

# Regularized randomized iterative algorithms for factorized linear systems

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# Outline

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- ① Solutions of Linear Systems
- ② Randomized Iterative Algorithms
- ③ Factorized Linear System
- ④ The Proposed Algorithms
- ⑤ Computed Examples
- ⑥ Summary

# The Pseudoinverse Solution of a Linear System

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- Consider a linear system of equations

$$\mathbf{Ax} = \mathbf{b}, \quad \mathbf{A} \in \mathbb{R}^{m \times n}, \quad \mathbf{b} \in \mathbb{R}^m.$$

The system is called *consistent* if  $\mathbf{b} \in \text{range}(\mathbf{A})$ , otherwise, *inconsistent*.

- The pseudoinverse solution  $\mathbf{A}^\dagger \mathbf{b}$ , where  $\mathbf{A}^\dagger$  denotes the Moore–Penrose pseudoinverse of  $\mathbf{A}$ .

$\mathbf{Ax} = \mathbf{b}$	$\text{rank}(\mathbf{A})$	$\mathbf{A}^\dagger \mathbf{b}$
consistent	$= n$	unique solution
consistent	$< n$	unique minimum 2-norm solution
inconsistent	$= n$	unique least-squares (LS) solution
inconsistent	$< n$	unique minimum 2-norm LS solution

## Sparse (Least Squares) Solutions of a Linear System

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- Sparsest solutions:

$$\text{minimize } \|\mathbf{x}\|_0 \quad \text{s.t.} \quad \mathbf{Ax} = \mathbf{b}$$

- The basis pursuit problem:

$$\text{minimize } \|\mathbf{x}\|_1 \quad \text{s.t.} \quad \mathbf{Ax} = \mathbf{b}$$

- The regularized basis pursuit problem

$$\text{minimize } \frac{1}{2}\|\mathbf{x}\|_2^2 + \lambda\|\mathbf{x}\|_1 \quad \text{s.t.} \quad \mathbf{Ax} = \mathbf{b}$$

- Sparse least squares solutions: replacing  $\mathbf{Ax} = \mathbf{b}$  with the normal equations

$$\mathbf{A}^\top \mathbf{Ax} = \mathbf{A}^\top \mathbf{b}.$$

## Sparsity-Promoting Property of $\ell_1$ Norm

- Comparison of  $\ell_0$ ,  $\ell_1$ , and  $\ell_2$  norms

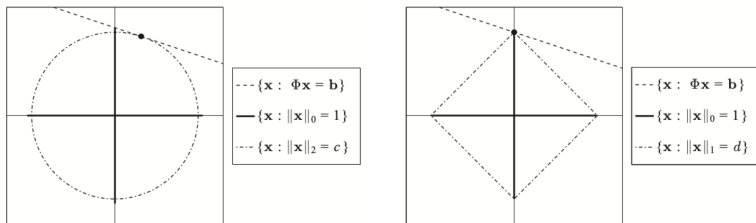


Figure 1.2 of [BL13]: Two-dimensional  $\ell_0$ ,  $\ell_1$ , and  $\ell_2$  balls and the solution set  $\{\mathbf{x} \mid \Phi \mathbf{x} = \mathbf{b}\}$ . Here  $c$  and  $d$  are constants with  $c$  a bit less than  $d$ . Note that the set  $\{\mathbf{x} \mid \|\mathbf{x}\|_0 = 1\}$  coincides with the coordinate axes.

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[BL13] K. Bryan and T. Leise. *Making Do with Less: An Introduction to Compressed Sensing*. SIAM Review, 55(3):547–566, 2013

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## Randomized Kaczmarz (RK)

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The RK algorithm for solving  $\mathbf{Ax} = \mathbf{b}$  [SV09]

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**Input:**  $\mathbf{A} \in \mathbb{R}^{m \times n}$ ,  $\mathbf{b} \in \mathbb{R}^m$ , and maximum number of iterations `maxit`.

**Output:** an approximation of the solution of  $\mathbf{Ax} = \mathbf{b}$ .

**Initialize:**  $\mathbf{x}^0 \in \mathbb{R}^n$ .

**for**  $k = 1, 2, \dots, \text{maxit}$  **do**

    Pick  $i \in [m]$  with probability  $\|\mathbf{A}_{i,:}\|_2^2 / \|\mathbf{A}\|_F^2$

    Set  $\mathbf{x}^k = \mathbf{x}^{k-1} - \frac{\mathbf{A}_{i,:}\mathbf{x}^{k-1} - b_i}{\|\mathbf{A}_{i,:}\|_2^2}(\mathbf{A}_{i,:})^\top$

**end**

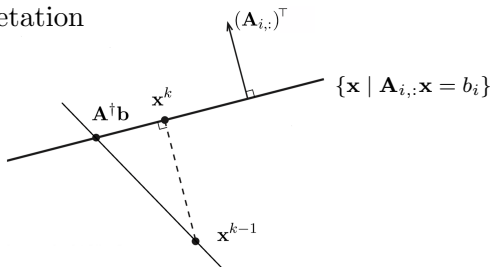
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[SV09] T. Strohmer and R. Vershynin. *A randomized Kaczmarz algorithm with exponential convergence*. J. Fourier Anal. Appl., 15(2):262–278, 2009.

# Geometric Interpretation and Convergence of RK

- Geometric interpretation



- Suppose that  $\mathbf{b} \in \text{range}(\mathbf{A})$ . The convergence result:

$$\mathbb{E} \left[ \|\mathbf{x}^k - \mathbf{x}_\star^0\|_2^2 \right] \leq \rho^k \|\mathbf{x}^0 - \mathbf{x}_\star^0\|_2^2,$$

where  $\rho = 1 - \frac{\sigma_{\min}^2(\mathbf{A})}{\|\mathbf{A}\|_F^2}$ ,  $\mathbf{x}_\star^0 = (\mathbf{I} - \mathbf{A}^\dagger \mathbf{A})\mathbf{x}^0 + \mathbf{A}^\dagger \mathbf{b}$ .

- RK fails to find least squares solutions for inconsistent case [Needell10].



## Randomized Gauss–Seidel (RGS)

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The RGS algorithm for solving  $\min_{\mathbf{x}} \|\mathbf{b} - \mathbf{Ax}\|_2$  [LL10]

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**Input:**  $\mathbf{A} \in \mathbb{R}^{m \times n}$ ,  $\mathbf{b} \in \mathbb{R}^m$ , and maximum number of iterations `maxit`.

**Output:** an approximation of the solution of  $\min_{\mathbf{x}} \|\mathbf{b} - \mathbf{Ax}\|_2$ .

**Initialize:**  $\mathbf{x}^0 \in \mathbb{R}^n$ .

**for**  $k = 1, 2, \dots, \text{maxit}$  **do**

    Pick  $j \in [n]$  with probability  $\|\mathbf{A}_{:,j}\|_2^2 / \|\mathbf{A}\|_F^2$

    Set  $\mathbf{x}^k = \mathbf{x}^{k-1} + \frac{(\mathbf{A}_{:,j})^\top (\mathbf{b} - \mathbf{Ax}^{k-1})}{\|\mathbf{A}_{:,j}\|_2^2} \mathbf{I}_{:,j}$

**end**

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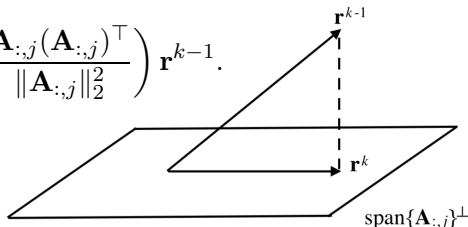
[LL10] D. J. Leventhal and A. S. Lewis. *Randomized methods for linear constraints: convergence rates and conditioning*. Math. Oper. Res., 35(3):641–654, 2010.

# Geometric Interpretation and Convergence of RGS

- Geometric interpretation

The residual  $\mathbf{r}^k = \left( \mathbf{I} - \frac{\mathbf{A}_{:,j}(\mathbf{A}_{:,j})^\top}{\|\mathbf{A}_{:,j}\|_2^2} \right) \mathbf{r}^{k-1}$ .

Here  $\mathbf{r}^k := \mathbf{b} - \mathbf{A}\mathbf{x}^k$ .



- For arbitrary  $\mathbf{A} \in \mathbb{R}^{m \times n}$  and  $\mathbf{b} \in \mathbb{R}^m$ , the convergence result:

$$\mathbb{E} \left[ \|\mathbf{A}\mathbf{x}^k - \mathbf{A}\mathbf{A}^\dagger \mathbf{b}\|_2^2 \right] \leq \left( 1 - \frac{\sigma_{\min}^2(\mathbf{A})}{\|\mathbf{A}\|_F^2} \right)^k \|\mathbf{A}(\mathbf{x}^0 - \mathbf{A}^\dagger \mathbf{b})\|_2^2.$$

- RGS finds a least squares solution, but usually not the minimum  $\ell_2$  norm one for rank-deficient case [MNR15].

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[MNR15] A. Ma, D. Needell, and A. Ramdas. *Convergence properties of the randomized extended Gauss–Seidel and Kaczmarz methods*. SIAM J. Matrix Anal. Appl., 36(4):1590–1604, 2015.

## Randomized Extended Kaczmarz (REK)

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- The normal equations  $\mathbf{A}^\top \mathbf{A} \mathbf{x} = \mathbf{A}^\top \mathbf{b}$  can be written as

$$\mathbf{A}^\top \mathbf{z} = \mathbf{0}, \quad \mathbf{A} \mathbf{x} = \mathbf{b} - \mathbf{z}.$$

- RK for  $\mathbf{A}^\top \mathbf{z} = \mathbf{0}$  with  $\mathbf{z}^0 \in \mathbf{b} + \text{range}(\mathbf{A})$  yields  $\{\mathbf{z}^k\}_0^\infty$  satisfying

$$\mathbf{z}^k \rightarrow (\mathbf{I} - \mathbf{A} \mathbf{A}^\dagger) \mathbf{b} \quad \text{as } k \rightarrow \infty.$$

Then  $\mathbf{A} \mathbf{x} = \mathbf{b} - \mathbf{z}^k \rightarrow \mathbf{A} \mathbf{x} = \mathbf{A} \mathbf{A}^\dagger \mathbf{b}$ , which is consistent.

- REK [ZF13] solves  $\mathbf{A}^\top \mathbf{A} \mathbf{x} = \mathbf{A}^\top \mathbf{b}$  via intertwining an iterate of RK on  $\mathbf{A}^\top \mathbf{z} = \mathbf{0}$  with an iterate of RK on  $\mathbf{A} \mathbf{x} = \mathbf{b} - \mathbf{z}^k$ :

$$\begin{aligned} \mathbf{z}^k &= \mathbf{z}^{k-1} - \frac{(\mathbf{A}_{:,j})^\top \mathbf{z}^{k-1}}{\|\mathbf{A}_{:,j}\|_2^2} \mathbf{A}_{:,j}, \\ \mathbf{x}^k &= \mathbf{x}^{k-1} - \frac{\mathbf{A}_{i,:} \mathbf{x}^{k-1} - \mathbf{b}_i + \mathbf{z}_i^k}{\|\mathbf{A}_{i,:}\|_2^2} (\mathbf{A}_{i,:})^\top. \end{aligned}$$

## Convergence of REK

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- The convergence result [Du19]:  $\forall \mathbf{z}^0 \in \mathbf{b} + \text{range}(\mathbf{A})$  and  $\mathbf{x}^0$

$$\mathbb{E} \left[ \|\mathbf{x}^k - \mathbf{x}_\star^0\|_2^2 \right] \leq \rho^k \|\mathbf{x}^0 - \mathbf{x}_\star^0\|_2^2 + \frac{k\rho^k}{\|\mathbf{A}\|_F^2} \|\mathbf{z}^0 - (\mathbf{I} - \mathbf{A}\mathbf{A}^\dagger)\mathbf{b}\|_2^2,$$

where

$$\rho = 1 - \frac{\sigma_{\min}^2(\mathbf{A})}{\|\mathbf{A}\|_F^2}, \quad \mathbf{x}_\star^0 = (\mathbf{I} - \mathbf{A}^\dagger \mathbf{A})\mathbf{x}^0 + \mathbf{A}^\dagger \mathbf{b}.$$

- REK works for arbitrary (consistent or inconsistent) linear systems (no assumptions about the dimensions or rank of  $\mathbf{A}$ ).
- REK is an RK-RK approach.

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[Du19] K. Du. *Tight upper bounds for the convergence of the randomized extended Kaczmarz and Gauss–Seidel algorithms*. Numer. Linear Algebra Appl., 26(3):e2233, 14pp, 2019.

## Randomized Extended Gauss–Seidel (REGS)

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- RGS with arbitrary  $\mathbf{z}^0$  for  $\min_{\mathbf{z}} \|\mathbf{b} - \mathbf{A}\mathbf{z}\|_2$  gives  $\mathbf{z}^k$  satisfying

$$\mathbf{A}\mathbf{z}^k \rightarrow \mathbf{A}\mathbf{A}^\dagger\mathbf{b} \quad \text{as } k \rightarrow \infty.$$

- REGS [MNR15] solves  $\min_{\mathbf{x}} \|\mathbf{b} - \mathbf{A}\mathbf{x}\|_2$  via intertwining an iterate of RGS on  $\min_{\mathbf{z}} \|\mathbf{b} - \mathbf{A}\mathbf{z}\|_2$  with an iterate of RK on  $\mathbf{A}\mathbf{x} = \mathbf{A}\mathbf{z}^k$ : [Du19]

$$\begin{aligned}\mathbf{z}^k &= \mathbf{z}^{k-1} - \frac{(\mathbf{A}_{:,j})^\top (\mathbf{A}\mathbf{z}^{k-1} - \mathbf{b})}{\|\mathbf{A}_{:,j}\|_2^2} \mathbf{I}_{:,j}, \\ \mathbf{x}^k &= \mathbf{x}^{k-1} - \frac{\mathbf{A}_{i,:}(\mathbf{x}^{k-1} - \mathbf{z}^k)}{\|\mathbf{A}_{i,:}\|_2^2} (\mathbf{A}_{i,:})^\top.\end{aligned}$$

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[MNR15] A. Ma, D. Needell, and A. Ramdas. *Convergence properties of the randomized extended Gauss–Seidel and Kaczmarz methods*. SIAM J. Matrix Anal. Appl., 36(4):1590–1604, 2015.

[Du19] K. Du. *Tight upper bounds for the convergence of the randomized extended Kaczmarz and Gauss–Seidel algorithms*. Numer. Linear Algebra Appl., 26(3):e2233, 14pp, 2019.

## Convergence of REGS

---

- The convergence result [Du19]:  $\forall \mathbf{z}^0$  and  $\mathbf{x}^0$ ,

$$\mathbb{E} \left[ \|\mathbf{x}^k - \mathbf{x}_\star^0\|_2^2 \right] \leq \rho^k \|\mathbf{x}^0 - \mathbf{x}_\star^0\|_2^2 + \frac{k\rho^k}{\|\mathbf{A}\|_F^2} \|\mathbf{A}(\mathbf{z}^0 - \mathbf{A}^\dagger \mathbf{b})\|_2^2,$$

where

$$\rho = 1 - \frac{\sigma_{\min}^2(\mathbf{A})}{\|\mathbf{A}\|_F^2}, \quad \mathbf{x}_\star^0 = (\mathbf{I} - \mathbf{A}^\dagger \mathbf{A})\mathbf{x}^0 + \mathbf{A}^\dagger \mathbf{b}.$$

- REGS works for arbitrary (consistent or inconsistent) linear systems (no assumptions about the dimensions or rank of  $\mathbf{A}$ ).
- REGS is an RGS-RK approach.

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[Du19] K. Du. *Tight upper bounds for the convergence of the randomized extended Kaczmarz and Gauss–Seidel algorithms*. Numer. Linear Algebra Appl., 26(3):e2233, 14pp, 2019.

## Convex Optimization Basics [Beck17]

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- **Subdifferential:** For a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , its subdifferential at  $\mathbf{x} \in \mathbb{R}^n$  is defined as

$$\partial f(\mathbf{x}) := \{\mathbf{z} \in \mathbb{R}^n \mid f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \mathbf{z}, \mathbf{y} - \mathbf{x} \rangle, \quad \forall \mathbf{y} \in \mathbb{R}^n\}.$$

- **$\gamma$ -strong convexity:** A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is called  $\gamma$ -strongly convex for a given  $\gamma > 0$  if the following inequality holds for all  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$  and  $\mathbf{z} \in \partial f(\mathbf{x})$ :

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \mathbf{z}, \mathbf{y} - \mathbf{x} \rangle + \frac{\gamma}{2} \|\mathbf{y} - \mathbf{x}\|_2^2.$$

As an example, the function  $f(\mathbf{x}) = \frac{1}{2} \|\mathbf{x}\|_2^2 + \lambda \|\mathbf{x}\|_1$  with  $\lambda \geq 0$  is 1-strongly convex.

## Convex Optimization Basics [Beck17]

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- **Conjugate function:** The conjugate function of  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  at  $\mathbf{x} \in \mathbb{R}^n$  is defined as

$$f^*(\mathbf{x}) := \sup_{\mathbf{y} \in \mathbb{R}^n} \{\langle \mathbf{x}, \mathbf{y} \rangle - f(\mathbf{y})\}.$$

If  $f(\mathbf{x})$  is  $\gamma$ -strongly convex, then  $f^*(\mathbf{x})$  is differentiable, and

$$\mathbf{z} \in \partial f(\mathbf{x}) \Leftrightarrow \mathbf{x} = \nabla f^*(\mathbf{z}).$$

- **Bregman distance:** For a convex function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , the Bregman distance between  $\mathbf{x}$  and  $\mathbf{y}$  with respect to  $f$  and  $\mathbf{z} \in \partial f(\mathbf{x})$  is defined as

$$D_{f,\mathbf{z}}(\mathbf{x}, \mathbf{y}) := f(\mathbf{y}) - f(\mathbf{x}) - \langle \mathbf{z}, \mathbf{y} - \mathbf{x} \rangle.$$

If  $f$  is  $\gamma$ -strongly convex, then it holds that

$$D_{f,\mathbf{z}}(\mathbf{x}, \mathbf{y}) \geq \frac{\gamma}{2} \|\mathbf{x} - \mathbf{y}\|_2^2.$$



# A Linear Equality Constrained Minimization Problem

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- Consider the linear equality constrained minimization problem

$$\text{minimize } f(\mathbf{x}), \quad \text{s.t. } \mathbf{Ax} = \mathbf{b},$$

where the objective function  $f$  is  $\gamma$ -strongly convex and the constraint  $\mathbf{Ax} = \mathbf{b}$  is consistent.

- The solution of the minimization problem is unique. The objective function  $f$  contains regularization terms for promoting certain structures of the underlying solutions.
- By combining the RK algorithm and the gradient of the conjugate function  $f^*$ , one obtains the regularized randomized Kaczmarz (RRK) algorithm [SL19][CQ21].

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[SL19] F. Schöpfer and D. A. Lorenz. *Linear convergence of the randomized sparse Kaczmarz method*. Math. Program., 173(1-2,Ser.A):509–536, 2019.

[CQ21] X. Chen and J. Qin. *Regularized Kaczmarz algorithms for tensor recovery*. SIAM J. Imaging Sci., 14(4):1439–1471, 2021.

## Regularized Randomized Kaczmarz (RRK)

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The RRK algorithm for solving  $\min_{\mathbf{x}} f(\mathbf{x})$  s.t.  $\mathbf{Ax} = \mathbf{b}$

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**Input:**  $\mathbf{A} \in \mathbb{R}^{m \times n}$ ,  $\mathbf{b} \in \mathbb{R}^m$ , and maximum number of iterations `maxit`.

**Output:** an approximation of the solution of  $\min_{\mathbf{Ax}=\mathbf{b}} f(\mathbf{x})$ .

**Initialize:**  $\mathbf{z}^0 \in \text{range}(\mathbf{A}^\top)$  and  $\mathbf{x}^0 = \nabla f^*(\mathbf{z}^0)$ .

**for**  $k = 1, 2, \dots, \text{maxit}$  **do**

    Pick  $i \in [m]$  with probability  $\|\mathbf{A}_{i,:}\|_2^2 / \|\mathbf{A}\|_F^2$

    Set  $\mathbf{z}^k = \mathbf{z}^{k-1} - \frac{\mathbf{A}_{i,:}\mathbf{x}^{k-1} - b_i}{\|\mathbf{A}_{i,:}\|_2^2} (\mathbf{A}_{i,:})^\top$

    Set  $\mathbf{x}^k = \nabla f^*(\mathbf{z}^k)$

**end**

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## Convergence of RRK [CQ21]

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- Assume that the objective function  $f$  is  $\gamma$ -strongly convex. Let  $\mathbf{x}_\star$  be the unique solution. If  $\gamma > 1/2$ , and for all  $\mathbf{x} \in \mathbb{R}^n$  and  $\mathbf{z} \in \partial f(\mathbf{x}) \cap \text{range}(\mathbf{A}^\top)$ ,

$$D_{f,\mathbf{z}}(\mathbf{x}, \mathbf{x}_\star) \leq \frac{1}{\nu_0} \|\mathbf{A}(\mathbf{x} - \mathbf{x}_\star)\|_2^2,$$

then for all  $\mathbf{z}^0 \in \text{range}(\mathbf{A}^\top)$ , the sequences  $\{\mathbf{x}^k\}$  and  $\{\mathbf{z}^k\}$  in the RRK algorithm satisfy

$$\mathbb{E} \left[ D_{f,\mathbf{z}^k}(\mathbf{x}^k, \mathbf{x}_\star) \right] \leq \beta_0^k D_{f,\mathbf{z}^0}(\mathbf{x}^0, \mathbf{x}_\star)$$

with

$$\beta_0 = 1 - \frac{(2\gamma - 1)\nu_0}{2\gamma \|\mathbf{B}\|_F^2}.$$

It follows that

$$\mathbb{E} \left[ \|\mathbf{x}^k - \mathbf{x}_\star\|_2^2 \right] \leq \beta_0^k \frac{2}{\gamma} D_{f,\mathbf{z}^0}(\mathbf{x}^0, \mathbf{x}_\star).$$

## Special Cases of RRG: RK and RSK

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- Case 1:  $f(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|_2^2$ . We have

$$\nabla f^*(\mathbf{x}) = \mathbf{x}.$$

The RRG algorithm becomes the RK algorithm.

- Case 2:  $f(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|_2^2 + \lambda\|\mathbf{x}\|_1$  with  $\lambda > 0$ . We have

$$\nabla f^*(\mathbf{x}) = S_\lambda(\mathbf{x}),$$

where  $S_\lambda(\mathbf{x})$  is the soft shrinkage function defined component-wise as

$$(S_\lambda(\mathbf{x}))_i = \max\{|x_i| - \lambda, 0\}\text{sign}(x_i).$$

The RRG algorithm becomes the randomized sparse Kaczmarz (RSK) algorithm [SL19].

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[SL19] F. Schöpfer and D. A. Lorenz. *Linear convergence of the randomized sparse Kaczmarz method*. Math. Program., 173(1-2,Ser.A):509–536, 2019.

## RRK: RCD for Dual Problem [Petra15]

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- The dual problem of  $\min_{\mathbf{Ax}=\mathbf{b}} f(\mathbf{x})$  with  $\mathbf{b} \in \text{range}(\mathbf{A})$  is the unconstrained problem

$$\min_{\mathbf{y} \in \mathbb{R}^m} g(\mathbf{y}) := f^*(\mathbf{A}^\top \mathbf{y}) - \langle \mathbf{y}, \mathbf{b} \rangle.$$

The gradient of  $g(\mathbf{y})$  is  $\nabla g(\mathbf{y}) = \mathbf{A} \nabla f^*(\mathbf{A}^\top \mathbf{y}) - \mathbf{b}$ .

The strong duality holds. The primal variable  $\mathbf{x}$  and the dual variable  $\mathbf{y}$  are related through the relation  $\mathbf{x} = \nabla f^*(\mathbf{A}^\top \mathbf{y})$ .

- Randomized coordinate descent (RCD) algorithm:

$$\mathbf{y}^k = \mathbf{y}^{k-1} - \frac{\mathbf{A}_{i,:} \nabla f^*(\mathbf{A}^\top \mathbf{y}^{k-1}) - b_i}{\|\mathbf{A}_{i,:}\|_2^2} \mathbf{I}_{:,i}$$

Introducing  $\mathbf{z}^k = \mathbf{A}^\top \mathbf{y}^k$  and  $\mathbf{x}^k = \nabla f^*(\mathbf{z}^k)$  yields RRK.

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[Petra15] S. Petra. *Randomized sparse block Kaczmarz as randomized dual block-coordinate descent*. An. Ştiinţ. Univ. “Ovidius” Constanţa Ser. Mat., 23(3):129–149, 2015.

## A Combined Optimization Problem

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- Consider the combined optimization problem:

$$\text{minimize } f(\mathbf{x}), \quad \text{s.t. } \mathbf{x} \in \underset{\mathbf{z} \in \mathbb{R}^n}{\operatorname{argmin}} \|\mathbf{b} - \mathbf{A}\mathbf{z}\|_2.$$

- The normal equations  $\mathbf{A}^\top \mathbf{A} \mathbf{x} = \mathbf{A}^\top \mathbf{b}$  can be written as

$$\mathbf{A}^\top \mathbf{y} = \mathbf{0}, \quad \mathbf{A} \mathbf{x} = \mathbf{b} - \mathbf{y}.$$

- An RK-RRK approach:

$$\begin{aligned} \mathbf{y}^k &= \mathbf{y}^{k-1} - \frac{(\mathbf{A}_{:,j})^\top \mathbf{y}^{k-1}}{\|\mathbf{A}_{:,j}\|_2^2} \mathbf{A}_{:,j}, \\ \mathbf{z}^k &= \mathbf{z}^{k-1} - \frac{\mathbf{A}_{i,:} \mathbf{x}^{k-1} - b_i + y_i^k}{\|\mathbf{A}_{i,:}\|_2^2} (\mathbf{A}_{i,:})^\top, \\ \mathbf{x}^k &= \nabla f^*(\mathbf{z}^k), \end{aligned}$$

with initial iterates

$$\mathbf{y}^0 \in \mathbf{b} + \operatorname{range}(\mathbf{A}), \quad \mathbf{z}^0 \in \operatorname{range}(\mathbf{A}^\top), \quad \mathbf{x}^0 = \nabla f^*(\mathbf{z}^0).$$

## Special Cases: REK and ExSRK

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- Case 1: For  $f(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|_2^2$ , by  $\nabla f^*(\mathbf{x}) = \mathbf{x}$ , we obtain the REK algorithm.
- Case 2: For  $f(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|_2^2 + \lambda\|\mathbf{x}\|_1$  with  $\lambda > 0$ , by

$$\nabla f^*(\mathbf{x}) = S_\lambda(\mathbf{x}),$$

we obtain the extended sparse randomized Kaczmarz (ExSRK) algorithm [SLTW22]:

$$\begin{aligned}\mathbf{y}^k &= \mathbf{y}^{k-1} - \frac{(\mathbf{A}_{:,j})^\top \mathbf{y}^{k-1}}{\|\mathbf{A}_{:,j}\|_2^2} \mathbf{A}_{:,j}, \\ \mathbf{z}^k &= \mathbf{z}^{k-1} - \frac{\mathbf{A}_{i,:} \mathbf{x}^{k-1} - b_i + y_i^k}{\|\mathbf{A}_{i,:}\|_2^2} (\mathbf{A}_{i,:})^\top, \\ \mathbf{x}^k &= S_\lambda(\mathbf{z}^k)\end{aligned}$$

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[SLTW22] F. Schöpfer, D. A. Lorenz, L. Tondji, and M. Winkler. *Extended randomized Kaczmarz method for sparse least squares and impulsive noise problems*. arXiv:2201.08620, 2022.

## Analogue: RSEGS

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- RGS with arbitrary  $\mathbf{y}^0$  for  $\min_{\mathbf{y}} \|\mathbf{b} - \mathbf{A}\mathbf{y}\|_2$  gives  $\mathbf{y}^k$  satisfying

$$\mathbf{A}\mathbf{y}^k \rightarrow \mathbf{A}\mathbf{A}^\dagger \mathbf{b} \quad \text{as } k \rightarrow \infty.$$

- A randomized sparse extended Gauss–Seidel (RSEGS) algorithm:

$$\begin{aligned}\mathbf{y}^k &= \mathbf{y}^{k-1} - \frac{(\mathbf{A}_{:,j})^\top (\mathbf{A}\mathbf{y}^{k-1} - \mathbf{b})}{\|\mathbf{A}_{:,j}\|_2^2} \mathbf{I}_{:,j}, \\ \mathbf{z}^k &= \mathbf{z}^{k-1} - \frac{\mathbf{A}_{i,:}(\mathbf{x}^{k-1} - \mathbf{y}^k)}{\|\mathbf{A}_{i,:}\|_2^2} (\mathbf{A}_{i,:})^\top, \\ \mathbf{x}^k &= S_\lambda(\mathbf{z}^k).\end{aligned}$$

with initial iterates

$$\mathbf{y}^0 \in \mathbb{R}^n, \quad \mathbf{z}^0 \in \text{range}(\mathbf{A}^\top), \quad \mathbf{x}^0 = \nabla f^*(\mathbf{z}^0).$$



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## Factorized Linear System

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- Consider a factorized linear system

$$\mathbf{A}\mathbf{B}\mathbf{x} = \mathbf{b},$$

where

$$\mathbf{A} \in \mathbb{R}^{m \times \ell}, \quad \mathbf{B} \in \mathbb{R}^{\ell \times n}, \quad \text{rank}(\mathbf{A}) = \text{rank}(\mathbf{B}) = \ell, \quad \mathbf{b} \in \mathbb{R}^m.$$

- The factorized linear system can be written as two individual subsystems

$$\mathbf{A}\mathbf{y} = \mathbf{b} \quad (\text{possibly inconsistent})$$

and

$$\mathbf{B}\mathbf{x} = \mathbf{y}. \quad (\text{always consistent})$$

- Is it feasible to solve each subsystem separately?

# RIAs for Factorized Linear Systems

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- Motivated by REK and REGS, interlaced randomized algorithms are proposed for solving factorized linear systems.
- The approach: ALG1-ALG2, where ALG1 is the algorithm used to solve subsystem  $\mathbf{A}\mathbf{y} = \mathbf{b}$  and ALG2 is the algorithm used to solve subsystem  $\mathbf{B}\mathbf{x} = \mathbf{y}$ . For example,
  - (1) The RK-RK algorithm [MNR18]
  - (2) The REK-RK algorithm [MNR18]
  - (3) The RGS-RK algorithm [ZWZ22]All find the minimum  $\ell_2$  norm (least squares) solution.
- How to find sparse solutions?

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[MNR18] A. Ma, D. Needell, and A. Ramdas. *Iterative methods for solving factorized linear systems*. SIAM J. Matrix Anal. Appl., 39(1):104–122, 2018.

[ZWZ22] J. Zhao, X. Wang, and J. Zhang. *A randomised iterative method for solving factorised linear systems*. Linear Multilinear Algebra, to appear, 2022.

# Outline

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- ① Solutions of Linear Systems
- ② Randomized Iterative Algorithms
- ③ Factorized Linear System
- ④ The Proposed Algorithms
- ⑤ Computed Examples
- ⑥ Summary

## A Combined Optimization Problem

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- Consider the combined optimization problem:

$$\text{minimize } f(\mathbf{x}), \quad \text{s.t. } \mathbf{x} \in \underset{\mathbf{z} \in \mathbb{R}^n}{\operatorname{argmin}} \|\mathbf{b} - \mathbf{A}\mathbf{B}\mathbf{z}\|_2.$$

- We consider the **ALG1-ALG2** approach. Specifically, we interlace the RK algorithm or the RGS algorithm for the subsystem

$$\mathbf{A}\mathbf{y} = \mathbf{b}$$

with the RRK algorithm for the linear equality constrained minimization problem

$$\text{minimize } f(\mathbf{x}), \quad \text{s.t. } \mathbf{B}\mathbf{x} = \mathbf{y}.$$

- The proposed algorithms become the RK-RK algorithm and the RGS-RK algorithm if  $f(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|_2^2$ .

## The RK-RRK algorithm: $\mathbf{b} \in \text{range}(\mathbf{AB})$

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The RK-RRK algorithm for solving  $\min_{\mathbf{ABx}=\mathbf{b}} f(\mathbf{x})$

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**Input:**  $\mathbf{A} \in \mathbb{R}^{m \times \ell}$ ,  $\mathbf{B} \in \mathbb{R}^{\ell \times n}$ ,  $\mathbf{b} \in \mathbb{R}^m$ , and maximum number of iterations **maxit**.

**Output:** an approximation of the solution of  $\min_{\mathbf{ABx}=\mathbf{b}} f(\mathbf{x})$ .

**Initialize:**  $\mathbf{y}^0 = \mathbf{0}$ ,  $\mathbf{z}^0 \in \text{range}(\mathbf{B}^\top)$ , and  $\mathbf{x}^0 = \nabla f^*(\mathbf{z}^0)$ .

**for**  $k = 1, 2, \dots, \text{maxit}$  **do**

    Pick  $j \in [m]$  with probability  $\|\mathbf{A}_{j,:}\|_2^2 / \|\mathbf{A}\|_F^2$

    Set  $\mathbf{y}^k = \mathbf{y}^{k-1} - \frac{\mathbf{A}_{j,:}\mathbf{y}^{k-1} - b_j}{\|\mathbf{A}_{j,:}\|_2^2} (\mathbf{A}_{j,:})^\top$

    Pick  $i \in [\ell]$  with probability  $\|\mathbf{B}_{i,:}\|_2^2 / \|\mathbf{B}\|_F^2$

    Set  $\mathbf{z}^k = \mathbf{z}^{k-1} - \frac{\mathbf{B}_{i,:}\mathbf{x}^{k-1} - y_i^k}{\|\mathbf{B}_{i,:}\|_2^2} (\mathbf{B}_{i,:})^\top$

    Set  $\mathbf{x}^k = \nabla f^*(\mathbf{z}^k)$

**end**

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## RK-RSK and ExSRK

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- The RK-RRK algorithm becomes the RK-RSK algorithm if

$$f(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|_2^2 + \lambda\|\mathbf{x}\|_1.$$

- The iterates of the ExSRK algorithm [SLTW22] for  $\mathbf{C}\mathbf{x} = \mathbf{b}$  are

$$\begin{aligned}\mathbf{y}^k &= \mathbf{y}^{k-1} - \frac{(\mathbf{C}_{:,j})^\top \mathbf{y}^{k-1}}{\|\mathbf{C}_{:,j}\|_2^2} \mathbf{C}_{:,j}, \\ \mathbf{z}^k &= \mathbf{z}^{k-1} - \frac{\mathbf{C}_{i,:} \mathbf{x}^{k-1} - b_i + y_i^k}{\|\mathbf{C}_{i,:}\|_2^2} (\mathbf{C}_{i,:})^\top, \\ \mathbf{x}^k &= S_\lambda(\mathbf{z}^k),\end{aligned}$$

with initial iterates  $\mathbf{y}^0 = \mathbf{b}$ ,  $\mathbf{z}^0 \in \text{range}(\mathbf{C}^\top)$ , and  $\mathbf{x}^0 = S_\lambda(\mathbf{z}^0)$ .

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[SLTW22] F. Schöpfer, D. A. Lorenz, L. Tondji, and M. Winkler. *Extended randomized Kaczmarz method for sparse least squares and impulsive noise problems*. arXiv:2201.08620, 2022.

## RK-RSK and ExSRK

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- Note that the normal equations

$$\mathbf{C}^\top \mathbf{C} \mathbf{x} = \mathbf{C}^\top \mathbf{b}$$

can be viewed as the factorized linear system

$$\hat{\mathbf{A}} \hat{\mathbf{B}} \mathbf{x} = \hat{\mathbf{b}}$$

with

$$\hat{\mathbf{A}} = \mathbf{C}^\top, \quad \hat{\mathbf{B}} = \mathbf{C}, \quad \hat{\mathbf{b}} = \mathbf{C}^\top \mathbf{b}.$$

We observe that the iterates  $\mathbf{x}^k$ ,  $\mathbf{y}^k$ , and  $\mathbf{z}^k$  of the ExSRK algorithm for

$$\mathbf{C} \mathbf{x} = \mathbf{b}$$

are equal to  $\hat{\mathbf{x}}^k$ ,  $\mathbf{b} - \hat{\mathbf{y}}^k$ , and  $\hat{\mathbf{z}}^k$ , respectively, where  $\hat{\mathbf{x}}^k$ ,  $\hat{\mathbf{y}}^k$ , and  $\hat{\mathbf{z}}^k$  are the iterates of the RK-RSK algorithm for

$$\hat{\mathbf{A}} \hat{\mathbf{B}} \mathbf{x} = \hat{\mathbf{b}}.$$



## The RGS-RRK algorithm

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The RGS-RRK algorithm for solving  $\min_{\mathbf{x} \in \arg\min_{\mathbf{z}} \|\mathbf{b} - \mathbf{A}\mathbf{B}\mathbf{z}\|_2} f(\mathbf{x})$

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**Input:**  $\mathbf{A} \in \mathbb{R}^{m \times \ell}$ ,  $\mathbf{B} \in \mathbb{R}^{\ell \times n}$ ,  $\mathbf{b} \in \mathbb{R}^m$ , and maximum number of iterations **maxit**.

**Output:** an approximation of the solution of  $\min_{\mathbf{x} \in \arg\min_{\mathbf{z}} \|\mathbf{b} - \mathbf{A}\mathbf{B}\mathbf{z}\|_2} f(\mathbf{x})$ .

**Initialize:**  $\mathbf{y}^0 = \mathbf{0}$ ,  $\mathbf{r}^0 = \mathbf{b}$ ,  $\mathbf{z}^0 \in \text{range}(\mathbf{B}^\top)$ , and  $\mathbf{x}^0 = \nabla f^*(\mathbf{z}^0)$ .

**for**  $k = 1, 2, \dots, \text{maxit}$  **do**

    Pick  $j \in [\ell]$  with probability  $\|\mathbf{A}_{:,j}\|_2^2 / \|\mathbf{A}\|_F^2$

    Compute  $d_k = (\mathbf{A}_{:,j})^\top \mathbf{r}^{k-1} / \|\mathbf{A}_{:,j}\|_2^2$

    Set  $y_j^k = y_j^{k-1} + d_k$ ,  $y_l^k = y_l^{k-1}$  for  $l \neq j$

    Set  $\mathbf{r}^k = \mathbf{r}^{k-1} - d_k \mathbf{A}_{:,j}$

    Pick  $i \in [\ell]$  with probability  $\|\mathbf{B}_{i,:}\|_2^2 / \|\mathbf{B}\|_F^2$

    Set  $\mathbf{z}^k = \mathbf{z}^{k-1} - \frac{\mathbf{B}_{i,:} \mathbf{x}^{k-1} - y_i^k}{\|\mathbf{B}_{i,:}\|_2^2} (\mathbf{B}_{i,:})^\top$

    Set  $\mathbf{x}^k = \nabla f^*(\mathbf{z}^k)$

**end**

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## Example 1: Settings

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- $\mathbf{A}=\text{randn}(m,1)$ ,  $\mathbf{B}=\text{randn}(1,n)$ .
- $\mathbf{x}_\star$  is an  $s$  sparse vector with normally distributed non-zero entries, whose support is randomly generated.
- $\mathbf{b} = \mathbf{ABx}_\star$  for  $\mathbf{b} \in \text{range}(\mathbf{AB})$ .
- $\mathbf{b} = \hat{\mathbf{b}} + \hat{\mathbf{b}}_\perp$  for  $\mathbf{b} \notin \text{range}(\mathbf{AB})$  with  $\hat{\mathbf{b}} = \mathbf{ABx}_\star$  and

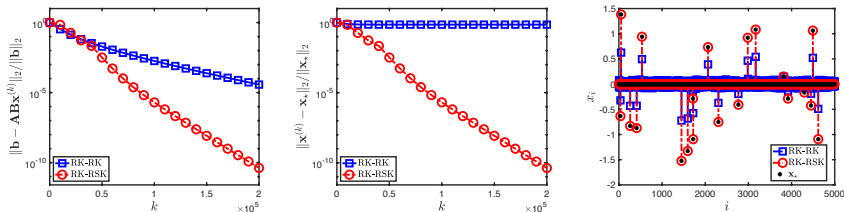
$$\hat{\mathbf{b}}_\perp = \mathbf{Nv}\|\hat{\mathbf{b}}\|_2/\|\mathbf{Nv}\|_2 \in \text{null}(\mathbf{B}^\top \mathbf{A}^\top) = \text{null}(\mathbf{A}^\top),$$

where the columns of  $\mathbf{N}$  form an orthonormal basis of  $\text{null}(\mathbf{A}^\top)$  and  $\mathbf{v}$  is a Gaussian vector generated by  $\mathbf{v}=\text{randn}(m-1,1)$ .

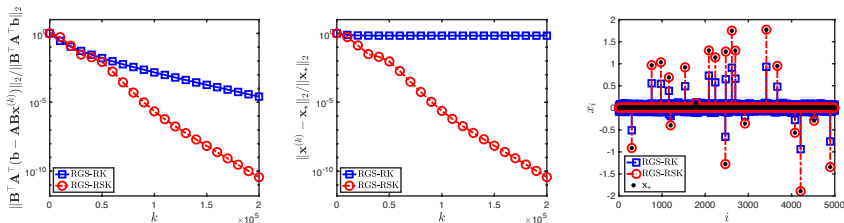
- For the proposed algorithms, we use  $\lambda = 1$ ,  $\mathbf{y}^0 = \mathbf{0}$ ,  $\mathbf{z}^0 = \mathbf{0}$ , and the maximum number of iterations  $\text{maxit}=20m$ .
- $m=10000$ ,  $l=2500$ ,  $n=5000$ ,  $s=20$ .

## Example 1: Results

- Comparison of RK-RK and RK-RSK



- Comparison of RGS-RK and RGS-RSK



## Example 2: Settings

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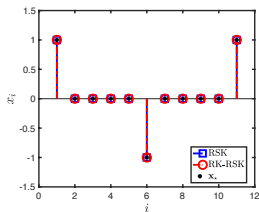
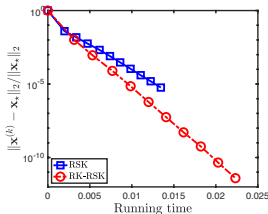
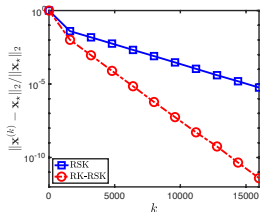
- Let  $\mathbf{X} \in \mathbb{R}^{m \times n}$  denote the wine quality data matrix (a sample of  $m = 1599$  red wines with  $n = 11$  physio-chemical properties of each wine) obtained from the UCI Machine Learning Repository [uci].
- The matrices  $\mathbf{A}$  and  $\mathbf{B}$  are obtained as follows:  
[A,B]=nnmf(X,5). We compute  $\mathbf{C} = \mathbf{AB}$  in MATLAB.
- The condition number of  $\mathbf{A}$ ,  $\mathbf{B}$ , and  $\mathbf{C}$  are 23.5, 4.4, and 45.4, respectively.
- Let  $\mathbf{x}_\star \in \mathbb{R}^{11}$  be a 3-sparse vector with support  $\{1, 6, 11\}$ . The three nonzero entries of  $\mathbf{x}_\star$  are set to be 1.
- For the proposed algorithms, we use  $\lambda = 1$ ,  $\mathbf{y}^0 = \mathbf{0}$ ,  $\mathbf{z}^0 = \mathbf{0}$ , and the maximum number of iterations `maxit=10m`.

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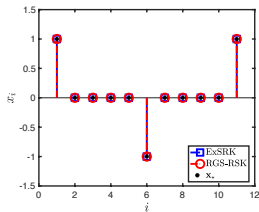
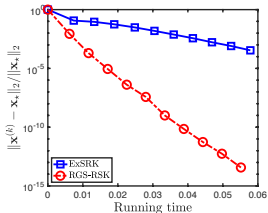
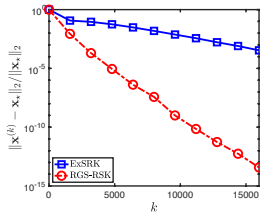
[uci] D. Dua and C. Graff. *UCI machine learning repository*, 2017. <http://archive.ics.uci.edu/ml>.

## Example 2: Results

- Comparison of RSK and RK-RSK



- Comparison of ExSRK and RGS-RSK



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## Summary

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- Two novel regularized randomized iterative algorithms using the ALG1-ALG2 approach are proposed to find (least squares) solutions with certain structures of factorized linear systems.
- Computed examples are given to illustrate that the new algorithms can find sparse (least squares) solutions of  $\mathbf{ABx} = \mathbf{b}$  and can be better than the existing randomized iterative algorithms for the corresponding full linear system  $\mathbf{Cx} = \mathbf{b}$  with  $\mathbf{C} = \mathbf{AB}$ .
- Existing acceleration strategies for RK and RGS can be integrated into our algorithms easily and the corresponding convergence analysis is straightforward.
- The extension to a factorized linear system with rank-deficient  $\mathbf{A}$  and  $\mathbf{B}$  will be the future work.



**Thanks!**