Lecture 7: Eigenvalue problem

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1. Eigenvalues

- The *eigenvalues* of a matrix $\mathbf{A} \in \mathbb{C}^{m\times m}$ are the m roots of its *characteristic polynomial*

$$p(z) = \det(z\mathbf{I} - \mathbf{A}).$$

- We have

$$\det(\mathbf{A}) = \lambda_1\lambda_2\cdots\lambda_m, \qquad \operatorname{tr}(\mathbf{A}) = \lambda_1 + \lambda_2 + \cdots + \lambda_m.$$

Theorem 1 (Gerschgorin's theorem)

Every eigenvalue of $\mathbf{A}$ lies in at least one of the m circular disks in the complex plane with centers a_{ii} and radii $\sum_{j\neq i}|a_{ij}|$. Moreover, if n of these disks form a connected domain that is disjoint from the other $m-n$ disks, then there are precisely n eigenvalues of $\mathbf{A}$ within this domain.

The proof is left as an exercise.

Theorem 2

Eigenvalues of $\mathbf{A}$ are continuous functions of entries of $\mathbf{A}$.

Proof.

See Demmel's book: Proposition 4.4, Page 149, Applied numerical linear algebra. □

Remark 3

Eigenvalues of $\mathbf{A}$ are not necessarily differentiable everywhere.

Example: Consider the $m \times m$ matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ \varepsilon & & & & 0 \end{bmatrix}. \quad \lambda_j(\varepsilon) = \varepsilon^{\frac{1}{m}} \exp\left(\frac{\mathrm{i}2j\pi}{m}\right).$$

2. Eigenvectors

- A nonzero vector $\mathbf{y} \in \mathbb{C}^m$ is called a *left eigenvector* of $\mathbf{A} \in \mathbb{C}^{m\times m}$ corresponding to $\lambda \in \Lambda(\mathbf{A})$ if $\mathbf{y}^*\mathbf{A} = \lambda\mathbf{y}^*$.
- A nonzero vector $\mathbf{x} \in \mathbb{C}^m$ is called a (*right*) *eigenvector* of $\mathbf{A} \in \mathbb{C}^{m\times m}$ corresponding to $\lambda \in \Lambda(\mathbf{A})$ if $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$.

Theorem 4

If $\mathbf{A} \in \mathbb{C}^{m\times m}$ and if $\lambda, \mu \in \Lambda(\mathbf{A})$, with $\lambda \neq \mu$, then any left eigenvector of $\mathbf{A}$ corresponding to μ is orthogonal to any right eigenvector of $\mathbf{A}$ corresponding to λ.

Proof.

Let $\mathbf{y}^*\mathbf{A} = \mu\mathbf{y}^*$ and $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$. We have

$$\mathbf{y}^*\mathbf{A}\mathbf{x} = \mathbf{y}^*(\lambda\mathbf{x}) = \lambda(\mathbf{y}^*\mathbf{x}), \quad \mathbf{y}^*\mathbf{A}\mathbf{x} = (\mu\mathbf{y}^*)\mathbf{x} = \mu(\mathbf{y}^*\mathbf{x}).$$

Then, $\mathbf{y}^*\mathbf{x} = 0$ follows from $\lambda \neq \mu$. $\square$

3. Geometric multiplicity and algebraic multiplicity

- The *geometric multiplicity* of an eigenvalue λ is the dimension of the null-space of $\mathbf{A} - \lambda\mathbf{I}$, which is an *eigenspace* corresponding to the eigenvalue λ.
- The *algebraic multiplicity* of an eigenvalue λ is its multiplicity as a root of the characteristic polynomial. The algebraic multiplicity of an eigenvalue is at least as great as its geometric multiplicity.
- An eigenvalue is *simple* if its algebraic multiplicity is 1. Otherwise, *multiple*.

Remark 5

Simple eigenvalue of $\mathbf{A}$ is differential at $\mathbf{A} \in \mathbb{C}^{m\times m}$.

Theorem 6

An eigenvalue is multiple if and only if it has a pair of orthogonal left and right eigenvectors.

The proof is left as an exercise.

4. Jordan form

Theorem 7

For any square matrix $\mathbf{A}$ *there exists a similar matrix* $\mathbf{J} = \mathbf{SAS}^{-1}$ *such that*

$$\mathbf{J} = \mathrm{diag}\{\mathbf{J}_1, \mathbf{J}_2, \cdots, \mathbf{J}_k\}$$

where each $\mathbf{J}_i$ *is a Jordan block:* $\mathbf{J}_i = \begin{bmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_i & 1 \\ & & & & \lambda_i \end{bmatrix}.$

- Up to permuting the order of the $\mathbf{J}_i$, the Jordan form is unique.
- Up to a nonzero constant, there are only one left eigenvector and one right eigenvector per $\mathbf{J}_i$.
- Discussion: How to determine the rank of $\mathbf{A}$ via its Jordan form?

- Jordan form is a discontinuous function of $\mathbf{A}$, so any rounding error can change it completely. Therefore, Jordan form is theoretically useful only.

  Example: Consider the matrix

$$\mathbf{A}(\varepsilon) = \begin{bmatrix} \varepsilon & 1 & & \\ & 2\varepsilon & \ddots & \\ & & \ddots & 1 \\ & & & m\varepsilon \end{bmatrix}.$$

  It is easy to show that

$$\lim_{\varepsilon \to 0} \mathbf{J}(\mathbf{A}(\varepsilon)) \neq \mathbf{J}(\mathbf{A}(0)) = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}.$$

5. Schur form

Theorem 8 (Schur factorization)

If $\mathbf{A} \in \mathbb{C}^{m\times m}$*, then there exists a unitary matrix* $\mathbf{Q} \in \mathbb{C}^{m\times m}$ *and an upper-triangular matrix* $\mathbf{T} \in \mathbb{C}^{m\times m}$ *such that* $\mathbf{A} = \mathbf{QTQ}^*$*.*

Proof. By induction on the dimension m of $\mathbf{A}$. □

Remark 9

See Demmel's book (Applied numerical linear algebra, Theorem 4.3, Page 147) *for real Schur form of a real matrix* $\mathbf{A}$*.*

Exercise: Let $\lambda_1, \cdots, \lambda_m$ be the m eigenvalues of $\mathbf{A} \in \mathbb{C}^{m\times m}$. Let

$$\mathbf{M} = \frac{\mathbf{A} + \mathbf{A}^*}{2}, \quad \mathbf{N} = \frac{\mathbf{A} - \mathbf{A}^*}{2}.$$

Prove that

$$\sum_{i=1}^{m} |\lambda_i|^2 \le \|\mathbf{A}\|_{\mathrm{F}}^2, \quad \sum_{i=1}^{m} |\mathrm{Re}\lambda_i|^2 \le \|\mathbf{M}\|_{\mathrm{F}}^2, \quad \sum_{i=1}^{m} |\mathrm{Im}\lambda_i|^2 \le \|\mathbf{N}\|_{\mathrm{F}}^2.$$

- Let $\mathbf{A} = \mathbf{QTQ}^*$ be a Schur factorization. If $\{\lambda, \mathbf{x}\}$ is an eigenpair of $\mathbf{T}$, then $\{\lambda, \mathbf{Qx}\}$ is an eigenpair of $\mathbf{A}$.

6. Unitary diagonalization

- A matrix $\mathbf{A}$ is called *unitarily diagonalizable* if there exists a unitary matrix $\mathbf{Q}$ and a diagonal matrix $\mathbf{\Lambda}$ such that $\mathbf{A} = \mathbf{Q\Lambda Q}^*$.

  Examples: Hermitian, skew-Hermitian, ...
- A matrix $\mathbf{A}$ is called *normal* if $\mathbf{A}^*\mathbf{A} = \mathbf{A}\mathbf{A}^*$.

  Examples: Hermitian, skew-Hermitian, ...

Theorem 10

A matrix is unitarily diagonalizable if and only if it is normal.

Proof.

"$\Rightarrow$": Easy. "$\Leftarrow$" By Schur factorization of $\mathbf{A}$. □

7. Eigenvalue perturbation theory

Theorem 11 (Bauer–Fike)

Suppose $\mathbf{A} \in \mathbb{C}^{m\times m}$ *is diagonalizable with* $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$, *and let* $\mathbf{\Delta} \in \mathbb{C}^{m\times m}$ *be arbitrary. For each eigenvalue* $\widehat{\lambda}$ *of* $\mathbf{A}+\mathbf{\Delta}$, *there exists an eigenvalue* λ *of* $\mathbf{A}$ *such that*

$$|\widehat{\lambda} - \lambda| \leq \|\mathbf{V}\|_2 \|\mathbf{V}^{-1}\|_2 \|\mathbf{\Delta}\|_2.$$

Proof. Assume that $\{\widehat{\lambda}, \mathbf{V}\mathbf{y}\}$ is an eigenpair of $\mathbf{A}+\mathbf{\Delta}$. Then we have

$$(\widehat{\lambda}\mathbf{I} - \mathbf{\Lambda})\mathbf{y} = \mathbf{V}^{-1}\mathbf{\Delta}\mathbf{V}\mathbf{y}.$$

Thus, $\displaystyle\min_{\lambda\in\Lambda(\mathbf{A})} |\widehat{\lambda} - \lambda| \leq \frac{\|(\widehat{\lambda}\mathbf{I} - \mathbf{\Lambda})\mathbf{y}\|_2}{\|\mathbf{y}\|_2} \leq \|\mathbf{V}\|_2 \|\mathbf{V}^{-1}\|_2 \|\mathbf{\Delta}\|_2.$ □

Corollary 12

If $\mathbf{A}$ *is normal, i.e.,* $\mathbf{A}\mathbf{A}^* = \mathbf{A}^*\mathbf{A}$, *then for each eigenvalue* $\widehat{\lambda}$ *of* $\mathbf{A}+\mathbf{\Delta}$, *there is an eigenvalue* λ *of* $\mathbf{A}$ *such that* $|\widehat{\lambda} - \lambda| \leq \|\mathbf{\Delta}\|_2$.

8. Hermitian matrix eigenvalues

Theorem 13 (Courant–Fisher)

If $\mathbf{A} \in \mathbb{C}^{m\times m}$ *is Hermitian, then the eigenvalues* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ *satisfy*

$$\begin{aligned}\lambda_k &= \max_{\mathcal{S}\subseteq\mathbb{C}^m, \dim(\mathcal{S})=k} \min_{\mathbf{0}\neq\mathbf{y}\in\mathcal{S}} \frac{\mathbf{y}^*\mathbf{A}\mathbf{y}}{\mathbf{y}^*\mathbf{y}} \\ &= \min_{\mathcal{S}\subseteq\mathbb{C}^m, \dim(\mathcal{S})=m-k+1} \max_{\mathbf{0}\neq\mathbf{y}\in\mathcal{S}} \frac{\mathbf{y}^*\mathbf{A}\mathbf{y}}{\mathbf{y}^*\mathbf{y}},\end{aligned}$$

for $k = 1, 2, \ldots, m$.

Theorem 14 (Interlacing property)

If $\mathbf{A} \in \mathbb{C}^{m\times m}$ *is Hermitian and* $\mathbf{A}_k = \mathbf{A}(1:k, 1:k)$, *then*

$$\begin{aligned}\lambda_{k+1}(\mathbf{A}_{k+1}) \leq \lambda_k(\mathbf{A}_k) \leq \lambda_k(\mathbf{A}_{k+1}) \leq \\ \cdots \leq \lambda_2(\mathbf{A}_{k+1}) \leq \lambda_1(\mathbf{A}_k) \leq \lambda_1(\mathbf{A}_{k+1})\end{aligned}$$

for $k = 1 : m-1$.

Theorem 15 (Weyl)

Let $\mathbf{A} \in \mathbb{C}^{m\times m}$ *and* $\mathbf{B} \in \mathbb{C}^{m\times m}$ *be Hermitian. Let* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ *be eigenvalues. Then*

$$|\lambda_k(\mathbf{A}) - \lambda_k(\mathbf{B})| \leq \|\mathbf{A} - \mathbf{B}\|_2, \quad k = 1, 2, \ldots, m.$$

Corollary 16

Let $\mathbf{A} \in \mathbb{C}^{m\times n}$ *and* $\mathbf{B} \in \mathbb{C}^{m\times n}$ *be arbitrary. Let* $p = \min\{m, n\}$ *and* $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p$ *be singular values. Then*

$$|\sigma_k(\mathbf{A}) - \sigma_k(\mathbf{B})| \leq \|\mathbf{A} - \mathbf{B}\|_2, \quad k = 1, 2, \ldots, p.$$

Theorem 17

Let $\mathbf{A} \in \mathbb{C}^{l\times m}$ *and* $\mathbf{B} \in \mathbb{C}^{m\times n}$ *be arbitrary. Let* $p = \min\{l, m, n\}$ *and* $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p$ *be singular values. Then*

$$\sigma_k(\mathbf{AB}) \leq \sigma_1(\mathbf{A})\sigma_k(\mathbf{B}), \quad k = 1, 2, \ldots, p.$$

9. Generalized eigenvalue problem

- For $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{m\times m}$, $\{\lambda, \mathbf{x}\}$ is called an eigenpair if $\{\lambda, \mathbf{x}\}$ satisfies $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{A}\mathbf{x} = \lambda\mathbf{B}\mathbf{x}$.
- If $\mathbf{A}$ and $\mathbf{B}$ are square and $\det(\mathbf{A} - \lambda\mathbf{B})$ is not identically zero, the pencil $\mathbf{A} - \lambda\mathbf{B}$ is called *regular*. Otherwise it is called *singular*.
- When $\mathbf{A} - \lambda\mathbf{B}$ is regular, $p(\lambda) = \det(\mathbf{A} - \lambda\mathbf{B})$ is called the *characteristic polynomial* of the pencil $\mathbf{A} - \lambda\mathbf{B}$ and the eigenvalues of the pencil $\mathbf{A} - \lambda\mathbf{B}$ are defined to be
  - (1) the roots of $p(\lambda) = 0$,
  - (2) ∞ (with multiplicity $m - \deg(p)$) if $\deg(p) < m$.
- Generalized Schur factorization for $(\mathbf{A}, \mathbf{B})$:

$$\mathbf{Q}^*\mathbf{A}\mathbf{Z} = \mathbf{S}, \quad \mathbf{Q}^*\mathbf{B}\mathbf{Z} = \mathbf{T},$$

  where $\mathbf{Q}$ and $\mathbf{Z}$ are unitary, and $\mathbf{S}$ and $\mathbf{T}$ are upper-triangular.
- Real generalized Schur forms for real matrices $\mathbf{A}$ and $\mathbf{B}$.
- QZ algorithm for generalized eigenvalue problem.

10. Matrix polynomial eigenvalue problem

- We consider the matrix polynomial

$$\mathbf{A}(\lambda) := \sum_{i=0}^{d} \lambda^i \mathbf{A}_i = \lambda^d \mathbf{A}_d + \lambda^{d-1}\mathbf{A}_{d-1} + \cdots + \lambda \mathbf{A}_1 + \mathbf{A}_0,$$

  where $\mathbf{A}_i \in \mathbb{C}^{m\times m}$ and $\mathbf{A}_d$ is nonsingular.

- The characteristic polynomial of the matrix polynomial $\mathbf{A}(\lambda)$ is

$$p(\lambda) = \det(\mathbf{A}(\lambda)).$$

  The roots of $p(\lambda) = 0$ are defined to be the eigenvalues. (How many eigenvalues?)

- Suppose that γ is an eigenvalue. A nonzero vector $\mathbf{x}$ satisfying $\mathbf{A}(\gamma)\mathbf{x} = \mathbf{0}$ is a right eigenvector for γ. A left eigenvector $\mathbf{y}$ is defined analogously by $\mathbf{y}^*\mathbf{A}(\gamma) = \mathbf{0}$.

Example: Consider the ODE system

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{B}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{0},$$

where $\mathbf{M}, \mathbf{B}, \mathbf{K} \in \mathbb{C}^{m\times m}$ and $\mathbf{M}$ is nonsingular. If we seek solutions of the form $\mathbf{x}(t) = \mathrm{e}^{\gamma t}\mathbf{x}(0)$, we get

$$\mathrm{e}^{\gamma t}(\gamma^2\mathbf{M}\mathbf{x}(0) + \gamma\mathbf{B}\mathbf{x}(0) + \mathbf{K}\mathbf{x}(0)) = \mathbf{0},$$

i.e.,

$$\gamma^2\mathbf{M}\mathbf{x}(0) + \gamma\mathbf{B}\mathbf{x}(0) + \mathbf{K}\mathbf{x}(0) = \mathbf{0}.$$

Thus γ is an eigenvalue and $\mathbf{x}(0)$ is an eigenvector of the matrix polynomial

$$\lambda^2\mathbf{M} + \lambda\mathbf{B} + \mathbf{K}.$$

- Linearize the matrix polynomial to get the generalized eigenvalue problem

$$\begin{bmatrix} -\mathbf{A}_{d-1} & -\mathbf{A}_{d-2} & \cdots & \cdots & -\mathbf{A}_0 \\ \mathbf{I} & & & & \\ & \mathbf{I} & & & \\ & & \ddots & & \\ & & & \mathbf{I} & \end{bmatrix} - \lambda \begin{bmatrix} \mathbf{A}_d & & & & \\ & \mathbf{I} & & & \\ & & \mathbf{I} & & \\ & & & \ddots & \\ & & & & \mathbf{I} \end{bmatrix}$$

- Linearize the matrix polynomial to get the standard eigenvalue problem

$$\begin{bmatrix} -\mathbf{A}_d^{-1}\mathbf{A}_{d-1} & -\mathbf{A}_d^{-1}\mathbf{A}_{d-2} & \cdots & \cdots & -\mathbf{A}_d^{-1}\mathbf{A}_0 \\ \mathbf{I} & & & & \\ & \mathbf{I} & & & \\ & & \ddots & & \\ & & & \mathbf{I} & \end{bmatrix} - \lambda \mathbf{I}$$