

Lecture 8: Power/Inverse iteration, Rayleigh quotient iteration



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1. Eigenvalue problem and polynomial rootfinding problem

- The eigenvalues of a matrix $\mathbf{A} \in \mathbb{C}^{m \times m}$ are the m roots of its characteristic polynomial $p(z) = \det(z\mathbf{I} - \mathbf{A})$.
- Suppose we have the monic polynomial

$$p(z) = z^m + a_{m-1}z^{m-1} + \cdots + a_1z + a_0.$$

It is not hard to verify that $p(z) = \det(z\mathbf{I} - \mathbf{A})$, where the $m \times m$ matrix \mathbf{A} is

$$\mathbf{A} = \begin{bmatrix} 0 & & & -a_0 \\ 1 & 0 & & -a_1 \\ & 1 & 0 & -a_2 \\ & & 1 & \ddots & \vdots \\ & & & \ddots & 0 & -a_{m-2} \\ & & & & 1 & -a_{m-1} \end{bmatrix}.$$

The matrix \mathbf{A} is called a *companion matrix* corresponding to $p(z)$.

- Any eigenvalue solver must be iterative because no explicit root expressing formula exists for polynomial of degree ≥ 5 . The goal of an eigenvalue solver is to produce sequences of numbers that converge rapidly towards eigenvalues.

- Convergence rate

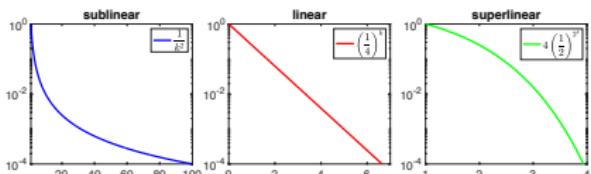
Let e_1, e_2, \dots be a sequence of nonnegative numbers representing errors in some iterative process that converge to zero, and suppose there are a positive constant c and an exponent α such that for all sufficiently large k , $e_{k+1} \leq c(e_k)^\alpha$. Then,

- (1) $\alpha = 1$ and $c < 1$, *linear convergence* or *geometric convergence*;
- (2) $\alpha = 2$, *quadratic convergence*;
- (3) $\alpha = 3$, *cubic convergence*; ...

2. Rayleigh quotient

- The *Rayleigh quotient* of a nonzero vector $\mathbf{x} \in \mathbb{C}^m$ with respect to \mathbf{A} is the scalar

$$r(\mathbf{x}) = \frac{\mathbf{x}^* \mathbf{A} \mathbf{x}}{\mathbf{x}^* \mathbf{x}}.$$



Theorem 1

Let $\{\lambda, \mathbf{v}\}$ be an eigenpair of $\mathbf{A} \in \mathbb{C}^{m \times m}$, i.e., $\mathbf{Av} = \lambda\mathbf{v}$ and $\mathbf{v} \neq 0$.

- (i) If \mathbf{A} is non-normal, then the Rayleigh quotient $r(\mathbf{x})$ is generally a linearly accurate estimate of the eigenvalue λ , i.e.,

$$|r(\mathbf{x}) - \lambda| = \mathcal{O}(\|\mathbf{x} - \mathbf{v}\|_2), \quad \text{as } \mathbf{x} \rightarrow \mathbf{v}.$$

- (ii) If \mathbf{A} is normal, then the Rayleigh quotient $r(\mathbf{x})$ is a quadratically accurate estimate of the eigenvalue λ , i.e.,

$$|r(\mathbf{x}) - \lambda| = \mathcal{O}(\|\mathbf{x} - \mathbf{v}\|_2^2), \quad \text{as } \mathbf{x} \rightarrow \mathbf{v}.$$

Hint:

For simplicity, we assume that $\|\mathbf{v}\|_2 = 1$ and consider a Schur form

$$\mathbf{T} = \mathbf{Q}^* \mathbf{A} \mathbf{Q}$$

with $t_{11} = \lambda$ and $\mathbf{Q}\mathbf{e}_1 = \mathbf{v}$. Let $\mathbf{y} = \mathbf{Q}^*\mathbf{x}$. Then $\mathbf{y} \rightarrow \mathbf{e}_1$ as $\mathbf{x} \rightarrow \mathbf{v}$. □

3. Power iteration

Algorithm 1: Power iteration

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 $\mathbf{v}^{(0)} = \text{some vector with } \|\mathbf{v}^{(0)}\|_2 = 1$ 
for  $k = 1, 2, 3, \dots,$ 
     $\mathbf{w} = \mathbf{A}\mathbf{v}^{(k-1)}$ 
     $\mathbf{v}^{(k)} = \mathbf{w}/\|\mathbf{w}\|_2$ 
     $\lambda^{(k)} = (\mathbf{v}^{(k)})^* \mathbf{A} \mathbf{v}^{(k)}$ 
end
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- Stopping criteria.
- One application: Google's Pagerank.
- Power iteration can find only an approximate eigenpair corresponding to the eigenvalue with the largest magnitude. The convergence is *linear*, which is very slow if the largest two eigenvalues are close in magnitude.

- Suppose $\mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^{-1}$ is diagonalizable with $\|\mathbf{v}_1\|_2 = 1$ and

$$\Lambda = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_m\}, \quad |\lambda_1| > |\lambda_2| \geq \dots \geq |\lambda_m|.$$

Let $\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} := \mathbf{V}^{-1}\mathbf{v}^{(0)}$. Suppose $\alpha_1 \neq 0$. Then we have

$$\mathbf{A}^k \mathbf{v}^{(0)} = \mathbf{V}\Lambda^k\mathbf{V}^{-1}\mathbf{v}^{(0)} = \mathbf{V} \begin{bmatrix} \alpha_1 \lambda_1^k \\ \alpha_2 \lambda_2^k \\ \vdots \\ \alpha_m \lambda_m^k \end{bmatrix} = \alpha_1 \lambda_1^k \mathbf{V} \begin{bmatrix} 1 \\ \frac{\alpha_2}{\alpha_1} \frac{\lambda_2^k}{\lambda_1^k} \\ \vdots \\ \frac{\alpha_m}{\alpha_1} \frac{\lambda_m^k}{\lambda_1^k} \end{bmatrix}.$$

By $\mathbf{v}^{(k)} = \frac{\mathbf{A}^k \mathbf{v}^{(0)}}{\|\mathbf{A}^k \mathbf{v}^{(0)}\|_2}$, we have $e^{-i\theta_k} \mathbf{v}^{(k)} \rightarrow \mathbf{v}_1$ and $\lambda^{(k)} \rightarrow \lambda_1$, where $\theta_k = k\theta + \theta_0$ with $e^{i\theta} = \lambda_1/|\lambda_1|$ and $e^{i\theta_0} = \alpha_1/|\alpha_1|$.

Theorem 2

Suppose \mathbf{A} is diagonalizable, i.e., $\mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^{-1}$ with

$$\Lambda = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_m\}.$$

Furthermore, suppose $\mathbf{e}_1^\top \mathbf{V}^{-1} \mathbf{v}^{(0)} \neq 0$, $\mathbf{e}_2^\top \mathbf{V}^{-1} \mathbf{v}^{(0)} \neq 0$, $\|\mathbf{v}_1\|_2 = 1$, and

$$|\lambda_1| > |\lambda_2| \geq \dots \geq |\lambda_m|.$$

Then the iterates of power iteration satisfy, as $k \rightarrow \infty$,

$$\|\mathbf{e}^{-i\theta_k} \mathbf{v}^{(k)} - \mathbf{v}_1\|_2 = \mathcal{O}\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right),$$

and

$$|\lambda^{(k)} - \lambda_1| = \mathcal{O}\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right) \quad \text{or} \quad \mathcal{O}\left(\left|\frac{\lambda_2}{\lambda_1}\right|^{2k}\right).$$

4. Inverse iteration

Proposition 3

For any μ that is not an eigenvalue, the eigenvectors of $(\mathbf{A} - \mu\mathbf{I})^{-1}$ are the same as the eigenvectors of \mathbf{A} , and the corresponding eigenvalues are $\{(\lambda_j - \mu)^{-1}\}$, where $\{\lambda_j\}$ are the eigenvalues of \mathbf{A} .

Algorithm 2: Inverse iteration

$\mathbf{v}^{(0)}$ = some vector with $\|\mathbf{v}^{(0)}\|_2 = 1$

for $k = 1, 2, 3, \dots$,

Solve $(\mathbf{A} - \mu\mathbf{I})\mathbf{w} = \mathbf{v}^{(k-1)}$ for \mathbf{w}

$\mathbf{v}^{(k)} = \mathbf{w}/\|\mathbf{w}\|_2$

$\lambda^{(k)} = (\mathbf{v}^{(k)})^* \mathbf{A} \mathbf{v}^{(k)}$

end

- We call μ the *shift* of inverse iteration. Like power iteration, inverse iteration exhibits only *linear* convergence.
- An important issue about numerical stability: TreBau Exercise 27.5

Theorem 4

Suppose \mathbf{A} is diagonalizable, i.e., $\mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^{-1}$ with

$$\Lambda = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_m\}.$$

Suppose λ_j is the closest eigenvalue to μ , λ_l is the second closest, and they satisfy

$$|\lambda_j - \mu| < |\lambda_l - \mu| \leq |\lambda_i - \mu|$$

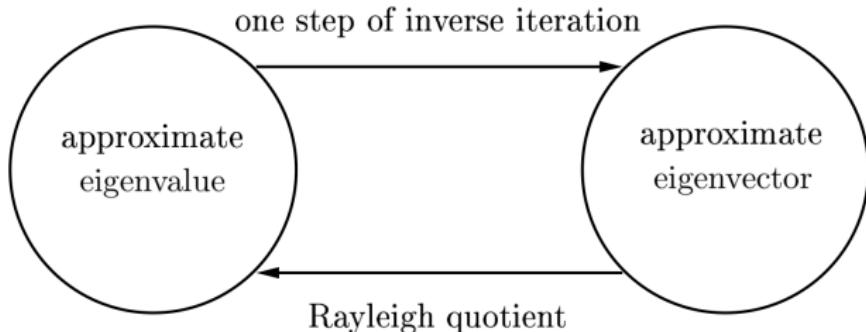
for each $i \neq j$. Furthermore, suppose $\mathbf{e}_j^\top \mathbf{V}^{-1} \mathbf{v}^{(0)} \neq 0$, $\mathbf{e}_l^\top \mathbf{V}^{-1} \mathbf{v}^{(0)} \neq 0$, and $\|\mathbf{v}_j\|_2 = 1$. Then the iterates of inverse iteration satisfy, as $k \rightarrow \infty$,

$$\|\mathbf{e}^{-i\theta_k} \mathbf{v}^{(k)} - \mathbf{v}_j\|_2 = \mathcal{O}\left(\left|\frac{\lambda_j - \mu}{\lambda_l - \mu}\right|^k\right), \quad (\text{Exercise : } \theta_k = ?)$$

and

$$|\lambda^{(k)} - \lambda_j| = \mathcal{O}\left(\left|\frac{\lambda_j - \mu}{\lambda_l - \mu}\right|^k\right) \quad \text{or} \quad \mathcal{O}\left(\left|\frac{\lambda_j - \mu}{\lambda_l - \mu}\right|^{2k}\right).$$

5. Rayleigh quotient iteration



Algorithm 3: Rayleigh quotient iteration

$\mathbf{v}^{(0)}$ = some vector with $\|\mathbf{v}^{(0)}\|_2 = 1$

$$\lambda^{(0)} = (\mathbf{v}^{(0)})^* \mathbf{A} \mathbf{v}^{(0)}$$

for $k = 1, 2, 3, \dots$,

Solve $(\mathbf{A} - \lambda^{(k-1)} \mathbf{I}) \mathbf{w} = \mathbf{v}^{(k-1)}$ for \mathbf{w}

$$\mathbf{v}^{(k)} = \mathbf{w} / \|\mathbf{w}\|_2$$

$$\lambda^{(k)} = (\mathbf{v}^{(k)})^* \mathbf{A} \mathbf{v}^{(k)}$$

end

Theorem 5

Rayleigh quotient iteration converges to an eigenpair

$$\{\lambda, \mathbf{v}\}, \quad \|\mathbf{v}\|_2 = 1,$$

for all except a set of measure zero of starting vectors $\mathbf{v}^{(0)}$. When it converges, the convergence is ultimately quadratic ($\alpha = 2$) for the non-normal case or cubic ($\alpha = 3$) for the normal case in the sense that if $e^{-i\theta_k} \mathbf{v}^{(k)}$ is sufficiently close to the eigenvector \mathbf{v} , then

$$\|e^{-i\theta_{k+1}} \mathbf{v}^{(k+1)} - \mathbf{v}\|_2 = \mathcal{O}(\|e^{-i\theta_k} \mathbf{v}^{(k)} - \mathbf{v}\|_2^\alpha)$$

and

$$|\lambda^{(k+1)} - \lambda| = \mathcal{O}(|\lambda^{(k)} - \lambda|^\alpha)$$

as $k \rightarrow \infty$.

Example:

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}, \quad \mathbf{v}^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} / \sqrt{3}.$$

The eigenvalue $\lambda = 5.214319743377$

Power iteration:

$$\lambda^{(0)} = 5$$

$$\lambda^{(1)} = 5.1818\dots$$

$$\lambda^{(2)} = 5.2081\dots$$

$$\lambda^{(3)} = 5.2130\dots$$

Rayleigh quotient iteration: $\lambda^{(0)} = 5$

$$\lambda^{(1)} = 5.2131\dots$$

$$\lambda^{(2)} = 5.214319743184\dots$$

The convergence of Rayleigh quotient iteration is spectacular:
each iteration triples the number of digits of accuracy.