

# Lecture 6: Stationary iterative methods



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# 1. Splitting and stationary iterative method

## Definition 1

A *splitting* of  $\mathbf{A} \in \mathbb{C}^{n \times n}$  is a decomposition  $\mathbf{A} = \mathbf{M} - \mathbf{K}$ , with  $\mathbf{M}$  nonsingular.

## Remark 2

A *splitting* yields a stationary iterative method as follows. The equation

$$\mathbf{A}\mathbf{x} = (\mathbf{M} - \mathbf{K})\mathbf{x} = \mathbf{b}$$

*implies*

$$\mathbf{x} = \mathbf{M}^{-1}\mathbf{K}\mathbf{x} + \mathbf{M}^{-1}\mathbf{b} := \mathbf{R}\mathbf{x} + \mathbf{c}.$$

Given a starting vector  $\mathbf{x}^{(0)}$ , we obtain a stationary iterative method

$$\mathbf{x}^{(m)} = \mathbf{R}\mathbf{x}^{(m-1)} + \mathbf{c}, \quad m = 1, 2, \dots$$

We note that  $\mathbf{R} = \mathbf{M}^{-1}\mathbf{K} = \mathbf{M}^{-1}(\mathbf{M} - \mathbf{A}) = \mathbf{I} - \mathbf{M}^{-1}\mathbf{A}$ .

## 2. Convergence criterion

### Definition 3

The spectral radius of a matrix  $\mathbf{A} \in \mathbb{C}^{n \times n}$  is  $\rho(\mathbf{A}) = \max_{\lambda \in \Lambda(\mathbf{A})} |\lambda|$ .

**Exercise.** Assume that  $\mathbf{A}$  is singular and has a splitting  $\mathbf{A} = \mathbf{M} - \mathbf{K}$  with  $\mathbf{M}$  nonsingular. Prove that the spectral radius  $\rho(\mathbf{M}^{-1}\mathbf{K}) \geq 1$ .

### Proposition 4

Let  $\|\cdot\|$  denote a matrix norm on  $\mathbb{C}^{n \times n}$  induced by a vector norm on  $\mathbb{C}^n$ . For any  $\mathbf{A} \in \mathbb{C}^{n \times n}$ , we have  $\rho(\mathbf{A}) \leq \|\mathbf{A}\|$ .

### Lemma 5

For any given  $\mathbf{A} \in \mathbb{C}^{n \times n}$  and  $\varepsilon > 0$  there exists an induced matrix norm  $\|\cdot\|_\star$  such that

$$\rho(\mathbf{A}) \leq \|\mathbf{A}\|_\star \leq \rho(\mathbf{A}) + \varepsilon.$$

The norm  $\|\cdot\|_\star$  depends on both  $\mathbf{A}$  and  $\varepsilon$ .

## Proof.

Let  $\mathbf{A} = \mathbf{SJS}^{-1}$  be a Jordan form of  $\mathbf{A}$ . Let

$$\mathbf{D}_\varepsilon = \text{diag}\{1, \varepsilon, \varepsilon^2, \dots, \varepsilon^{n-1}\}.$$

Now for all  $\mathbf{x} \in \mathbb{C}^n$  and for all  $\mathbf{B} \in \mathbb{C}^{n \times n}$ , define the vector norm

$$\|\mathbf{x}\|_\star := \|(\mathbf{SD}_\varepsilon)^{-1}\mathbf{x}\|_\infty$$

and the corresponding induced matrix norm

$$\begin{aligned}\|\mathbf{B}\|_\star &:= \sup_{\mathbf{x} \in \mathbb{C}^n, \mathbf{x} \neq \mathbf{0}} \frac{\|\mathbf{Bx}\|_\star}{\|\mathbf{x}\|_\star} = \sup_{\mathbf{x} \in \mathbb{C}^n, \mathbf{x} \neq \mathbf{0}} \frac{\|(\mathbf{SD}_\varepsilon)^{-1}\mathbf{Bx}\|_\infty}{\|(\mathbf{SD}_\varepsilon)^{-1}\mathbf{x}\|_\infty} \\ &= \sup_{\mathbf{y} \in \mathbb{C}^n, \mathbf{y} \neq \mathbf{0}} \frac{\|(\mathbf{SD}_\varepsilon)^{-1}\mathbf{B}(\mathbf{SD}_\varepsilon)\mathbf{y}\|_\infty}{\|\mathbf{y}\|_\infty} \\ &= \|\mathbf{D}_\varepsilon^{-1}\mathbf{S}^{-1}\mathbf{B}\mathbf{SD}_\varepsilon\|_\infty.\end{aligned}$$

The statement follows from  $\|\mathbf{A}\|_\star = \|\mathbf{D}_\varepsilon^{-1}\mathbf{JD}_\varepsilon\|_\infty \leq \rho(\mathbf{A}) + \varepsilon$ . □

## Theorem 6

The iteration  $\mathbf{x}^{(m)} = \mathbf{R}\mathbf{x}^{(m-1)} + \mathbf{c}$  converges to the solution of  $\mathbf{A}\mathbf{x} = \mathbf{b}$  for all starting vectors  $\mathbf{x}^{(0)}$  if and only if  $\rho(\mathbf{R}) < 1$ .

## Proof.

For all  $\mathbf{x}^{(0)}$ , we have  $\mathbf{x}^{(m)} - \mathbf{x} = \mathbf{R}(\mathbf{x}^{(m-1)} - \mathbf{x}) = \cdots = \mathbf{R}^m(\mathbf{x}^{(0)} - \mathbf{x})$ .

“ $\Leftarrow$ ” : If  $\rho(\mathbf{R}) < 1$ , by Lemma 5 there exists an induced matrix norm  $\|\cdot\|_*$  such that  $\|\mathbf{R}\|_* < 1$ , then  $\|\mathbf{x}^{(m)} - \mathbf{x}\|_* \leq \|\mathbf{R}\|_*^m \|\mathbf{x}^{(0)} - \mathbf{x}\|_* \rightarrow 0$  for all  $\mathbf{x}^{(0)}$ , i.e.,  $\mathbf{x}^{(m)} \rightarrow \mathbf{x}$  as  $m \rightarrow \infty$ .

“ $\Rightarrow$ ” : Proof by contradiction. Let  $\mathbf{x}^{(0)} - \mathbf{x}$  be an eigenvector of  $\mathbf{R}$  corresponding to eigenvalue  $\lambda$  with  $|\lambda| = \rho(\mathbf{R})$ . If  $\rho(\mathbf{R}) \geq 1$ , then  $\mathbf{x}^{(m)} - \mathbf{x} = \lambda^m(\mathbf{x}^{(0)} - \mathbf{x})$  will not approach  $\mathbf{0}$ . This is a contradiction.  $\square$

## Remark 7

The goal is to choose a splitting  $\mathbf{A} = \mathbf{M} - \mathbf{K}$  so that both

- (1)  $\mathbf{R}\mathbf{v} = \mathbf{M}^{-1}\mathbf{K}\mathbf{v}$  and  $\mathbf{c} = \mathbf{M}^{-1}\mathbf{b}$  are easy to evaluate, and
- (2)  $\rho(\mathbf{R})$  is small ( $< 1$ ).

- (1) and (2) are conflicting goals, and need to be balanced.
- If  $\Lambda(\mathbf{R}) \subset (-\rho(\mathbf{R}), \rho(\mathbf{R}))$ , then Chebyshev acceleration technique can be used. See Demmel's book ANLA, section 6.5.6.

### 3. Classical stationary iterative methods

- Let  $\mathbf{A} = \mathbf{D} - \mathbf{L} - \mathbf{U}$ , where
  - $\mathbf{D}$  is the diagonal matrix with diagonal entries  $d_{ii} = a_{ii}$ ,
  - $-\mathbf{L}$  is the strictly lower triangular part of  $\mathbf{A}$ ,
  - $-\mathbf{U}$  is the strictly upper triangular part of  $\mathbf{A}$ .
- Assume that  $\mathbf{A}$  has no zero diagonal entries. We will introduce
  - (1) Jacobi's method,
  - (2) Gauss–Seidel method,
  - (3) Successive overrelaxation:  $\text{SOR}(\omega)$ ,
  - (4) Symmetric successive overrelaxation:  $\text{SSOR}(\omega)$ .

### 3.1. Jacobi's method

- The splitting is

$$\mathbf{A} = \mathbf{D} - (\mathbf{L} + \mathbf{U}),$$

and the corresponding

$$\mathbf{R} = \mathbf{D}^{-1}(\mathbf{L} + \mathbf{U}) = \mathbf{I} - \mathbf{D}^{-1}\mathbf{A} \quad \text{and} \quad \mathbf{c} = \mathbf{D}^{-1}\mathbf{b}.$$

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**Algorithm 1:** Jacobi's method

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**for**  $j = 1$  **to**  $n$

$$x_j^{(m+1)} = \frac{1}{a_{jj}} \left( b_j - \sum_{k \neq j} a_{jk} x_k^{(m)} \right)$$

**end**

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#### Theorem 8

*If  $\mathbf{A}$  is Hermitian and  $a_{ii} > 0$  for all  $i$ , then Jacobi's method converges for all starting vectors if and only if  $\mathbf{A}$  and  $2\mathbf{D} - \mathbf{A}$  are both HPD.*

### 3.2. Gauss–Seidel method

- The splitting is

$$\mathbf{A} = (\mathbf{D} - \mathbf{L}) - \mathbf{U},$$

and the corresponding

$$\mathbf{R} = (\mathbf{D} - \mathbf{L})^{-1}\mathbf{U},$$

and

$$\mathbf{c} = (\mathbf{D} - \mathbf{L})^{-1}\mathbf{b}.$$

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**Algorithm 2:** Gauss–Seidel method

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**for**  $j = 1$  **to**  $n$

$$x_j^{(m+1)} = \frac{1}{a_{jj}} \left( b_j - \sum_{k=1}^{j-1} a_{jk} x_k^{(m+1)} - \sum_{k=j+1}^n a_{jk} x_k^{(m)} \right)$$

**end**

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### 3.3. Successive overrelaxation: $\text{SOR}(\omega)$ , $\omega \in \mathbb{R}$

- The splitting is  $\omega \mathbf{A} = (\mathbf{D} - \omega \mathbf{L}) - ((1 - \omega)\mathbf{D} + \omega \mathbf{U})$ , and the corresponding

$$\mathbf{R} = (\mathbf{D} - \omega \mathbf{L})^{-1}((1 - \omega)\mathbf{D} + \omega \mathbf{U}),$$

and

$$\mathbf{c} = \omega(\mathbf{D} - \omega \mathbf{L})^{-1}\mathbf{b}.$$

- $\omega = 1$ : Gauss–Seidel method
- $0 < \omega < 2$ : Necessary in some sense (see Theorem 13)
- Optimal  $\omega$ :

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**Algorithm 3:**  $\text{SOR}(\omega)$ , here  $\omega$  is the relaxation parameter

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**for**  $j = 1$  **to**  $n$

$$x_j^{(m+1)} = (1 - \omega)x_j^{(m)} + \frac{\omega}{a_{jj}} \left( b_j - \sum_{k=1}^{j-1} a_{jk}x_k^{(m+1)} - \sum_{k=j+1}^n a_{jk}x_k^{(m)} \right)$$

**end**

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### 3.4. Symmetric successive overrelaxation: SSOR( $\omega$ ), $\omega \in \mathbb{R}$

- This method uses two splittings:

$$\begin{aligned}\omega \mathbf{A} &= (\mathbf{D} - \omega \mathbf{L}) - ((1 - \omega)\mathbf{D} + \omega \mathbf{U}) \\ &= (\mathbf{D} - \omega \mathbf{U}) - ((1 - \omega)\mathbf{D} + \omega \mathbf{L}),\end{aligned}$$

and the corresponding

$$\begin{aligned}\mathbf{R} &= (\mathbf{D} - \omega \mathbf{U})^{-1}((1 - \omega)\mathbf{D} + \omega \mathbf{L})(\mathbf{D} - \omega \mathbf{L})^{-1}((1 - \omega)\mathbf{D} + \omega \mathbf{U}), \\ \mathbf{c} &= \omega(2 - \omega)(\mathbf{D} - \omega \mathbf{U})^{-1}\mathbf{D}(\mathbf{D} - \omega \mathbf{L})^{-1}\mathbf{b}.\end{aligned}$$

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**Algorithm 4:** SSOR( $\omega$ )

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**for**  $j = 1$  **to**  $n$

$$x_j^{(m+1/2)} = (1 - \omega)x_j^{(m)} + \frac{\omega}{a_{jj}} \left( b_j - \sum_{k=1}^{j-1} a_{jk}x_k^{(m+1/2)} - \sum_{k=j+1}^n a_{jk}x_k^{(m)} \right)$$

**end**

**for**  $j = n$  **to**  $1$

$$x_j^{(m+1)} = (1 - \omega)x_j^{(m+1/2)} + \frac{\omega}{a_{jj}} \left( b_j - \sum_{k=1}^{j-1} a_{jk}x_k^{(m+1/2)} - \sum_{k=j+1}^n a_{jk}x_k^{(m+1)} \right)$$

**end**

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### 3.5. Convergence (see Demmel's book ANLA, section 6.5.5)

#### Definition 9

A matrix  $\mathbf{A}$  is an irreducible matrix if there is no permutation matrix such that

$$\mathbf{PAP}^\top = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix}.$$

#### Definition 10

A matrix  $\mathbf{A} \in \mathbb{C}^{n \times n}$  is **weakly** row diagonally dominant if for all  $i$ ,

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|$$

with strict inequality at least once. A matrix  $\mathbf{A}$  is **strictly** row diagonally dominant if for all  $i$ :

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|.$$

### Theorem 11

*If  $\mathbf{A}$  is strictly row diagonally dominant, then both Jacobi's and Gauss-Seidel methods converge. In fact  $\|\mathbf{R}_{\text{GS}}\|_{\infty} \leq \|\mathbf{R}_{\text{J}}\|_{\infty} < 1$ .*

### Theorem 12

*If  $\mathbf{A}$  is irreducible and weakly row diagonally dominant, then both Jacobi's and Gauss-Seidel methods converge, and  $\rho(\mathbf{R}_{\text{GS}}) < \rho(\mathbf{R}_{\text{J}}) < 1$ .*

### Theorem 13

*For any matrix  $\mathbf{A}$ , it holds  $\rho(\mathbf{R}_{\text{SOR}(\omega)}) \geq |\omega - 1|$ . Therefore  $0 < \omega < 2$  is required for the convergence of  $\text{SOR}(\omega)$  for all starting vectors.*

### Theorem 14

*If  $\mathbf{A}$  is Hermitian positive definite, then  $\rho(\mathbf{R}_{\text{SOR}(\omega)}) < 1$  for all  $0 < \omega < 2$ , i.e.,  $\text{SOR}(\omega)$  converges for all  $0 < \omega < 2$ . Gauss-Seidel ( $\text{SOR}(1)$ ) converges for Hermitian positive definite  $\mathbf{A}$ .*