Data Analysis and Matrix Computations Assignment 2

Exercise 1.

Let $\mathcal{T}$ denote a random subset of $\{1, 2, \ldots, N\}$ whose cardinality depends on the random integer variable T that takes values from 1 to N, where $\mathcal{T}$ is sampled uniformly. Let $\mathbf{D}_{\mathcal{T}}$ denote the diagonal random matrix formed by the summation of T canonical outer products

$$\mathbf{D}_{\mathcal{T}} = \sum_{i \in \mathcal{T}} \mathbf{e}_i \mathbf{e}_i^\top.$$

Prove that

$$\mathbb{E}(\mathbf{D}_{\mathcal{T}}) = \frac{\mathbb{E}(T)}{N} \mathbf{I}.$$

Exercise 2.

Consider the following relaxed RK algorithm.

Algorithm: Relaxed RK for $\mathbf{Ax} = \mathbf{b}$

Initialize $\mathbf{x}^0 \in \mathbb{R}^n$ and $0 < \alpha < 2$
for $k = 1, 2, \ldots,$ **do**
  Pick $i \in [m]$ with probability $\dfrac{\|\mathbf{A}_{i,:}\|_2^2}{\|\mathbf{A}\|_{\mathrm{F}}^2}$
  Set $\mathbf{x}^k = \mathbf{x}^{k-1} - \alpha \dfrac{\mathbf{A}_{i,:}\mathbf{x}^{k-1} - \mathbf{b}_i}{\|\mathbf{A}_{i,:}\|_2^2} (\mathbf{A}_{i,:})^\top$
end

Prove a convergence result of the relaxed RK for consistent linear systems.

Exercise 3.

Prove the convergence result of the REK algorithm:

$$\mathbb{E}\left[\|\mathbf{x}^k - \mathbf{x}_\star^0\|_2^2\right] \leq \rho^k \|\mathbf{x}^0 - \mathbf{x}_\star^0\|_2^2 + \frac{k\rho^k}{\|\mathbf{A}\|_{\mathrm{F}}^2} \|\mathbf{z}^0 - (\mathbf{I} - \mathbf{A}\mathbf{A}^\dagger)\mathbf{b}\|_2^2.$$