Numerical Linear Algebra Assignment 4

Exercise 1. (10 points)

Let $\mathbf{x} \in \mathbb{C}^m$ and $x_1 = \mathbf{e}_1^\top \mathbf{x} \neq 0$. Show that the matrix

$$\mathbf{H} = \mathbf{I} - 2\frac{\mathbf{v}\mathbf{v}^*}{\mathbf{v}^*\mathbf{v}}, \qquad \mathbf{v} = \pm\frac{x_1}{|x_1|}\|\mathbf{x}\|_2\mathbf{e}_1 - \mathbf{x},$$

satisfies that

$$\mathbf{H}\mathbf{x} = \pm\frac{x_1}{|x_1|}\|\mathbf{x}\|_2\mathbf{e}_1.$$

Exercise 2. (10 points)

Prove that $\mathbf{I} - \dfrac{\mathbf{v}\mathbf{v}^*}{\mathbf{v}^*\mathbf{v}}$ is the orthogonal projector which projects $\mathbb{C}^m$ onto the *hyperplane* $\operatorname{span}\{\mathbf{v}\}^\perp$ along $\operatorname{span}\{\mathbf{v}\}$.

Exercise 3. (10 points)

Let $\mathbf{x} = \begin{bmatrix} 3 & 4 & 0 \end{bmatrix}^\top$ and $\mathbf{e}_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top$. Try to find all possible numbers $z \in \mathbb{C}$ and the corresponding Householder reflectors $\mathbf{H}$ such that $\mathbf{H}\mathbf{x} = z\mathbf{e}_3$.

Exercise 4. (10 points)

Let $m > n$, $\mathbf{A} \in \mathbb{C}^{m\times n}$ of rank n, $\mathbf{b} \in \mathbb{C}^m$, $\mathbf{b} \notin \operatorname{range}(\mathbf{A})$ and $\mathbf{Q}\mathbf{R} = \begin{bmatrix} \mathbf{A} & \mathbf{b} \end{bmatrix}$ (i.e., full QR factorization of $\begin{bmatrix} \mathbf{A} & \mathbf{b} \end{bmatrix}$). Show that (we use Matlab's notation for convenience)

$$\min_{\mathbf{x}\in\mathbb{C}^n} \|\mathbf{b} - \mathbf{A}\mathbf{x}\|_2 = |\mathbf{R}(n+1, n+1)|,$$

and the least squares solution is given by

$$\mathbf{x} = \mathbf{R}(1\colon n, 1\colon n)\backslash\mathbf{R}(1\colon n, n+1).$$

Exercise 5. (TreBau Exercise 19.1, 10 points)

Given $\mathbf{A} \in \mathbb{C}^{m\times n}$ of rank n and $\mathbf{b} \in \mathbb{C}^m$, consider the block 2×2 system of equations

$$\begin{bmatrix} \mathbf{I} & \mathbf{A} \\ \mathbf{A}^* & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix},$$

where $\mathbf{I}$ is the $m \times m$ identity. Show that this system has a unique solution $\begin{bmatrix} \mathbf{r} \\ \mathbf{x} \end{bmatrix}$, and that the vectors $\mathbf{r}$ and $\mathbf{x}$ are the residual and the least squares solution of the least squares problem: Given $\mathbf{A} \in \mathbb{C}^{m\times n}$ of full rank, $m \geq n$, $\mathbf{b} \in \mathbb{C}^m$, find $\mathbf{x} \in \mathbb{C}^n$ such that $\|\mathbf{b} - \mathbf{A}\mathbf{x}\|_2$ is minimized.

Exercise 6. (Demmel Question 3.11, 10 points)

Let $\mathbf{A} \in \mathbb{C}^{m\times n}$. Show that $\mathbf{X} = \mathbf{A}^\dagger$ (the Moore–Penrose pseudoinverse) minimizes $\|\mathbf{A}\mathbf{X} - \mathbf{I}\|_{\mathrm{F}}$ over all $n \times m$ matrices. What is the value (positive square root of some integer) of this minimum?

Exercise 7. (10 points)

Let $\mathbf{A} \in \mathbb{C}^{m\times n}$ and $\mathbf{b} \in \mathbb{C}^m$. Solve the *penalized* problem

$$\min_{\mathbf{x}\in\mathbb{C}^n} \left\{ \|\mathbf{b} - \mathbf{A}\mathbf{x}\|_2^2 + \lambda^2\|\mathbf{x}\|_2^2 \right\},$$

where $\lambda > 0$. Hint: consider the LSP

$$\min_{\mathbf{x}\in\mathbb{C}^n} \left\| \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{A} \\ \lambda\mathbf{I} \end{bmatrix} \mathbf{x} \right\|_2.$$

Exercise 8. (10 points)

Suppose a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ satisfies

$$\mathbf{A} = \mathbf{U}\mathbf{V}, \qquad \mathbf{U} \in \mathbb{R}^{m\times l}, \qquad \mathbf{V} \in \mathbb{R}^{l\times n}.$$

Prove the following statements:

(i) If $\operatorname{rank}(\mathbf{U}) = \operatorname{rank}(\mathbf{V}) = l$ (i.e., $\mathbf{A} = \mathbf{U}\mathbf{V}$ is a full-rank factorization), then $\mathbf{A}^\dagger = \mathbf{V}^\dagger\mathbf{U}^\dagger$.

(ii) For all $\mathbf{b} \in \operatorname{range}(\mathbf{A})$, if $\operatorname{rank}(\mathbf{U}) = l$ and $\operatorname{rank}(\mathbf{V}) = n$, then $\mathbf{A}^\dagger\mathbf{b} = \mathbf{V}^\dagger\mathbf{U}^\dagger\mathbf{b}$.

Exercise 9. (10 points)

For a given $\mathbf{b} \in \mathbb{C}^m$ and any $\mathbf{A} \in \mathbb{C}^{m\times n}$, let $\mathbf{y}$ be the closest point (we use $\|\cdot\|_2$) to $\mathbf{b}$ in $\operatorname{range}(\mathbf{A})$. Prove that $\mathbf{y}$ is located on the sphere of radius $\|\mathbf{b}\|_2/2$ centered at $\mathbf{b}/2$.

Exercise 10. (10 points)

Let $\mathbf{A} \in \mathbb{C}^{m\times n}$, $\mathbf{b} \in \mathbb{C}^n$, and $\mathbf{y} \in \mathbb{C}^m$ be given. Assume that $\operatorname{rank}(\mathbf{A}) < \min\{m, n\}$ and $\mathbf{y} \in \operatorname{range}(\mathbf{A})$. Solve the following problem

$$\min_{\mathbf{x}\in\mathbb{C}^n,\ \mathbf{A}\mathbf{x}=\mathbf{y}} \|\mathbf{b} - \mathbf{x}\|_2.$$

Exercise 11. (Programming, TreBau Exercises 10.2, 10 points)

Exercise 12. (Programming, TreBau Exercises 10.3, 10 points)

Additional Exercise 1. (Carl D. Meyer)

For a vector $\mathbf{x} \in \mathbb{C}^m$ with $\|\mathbf{x}\|_2 = 1$, partition $\mathbf{x}$ as $\mathbf{x} = \begin{bmatrix} x_1 \\ \widetilde{\mathbf{x}} \end{bmatrix}$, where $\widetilde{\mathbf{x}} \in \mathbb{C}^{m-1}$. Show that if $|x_1| \neq 0, 1$, and if

$$\alpha = \frac{1}{1 - |x_1|} \quad \text{and} \quad \beta = \frac{x_1}{|x_1|},$$

then for some $\mathbf{v} \in \mathbb{C}^m$,

$$\mathbf{Q} = \begin{bmatrix} x_1 & \beta^2\widetilde{\mathbf{x}}^* \\ \widetilde{\mathbf{x}} & \beta(\mathbf{I} - \alpha\widetilde{\mathbf{x}}\widetilde{\mathbf{x}}^*) \end{bmatrix} = \beta\left(\mathbf{I} - 2\frac{\mathbf{v}\mathbf{v}^*}{\mathbf{v}^*\mathbf{v}}\right).$$

This result provides an easy way to extend a vector $\mathbf{x}$ to a complete orthonormal set.