

# Numerical Linear Algebra Assignment 3

## Exercise 1. (10 points)

- (1) Let  $\mathbf{P}$  be a projector. Given an explicit expression for the inverse of  $\lambda\mathbf{I} - \mathbf{P}$ , where  $\lambda \neq 0, 1$ .
- (2) Suppose  $\mathbf{A} \in \mathbb{C}^{m \times n}$  has a full SVD  $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^*$ , where

$$\mathbf{U} = [\mathbf{U}_r \quad \mathbf{U}_c], \quad \mathbf{V} = [\mathbf{V}_r \quad \mathbf{V}_c], \quad r = \text{rank}(\mathbf{A}).$$

What are the orthogonal projections onto  $\text{null}(\mathbf{A})^\perp$ ,  $\text{null}(\mathbf{A})$ ,  $\text{range}(\mathbf{A})$  and  $\text{range}(\mathbf{A})^\perp$ ?

## Exercise 2. (Carl D. Meyer, 10 points)

Let  $\mathbf{P}$  and  $\mathbf{Q}$  be projectors (oblique or orthogonal).

- (i) Prove that  $\text{range}(\mathbf{P}) = \text{range}(\mathbf{Q})$  if and only if  $\mathbf{PQ} = \mathbf{Q}$  and  $\mathbf{QP} = \mathbf{P}$ .
- (ii) Prove that  $\text{null}(\mathbf{P}) = \text{null}(\mathbf{Q})$  if and only if  $\mathbf{PQ} = \mathbf{P}$  and  $\mathbf{QP} = \mathbf{Q}$ .

## Exercise 3. (10 points)

Two subspaces  $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathbb{C}^m$  are called *complementary subspaces* if they satisfy

$$\mathcal{S}_1 \cap \mathcal{S}_2 = \{\mathbf{0}\}, \quad \mathcal{S}_1 + \mathcal{S}_2 = \mathbb{C}^m.$$

Let  $\mathcal{S}_1$  and  $\mathcal{S}_2$  be complementary subspaces. Prove that there exists a projector  $\mathbf{P}$  with

$$\text{range}(\mathbf{P}) = \mathcal{S}_1, \quad \text{null}(\mathbf{P}) = \mathcal{S}_2.$$

## Exercise 4. (TreBau Exercise 6.1, 10 points)

If  $\mathbf{P}$  is an orthogonal projector, then  $\mathbf{I} - 2\mathbf{P}$  is unitary. Prove this algebraically, and give a geometric interpretation.

## Exercise 5. (TreBau Exercise 6.5, 10 points)

Let  $\mathbf{P} \in \mathbb{C}^{m \times m}$  be a nonzero projector. Show that  $\|\mathbf{P}\|_2 \geq 1$ , with equality if and only if  $\mathbf{P}$  is an orthogonal projector.

## Exercise 6. (10 points)

Let  $\mathcal{S} \subseteq \mathbb{C}^m$  and  $\mathcal{T} \subseteq \mathbb{C}^m$ . Let  $\mathbf{P}_{\mathcal{S}}$  and  $\mathbf{P}_{\mathcal{T}}$  be orthogonal projectors onto  $\mathcal{S}$  and  $\mathcal{T}$ , respectively. Assume that  $\mathcal{S} \subseteq \mathcal{T}$ .

- (i) Prove that  $\mathbf{P}_{\mathcal{S}}\mathbf{P}_{\mathcal{T}} = \mathbf{P}_{\mathcal{T}}\mathbf{P}_{\mathcal{S}} = \mathbf{P}_{\mathcal{S}}$ .
- (ii) Prove that  $\mathbf{P}_{\mathcal{T}} - \mathbf{P}_{\mathcal{S}}$  is also an orthogonal projection.
- (iii)  $\text{range}(\mathbf{P}_{\mathcal{T}} - \mathbf{P}_{\mathcal{S}}) = ?$     $\text{null}(\mathbf{P}_{\mathcal{T}} - \mathbf{P}_{\mathcal{S}}) = ?$

## Exercise 7. (TreBau Exercise 7.3, 10 points)

Let  $\mathbf{A}$  be an  $m \times m$  matrix, and let  $\mathbf{a}_j$  be its  $j$ th column. Give an algebraic proof of *Hadamard's inequality*:

$$|\det(\mathbf{A})| \leq \prod_{j=1}^m \|\mathbf{a}_j\|_2.$$

Also give a geometric interpretation of this result, making use of the fact that the determinant equals the volume of a parallelepiped.

**Exercise 8. (TreBau Exercise 7.5, 10 points)**

Let  $\mathbf{A}$  be an  $m \times n$  matrix ( $m \geq n$ ), and let  $\mathbf{A} = \mathbf{Q}_n \mathbf{R}_n$  be a reduced QR factorization.

- Show that  $\mathbf{A}$  has rank  $n$  if and only if all the diagonal entries of  $\mathbf{R}_n$  are nonzero.
- Suppose  $\mathbf{R}_n$  has  $k$  nonzero diagonal entries for some  $k$  with  $0 \leq k \leq n$ . What does this imply about the rank of  $\mathbf{A}$ ? Exactly  $k$ ? At least  $k$ ? At most  $k$ ? Give a precise answer, and prove it.

**Exercise 9. (10 points)**

Compute a QR factorization of the matrix  $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ \sqrt{2} & 1 + \sqrt{2} & 1 \\ 1 & 2 & 1 \end{bmatrix}$ .

**Exercise 10. (Programming, TreBau Exercise 8.2, 10 points)****Additional Exercise 1.**

Let  $C[-1, 1]$  denote the linear space of real-valued continuous functions on  $[-1, 1]$  with the inner product

$$\forall f, g \in C[-1, 1], \quad \langle f, g \rangle_w = \int_{-1}^1 w(x) f(x) g(x) dx,$$

where  $w(x) \geq 0 (\neq 0)$  is a weight function (continuous). For the case  $w(x) = 1 + x^2$ , complete the following:

- Write Matlab code to compute the first six orthogonal (with respect to the inner product  $\langle \cdot, \cdot \rangle_w$ ) polynomials  $(P_j(x), j = 0, 1, 2, 3, 4, 5)$ , which are conventionally normalized so that  $P_j(1) = 1$ . Hint: you can use Matlab's symbolic toolbox. For your reference, the polynomials are given by:

P =

$$\begin{aligned} & 1 \\ & x \\ & (5x^2)/3 - 2/3 \\ & (14x^3)/5 - (9x)/5 \\ & (119x^4)/24 - (161x^2)/36 + 37/72 \\ & (1221x^5)/136 - (705x^3)/68 + (325x)/136 \end{aligned}$$

- Modify the code we used for discrete Legendre polynomials to plot the discrete polynomials corresponding to those obtained in (i).