Lecture 20: Backward stability of an algorithm

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1. Floating point number

- For given integers p and β, in the IEEE floating point standard (founded in 1985, updated in 2008, being undated again now), the elements of the floating point number system $\mathbf{F}$ are the number 0 together with numbers of the form

$$x = \pm d_1.d_2 \cdots d_p \times \beta^e$$

where the integers d_i, e satisfy

$$0 \leqslant d_i \leqslant \beta - 1, \quad d_1 \neq 0, \quad e_{\min} \leqslant e \leqslant e_{\max}.$$

- One need to store sign bit ($\pm$), exponent (e), and mantissa ($d_1.d_2 \cdots d_p$); but not the *base* or *radix* ($\beta \geqslant 2$). Floating point number system usually uses $\beta = 2$ (10 sometimes, 16 historically).

Precision	β	Bits	p	$e_{\min}$	$e_{\max}$	$\epsilon_{\text{machine}}$
Single (32)	2	1+8+23	24	-126	127	2^{-24}
Double (64)	2	1+11+52	53	-1022	1023	2^{-53}

1.1. Limitations of digital representations

- Only a finite subset of the real numbers (or the complex numbers) can be represented. Therefore,

  (i) the represented numbers cannot be arbitrarily large or small;

  (ii) there are gaps between these numbers.

1.2. Floating point number machine accuracy

- In IEEE double precision arithmetic, the interval $[1, 2]$ is represented by the discrete subset

$$1, \quad 1 + 2^{-52}, \quad 1 + 2 \times 2^{-52}, \quad 1 + 3 \times 2^{-52}, \quad \cdots, \quad 2.$$

  The interval $[2, 4]$ is represented by the same numbers multiplied by 2,

$$2, \quad 2 + 2^{-51}, \quad 2 + 2 \times 2^{-51}, \quad 2 + 3 \times 2^{-51}, \quad \cdots, \quad 4.$$

In general, the interval $[2^j, 2^{j+1}]$ is represented by the numbers for $[1, 2]$ times 2^j.

- For floating point number system, the machine accuracy, denoted by $\epsilon_{\text{machine}}$, is defined as: *half the distance between* 1 *and the next larger floating point number*. We have

$$\forall x \in [\theta, \Theta], \quad \exists x' \in \mathbf{F} \quad s.t., \quad |x - x'| \leqslant \epsilon_{\text{machine}} |x|.$$

  In Matlab, `eps` $= 2\epsilon_{\text{machine}} = 2^{-52}$ in double precision.

- Let fl: $\mathbb{R} \to \mathbf{F}$ denote the function giving the closest floating point approximation. We have

$$\forall x \in [\theta, \Theta], \quad \exists \epsilon \in \mathbb{R} \quad s.t., \quad |\epsilon| \leqslant \epsilon_{\text{machine}} \quad and \quad \mathrm{fl}(x) = x(1+\epsilon).$$

  Exercise. (James Demmel) Prove the following: If floating point numbers x and y satisfy $2y \geqslant x \geqslant y \geqslant 0$, then $\mathrm{fl}(x - y) = x - y$, i.e., $x - y$ is an exact floating point number.

1.3. Floating point arithmetic

- $*$ $(+, -, \times, \div)$ in $\mathbb{R}$; $\circledast$ $(\oplus, \ominus, \otimes, \oslash)$ in $\mathbf{F}$; $x \circledast y = \mathrm{fl}(x{*}y)$.

Fundamental Axiom of Floating Point Arithmetic

For all $x, y \in \mathbf{F}$, there exists ϵ with $|\epsilon| \leq \epsilon_{\text{machine}}$ such that

$$x \circledast y \;=\; (x * y)(1 + \epsilon).$$

1.4. Programming exercise

- TreBau Exercise 13.3 (Horner's rule for polynomial evaluation).

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

Algorithm Horner's rule for $p(x) = \sum_{i=0}^{n} a_i x^i$.

```
p = a_n
for i = n - 1 : -1 : 0
    p = x p + a_i;
end
```

2. Algorithm

- Given a *problem* $f : \mathbb{X} \to \mathbb{Y}$. An *algorithm* for the problem f can be viewed as a map $\widetilde{f} : \mathbb{X} \to \mathbb{Y}$.
- More precisely, assume that a problem f, a computer with floating point system, and a program for solving the problem are fixed:
  (1) given $x \in \mathbb{X}$, let $\mathrm{fl}(x)$ be the corresponding floating point representation;
  (2) input $\mathrm{fl}(x)$ to the program and run it in the computer;
  (3) the output (computed result) of the program belongs to $\mathbb{Y}$ and is called $\widetilde{f}(x)$.
- A problem may have different algorithms (due to different programs). For example: the problem of sum of three numbers: $a + b + c$. Programs: $(a + b) + c$, $a + (b + c)$, and $(a + c) + b$.
- What can happen for an ill-conditioned problem? Since x is perturbed to $\mathrm{fl}(x)$, then $\|\widetilde{f}(x) - f(x)\|$ maybe large.

2.1. Accuracy

- An algorithm $\widetilde{f}$ for a problem f is *accurate* if for each $x \in \mathbb{X}$,

$$\frac{\|\widetilde{f}(x) - f(x)\|}{\|f(x)\|} = \mathcal{O}(\epsilon_{\text{machine}}).$$

- The meaning of $\mathcal{O}(\cdot)$: $\phi(t) = \mathcal{O}(\psi(t))$ means there exists a positive constant C such that $|\phi(t)| \leqslant C\psi(t)$ for all t sufficiently close to an understood limit (e.g., $t \to 0$ or $t \to \infty$).

2.2. Stability

- An algorithm $\widetilde{f}$ for a problem f is *stable* if for each $x \in \mathbb{X}$, there exists $\widetilde{x} \in \mathbb{X}$, such that

$$\frac{\|\widetilde{x} - x\|}{\|x\|} = \mathcal{O}(\epsilon_{\text{machine}}), \quad \frac{\|\widetilde{f}(x) - f(\widetilde{x})\|}{\|f(\widetilde{x})\|} = \mathcal{O}(\epsilon_{\text{machine}}).$$

Remark 1

A stable algorithm gives nearly the right answer to nearly the right question.

2.3. Backward stability

- An algorithm $\widetilde{f}$ for a problem f is *backward stable* if for each $x \in \mathbb{X}$, there exists $\widetilde{x} \in \mathbb{X}$, such that

$$\frac{\|\widetilde{x} - x\|}{\|x\|} = \mathcal{O}(\epsilon_{\text{machine}}), \quad \widetilde{f}(x) = f(\widetilde{x}).$$

Remark 2

A backward stable algorithm gives exactly the right answer to nearly the right question.

Remark 3

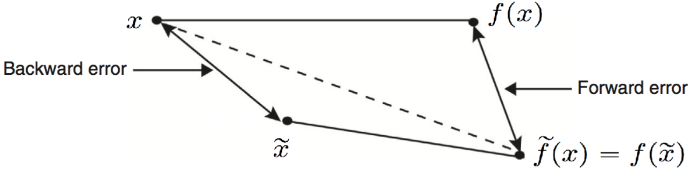
Backward stability obviously implies stability.

Remark 4

Backward stability is both stronger and simpler than stability. Many algorithms of NLA *are backward stable.*

3. Backward error analysis

- The first step is to investigate the conditioning of the problem. The second step is to investigate the backward stability of the corresponding algorithm.
- Forward error $\lesssim$ Condition number $\times$ Backward error.



Theorem 5 (Accuracy of a backward stable algorithm)

Suppose $\tilde{f}$ is backward stable for f. Let $\kappa(f(x))$ denote the condition number of the problem $f(x)$. Then the relative errors satisfy

$$\frac{\|\tilde{f}(x) - f(x)\|}{\|f(x)\|} = \mathcal{O}(\kappa(f(x))\epsilon_{\text{machine}}).$$

Proof.

By the definition of backward stability, we have there exists $\tilde{x}$ such that

$$\frac{\|\tilde{x} - x\|}{\|x\|} = \mathcal{O}(\epsilon_{\text{machine}}), \quad \tilde{f}(x) = f(\tilde{x}).$$

By the definition of $\kappa(f(x))$,

$$\kappa(f(x)) = \lim_{\varepsilon \to 0^+} \sup_{\|\delta x\| \leqslant \varepsilon} \left(\frac{\|\delta f\|}{\|f(x)\|} \Big/ \frac{\|\delta x\|}{\|x\|} \right),$$

we have

$$\frac{\|\tilde{f}(x) - f(x)\|}{\|f(x)\|} \leqslant (\kappa(f(x)) + o(1)) \frac{\|\tilde{x} - x\|}{\|x\|},$$

where $o(1)$ denotes a quantity that converges to zero as $\epsilon_{\text{machine}} \to 0$. Then the statement follows. $\square$

4. Examples

- **Floating point arithmetic**: The floating point operations $\oplus, \ominus, \otimes, \oslash$ are all backward stable. Let $x_1, x_2 \in \mathbb{R}$. Consider the problem $f(x_1, x_2) = x_1 * x_2$ and the corresponding algorithm $\widetilde{f}(x_1, x_2) = \mathrm{fl}(x_1) \circledast \mathrm{fl}(x_2)$. There exist $|\epsilon_1|$, $|\epsilon_2|$, $|\epsilon_3| \leqslant \epsilon_{\text{machine}}$ and $|\epsilon_4|$, $|\epsilon_5| \leqslant 2\epsilon_{\text{machine}} + \mathcal{O}(\epsilon_{\text{machine}}^2)$ such that (except $\otimes$ and $\oslash$)

$$\begin{aligned}
\widetilde{f}(x_1, x_2) &= \mathrm{fl}(x_1) \circledast \mathrm{fl}(x_2) \\
&= ([x_1(1+\epsilon_1)] * [x_2(1+\epsilon_2)])(1+\epsilon_3) \\
&= [x_1(1+\epsilon_1)(1+\epsilon_3)]*[x_2(1+\epsilon_2)(1+\epsilon_3)] \\
&= [x_1(1+\epsilon_4)]*[x_2(1+\epsilon_5)] \\
&= \widetilde{x}_1 * \widetilde{x}_2 = f(\widetilde{x}_1, \widetilde{x}_2).
\end{aligned}$$

Backward stability follows from

$$\frac{|\widetilde{x}_1 - x_1|}{|x_1|} = \mathcal{O}(\epsilon_{\text{machine}}), \qquad \frac{|\widetilde{x}_2 - x_2|}{|x_2|} = \mathcal{O}(\epsilon_{\text{machine}}).$$

- **Inner product**

  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, $\alpha = \mathbf{x}^\top \mathbf{y}$. $\widetilde{\alpha}$ by $\otimes$ and $\oplus$. Backward stable.

- **Outer product**

  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, $\mathbf{A} = \mathbf{x}\mathbf{y}^\top$. $\widetilde{\mathbf{A}}$ by $\otimes$. Stable but not backward stable.

  Explanation: the matrix $\widetilde{\mathbf{A}}$ will be most unlikely to have rank exactly 1, i.e., cannot be written as $(\mathbf{x} + \delta\mathbf{x})(\mathbf{y} + \delta\mathbf{y})^*$.

  As a rule, for problems where the dimension of the space $\mathbb{Y}$ is greater than that of the space $\mathbb{X}$, backward stability is rare.

- **Compute** $f(x) = x + 1$

  By $\oplus$, $\widetilde{f}(x) = \mathrm{fl}(x) \oplus 1$. Stable but not backward stable.

  We have

$$\begin{aligned}\widetilde{f}(x) = \mathrm{fl}(x) \oplus 1 &= (x(1 + \epsilon_1) + 1)(1 + \epsilon_2) \\ &= x(1 + \epsilon_1 + \epsilon_2 + \epsilon_1\epsilon_2) + \epsilon_2 + 1.\end{aligned}$$

Obviously, $x(1+\epsilon_1+\epsilon_2+\epsilon_1\epsilon_2)+\epsilon_2$ is not small compared with $x \to 0$, i.e., for $x \to 0$,

$$\frac{|x(1+\epsilon_1+\epsilon_2+\epsilon_1\epsilon_2)+\epsilon_2|}{|x|} \neq \mathcal{O}(\epsilon_{\text{machine}}).$$

Explanation: For $x \approx 0$, $\oplus$ introduces absolute errors of size $\mathcal{O}(\epsilon_{\text{machine}})$, which cannot be interpreted as caused by small relative perturbations in x. Therefore, not backward stable.

To show stability, for all x, let $\tilde{x} = x(1+\epsilon_1)$. Note that

$$\frac{|\tilde{f}(x) - f(\tilde{x})|}{|f(\tilde{x})|} = \frac{|\epsilon_2(x(1+\epsilon_1)+1)|}{|x(1+\epsilon_1)+1|} = |\epsilon_2| = \mathcal{O}(\epsilon_{\text{machine}}).$$

Then stability follows.

Comparison: Let $x, y \in \mathbb{R}$. Consider $f(x,y) = x + y$ and the corresponding backward stable algorithm $\tilde{f}(x,y) = \text{fl}(x) \oplus \text{fl}(y)$.

4.1. Unitary matrix multiplication: (see also TreBau Exercise 16.1)

- In the rest of this lecture, for simplicity, we always assume that the given data are floating point numbers already if not explicitly stated.

Theorem 6

Left and/or right unitary matrix multiplications are backward stable in the sense: Let $\mathbf{Q}$ *be a unitary matrix. The computed quantity* $\widetilde{\mathbf{B}}$ *for* $\mathbf{B} = \mathbf{QA}$ *or* $\mathbf{B} = \mathbf{AQ}$ *satisfies*

$$\widetilde{\mathbf{B}} = \mathbf{Q}(\mathbf{A} + \delta\mathbf{A}), \quad \text{or} \quad \widetilde{\mathbf{B}} = (\mathbf{A} + \delta\mathbf{A})\mathbf{Q}, \qquad \frac{\|\delta\mathbf{A}\|}{\|\mathbf{A}\|} = \mathcal{O}(\epsilon_{\text{machine}}).$$

Proof.

We only prove the real case. The complex case is similar. Consider the algorithm for the inner product $\mathbf{q}^\top\mathbf{a}$*, then matrix-vector product* $\mathbf{Qa}$*, and then matrix-matrix product* $\mathbf{QA}$*.* □

4.2. An unstable algorithm for computing eigenvalues

- Since z is an eigenvalue of $\mathbf{A}$ if and only if $p(z) = 0$, where $p(z)$ is the characteristic polynomial $\det(z\mathbf{I} - \mathbf{A})$, the roots of $p(z)$ are the eigenvalues of $\mathbf{A}$. This suggests the following algorithm:

  (1). Find the coefficients of the characteristic polynomial.

  (2). Find its roots.

- This algorithm is unstable due to the second step.

  Explanation: The problem of finding the roots of a polynomial, given the coefficients, is generally ill-conditioned. Therefore, although only small errors exist in the coefficients of the polynomials, the difference between their roots, $\|r(p) - r(\widetilde{p})\|$, maybe vastly larger than $\epsilon_{\text{machine}}\|r(p)\|$. Instability follows.

- See the discussion of TreBau's book – Numerical linear algebra, page 110–111.

4.3. Backward stability of back substitution

- The solution of the nonsingular upper-triangular system

$$\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1m} \\ & r_{22} & \ddots & \vdots \\ & & \ddots & \vdots \\ & & & r_{mm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$

can be obtained by the following back substitution algorithm

Algorithm: Back substitution

$$\begin{aligned} x_m &= b_m / r_{mm} \\ x_{m-1} &= (b_{m-1} - x_m r_{m-1,m}) / r_{m-1,m-1} \\ x_{m-2} &= (b_{m-2} - x_{m-1} r_{m-2,m-1} - x_m r_{m-2,m}) / r_{m-2,m-2} \\ &\vdots \\ x_j &= (b_j - \textstyle\sum_{k=j+1}^{m} x_k r_{jk}) / r_{jj} \end{aligned}$$

Theorem 7

Back substitution is backward stable in the sense that the computed solution $\tilde{\mathbf{x}} \in \mathbb{C}^m$ *satisfies*

$$(\mathbf{R} + \delta\mathbf{R})\tilde{\mathbf{x}} = \mathbf{b},$$

for some upper-triangular $\delta\mathbf{R} \in \mathbb{C}^{m\times m}$ *with*

$$\frac{\|\delta\mathbf{R}\|}{\|\mathbf{R}\|} = \mathcal{O}(\epsilon_{\text{machine}}).$$

Specifically, for each i, j,

$$\frac{|\delta r_{ij}|}{|r_{ij}|} \leqslant m\epsilon_{\text{machine}} + \mathcal{O}(\epsilon_{\text{machine}}^2).$$

- Our task is to express every floating point error as a perturbation of the input.

(i) The case $m = 1$:

$$\widetilde{x}_1 = b_1 \oplus r_{11} = \frac{b_1(1+\epsilon_1)}{r_{11}}, \quad |\epsilon_1| \leqslant \epsilon_{\text{machine}}$$

Set $1 + \epsilon_1' = 1/(1+\epsilon_1)$. We have

$$\epsilon_1' = -\frac{\epsilon_1}{1+\epsilon_1} \Rightarrow \widetilde{x}_1 = \frac{b_1}{r_{11}(1+\epsilon_1')}, \quad |\epsilon_1'| \leqslant \epsilon_{\text{machine}} + \mathcal{O}(\epsilon_{\text{machine}}^2).$$

Therefore

$$(r_{11} + \delta r_{11})\widetilde{x}_1 = b_1; \quad \delta r_{11} = \epsilon_1' r_{11}; \quad \frac{|\delta r_{11}|}{|r_{11}|} \leqslant \epsilon_{\text{machine}} + \mathcal{O}(\epsilon_{\text{machine}}^2).$$

(ii) The case $m = 2$. The first step is the same as in $m = 1$ case,

$$\widetilde{x}_2 = b_2 \oplus r_{22} = \frac{b_2}{r_{22}(1+\epsilon_1)}, \quad |\epsilon_1| \leqslant \epsilon_{\text{machine}} + \mathcal{O}(\epsilon_{\text{machine}}^2).$$

The second step: there exist $|\epsilon_2|, |\epsilon_3|, |\epsilon_4| \leqslant \epsilon_{\text{machine}}$,

$$
\begin{aligned}
\widetilde{x}_1 &= (b_1 \ominus (\widetilde{x}_2 \otimes r_{12})) \oplus r_{11} = (b_1 \ominus \widetilde{x}_2 r_{12}(1+\epsilon_2)) \oplus r_{11} \\
&= (b_1 - \widetilde{x}_2 r_{12}(1+\epsilon_2))(1+\epsilon_3) \oplus r_{11} \\
&= \frac{(b_1 - \widetilde{x}_2 r_{12}(1+\epsilon_2))(1+\epsilon_3)}{r_{11}}(1+\epsilon_4).
\end{aligned}
$$

Shift ϵ_3 and ϵ_4 to the denominator

$$
\widetilde{x}_1 = \frac{b_1 - \widetilde{x}_2 r_{12}(1+\epsilon_2)}{r_{11}(1+\epsilon_3')(1+\epsilon_4')},
$$

or equivalently,

$$
\widetilde{x}_1 = \frac{b_1 - \widetilde{x}_2 r_{12}(1+\epsilon_2)}{r_{11}(1+2\epsilon_5)}, \quad |\epsilon_3'|, |\epsilon_4'|, |\epsilon_5| \leqslant \epsilon_{\text{machine}} + \mathcal{O}(\epsilon_{\text{machine}}^2).
$$

Obviously, $\widetilde{x}_1$ is exactly correct if r_{22}, r_{12} and r_{11} perturbed by factors $(1+\epsilon_1)$, $(1+\epsilon_2)$ and $(1+2\epsilon_5)$, respectively. Thus,

$$
(\mathbf{R} + \delta\mathbf{R})\widetilde{\mathbf{x}} = \mathbf{b},
$$

where the entries δr_{ij} of $\delta\mathbf{R}$ satisfy

$$\begin{bmatrix} \frac{|\delta r_{11}|}{|r_{11}|} & \frac{|\delta r_{12}|}{|r_{12}|} \\ & \frac{|\delta r_{22}|}{|r_{22}|} \end{bmatrix} = \begin{bmatrix} 2|\epsilon_5| & |\epsilon_2| \\ & |\epsilon_1| \end{bmatrix} \leqslant \begin{bmatrix} 2 & 1 \\ & 1 \end{bmatrix} \epsilon_{\text{machine}} + \mathcal{O}(\epsilon_{\text{machine}}^2).$$

The last formula guarantees $\|\delta\mathbf{R}\|/\|\mathbf{R}\| = \mathcal{O}(\epsilon_{\text{machine}})$ in any norm.
(iii) The case $m = 3$. The first two steps are the same as before:

$$\widetilde{x}_3 = b_3 \oplus\!\!\!\!\div\, r_{33} = \frac{b_3}{r_{33}(1+\epsilon_1)},$$

$$\widetilde{x}_2 = (b_2 \ominus (\widetilde{x}_3 \otimes r_{23})) \oplus\!\!\!\!\div\, r_{22} = \frac{b_2 - \widetilde{x}_3 r_{23}(1+\epsilon_2)}{r_{22}(1+2\epsilon_3)},$$

where

$$\begin{bmatrix} 2|\epsilon_3| & |\epsilon_2| \\ & |\epsilon_1| \end{bmatrix} \leqslant \begin{bmatrix} 2 & 1 \\ & 1 \end{bmatrix} \epsilon_1 + \mathcal{O}(\epsilon_{\text{machine}}^2)$$

The third step:

$$\begin{aligned}
\widetilde{x}_1 &= [(b_1 \ominus (\widetilde{x}_2 \otimes r_{12})) \ominus (\widetilde{x}_3 \otimes r_{13})] \oslash r_{11} \\
&= [(b_1 - \widetilde{x}_2 r_{12}(1+\epsilon_4))(1+\epsilon_6) - \widetilde{x}_3 r_{13}(1+\epsilon_5)](1+\epsilon_7) \oslash r_{11} \\
&= \frac{[(b_1 - \widetilde{x}_2 r_{12}(1+\epsilon_4))(1+\epsilon_6) - \widetilde{x}_3 r_{13}(1+\epsilon_5)](1+\epsilon_7)}{r_{11}(1+\epsilon_8')} \\
&= \frac{b_1 - \widetilde{x}_2 r_{12}(1+\epsilon_4) - \widetilde{x}_3 r_{13}(1+\epsilon_5)(1+\epsilon_6')}{r_{11}(1+\epsilon_6')(1+\epsilon_7')(1+\epsilon_8')},
\end{aligned}$$

r_{13} has two perturbations of size at most $\epsilon_{\text{machine}}$, r_{11} has three. Then we have $(\mathbf{R} + \delta\mathbf{R})\tilde{\mathbf{x}} = \mathbf{b}$ with the entries δr_{ij} satisfying

$$\begin{bmatrix} \frac{|\delta r_{11}|}{|r_{11}|} & \frac{|\delta r_{12}|}{|r_{12}|} & \frac{|\delta r_{13}|}{|r_{13}|} \\ & \frac{|\delta r_{22}|}{|r_{22}|} & \frac{|\delta r_{23}|}{|r_{23}|} \\ & & \frac{|\delta r_{33}|}{|r_{33}|} \end{bmatrix} \leqslant \begin{bmatrix} 3 & 1 & 2 \\ & 2 & 1 \\ & & 1 \end{bmatrix} \epsilon_{\text{machine}} + \mathcal{O}(\epsilon_{\text{machine}}^2).$$

(iv) General m: Higher-dimensional cases are similar. For example, 5×5 case:

$$\frac{|\delta \mathbf{R}|}{|\mathbf{R}|} \leqslant \begin{bmatrix} 5 & 1 & 2 & 3 & 4 \\ & 4 & 1 & 2 & 3 \\ & & 3 & 1 & 2 \\ & & & 2 & 1 \\ & & & & 1 \end{bmatrix} \epsilon_{\text{machine}} + \mathcal{O}(\epsilon_{\text{machine}}^2).$$

The entries of the matrix in this formula are obtained from three components. The multiplications $\widetilde{x}_k r_{jk}$ introduce $\epsilon_{\text{machine}}$ perturbations in the pattern

$$\otimes : \widetilde{x}_k r_{jk} \qquad \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ & 0 & 1 & 1 & 1 \\ & & 0 & 1 & 1 \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}. \quad \text{(inner level)}$$

The division by r_{kk} introduce perturbations in the pattern

$$\oplus : \text{ divisions by } r_{kk} \qquad \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}. \quad \text{(outer level)}$$

Finally, the subtractions also occur in the pattern for $\otimes$, and, due to the decision to compute from left to right, each one introduces a perturbation on the diagonal and at each position to the right. This adds up to the pattern

$$\ominus : \qquad \begin{bmatrix} 4 & 0 & 1 & 2 & 3 \\ & 3 & 0 & 1 & 2 \\ & & 2 & 0 & 1 \\ & & & 1 & 0 \\ & & & & 0 \end{bmatrix}.$$

Remark 8

Perturbations of order $\epsilon_{\text{machine}}$ are composed additively and moved freely between numerators and denominators since the difference is of order $\epsilon_{\text{machine}}^2$.

Remark 9

More than one error bound can be derived for a given algorithm. In the present case, we could have perturbed b_j as well as r_{ij}, avoiding the need for the trickery represented pattern for $\ominus$. On the other hand, a final result in which only $\mathbf{R}$ is perturbed is appealing clean.

Remark 10

We have done componentwise backward error bound. If $r_{ij} = 0$, this entry undergoes no perturbation at all: $\delta\mathbf{R}$ has the same sparsity pattern as $\mathbf{R}$.