Numerical Linear Algebra Assignment 11

Exercise 1. (10 points)

Let $z \in \mathbb{C}$, $\mathbf{A} \in \mathbb{C}^{m\times m}$, and $\mathbf{B} = \mathbf{A} + z\mathbf{I}$. Prove the translation-invariance of Krylov subspaces, i.e., $\forall j \in \mathbb{N}$,

$$\mathcal{K}_j(\mathbf{A}, \mathbf{r}) = \mathcal{K}_j(\mathbf{B}, \mathbf{r}).$$

Exercise 2. (10 points)

If the minimal polynomial of the nonsingular matrix $\mathbf{A}$ has degree n, then the solution to $\mathbf{Ax} = \mathbf{b}$ lies in the space $\mathcal{K}_n(\mathbf{A}, \mathbf{b})$. (Hint: Let $q(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_{n-1}z^{n-1} + z^n$ denote the minimal polynomial of $\mathbf{A}$. Then $\alpha_0 \neq 0$.)

Exercise 3. (10 points)

Suppose the minimal polynomial of the matrix $\mathbf{A} \in \mathbb{C}^{m\times m}$ has degree n and the Arnoldi process for $\mathbf{A}$ and a nonzero $\mathbf{r}$ breaks down at step k, i.e., $h_{k+1,k} = 0$ is encountered. Prove the following

(i) $k \leq n$.

(ii) $\mathcal{K}_k(\mathbf{A}, \mathbf{r}) = \mathcal{K}_{k+1}(\mathbf{A}, \mathbf{r}) = \mathcal{K}_{k+2}(\mathbf{A}, \mathbf{r}) = \cdots$.

(iii) Each eigenvalue of $\mathbf{H}_k$ is an eigenvalue of $\mathbf{A}$.

(iv) If $\mathbf{A}$ is nonsingular, then the solution $\mathbf{x}$ of $\mathbf{Ax} = \mathbf{r}$ lies in $\mathcal{K}_k(\mathbf{A}, \mathbf{r})$.

Exercise 4. (10 points)

Assume $c_0 \neq 0$. Let $\mathbf{r}_0 = \mathbf{e}_1$ and

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_{m-1} \\ -c_0 & -\mathbf{c}_{m-1} \end{bmatrix} \in \mathbb{C}^{m\times m}, \quad \mathbf{c}_{m-1} = \begin{bmatrix} c_1 & c_2 & \cdots & c_{m-1} \end{bmatrix}.$$

Prove that

$$\|\mathbf{r}_0\|_2 = \|\mathbf{r}_1\|_2 = \cdots = \|\mathbf{r}_{m-1}\|_2, \qquad \|\mathbf{r}_m\|_2 = 0.$$

The above example implies that GMRES can completely stagnate, i.e., the residual norm can be nondecreasing at the first $m-1$ steps, and "convergence" occurs in the last step.

Exercise 5. (10 points)

Assume the Arnoldi process for $\{\mathbf{A}, \mathbf{r}_0\}$ breaks down at step $k > 1$. For $1 \leq j < k$, we have $\mathbf{AQ}_j = \mathbf{Q}_{j+1}\widetilde{\mathbf{H}}_j$ and $\mathbf{H}_j = \mathbf{Q}_j^*\mathbf{AQ}_j$. For $1 \leq j < k$, prove the following:

(a) The jth residual vector $\mathbf{r}_j$ of GMRES can be *uniquely* expressed as

$$\mathbf{r}_j = p_j(\mathbf{A})\mathbf{r}_0, \qquad \deg(p_j) \leq j, \qquad p_j(0) = 1.$$

(b) The unique polynomial p_j in (a) is given by

$$p_j(z) = \prod_{i=1}^{j} \left(1 - \theta_i^{(j)} z\right),$$

where $\theta_i^{(j)}$, $i = 1, 2, \ldots, j$, are the eigenvalues of $(\widetilde{\mathbf{H}}_j^*\widetilde{\mathbf{H}}_j)^{-1}\mathbf{H}_j^*$.

Exercise 6. (Programming, 10 points)

Write MATLAB code to plot the four pictures in Lecture 11.