Lecture 18: Multigrid

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1. Finite difference discretization of a BVP

- Consider the following 1-D Dirichlet boundary value problem

$$\begin{cases} -u''(x) = f(x), & x \in (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

- For $n \in \mathbb{N}$, let

$$h = \frac{1}{n+1}, \quad x_i = ih = \frac{i}{n+1}, \quad 0 \le i \le n+1.$$

- The finite difference method is: let $u_0^h = u_{n+1}^h = 0$, $f_i^h = f(x_i)$, $1 \le i \le n$, find

$$\mathbf{u}^h = \begin{bmatrix} u_1^h & u_2^h & \cdots & u_n^h \end{bmatrix}^\top$$

such that

$$-\frac{u_{i+1}^h - 2u_i^h + u_{i-1}^h}{h^2} = f_i^h, \quad 1 \le i \le n.$$

- The FD system $\mathbf{A}_h \mathbf{u}^h = \mathbf{f}^h$:

$$\frac{1}{h^2}\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}\begin{bmatrix} u_1^h \\ u_2^h \\ \vdots \\ u_{n-1}^h \\ u_n^h \end{bmatrix} = \begin{bmatrix} f_1^h \\ f_2^h \\ \vdots \\ f_{n-1}^h \\ f_n^h \end{bmatrix}.$$

2. Classical stationary iterative methods (Lecture 6)

- Given a starting vector $\mathbf{u}^{(0)}$,

$$\mathbf{u}^{(j)} = \mathbf{R}\mathbf{u}^{(j-1)} + \mathbf{c}, \quad j = 1, 2, \ldots$$

  - Jacobi's method
  - Gauss–Seidel method
  - Successive overrelaxation: SOR(ω)
  - Symmetric successive overrelaxation: SSOR(ω)

2.1. Jacobi's method and its relaxation for the FD system

- The iteration matrix of Jacobi's method

$$\mathbf{R} = \mathbf{D}^{-1}(\mathbf{D} - \mathbf{A}) = \begin{bmatrix} \frac{1}{2} & & & & \\ & \frac{1}{2} & & & \\ & & \ddots & & \\ & & & \frac{1}{2} & \\ & & & & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & 1 \\ & & & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{2} & 0 & \frac{1}{2} \\ & & & \frac{1}{2} & 0 \end{bmatrix}.$$

- The relaxation of Jacobi's method:

$$\mathbf{u}^{(j)} = \mathbf{R}(\omega)\mathbf{u}^{(j-1)} + \omega\mathbf{c}, \quad \mathbf{R}(\omega) = (1-\omega)\mathbf{I} + \omega\mathbf{R} = \mathbf{I} - \omega\mathbf{D}^{-1}\mathbf{A}.$$

- The eigenvalues of $\mathbf{R}$ are given by

$$\lambda_k = \cos(k\pi h), \quad 1 \leq k \leq n,$$

and the corresponding eigenvectors are given by

$$\mathbf{v}_k = \begin{bmatrix} \sin(k\pi h) & \sin(2k\pi h) & \cdots & \sin(nk\pi h) \end{bmatrix}^\top, \quad 1 \leq k \leq n.$$

The convergence of the Jacobi's method becomes worse for larger n since the spectral radius approaches 1 in this situation.

- The eigenvalues of $\mathbf{R}(\omega)$ are given by

$$\lambda_k(\omega) = 1 - \omega + \omega\lambda_k = 1 - \omega + \omega\cos(k\pi h), \quad 1 \leq k \leq n.$$

Note that relaxation does not lead to an improved convergence, since, in this case, the optimal relaxation parameter is $\omega_\star = 1$. (why?)

- $\mathbf{R}(\omega)$ and $\mathbf{R}$ have the same eigenvectors.

2.2. What makes the convergence of Jacobi's method slow?

- Recall that the solution $\mathbf{u}$ of the linear system is a fixed point, i.e.,

$$\mathbf{u} = \mathbf{R}\mathbf{u} + \mathbf{c}.$$

This leads to

$$\mathbf{u} - \mathbf{u}^{(j)} = \mathbf{R}(\mathbf{u} - \mathbf{u}^{(j-1)}) = \cdots = \mathbf{R}^j(\mathbf{u} - \mathbf{u}^{(0)}).$$

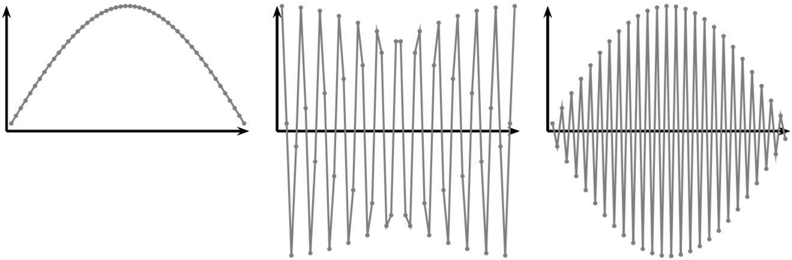
We expand $\mathbf{u} - \mathbf{u}^{(0)}$ in the basis consisting of the eigenvectors:

$$\mathbf{u} - \mathbf{u}^{(0)} = \sum_{k=1}^{n} \alpha_k \mathbf{v}_k.$$

This gives

$$\mathbf{u} - \mathbf{u}^{(j)} = \sum_{k=1}^{n} \alpha_k \lambda_k^j \mathbf{v}_k.$$

- If $|\lambda_k|$ is small then the component of $\mathbf{u} - \mathbf{u}^{(j)}$ in the direction of $\mathbf{v}_k$ vanishes quickly.
- After only a few iterations, the error is dominated by those components in direction $\mathbf{v}_k$, where $|\lambda_k| \approx 1$.
- The eigenvectors of $\mathbf{R}(\omega)$ with $n = 50$. From left to right: $\mathbf{v}_1$, $\mathbf{v}_{25}$, and $\mathbf{v}_{50}$. Each graph shows the points $(ih, (\mathbf{v}_k)_i)$ for $1 \leq i \leq n$ linearly connected.



Low-frequency $(k \leq n/2)$, high-frequency $(k > n/2)$

- Since $|\lambda_1| = |\lambda_n| \approx 1$, the error in direction $\mathbf{v}_1$ and direction $\mathbf{v}_n$ is large, which means no matter how many steps in Jacobi's method we compute, the error will always contain both low-frequency and high-frequency eigenvectors.
- To avoid this, let us have another look at the relaxation of Jacobi's method. Choosing $\omega = 1/2$ yields the eigenvalues

$$\lambda_k(1/2) = (1 + \cos(k\pi h))/2, \quad 1 \leq k \leq n.$$

For large k this means that $\lambda_k(1/2)$ is very close to zero, while for small k we have $\lambda_k(1/2)$ very close to 1.
- Consider the error after j iterations

$$\mathbf{u} - \mathbf{u}^{(j)} = [\mathbf{R}(1/2)]^j(\mathbf{u} - \mathbf{u}^{(0)}) = \sum_{k=1}^{n} \alpha_k [\lambda_k(1/2)]^j \mathbf{v}_k.$$

The low-frequency eigenvectors dominate and the influence of the high-frequency eigenvectors tends to zero.

- The error, in a certain way, is "smoothed" during the process.
- A "smoother" error can be represented using a smaller n and this gives the idea of the two-grid method, as follows.
  - Compute j steps of the relaxation, resulting in an error

$$\boldsymbol{\varepsilon}^{(j)} = \mathbf{u} - \mathbf{u}^{(j)}$$

  which is much "smoother" than $\boldsymbol{\varepsilon}^{(0)}$.
  - We have $\mathbf{u} = \mathbf{u}^{(j)} + \boldsymbol{\varepsilon}^{(j)}$ and $\boldsymbol{\varepsilon}^{(j)}$ satisfies

$$\mathbf{A}\boldsymbol{\varepsilon}^{(j)} = \mathbf{A}(\mathbf{u} - \mathbf{u}^{(j)}) = \mathbf{f} - \mathbf{A}\mathbf{u}^{(j)} =: \mathbf{r}^{(j)}.$$

  Hence, if we can solve $\mathbf{A}\boldsymbol{\varepsilon}^{(j)} = \mathbf{r}^{(j)}$ then the overall solution is given by $\mathbf{u} = \mathbf{u}^{(j)} + \boldsymbol{\varepsilon}^{(j)}$.
  - Since we expect the error $\boldsymbol{\varepsilon}^{(j)}$ to be "smooth", we will solve the equation $\mathbf{A}\boldsymbol{\varepsilon}^{(j)} = \mathbf{r}^{(j)}$ somehow on a coarser grid to save computational time and transfer the solution back to the finer grid.

3. Two-grid, V-cycle, and Multigrid

- Assume that we are given two grids: a fine grid X_h with n_h points and a coarse grid X_H with $n_H < n_h$ points. Associated with these grids are discrete solution spaces $V_h = \mathbb{R}^{n_h}$ and $V_H = \mathbb{R}^{n_H}$.
- We need a *prolongation* operator $\mathbf{I}_H^h : V_H \mapsto V_h$ which maps from coarse to fine, and a *restriction* operator $\mathbf{I}_h^H : V_h \mapsto V_H$ which maps from fine to coarse.
- For simplicity, suppose the coarse grid is given by

$$X_H = \{jH : 0 \leq j \leq n_H - 1\}$$

with $n_H = 2^m + 1$, $m \in \mathbb{N}$, and $H = 1/(n_H - 1)$. Then the natural fine grid X_h would consist of X_H and all points in the middle between two points from X_H, i.e.,

$$X_h = \{jh : 0 \leq j \leq n_h - 1\}$$

with $h = H/2$ and $n_h = 2^{m+1} + 1$.

- In this case we could define the prolongation and restriction operators as follows. The prolongation $\mathbf{v}^h = \mathbf{I}_H^h \mathbf{v}^H$ is defined by linear interpolation on the "in-between" points:

$$\begin{aligned} v_{2j}^h &:= v_j^H, && 0 \le j \le n_H - 1; \\ v_{2j+1}^h &:= \frac{v_j^H + v_{j+1}^H}{2}, && 0 \le j \le n_H - 2. \end{aligned}$$

In matrix form we have

$$\mathbf{I}_H^h = \frac{1}{2} \begin{bmatrix} 2 & & & & & \\ 1 & 1 & & & & \\ & 2 & & & & \\ & 1 & 1 & & & \\ & & \vdots & \vdots & \vdots & \\ & & & & 1 & 1 \\ & & & & & 2 \end{bmatrix}.$$

- For the restriction, $\mathbf{v}^H = \mathbf{I}_h^H \mathbf{v}^h$ we could use the natural inclusion, i.e., we could simply define $v_j^H := v_{2j}^h$, $0 \le j \le n_H - 1$.

  We could, however, also use a so-called full weighting, which is given by

$$v_j^H = \frac{1}{4}(v_{2j-1}^h + 2v_{2j}^h + v_{2j+1}^h), \quad 0 \le j \le n_H - 1,$$

  where we have implicitly set $v_{-1}^h = v_{n_h}^h = 0$. In matrix form (for the full weighting case) we have

$$\mathbf{I}_h^H = \frac{1}{4}\begin{bmatrix} 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & \\ & & & 1 & \cdots & & \\ & & & & \cdots & & \\ & & & & \cdots & 1 & \\ & & & & & 1 & 2 \end{bmatrix}.$$

  For this case, we have $\mathbf{I}_H^h = 2(\mathbf{I}_h^H)^\top$.

- Our goal is to solve the finite difference system $\mathbf{A}_h\mathbf{u}^h = \mathbf{f}^h$ on the fine level, using the possibility of solving a system $\mathbf{A}_H\boldsymbol{\varepsilon}^H = \mathbf{r}^H$ on a coarse level.

  Note that the matrix $\mathbf{A}_h$ and $\mathbf{A}_H$ only refer to interior nodes. Hence, we delete the first and last columns and rows in the matrix representation of $\mathbf{I}_H^h$ and $\mathbf{I}_h^H$. We still use $\mathbf{I}_H^h$ and $\mathbf{I}_h^H$ to denote the resulting matrices, i.e.,

$$\mathbf{I}_H^h = \frac{1}{2}\begin{bmatrix} 1 & & & \\ 2 & & & \\ 1 & 1 & & \\ & \vdots & \vdots & \vdots \\ & & & 1 \end{bmatrix}, \quad \mathbf{I}_h^H = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 & & \\ & & 1 & \cdots & \\ & & & \cdots & \\ & & & \cdots & 1 \end{bmatrix}.$$

  We can prove that (Exercise)

$$\mathbf{A}_H = \mathbf{I}_h^H\mathbf{A}_h\mathbf{I}_H^h.$$

- We will use a stationary iterative method as a "smoother". Recall that such an iterative method for solving $\mathbf{A}_h \mathbf{u}^h = \mathbf{f}^h$ is given by

$$\mathbf{u}^h_{(j+1)} = \mathbf{S}_h(\mathbf{u}^h_{(j)}) := \mathbf{R}_h \mathbf{u}^h_{(j)} + \mathbf{c}^h,$$

  where the solution $\mathbf{u}^h$ of the linear system is a fixed point of $\mathbf{S}_h(\cdot)$, i.e., it satisfies

$$\mathbf{u}^h = \mathbf{S}_h(\mathbf{u}^h) := \mathbf{R}_h \mathbf{u}^h + \mathbf{c}^h.$$

  Sometimes we call it a consistent iterative method.

- If we apply $\ell \in \mathbb{N}$ iterations of such a smoother with initial data $\mathbf{u}^h_{(0)}$, it is easy to see that the result has the form

$$\mathbf{S}^\ell_h(\mathbf{u}^h_{(0)}) = \mathbf{R}^\ell_h \mathbf{u}^h_{(0)} + \sum_{j=0}^{\ell-1} \mathbf{R}^j_h \mathbf{c}^h := \mathbf{R}^\ell_h \mathbf{u}^h_{(0)} + \mathbf{s}^h.$$

- Note that we can use any consistent method as the smoother $\mathbf{S}_h(\cdot)$.

Algorithm: Two-grid for $\mathbf{A}_h\mathbf{u}^h = \mathbf{f}^h$, $\mathrm{TG}(\mathbf{A}_h, \mathbf{f}^h, \mathbf{u}^h_{(0)}, \ell_1, \ell_2)$

Input: $\mathbf{A}_h$, $\mathbf{f}^h$, $\mathbf{u}^h_{(0)}$, $\ell_1, \ell_2 \in \mathbb{N}$.

Output: Approximation to $\mathbf{A}_h^{-1}\mathbf{f}^h$.

1. `Presmooth :` $\mathbf{u}^h := \mathbf{S}_h^{\ell_1}(\mathbf{u}^h_{(0)})$
2. `Get residual :` $\mathbf{r}^h := \mathbf{f}^h - \mathbf{A}_h\mathbf{u}^h$
3. `Coarsen :` $\mathbf{r}^H := \mathbf{I}_h^H\mathbf{r}^h$
4. `Solve :` $\boldsymbol{\varepsilon}^H := \mathbf{A}_H^{-1}\mathbf{r}^H$
5. `Prolong :` $\boldsymbol{\varepsilon}^h := \mathbf{I}_H^h\boldsymbol{\varepsilon}^H$
6. `Correct :` $\mathbf{u}^h := \mathbf{u}^h + \boldsymbol{\varepsilon}^h$
7. `Postsmooth :` $\mathbf{u}^h := \mathbf{S}_h^{\ell_2}(\mathbf{u}^h)$

- The two-grid method can be seen as only one update of a new stationary iterative method.

Theorem 1

Assume that $\mathbf{A}_H$ *is invertible. Assume that* $\mathbf{u}^h_{(j)}$ *is the input vector and* $\mathbf{u}^h_{(j+1)}$ *is the resulting output vector of the two-grid method. Then, we have*

$$\mathbf{u}^h_{(j+1)} = \mathbf{T}_h \mathbf{u}^h_{(j)} + \mathbf{d}^h \quad \textit{and} \quad \mathbf{u}^h = \mathbf{T}_h \mathbf{u}^h + \mathbf{d}^h,$$

where the iteration matrix $\mathbf{T}_h$ *is given by*

$$\mathbf{T}_h = \mathbf{R}_h^{\ell_2} \mathbf{T}_{h,H} \mathbf{R}_h^{\ell_1} \quad \textit{with} \quad \mathbf{T}_{h,H} = \mathbf{I} - \mathbf{I}_H^h \mathbf{A}_H^{-1} \mathbf{I}_h^H \mathbf{A}_h.$$

Moreover, we have the error representation

$$\mathbf{u}^h_{(j+1)} - \mathbf{u}^h = \mathbf{T}_h (\mathbf{u}^h_{(j)} - \mathbf{u}^h) = \mathbf{T}_h^{j+1} (\mathbf{u}^h_{(0)} - \mathbf{u}^h),$$

showing that the method converges if the spectral radius $\rho(\mathbf{T}_h) < 1$.

Proof. We go through the two-grid method step by step. With the first step $\mathbf{u}_{(j)}^h$ is mapped to $\mathbf{S}_h^{\ell_1}(\mathbf{u}_{(j)}^h)$, which is the input to the second step. After the second and third steps we have

$$\mathbf{r}_{(j)}^H = \mathbf{I}_h^H(\mathbf{f}^h - \mathbf{A}_h\mathbf{S}_h^{\ell_1}(\mathbf{u}_{(j)}^h)),$$

which is the input for the fourth step, so that the results after the fourth and fifth steps become

$$\boldsymbol{\varepsilon}_{(j)}^h = \mathbf{I}_H^h\mathbf{A}_H^{-1}\mathbf{I}_h^H(\mathbf{f}^h - \mathbf{A}_h\mathbf{S}_h^{\ell_1}(\mathbf{u}_{(j)}^h)).$$

Applying steps 6 and 7 to this finally results in the new iteration

$$\begin{aligned}
\mathbf{u}_{(j+1)}^h &= \mathbf{S}_h^{\ell_2}\left(\mathbf{S}_h^{\ell_1}(\mathbf{u}_{(j)}^h) + \mathbf{I}_H^h\mathbf{A}_H^{-1}\mathbf{I}_h^H(\mathbf{f}^h - \mathbf{A}_h\mathbf{S}_h^{\ell_1}(\mathbf{u}_{(j)}^h))\right) \\
&= \mathbf{S}_h^{\ell_2}\left((\mathbf{I} - \mathbf{I}_H^h\mathbf{A}_H^{-1}\mathbf{I}_h^H\mathbf{A}_h)\mathbf{S}_h^{\ell_1}(\mathbf{u}_{(j)}^h) + \mathbf{I}_H^h\mathbf{A}_H^{-1}\mathbf{I}_h^H\mathbf{f}^h\right) \\
&= \mathbf{S}_h^{\ell_2}\left(\mathbf{T}_{h,H}\mathbf{S}_h^{\ell_1}(\mathbf{u}_{(j)}^h) + \mathbf{I}_H^h\mathbf{A}_H^{-1}\mathbf{I}_h^H\mathbf{f}^h\right).
\end{aligned}$$

Note that the operator $\mathbf{S}_h(\cdot)$ is only affine and not linear. Define

$$\widetilde{\mathbf{T}}_h(\cdot) := \mathbf{S}_h^{\ell_2}(\mathbf{T}_{h,H}\mathbf{S}_h^{\ell_1}(\cdot)), \qquad \widetilde{\mathbf{d}}^h := \mathbf{R}_h^{\ell_2}\mathbf{I}_H^h\mathbf{A}_H^{-1}\mathbf{I}_h^H\mathbf{f}^h.$$

By straightforward calculations, we have

$$\mathbf{u}_{(j+1)}^h = \widetilde{\mathbf{T}}_h(\mathbf{u}_{(j)}^h) + \widetilde{\mathbf{d}}^h,$$

and

$$\begin{aligned}
\widetilde{\mathbf{T}}_h(\mathbf{u}) &= \mathbf{S}_h^{\ell_2}(\mathbf{T}_{h,H}\mathbf{S}_h^{\ell_1}(\mathbf{u})) \\
&= \mathbf{S}_h^{\ell_2}\left(\mathbf{T}_{h,H}\left(\mathbf{R}_h^{\ell_1}\mathbf{u} + \sum_{j=0}^{\ell_1-1}\mathbf{R}_h^j\mathbf{c}^h\right)\right) \\
&= \mathbf{R}_h^{\ell_2}\mathbf{T}_{h,H}\mathbf{R}_h^{\ell_1}\mathbf{u} + \mathbf{R}_h^{\ell_2}\mathbf{T}_{h,H}\sum_{j=0}^{\ell_1-1}\mathbf{R}_h^j\mathbf{c}^h + \sum_{j=0}^{\ell_2-1}\mathbf{R}_h^j\mathbf{c}^h \\
&=: \mathbf{T}_h\mathbf{u} + \widehat{\mathbf{d}}^h,
\end{aligned}$$

which shows

$$\mathbf{u}^h_{(j+1)} = \mathbf{T}_h \mathbf{u}^h_{(j)} + \mathbf{d}^h \quad \text{with} \quad \mathbf{d}^h = \widehat{\mathbf{d}}^h + \widetilde{\mathbf{d}}^h.$$

Hence, the iteration matrix is indeed given by

$$\mathbf{T}_h = \mathbf{R}_h^{\ell_2} \mathbf{T}_{h,H} \mathbf{R}_h^{\ell_1}.$$

As $\mathbf{S}_h(\cdot)$ is consistent, i.e., $\mathbf{S}_h(\mathbf{u}^h) = \mathbf{u}^h$, we have

$$\begin{aligned}
\mathbf{T}_h \mathbf{u}^h + \mathbf{d}^h &= \mathbf{T}_h \mathbf{u}^h + \widehat{\mathbf{d}}^h + \widetilde{\mathbf{d}}^h = \widetilde{\mathbf{T}}_h(\mathbf{u}^h) + \widetilde{\mathbf{d}}^h \\
&= \mathbf{S}_h^{\ell_2}((\mathbf{I} - \mathbf{I}_H^h \mathbf{A}_H^{-1} \mathbf{I}_h^H \mathbf{A}_h)\mathbf{u}^h) + \mathbf{R}_h^{\ell_2} \mathbf{I}_H^h \mathbf{A}_H^{-1} \mathbf{I}_h^H \mathbf{f}^h \\
&= \mathbf{S}_h^{\ell_2}(\mathbf{u}^h) - \mathbf{R}_h^{\ell_2} \mathbf{I}_H^h \mathbf{A}_H^{-1} \mathbf{I}_h^H \mathbf{f}^h + \mathbf{R}_h^{\ell_2} \mathbf{I}_H^h \mathbf{A}_H^{-1} \mathbf{I}_h^H \mathbf{f}^h \\
&= \mathbf{u}^h.
\end{aligned}$$

This shows

$$\mathbf{u}^h_{(j+1)} - \mathbf{u}^h = \mathbf{T}_h(\mathbf{u}^h_{(j)} - \mathbf{u}^h) = \mathbf{T}_h^{j+1}(\mathbf{u}^h_{(0)} - \mathbf{u}^h). \quad \square$$

Proposition 2

Assume the following four conditions hold.

(1) *The matrix* $\mathbf{A}_h$ *is symmetric and positive definite.*

(2) *The prolongation and restriction operators are connected by*

$$\mathbf{I}_H^h = \gamma(\mathbf{I}_h^H)^\top$$

with $\gamma > 0$.

(3) *The prolongation operator* $\mathbf{I}_H^h$ *is injective.*

(4) *The coarse grid matrix is given by* $\mathbf{A}_H := \mathbf{I}_h^H \mathbf{A}_h \mathbf{I}_H^h$.

Then we have:

(i) *The coarse-grid correction operator* $\mathbf{T}_{h,H}$ *is an orthogonal projector with respect to the inner product* $\langle \cdot, \cdot \rangle_{\mathbf{A}_h}$.

(ii) *The range of* $\mathbf{T}_{h,H}$ *is* $\langle \cdot, \cdot \rangle_{\mathbf{A}_h}$*-orthogonal to the range of* $\mathbf{I}_H^h$.

Proof. We start by showing that $\mathbf{A}_H$ is symmetric and positive definite. It is symmetric since

$$\mathbf{A}_H^\top = (\mathbf{I}_H^h)^\top \mathbf{A}_h^\top (\mathbf{I}_h^H)^\top = \gamma \mathbf{I}_h^H \mathbf{A}_h \frac{1}{\gamma} \mathbf{I}_H^h = \mathbf{I}_h^H \mathbf{A}_h \mathbf{I}_H^h = \mathbf{A}_H.$$

It is positive definite since we have for any $\mathbf{x} \neq \mathbf{0}$ that $\mathbf{I}_H^h \mathbf{x} \neq \mathbf{0}$ because of the injectivity of $\mathbf{I}_H^h$ and hence

$$\mathbf{x}^\top \mathbf{A}_H \mathbf{x} = \mathbf{x}^\top \mathbf{I}_h^H \mathbf{A}_h \mathbf{I}_H^h \mathbf{x} = \frac{1}{\gamma} (\mathbf{I}_H^h \mathbf{x})^\top \mathbf{A}_h (\mathbf{I}_H^h \mathbf{x}) > 0.$$

This means in particular that the coarse grid correction operator

$$\mathbf{T}_{h,H} = \mathbf{I} - \mathbf{I}_H^h \mathbf{A}_H^{-1} \mathbf{I}_h^H \mathbf{A}_h$$

is well-defined. Next, let

$$\mathbf{Q}_{h,H} := \mathbf{I} - \mathbf{T}_{h,H} = \mathbf{I}_H^h \mathbf{A}_H^{-1} \mathbf{I}_h^H \mathbf{A}_h.$$

Actually, the mapping $\mathbf{Q}_{h,H}$ is a projection since we have

$$\begin{aligned}
\mathbf{Q}_{h,H}^2 &= (\mathbf{I}_H^h \mathbf{A}_H^{-1} \mathbf{I}_h^H \mathbf{A}_h)(\mathbf{I}_H^h \mathbf{A}_H^{-1} \mathbf{I}_h^H \mathbf{A}_h) \\
&= \mathbf{I}_H^h \mathbf{A}_H^{-1} (\mathbf{I}_h^H \mathbf{A}_h \mathbf{I}_H^h) \mathbf{A}_H^{-1} \mathbf{I}_h^H \mathbf{A}_h \\
&= \mathbf{I}_H^h \mathbf{A}_H^{-1} \mathbf{I}_h^H \mathbf{A}_h \\
&= \mathbf{Q}_{h,H}.
\end{aligned}$$

It is also self-adjoint and hence an orthogonal projector:

$$\begin{aligned}
\langle \mathbf{Q}_{h,H}\mathbf{x}, \mathbf{y} \rangle_{\mathbf{A}_h} &= \mathbf{x}^\top \mathbf{Q}_{h,H}^\top \mathbf{A}_h \mathbf{y} = \mathbf{x}^\top \mathbf{A}_h^\top (\mathbf{I}_h^H)^\top (\mathbf{A}_H^{-1})^\top (\mathbf{I}_H^h)^\top \mathbf{A}_h \mathbf{y} \\
&= \mathbf{x}^\top \mathbf{A}_h \left(\frac{1}{\gamma} \mathbf{I}_H^h \right) \mathbf{A}_H^{-1} \gamma \mathbf{I}_h^H \mathbf{A}_h \mathbf{y} \\
&= \mathbf{x}^\top \mathbf{A}_h \mathbf{Q}_{h,H} \mathbf{y} \\
&= \langle \mathbf{x}, \mathbf{Q}_{h,H} \mathbf{y} \rangle_{\mathbf{A}_h}.
\end{aligned}$$

Then $\mathbf{T}_{h,H}$ is also an orthogonal projector with respect to $\langle \cdot, \cdot \rangle_{\mathbf{A}_h}$.

It remains to show that range$(\mathbf{T}_{h,H})$ is $\langle\cdot,\cdot\rangle_{\mathbf{A}_h}$-orthogonal to range$(\mathbf{I}_H^h)$. This is true because we have

$$\begin{aligned}
\langle \mathbf{T}_{h,H}\mathbf{x}, \mathbf{I}_H^h\mathbf{y}\rangle_{\mathbf{A}_h} &= \mathbf{x}^\top(\mathbf{I} - \mathbf{I}_H^h\mathbf{A}_H^{-1}\mathbf{I}_h^H\mathbf{A}_h)^\top\mathbf{A}_h\mathbf{I}_H^h\mathbf{y} \\
&= \mathbf{x}^\top(\mathbf{I} - \mathbf{A}_h\mathbf{I}_H^h\mathbf{A}_H^{-1}\mathbf{I}_h^H)\mathbf{A}_h\mathbf{I}_H^h\mathbf{y} \\
&= \mathbf{x}^\top(\mathbf{A}_h\mathbf{I}_H^h - \mathbf{A}_h\mathbf{I}_H^h\mathbf{A}_H^{-1}\mathbf{I}_h^H\mathbf{A}_h\mathbf{I}_H^h)\mathbf{y} \\
&= \mathbf{x}^\top(\mathbf{A}_h\mathbf{I}_H^h - \mathbf{A}_h\mathbf{I}_H^h)\mathbf{y} \\
&= 0
\end{aligned}$$

for all $\mathbf{x}$ and $\mathbf{y}$. □

- Let $\mathbf{D}_h$ be the diagonal part of $\mathbf{A}_h$. We say that $\mathbf{S}_h(\cdot)$ has the *smoothing property* if there is a constant $\alpha > 0$ such that

$$\|\mathbf{R}_h\mathbf{v}^h\|_{\mathbf{A}_h}^2 \le \|\mathbf{v}^h\|_{\mathbf{A}_h}^2 - \alpha\|\mathbf{A}_h\mathbf{v}^h\|_{\mathbf{D}_h^{-1}}^2, \quad \forall\ \mathbf{v}^h.$$

  We say that the prolongation operator $\mathbf{I}_H^h$ has the *approximation property* if there is a constant $\beta > 0$ such that

$$\min_{\mathbf{v}^H}\|\mathbf{v}^h - \mathbf{I}_H^h\mathbf{v}^H\|_{\mathbf{D}_h} \le \beta\|\mathbf{A}_h\mathbf{v}^h\|_{\mathbf{D}_h^{-1}}, \quad \forall\ \mathbf{v}^h.$$

Theorem 3

Let the conditions of Proposition 2 be satisfied. Assume:

(1) *the smoothing process* $\mathbf{S}_h(\cdot)$ *has the smoothing property,*

(2) *the prolongation operator* $\mathbf{I}_H^h$ *has the approximation property.*

Then, we have

$$\alpha \leq \beta$$

and for the iteration matrix $\mathbf{T}_h$ *of the two-grid method,*

$$\|\mathbf{T}_h\|_{\mathbf{A}_h} < \sqrt{1 - \alpha/\beta} \text{ if } \alpha < \beta \text{ and } \|\mathbf{T}_h\|_{\mathbf{A}_h} = 0 \text{ if } \alpha = \beta.$$

Hence, as an iterative scheme, the two-grid method converges.

Proof. Since range($\mathbf{T}_{h,H}$) is $\langle \cdot, \cdot \rangle_{\mathbf{A}_h}$-orthogonal to range($\mathbf{I}_H^h$), we have

$$\langle \mathbf{v}^h, \mathbf{I}_H^h \mathbf{v}^H \rangle_{\mathbf{A}_h} = 0, \quad \forall\, \mathbf{v}^h \in \text{range}(\mathbf{T}_{h,H}), \quad \forall\, \mathbf{v}^H.$$

From this, we can conclude for all $\mathbf{v}^h \in \text{range}(\mathbf{T}_{h,H})$ and $\mathbf{v}^H$ that

$$\begin{aligned}
\|\mathbf{v}^h\|_{\mathbf{A}_h}^2 &= \langle \mathbf{v}^h, \mathbf{v}^h - \mathbf{I}_H^h \mathbf{v}^H \rangle_{\mathbf{A}_h} = \langle \mathbf{A}_h \mathbf{v}^h, \mathbf{v}^h - \mathbf{I}_H^h \mathbf{v}^H \rangle \\
&= \langle \mathbf{D}_h^{-1/2} \mathbf{A}_h \mathbf{v}^h, \mathbf{D}_h^{1/2} (\mathbf{v}^h - \mathbf{I}_H^h \mathbf{v}^H) \rangle \\
&\leq \|\mathbf{D}_h^{-1/2} \mathbf{A}_h \mathbf{v}^h\|_2 \|\mathbf{D}_h^{1/2} (\mathbf{v}^h - \mathbf{I}_H^h \mathbf{v}^H)\|_2 \\
&= \|\mathbf{A}_h \mathbf{v}^h\|_{\mathbf{D}_h^{-1}} \|\mathbf{v}^h - \mathbf{I}_H^h \mathbf{v}^H\|_{\mathbf{D}_h}.
\end{aligned}$$

Going over to the infimum over all $\mathbf{v}^H$ and using the approximation property leads to

$$\|\mathbf{v}^h\|_{\mathbf{A}_h} \leq \sqrt{\beta} \|\mathbf{A}_h \mathbf{v}^h\|_{\mathbf{D}_h^{-1}}, \qquad \forall\ \mathbf{v}^h \in \text{range}(\mathbf{T}_{h,H}).$$

This is equivalent to

$$\|\mathbf{T}_{h,H} \mathbf{v}^h\|_{\mathbf{A}_h} \leq \sqrt{\beta} \|\mathbf{A}_h \mathbf{T}_{h,H} \mathbf{v}^h\|_{\mathbf{D}_h^{-1}}, \qquad \forall\ \mathbf{v}^h.$$

Using this and the smoothing property then leads to

$$\begin{aligned}
0 \leq \|\mathbf{R}_h \mathbf{T}_{h,H} \mathbf{v}^h\|_{\mathbf{A}_h}^2 &\leq \|\mathbf{T}_{h,H} \mathbf{v}^h\|_{\mathbf{A}_h}^2 - \alpha \|\mathbf{A}_h \mathbf{T}_{h,H} \mathbf{v}^h\|_{\mathbf{D}_h^{-1}}^2 \\
&\leq \|\mathbf{T}_{h,H} \mathbf{v}^h\|_{\mathbf{A}_h}^2 - \frac{\alpha}{\beta} \|\mathbf{T}_{h,H} \mathbf{v}^h\|_{\mathbf{A}_h}^2 \\
&= (1 - \alpha/\beta) \|\mathbf{T}_{h,H} \mathbf{v}^h\|_{\mathbf{A}_h}^2 \\
&\leq (1 - \alpha/\beta) \|\mathbf{v}^h\|_{\mathbf{A}_h}^2,
\end{aligned}$$

where the last inequality follows from the fact that $\mathbf{T}_{h,H}$ is an orthogonal projector with respect to $\langle \cdot, \cdot \rangle_{\mathbf{A}_h}$. This means, first of all, $\alpha \leq \beta$ and secondly

$$\|\mathbf{R}_h \mathbf{T}_{h,H}\|_{\mathbf{A}_h} \leq \sqrt{1 - \alpha/\beta}.$$

By the smoothing property, we have $\|\mathbf{R}_h\|_{\mathbf{A}_h} < 1$. Then we finally derive $\|\mathbf{T}_h\|_{\mathbf{A}_h} = 0$ if $\alpha = \beta$, and if $\alpha < \beta$,

$$\|\mathbf{T}_h\|_{\mathbf{A}_h} = \|\mathbf{R}_h^{\ell_2} \mathbf{T}_{h,H} \mathbf{R}_h^{\ell_1}\|_{\mathbf{A}_h} < \|\mathbf{R}_h \mathbf{T}_{h,H}\|_{\mathbf{A}_h} \leq \sqrt{1 - \alpha/\beta}. \quad \square$$

Algorithm: V-cycle for $\mathbf{A}_h\mathbf{u}^h = \mathbf{f}^h$, V-cycle$(\mathbf{A}_h, \mathbf{f}^h, \mathbf{u}^h_{(0)}, \ell_1, \ell_2, h_0)$

Input: $\mathbf{A}_h$, $\mathbf{f}^h$, $\mathbf{u}^h_{(0)}$, $\ell_1, \ell_2 \in \mathbb{N}$, h_0.

Output: Approximation to $\mathbf{A}_h^{-1}\mathbf{f}^h$.

1. `Presmooth :` $\quad \mathbf{u}^h := \mathbf{S}_h^{\ell_1}(\mathbf{u}^h_{(0)})$
2. `Get residual :` $\quad \mathbf{r}^h := \mathbf{f}^h - \mathbf{A}_h\mathbf{u}^h$
3. `Coarsen :` $\quad H := 2h, \quad \mathbf{r}^H := \mathbf{I}_h^H\mathbf{r}^h$
4. **if** $H = h_0$

   Solve $\mathbf{A}_H\boldsymbol{\varepsilon}^H = \mathbf{r}^H$

   else

   $\boldsymbol{\varepsilon}^H :=$V-cycle$(\mathbf{A}_H, \mathbf{r}^H, \mathbf{0}, \ell_1, \ell_2, h_0)$

   end
5. `Prolong :` $\quad \boldsymbol{\varepsilon}^h := \mathbf{I}_H^h\boldsymbol{\varepsilon}^H$
6. `Correct :` $\quad \mathbf{u}^h := \mathbf{u}^h + \boldsymbol{\varepsilon}^h$
7. `Postsmooth :` $\quad \mathbf{u}^h := \mathbf{S}_h^{\ell_2}(\mathbf{u}^h)$

Algorithm: Multigrid for $\mathbf{A}_h\mathbf{u}^h = \mathbf{f}^h$, $\mathrm{MG}(\mathbf{A}_h, \mathbf{f}^h, \mathbf{u}^h_{(0)}, \ell_1, \ell_2, \ell, h_0)$

Input: $\mathbf{A}_h$, $\mathbf{f}^h$, $\mathbf{u}^h_{(0)}$, $\ell_1, \ell_2, \ell \in \mathbb{N}$, h_0.

Output: Approximation to $\mathbf{A}_h^{-1}\mathbf{f}^h$.

1. `Presmooth :` $\mathbf{u}^h := \mathbf{S}_h^{\ell_1}(\mathbf{u}^h_{(0)})$
2. `Get residual :` $\mathbf{r}^h := \mathbf{f}^h - \mathbf{A}_h\mathbf{u}^h$
3. `Coarsen :` $H := 2h, \quad \mathbf{r}^H := \mathbf{I}_h^H\mathbf{r}^h$
4. **if** $H = h_0$

    Solve $\mathbf{A}_H\boldsymbol{\varepsilon}^H = \mathbf{r}^H$

    else

    $\boldsymbol{\varepsilon}^H := \mathbf{0}$

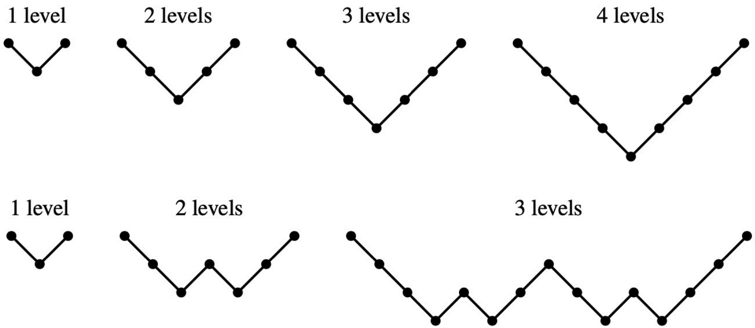
    for $j = 1 : \ell$

    $\boldsymbol{\varepsilon}^H := \mathrm{MG}(\mathbf{A}_H, \mathbf{r}^H, \boldsymbol{\varepsilon}^H, \ell_1, \ell_2, \ell, h_0)$

    end

    end
5. `Prolong :` $\boldsymbol{\varepsilon}^h := \mathbf{I}_H^h\boldsymbol{\varepsilon}^H$
6. `Correct :` $\mathbf{u}^h := \mathbf{u}^h + \boldsymbol{\varepsilon}^h$
7. `Postsmooth :` $\mathbf{u}^h := \mathbf{S}_h^{\ell_2}(\mathbf{u}^h)$

- It is helpful to visualize the recursion, depending on the choice of ℓ and how many levels there are, meaning how many grids we use. Assume that k levels are used, i.e., k satisfies $h_0 = 2^k h$.



The recursion of multigrid with $\ell = 1$ (top, V-cycle) and $\ell = 2$ (bottom, W-cycle).

- To determine the iteration matrix $\mathbf{M}_h$ of the multigrid method, we start with the iteration matrix of the two-grid method

$$\mathbf{T}_h = \mathbf{R}_h^{\ell_2}(\mathbf{I} - \mathbf{I}_H^h \mathbf{A}_H^{-1} \mathbf{I}_h^H \mathbf{A}_h)\mathbf{R}_h^{\ell_1}$$

and recall that the term $\mathbf{A}_H^{-1}$ came from step 4 in two-grid cycle and hence has now to be replaced by ℓ steps of the multigrid method on grid X_H:

$$\begin{aligned} \boldsymbol{\varepsilon}_{(0)}^H &:= \mathbf{0}, \\ \boldsymbol{\varepsilon}_{(j)}^H &:= \mathbf{M}_H \boldsymbol{\varepsilon}_{(j-1)}^H + \mathbf{d}^H, \quad 1 \le j \le \ell. \end{aligned}$$

As this is a consistent method for solving $\mathbf{A}_H \boldsymbol{\varepsilon}^H = \mathbf{r}^H$, we have

$$\boldsymbol{\varepsilon}_{(\ell)}^H - \boldsymbol{\varepsilon}^H = \mathbf{M}_H(\boldsymbol{\varepsilon}_{(\ell-1)}^H - \boldsymbol{\varepsilon}^H) = \mathbf{M}_H^\ell(\boldsymbol{\varepsilon}_{(0)}^H - \boldsymbol{\varepsilon}^H) = -\mathbf{M}_H^\ell \boldsymbol{\varepsilon}^H.$$

Then, we have

$$\boldsymbol{\varepsilon}_{(\ell)}^H = (\mathbf{I} - \mathbf{M}_H^\ell)\boldsymbol{\varepsilon}^H = (\mathbf{I} - \mathbf{M}_H^\ell)\mathbf{A}_H^{-1}\mathbf{r}^H.$$

Replacing $\mathbf{A}_H^{-1}$ by $(\mathbf{I} - \mathbf{M}_H^\ell)\mathbf{A}_H^{-1}$ yields

$$\begin{aligned}
\mathbf{M}_h &= \mathbf{R}_h^{\ell_2}[\mathbf{I} - \mathbf{I}_H^h(\mathbf{I} - \mathbf{M}_H^\ell)\mathbf{A}_H^{-1}\mathbf{I}_h^H\mathbf{A}_h]\mathbf{R}_h^{\ell_1} \\
&= \mathbf{R}_h^{\ell_2}[\mathbf{I} - \mathbf{I}_H^h\mathbf{A}_H^{-1}\mathbf{I}_h^H\mathbf{A}_h + \mathbf{I}_H^h\mathbf{M}_H^\ell\mathbf{A}_H^{-1}\mathbf{I}_h^H\mathbf{A}_h]\mathbf{R}_h^{\ell_1} \\
&= \mathbf{T}_h + \mathbf{R}_h^{\ell_2}\mathbf{I}_H^h\mathbf{M}_H^\ell\mathbf{A}_H^{-1}\mathbf{I}_h^H\mathbf{A}_h\mathbf{R}_h^{\ell_1},
\end{aligned}$$

which can be seen as a perturbation of $\mathbf{T}_h$.

- *Algebraic multigrid* (AMG): $\mathbf{A}_h\mathbf{u}^h = \mathbf{f}^h$ with $\mathbf{A}_h \in \mathbb{R}^{n_h \times n_h}$
  - define the coarse subset $\mathbb{R}^{n_H}$ from the fine set $\mathbb{R}^{n_h}$,
  - define the coarsening operator $\mathbf{I}_h^H$ from $\mathbb{R}^{n_h}$ to $\mathbb{R}^{n_H}$
  - use the abstract definitions

$$\mathbf{I}_H^h = (\mathbf{I}_h^H)^\top, \quad \mathbf{A}_H = \mathbf{I}_h^H\mathbf{A}_h\mathbf{I}_H^h, \quad \mathbf{f}^H = \mathbf{I}_h^H\mathbf{f}^h$$

to complete the set-up.

4. Further reading

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  John Wiley & Sons, 1992