

Lecture 1: Inner product, Orthogonality, Vector/Matrix norms



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1. Inner product on a linear space \mathbb{V} over a number field \mathbb{F} (\mathbb{C} or \mathbb{R})

- **Definition:** A function $\langle \cdot, \cdot \rangle : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{F}$ is called an *inner product*, if it satisfies the following three conditions ($\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{V}, \forall \alpha \in \mathbb{F}$):

(1) Conjugate symmetry:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$$

(2) Positive definiteness:

$$\langle \mathbf{x}, \mathbf{x} \rangle \geq 0, \quad \langle \mathbf{x}, \mathbf{x} \rangle = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$$

(3) Linearity in the first variable:

$$\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle, \quad \langle \alpha \mathbf{x}, \mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$$

Example: the standard inner product on the space $\mathbb{V} = \mathbb{C}^m$:

$$\forall \mathbf{x}, \mathbf{y} \in \mathbb{C}^m, \quad \langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^* \mathbf{x} = \sum_{i=1}^m x_i \bar{y}_i.$$

Example: the \mathbf{A} -inner product on the space $\mathbb{V} = \mathbb{C}^m$:

$$\forall \mathbf{x}, \mathbf{y} \in \mathbb{C}^m, \quad \langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^* \mathbf{A} \mathbf{x},$$

where \mathbf{A} is a given Hermitian positive definite matrix.

2. Orthogonality

- Orthogonality is a mathematical concept with respect to a given inner product $\langle \cdot, \cdot \rangle$.
 - (1) Two vectors \mathbf{x} and \mathbf{y} are called *orthogonal* if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.
 - (2) Two sets of vectors \mathcal{X} and \mathcal{Y} are called orthogonal if $\forall \mathbf{x} \in \mathcal{X}$ and $\mathbf{y} \in \mathcal{Y}$, $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.
 - (3) A set of nonzero vectors \mathcal{S} is orthogonal if $\forall \mathbf{x}, \mathbf{y} \in \mathcal{S}$ and $\mathbf{x} \neq \mathbf{y}$, $\langle \mathbf{x}, \mathbf{y} \rangle = 0$; if further $\forall \mathbf{x} \in \mathcal{S}$, $\langle \mathbf{x}, \mathbf{x} \rangle = 1$, \mathcal{S} is called *orthonormal*.

Proposition 1

The vectors in an orthogonal set \mathcal{S} are linearly independent.

2.1. Orthogonal components of a vector

- Inner products can be used to decompose arbitrary vectors into orthogonal components. Given an *orthonormal* set $\{\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n\}$ and an arbitrary vector \mathbf{v} , let

$$\mathbf{r} = \mathbf{v} - \langle \mathbf{v}, \mathbf{q}_1 \rangle \mathbf{q}_1 - \langle \mathbf{v}, \mathbf{q}_2 \rangle \mathbf{q}_2 - \dots - \langle \mathbf{v}, \mathbf{q}_n \rangle \mathbf{q}_n.$$

Obviously,

$$\mathbf{r} \in \text{span}\{\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n\}^\perp.$$

Thus we see that \mathbf{v} can be decomposed into $n + 1$ orthogonal components:

$$\mathbf{v} = \mathbf{r} + \langle \mathbf{v}, \mathbf{q}_1 \rangle \mathbf{q}_1 + \langle \mathbf{v}, \mathbf{q}_2 \rangle \mathbf{q}_2 + \dots + \langle \mathbf{v}, \mathbf{q}_n \rangle \mathbf{q}_n.$$

We call $\langle \mathbf{v}, \mathbf{q}_i \rangle \mathbf{q}_i$ the part of \mathbf{v} in the direction of \mathbf{q}_i , and \mathbf{r} the part of \mathbf{v} orthogonal to the subspace $\text{span}\{\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n\}$.

Exercise: Write the expression for \mathbf{v} when the set $\{\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n\}$ is only orthogonal.

- **Cauchy–Schwarz inequality:** For any given inner product $\langle \cdot, \cdot \rangle$,

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} \sqrt{\langle \mathbf{y}, \mathbf{y} \rangle}.$$

The equality holds if and only if \mathbf{x} and \mathbf{y} are linearly dependent.

Exercise: Prove the inequality. Hint: write

$$\mathbf{x} = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\langle \mathbf{y}, \mathbf{y} \rangle} \mathbf{y} + \mathbf{z}.$$

Then $\langle \mathbf{z}, \mathbf{y} \rangle = 0$. Consider $\langle \mathbf{x}, \mathbf{x} \rangle$.

Application: For any Hermitian positive definite matrix \mathbf{A} ,

$$|\mathbf{y}^* \mathbf{A} \mathbf{x}|^2 \leq (\mathbf{x}^* \mathbf{A} \mathbf{x})(\mathbf{y}^* \mathbf{A} \mathbf{y}).$$

3. Norm on a linear space \mathbb{V} over a number field \mathbb{F} (\mathbb{C} or \mathbb{R})

- **Definition:** A function $\|\cdot\| : \mathbb{V} \rightarrow \mathbb{R}$ is called a *norm* if it satisfies the following three conditions ($\forall \mathbf{x}, \mathbf{y} \in \mathbb{V}$ and $\forall \alpha \in \mathbb{F}$):

(1) Nonnegativity:

$$\|\mathbf{x}\| \geq 0, \quad \|\mathbf{x}\| = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$$

(2) Positive homogeneity:

$$\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$$

(3) Triangle inequality:

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$$

Exercise: Show that any norm is continuous.

- **More on metric, norm, and inner product** (click!)



Exercise: For any given inner product $\langle \cdot, \cdot \rangle$, let $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$.

(1) Prove that the function $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$ is a norm.

(2) Prove the parallelogram law

$$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2.$$

(3) For a set of n orthogonal (with respect to the inner product $\langle \cdot, \cdot \rangle$) vectors $\{\mathbf{x}_i\}$, prove that

$$\left\| \sum_{i=1}^n \mathbf{x}_i \right\|^2 = \sum_{i=1}^n \|\mathbf{x}_i\|^2.$$

The function $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$ is called the norm *induced* by the inner product $\langle \cdot, \cdot \rangle$. Using this norm, we can write the Cauchy–Schwarz inequality as

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \|\mathbf{y}\|.$$

Theorem 2 (Equivalence of norms)

For each pair of norms $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ on a *finite-dimensional* linear space \mathbb{V} , there exist positive constants $a > 0$ and $b > 0$ (depending only on the norms) such that

$$a\|\mathbf{x}\|_\beta \leq \|\mathbf{x}\|_\alpha \leq b\|\mathbf{x}\|_\beta, \quad \forall \mathbf{x} \in \mathbb{V}.$$

Proof.

For all $\mathbf{x} = \sum_i x_i \mathbf{v}_i$, where $\{\mathbf{v}_i\}$ is a basis of \mathbb{V} , $\|\mathbf{x}\| = \sum_i |x_i|$ is a norm on \mathbb{V} . By $\|\mathbf{x}\|_\alpha = \|\sum_i x_i \mathbf{v}_i\|_\alpha \leq \sum_i |x_i| \|\mathbf{v}_i\|_\alpha \leq \|\mathbf{x}\| \cdot \max_i \|\mathbf{v}_i\|_\alpha$, we know $\|\cdot\|_\alpha$ is a continuous function with respect to $\|\cdot\|$, which attains its minimum c and maximum C on the unit sphere $\{\mathbf{x} \in \mathbb{V}, \|\mathbf{x}\| = 1\}$ (because it is a compact set). Then, $\forall \mathbf{x} \in \mathbb{V}$, $c\|\mathbf{x}\| \leq \|\mathbf{x}\|_\alpha \leq C\|\mathbf{x}\|$. \square

- **Convergence of a sequence** $\{\mathbf{x}_k\} \subset \mathbb{V}$: $\mathbf{x}_k \rightarrow \mathbf{x}$

We say \mathbf{x}_k converges to \mathbf{x} if $\lim_{k \rightarrow \infty} \|\mathbf{x}_k - \mathbf{x}\| = 0$.

3.1. Vector norms on \mathbb{C}^m

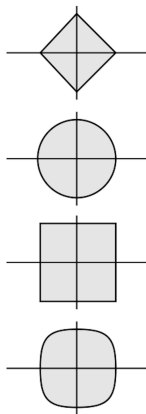
- ℓ_p -norm

$$\|\mathbf{x}\|_1 = \sum_{i=1}^m |x_i|,$$

$$\|\mathbf{x}\|_2 = \left(\sum_{i=1}^m |x_i|^2 \right)^{1/2} = \sqrt{\mathbf{x}^* \mathbf{x}},$$

$$\|\mathbf{x}\|_\infty = \max_{1 \leq i \leq m} |x_i|,$$

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^m |x_i|^p \right)^{1/p}, \quad (1 \leq p < \infty)$$



- Inequalities: $\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1 \leq \sqrt{m} \|\mathbf{x}\|_2,$

$$\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2 \leq \sqrt{m} \|\mathbf{x}\|_\infty, \quad \|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_1 \leq m \|\mathbf{x}\|_\infty.$$

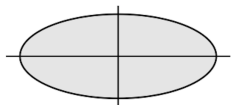
- Weighted norm

Let $\|\cdot\|$ denote any norm on \mathbb{C}^m . Suppose a diagonal matrix $\mathbf{W} = \text{diag}\{w_1, \dots, w_m\}$, $w_i \neq 0$. Then

$$\|\mathbf{x}\|_{\mathbf{W}} = \|\mathbf{W}\mathbf{x}\|$$

is a norm, called *weighted norm*. For example, weighted 2-norm

$$\|\mathbf{x}\|_{\mathbf{W}} = \|\mathbf{W}\mathbf{x}\|_2 = \left(\sum_{i=1}^m |w_i x_i|^2 \right)^{1/2}.$$



- Dual norm

Let $\|\cdot\|$ denote any norm on \mathbb{C}^m . The corresponding *dual norm* $\|\cdot\|'$ (with respect to an inner product $\langle \cdot, \cdot \rangle$) is defined by

$$\|\mathbf{x}\|' = \sup_{\mathbf{y} \in \mathbb{C}^m, \|\mathbf{y}\|=1} |\langle \mathbf{x}, \mathbf{y} \rangle|.$$

Exercise: If $p, q \in [1, \infty]$ with $1/p + 1/q = 1$, then $\|\cdot\|'_p = \|\cdot\|_q$.

3.2. Matrix norms on $\mathbb{C}^{m \times n}$

- Frobenius norm: $\forall \mathbf{A} = [\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n] \in \mathbb{C}^{m \times n}$, define

$$\|\mathbf{A}\|_F := \left(\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2 \right)^{1/2} = \left(\sum_{j=1}^n \|\mathbf{a}_j\|_2^2 \right)^{1/2}$$

or

$$\|\mathbf{A}\|_F = \sqrt{\text{tr}(\mathbf{A}^* \mathbf{A})} = \sqrt{\text{tr}(\mathbf{A} \mathbf{A}^*)}.$$

- Max norm:

$$\|\mathbf{A}\|_{\max} := \max_{i,j} |a_{ij}|.$$

- Induced matrix norm (operator norm): $\forall \mathbf{A} \in \mathbb{C}^{m \times n}$, define

$$\|\mathbf{A}\|_{\alpha,\beta} := \sup_{\substack{\mathbf{x} \in \mathbb{C}^n \\ \mathbf{x} \neq \mathbf{0}}} \frac{\|\mathbf{A}\mathbf{x}\|_{\alpha}}{\|\mathbf{x}\|_{\beta}} = \sup_{\substack{\mathbf{x} \in \mathbb{C}^n \\ \|\mathbf{x}\|_{\beta}=1}} \|\mathbf{A}\mathbf{x}\|_{\alpha} = \sup_{\substack{\mathbf{x} \in \mathbb{C}^n \\ \|\mathbf{x}\|_{\beta} \leq 1}} \|\mathbf{A}\mathbf{x}\|_{\alpha},$$

where $\|\cdot\|_{\alpha}$ is a norm on \mathbb{C}^m and $\|\cdot\|_{\beta}$ is a norm on \mathbb{C}^n . We say that $\|\cdot\|_{\alpha,\beta}$ is the matrix norm induced by $\|\cdot\|_{\alpha}$ and $\|\cdot\|_{\beta}$.

Exercise: $\forall \mathbf{x} \in \mathbb{C}^n$, prove that

$$\|\mathbf{Ax}\|_\alpha \leq \|\mathbf{A}\|_{\alpha,\beta} \|\mathbf{x}\|_\beta.$$

Exercise: Let $\mathbf{A} \in \mathbb{C}^{m \times n}$, $\mathbf{B} \in \mathbb{C}^{n \times r}$ and let $\|\cdot\|_\alpha$, $\|\cdot\|_\beta$, and $\|\cdot\|_\gamma$ be norms on \mathbb{C}^m , \mathbb{C}^n , and \mathbb{C}^r , respectively. Prove the induced matrix norms $\|\cdot\|_{\alpha,\gamma}$, $\|\cdot\|_{\alpha,\beta}$, and $\|\cdot\|_{\beta,\gamma}$ satisfy

$$\|\mathbf{AB}\|_{\alpha,\gamma} \leq \|\mathbf{A}\|_{\alpha,\beta} \|\mathbf{B}\|_{\beta,\gamma}.$$

Exercise: Prove that

$$\|\mathbf{A}\|_{\infty,1} = \max_{i,j} |a_{ij}|,$$

i.e., $\|\mathbf{A}\|_{\max}$ is the matrix norm induced by $\|\cdot\|_\infty$ and $\|\cdot\|_1$.

- The Frobenius norm $\|\cdot\|_F$ on $\mathbb{C}^{m \times n}$ is not induced by norms on \mathbb{C}^m and \mathbb{C}^n . (See [Ref. 1](#) and [Ref. 2](#))

- Induced matrix p -norm of $\mathbf{A} \in \mathbb{C}^{m \times n}$: For $p \in [1, +\infty]$,

$$\|\mathbf{A}\|_p := \|\mathbf{A}\|_{p,p} = \sup_{\mathbf{x} \in \mathbb{C}^n, \|\mathbf{x}\|_p=1} \|\mathbf{A}\mathbf{x}\|_p.$$

Example: p -norm of a diagonal matrix $\mathbf{D} = \text{diag}\{d_1, \dots, d_m\}$

$$\|\mathbf{D}\|_p = \max_{1 \leq i \leq m} |d_i|$$

Example: $1, 2, \infty$ -norm

$$\|\mathbf{A}\|_1 = \max_j \sum_i |a_{ij}|, \quad \|\mathbf{A}\|_\infty = \max_i \sum_j |a_{ij}|$$

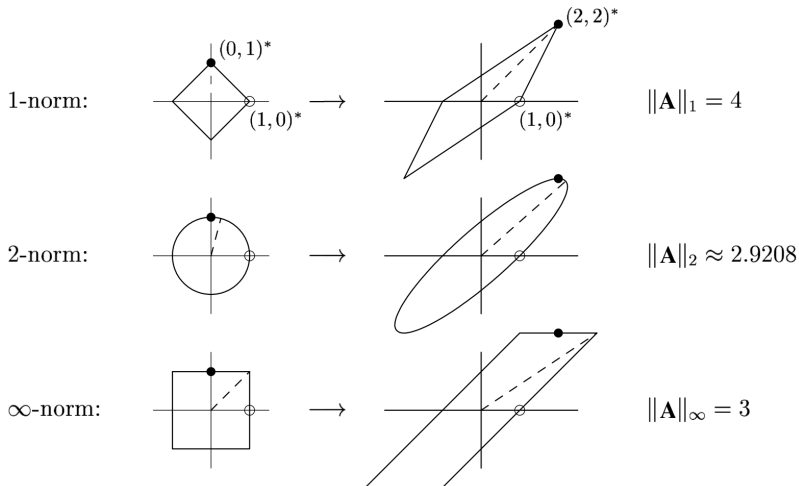
$$\|\mathbf{A}\|_2 = \sqrt{\lambda_{\max}(\mathbf{A}^* \mathbf{A})} = \sqrt{\lambda_{\max}(\mathbf{A} \mathbf{A}^*)} \leq \|\mathbf{A}\|_F$$

The norm $\|\cdot\|_2$ on $\mathbb{C}^{m \times n}$ is also called the spectral norm.

Inequalities: $\|\mathbf{A}\|_\infty \leq \sqrt{n} \|\mathbf{A}\|_2, \quad \|\mathbf{A}\|_2 \leq \sqrt{m} \|\mathbf{A}\|_\infty.$

- Matlab: `norm` for $1, 2, \infty$ -norm

Example: $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$



3.3. Unitary invariance of $\|\cdot\|_2$ and $\|\cdot\|_F$: $\forall \mathbf{A} \in \mathbb{C}^{m \times n}$

- If \mathbf{P} has orthonormal columns, i.e.,

$$\mathbf{P} \in \mathbb{C}^{p \times m}, \quad p \geq m, \quad \mathbf{P}^* \mathbf{P} = \mathbf{I}_m,$$

then

$$\|\mathbf{P}\mathbf{A}\|_2 = \|\mathbf{A}\|_2, \quad \|\mathbf{P}\mathbf{A}\|_F = \|\mathbf{A}\|_F.$$

- If \mathbf{Q} has orthonormal rows, i.e.,

$$\mathbf{Q} \in \mathbb{C}^{n \times q}, \quad n \leq q, \quad \mathbf{Q}\mathbf{Q}^* = \mathbf{I}_n,$$

then

$$\|\mathbf{A}\mathbf{Q}\|_2 = \|\mathbf{A}\|_2, \quad \|\mathbf{A}\mathbf{Q}\|_F = \|\mathbf{A}\|_F.$$

4. Unitary matrix

- For $\mathbf{Q} \in \mathbb{C}^{m \times m}$, if $\mathbf{Q}^* = \mathbf{Q}^{-1}$, i.e., $\mathbf{Q}^* \mathbf{Q} = \mathbf{I}$, \mathbf{Q} is called *unitary* (or *orthogonal* in the real case).

$$\begin{bmatrix} \mathbf{q}_1^* \\ \mathbf{q}_2^* \\ \vdots \\ \mathbf{q}_m^* \end{bmatrix} \begin{bmatrix} | & | & & | \\ \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_m \\ | & | & & | \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

Exercise: Let $\mathbf{Q} \in \mathbb{C}^{m \times m}$ be a unitary matrix. Prove

$$\|\mathbf{Q}\|_2 = 1, \quad \|\mathbf{Q}\|_F = \sqrt{m}.$$

- A unitary matrix has both orthonormal rows and orthonormal columns.
- The columns of a unitary matrix form an orthonormal basis of \mathbb{C}^m and vice versa.