recover $\mathbf{x}$ from the random linear measurements given by the vector

$$\mathbf{b} = \boldsymbol{A}\mathbf{x},$$

  where $\boldsymbol{A}$ is an $m \times n$ random matrix.

- The number of nonzero entries of a vector $\mathbf{x} \in \mathbb{R}^n$, or the sparsity of $\mathbf{x}$, is often denoted $\|\mathbf{x}\|_0$. Recall that $\|\mathbf{x}\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$. You can quickly check that

$$\|\mathbf{x}\|_0 = \lim_{p \to 0} \|\mathbf{x}\|_p.$$

- Keep in mind that neither $\|\mathbf{x}\|_0$ nor $\|\mathbf{x}\|_p$ for $0 < p < 1$ are actually norms on $\mathbb{R}^n$, since they fail triangle inequality.

- The problem is:

$$\min \|\mathbf{x}\|_0 \quad \text{subject to} \quad \mathbf{b} = \boldsymbol{A}\mathbf{x},$$

  which selects the sparsest feasible solution. The function $\|\mathbf{x}\|_0$ is highly non-convex and even discontinuous. No known algorithm exists to solve this problem efficiently.

- To overcome this difficulty, we use

$$\min \|\mathbf{x}\|_1 \quad \text{subject to} \quad \mathbf{b} = \boldsymbol{A}\mathbf{x}.$$

  This is a convexification of the non-convex problem, and a variety of numeric convex optimization methods are available to solve it efficiently.

- We will now show that an s-sparse signal $\mathbf{x} \in \mathbb{R}^n$ can be efficiently recovered from $m \sim s \log n$ random linear measurements.

Theorem 4 (Sparse recovery by optimization)

Assume $\boldsymbol{A}$ *is a random matrix as in Theorem 3. If an unknown vector* $\mathbf{x} \in \mathbb{R}^n$ *has at most* s *non-zero coordinates, i.e.* $\|\mathbf{x}\|_0 \le s$*, then the solution* $\widehat{\mathbf{x}}$ *of the* ℓ_1 *optimization problem satisfies*

$$\mathbb{E}\|\widehat{\mathbf{x}} - \mathbf{x}\|_2 \lesssim \sqrt{(s \log n)/m}\|\mathbf{x}\|_2.$$

Proof. Cauchy–Schwarz inequality shows that $\|\mathbf{x}\|_1 \le \sqrt{s}\|\mathbf{x}\|_2$. Denote the unit ball of the ℓ_1 norm in $\mathbb{R}^n$ by $\mathcal{B}_1^n$. Then we can rewrite $\|\mathbf{x}\|_1 \le \sqrt{s}\|\mathbf{x}\|_2$ as the inclusion

$$\mathbf{x} \in \sqrt{s}\|\mathbf{x}\|_2 \cdot \mathcal{B}_1^n := \mathcal{K}.$$

By the Gaussian complexity $\gamma(\mathcal{B}_1^n) \lesssim \sqrt{\log n}$, we have

$$\omega(\mathcal{K}) = \sqrt{s}\|\mathbf{x}\|_2 \cdot \omega(\mathcal{B}_1^n) \le \sqrt{s}\|\mathbf{x}\|_2 \cdot \gamma(\mathcal{B}_1^n) \lesssim \sqrt{s}\|\mathbf{x}\|_2 \cdot \sqrt{\log n}.$$

Substitute this in Theorem 3 and complete the proof. □