

# Lecture 10: Jacobi method, Bisection method, Divide-and-conquer method



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## 1. Jacobi method

- The method is based on the fact that a  $2 \times 2$  real symmetric matrix  $\mathbf{A}$  can be diagonalized in the form

$$\mathbf{J}_2 = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}, \quad \mathbf{J}_2^\top \begin{bmatrix} a & d \\ d & b \end{bmatrix} \mathbf{J}_2 = \begin{bmatrix} \times & 0 \\ 0 & \times \end{bmatrix},$$

where  $\theta$  satisfies  $\tan(2\theta) = \frac{2d}{b-a}$ .

- Define the  $m \times m$  Jacobi rotation matrix  $\mathbf{J}(i, j; \theta)$ ,  $i < j$

$$\begin{aligned} \mathbf{J} = \mathbf{J}(i, j; \theta) &:= \mathbf{I} + \begin{bmatrix} \mathbf{e}_i & \mathbf{e}_j \end{bmatrix} \begin{bmatrix} \cos \theta - 1 & \sin \theta \\ -\sin \theta & \cos \theta - 1 \end{bmatrix} \begin{bmatrix} \mathbf{e}_i^\top \\ \mathbf{e}_j^\top \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{I} & & & \\ & \cos \theta & \sin \theta & \\ & & \mathbf{I} & \\ & -\sin \theta & \cos \theta & \\ & & & \mathbf{I} \end{bmatrix} \begin{matrix} \text{row } i \\ \text{row } j \end{matrix}. \end{aligned}$$

The  $(i, j)$  and  $(j, i)$  entries of  $\mathbf{B} := \mathbf{J}^\top \mathbf{A} \mathbf{J}$  are zeros via appropriate  $\theta$ .

## Remark 1

*The Jacobi rotation matrix  $\mathbf{J}$  is orthogonal.*

## Remark 2

*We have  $\|\mathbf{B}\|_F = \|\mathbf{A}\|_F$  ( $\|\cdot\|_F$  is invariant under orthogonal  $\mathbf{J}$ ).*

## Theorem 3

*Suppose that  $a_{ij} = a_{ji} \neq 0$  and  $i < j$ . Let  $\mathbf{J}(i, j; \theta)$  be the Jacobi rotation matrix such that the  $(i, j)$  and  $(j, i)$  entries of  $\mathbf{B} = \mathbf{J}^\top \mathbf{A} \mathbf{J}$  are zeros. Then, for  $k \neq i, j$ ,  $b_{kk} = a_{kk}$  and*

$$b_{ii}^2 + b_{jj}^2 = a_{ii}^2 + a_{jj}^2 + a_{ij}^2 + a_{ji}^2.$$

## Remark 4

*At each step a symmetric pair of zeros is introduced into the matrix (note that previous zeros maybe destroyed). The usual effect is that the sum of the squares of magnitude of off-diagonal entries shrink steadily.*

## 2. Bisection method

- Consider an unreduced (all of its  $(i+1, i)$  and  $(i, i+1)$  entries are nonzero) tridiagonal real symmetric matrix  $\mathbf{A}$ :

$$\mathbf{A} = \begin{bmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & b_2 & a_3 & \ddots & \\ & & \ddots & \ddots & b_{m-1} \\ & & & b_{m-1} & a_m \end{bmatrix}, \quad b_j \neq 0.$$

- Let  $\mathbf{A}^{(1)}, \dots, \mathbf{A}^{(m)}$  denote its leading square principal submatrices of dimension  $1, \dots, m$ .

### Proposition 5

*The eigenvalues of  $\mathbf{A}^{(k)}$  are distinct:  $\lambda_1^{(k)} > \lambda_2^{(k)} > \dots > \lambda_k^{(k)}$ .*

## Proposition 6

The eigenvalues of  $\mathbf{A}^{(1)}, \dots, \mathbf{A}^{(m)}$  strictly interlace, i.e.,

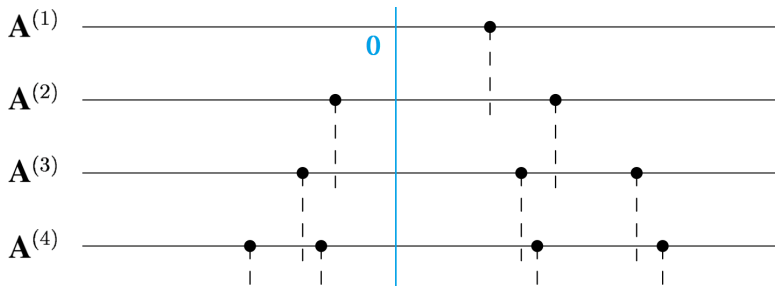
$$\lambda_j^{(k+1)} > \lambda_j^{(k)} > \lambda_{j+1}^{(k+1)},$$

for  $k = 1, 2, \dots, m-1$  and  $j = 1, 2, \dots, k$ .

Proof: See Golub and van Loan's book: Theorem 8.4.1, Page 468, [Matrix computations](#), 4th edition.

- The interlacing property makes it possible to count the exact number of negative eigenvalues of a real symmetric tridiagonal matrix. For example,

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & & \\ 1 & 0 & 1 & \\ & 1 & 2 & 1 \\ & & 1 & -1 \end{bmatrix}, \quad \begin{aligned} \det(\mathbf{A}^{(1)}) &= 1, \\ \det(\mathbf{A}^{(2)}) &= -1, \\ \det(\mathbf{A}^{(3)}) &= -3, \\ \det(\mathbf{A}^{(4)}) &= 4. \end{aligned}$$



### Remark 7

*In general, for any unreduced tridiagonal real symmetric  $\mathbf{A}$ , the number of negative eigenvalues is equal to the number of sign changes in the sequence*

$$1, \det(\mathbf{A}^{(1)}), \det(\mathbf{A}^{(2)}), \dots, \det(\mathbf{A}^{(m)}),$$

*which is known as a Sturm sequence. Here, we define a “sign change” to mean a transition from  $+$  or  $0$  to  $-$  or from  $-$  or  $0$  to  $+$  but not from  $+$  or  $-$  to  $0$ .*

## Remark 8

*By shifting  $\mathbf{A}$  by a multiple of the identity, we can determine the number of eigenvalues in any interval  $[a, b)$ : it is the number of eigenvalues in  $(-\infty, b)$  minus the number in  $(-\infty, a)$ ; i.e., we only need consider two matrices  $\mathbf{A} - b\mathbf{I}$  and  $\mathbf{A} - a\mathbf{I}$ .*

## Remark 9

*The determinants of the matrices  $\{\mathbf{A}^{(k)}\}$  are related by a three-term recurrence relation:*

$$\det(\mathbf{A}^{(k)}) = a_k \det(\mathbf{A}^{(k-1)}) - b_{k-1}^2 \det(\mathbf{A}^{(k-2)}).$$

*Introducing the shift by  $z\mathbf{I}$  and writing  $p^{(k)}(z) = \det(\mathbf{A}^{(k)} - z\mathbf{I})$ , we get*

$$p^{(k)}(z) = (a_k - z)p^{(k-1)}(z) - b_{k-1}^2 p^{(k-2)}(z),$$

*where  $p^{(-1)}(z) = 0$ ,  $p^{(0)}(z) = 1$ .*

### 3. Secular equation

#### Proposition 10

Let  $\mathbf{D} \in \mathbb{R}^{m \times m}$  be a diagonal matrix with *distinct* diagonal entries  $\{d_j\}$  and  $\mathbf{w} \in \mathbb{R}^m$  be a vector with  $w_j \neq 0$  for all  $j$ . Assume  $\beta \in \mathbb{R}$  and  $\beta \neq 0$ . The eigenvalues of  $\mathbf{D} + \beta \mathbf{w} \mathbf{w}^\top$  are the roots of the rational function

$$f(\lambda) = 1 + \beta \sum_{j=1}^m \frac{w_j^2}{d_j - \lambda}.$$

#### Proof.

Suppose  $\mathbf{q}$  is an eigenvector of  $\mathbf{D} + \beta \mathbf{w} \mathbf{w}^\top$ . The statement follows from  $\mathbf{w}^\top \mathbf{q} \neq 0$ ,  $\lambda \neq d_j$  (why?) and  $\mathbf{w}^\top \mathbf{q} (1 + \beta \mathbf{w}^\top (\mathbf{D} - \lambda \mathbf{I})^{-1} \mathbf{w}) = 0$ .  $\square$

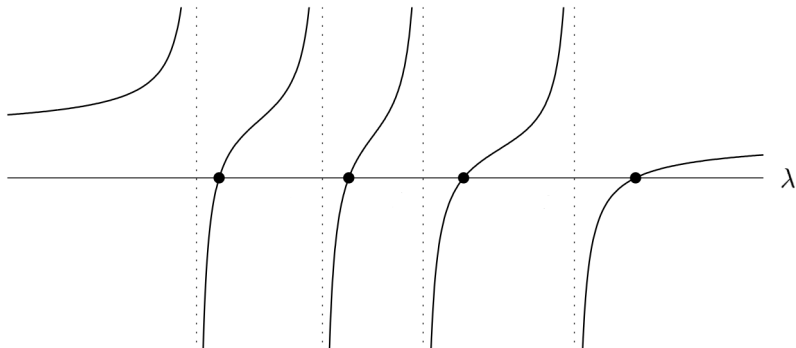
#### Remark 11

The equation  $f(\lambda) = 0$  is known as the secular equation.



**Exercise:** Assume  $\mathbf{D}$ ,  $\beta$  and  $\mathbf{w}$  are those in Proposition 10. If  $\lambda$  is an eigenvalue of  $\mathbf{D} + \beta\mathbf{w}\mathbf{w}^\top$ , then  $(\mathbf{D} - \lambda\mathbf{I})^{-1}\mathbf{w}$  is a corresponding eigenvector.

- Plot of the function  $f(\lambda)$  for a problem of dimension 4. The poles of  $f(\lambda)$  are the eigenvalues  $\{d_j\}$  of  $\mathbf{D}$ , and the roots of  $f(\lambda)$  (solid dots) are the eigenvalues of  $\mathbf{D} + \beta\mathbf{w}\mathbf{w}^\top$ . These roots can be determined rapidly.



## Proposition 12

Let  $\mathbf{D} \in \mathbb{R}^{m \times m}$  be a diagonal matrix and  $\mathbf{w} \in \mathbb{R}^m$  be a vector. Assume  $\beta \in \mathbb{R}$  and  $\beta \neq 0$ . Then there exist a permutation matrix  $\mathbf{P}$  and an orthogonal matrix  $\mathbf{V}$  such that

$$\mathbf{P}^\top \mathbf{V}^\top (\mathbf{D} + \beta \mathbf{w} \mathbf{w}^\top) \mathbf{V} \mathbf{P} = \begin{bmatrix} \mathbf{D}_1 + \beta \mathbf{w}_1 \mathbf{w}_1^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_2 \end{bmatrix},$$

where  $\mathbf{D}_1 \in \mathbb{R}^{r \times r}$  is a diagonal matrix with distinct diagonal entries,  $\mathbf{D}_2 \in \mathbb{R}^{(m-r) \times (m-r)}$  is a diagonal matrix, and  $\mathbf{w}_1 \in \mathbb{R}^r$  is a vector with nonzero entries. More precisely,

$$\mathbf{P}^\top \mathbf{V}^\top \mathbf{D} \mathbf{V} \mathbf{P} = \begin{bmatrix} \mathbf{D}_1 & \\ & \mathbf{D}_2 \end{bmatrix}, \quad \mathbf{P}^\top \mathbf{V}^\top \mathbf{w} = \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{0} \end{bmatrix}.$$

- **Exercise:** Prove Proposition 12.

## 4. Divide-and-conquer

### Remark 13

*A symmetric tridiagonal matrix can be written as the sum of a  $2 \times 2$  block diagonal matrix with tridiagonal blocks and a rank-one correction.*

- Let  $\mathbf{T} \in \mathbb{R}^{m \times m}$  be symmetric, tridiagonal, and unreduced. For any  $n$  in the range  $1 \leq n < m$ , we can write

$$\mathbf{T} = \begin{bmatrix} \hat{\mathbf{T}}_1 & \\ & \hat{\mathbf{T}}_2 \end{bmatrix} + \beta \begin{bmatrix} \mathbf{e}_n \\ \mathbf{e}_1 \end{bmatrix} \begin{bmatrix} \mathbf{e}_n^\top & \mathbf{e}_1^\top \end{bmatrix}.$$

$$\mathbf{T} = \begin{array}{|c|c|} \hline \mathbf{T}_1 & \\ \hline \beta & \mathbf{T}_2 \\ \hline \end{array} = \begin{array}{|c|c|} \hline \hat{\mathbf{T}}_1 & \\ \hline & \hat{\mathbf{T}}_2 \\ \hline \end{array} + \begin{array}{|c|c|} \hline & \\ \hline \beta & \beta \\ \hline \beta & \beta \\ \hline \end{array}$$

- Suppose that the eigen decompositions  $\hat{\mathbf{T}}_1 = \mathbf{Q}_1 \mathbf{D}_1 \mathbf{Q}_1^\top$  and  $\hat{\mathbf{T}}_2 = \mathbf{Q}_2 \mathbf{D}_2 \mathbf{Q}_2^\top$  have been computed ( $\mathbf{D}_1$  and  $\mathbf{D}_2$  are diagonal, and  $\mathbf{Q}_1$  and  $\mathbf{Q}_2$  are orthogonal). Then we have

$$\mathbf{T} = \begin{bmatrix} \mathbf{Q}_1 & \\ & \mathbf{Q}_2 \end{bmatrix} \left( \begin{bmatrix} \mathbf{D}_1 & \\ & \mathbf{D}_2 \end{bmatrix} + \beta \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} \begin{bmatrix} \mathbf{u}^\top & \mathbf{v}^\top \end{bmatrix} \right) \begin{bmatrix} \mathbf{Q}_1^\top & \\ & \mathbf{Q}_2^\top \end{bmatrix},$$

where  $\mathbf{u} := \mathbf{Q}_1^\top \mathbf{e}_n$  and  $\mathbf{v} := \mathbf{Q}_2^\top \mathbf{e}_1$ . The problem is reduced to find the eigenvalues of a diagonal matrix plus a rank-one correction.

### Remark 14

*Suppose that the eigenvalues of  $\hat{\mathbf{T}}_1$  and  $\hat{\mathbf{T}}_2$  are known. A nonlinear but rapid calculation can be used to get from the eigenvalues of  $\hat{\mathbf{T}}_1$  and  $\hat{\mathbf{T}}_2$  to those of  $\mathbf{T}$  itself by the secular equation. The divide-and-conquer algorithm is based on recursive use of this idea.*