Numerical Linear Algebra Assignment 7

Exercise 1. (10 points)

Let $\lambda_1, \cdots \lambda_m$ be the m eigenvalues of $\mathbf{A} \in \mathbb{C}^{m\times m}$. Let

$$\mathbf{M} = \frac{\mathbf{A}+\mathbf{A}^*}{2}, \quad \mathbf{N} = \frac{\mathbf{A}-\mathbf{A}^*}{2}.$$

Prove that

$$\sum_{i=1}^{m} |\lambda_i|^2 \le \|\mathbf{A}\|_{\mathrm{F}}^2, \quad \sum_{i=1}^{m} |\mathrm{Re}\lambda_i|^2 \le \|\mathbf{M}\|_{\mathrm{F}}^2, \quad \sum_{i=1}^{m} |\mathrm{Im}\lambda_i|^2 \le \|\mathbf{N}\|_{\mathrm{F}}^2.$$

Exercise 2. (TreBau Exercise 25.1, 10 points)

(a) Let $\mathbf{A} \in \mathbb{C}^{m\times m}$ be tridiagonal and Hermitian, with all its sub- and superdiagonal entries nonzero. Prove that the eigenvalues of $\mathbf{A}$ are distinct. (Hint: Show that for any $\lambda \in \mathbb{C}$, $\mathbf{A} - \lambda\mathbf{I}$ has rank at least $m-1$.)

(b) On the other hand, let $\mathbf{A}$ be upper-Hessenberg, with all its subdiagonal entries nonzero. Give an example that shows that the eigenvalues of $\mathbf{A}$ are not necessarily distinct.

Exercise 3. (10 points)

Suppose that $\mathbf{A}$ is normal and triangular. Show that $\mathbf{A}$ must be diagonal.

Exercise 4. (10 points)

Let $\mathbf{A} \in \mathbb{C}^{m\times m}$, $\mathbf{B} \in \mathbb{C}^{n\times n}$ and $\mathbf{C} \in \mathbb{C}^{m\times n}$ be given. Prove that the Sylvester equation $\mathbf{AX}-\mathbf{XB} = \mathbf{C}$ has a unique solution $\mathbf{X} \in \mathbb{C}^{m\times n}$ if and only if $\Lambda(\mathbf{A}) \cap \Lambda(\mathbf{B}) = \emptyset$. (You MUST use Schur factorizations of $\mathbf{A}$ and $\mathbf{B}$ to prove the conclusion. Any other approach is NOT accepted.)

Exercise 5. (10 points)

Let $\mathbf{H} \in \mathbb{C}^{m\times m}$ be an irreducible ($h_{i+1,i} \ne 0$ for $i = 1 : m-1$) upper Hessenberg matrix. Prove that any eigenvalue of $\mathbf{H}$ has geometric multiplicity 1.

Exercise 6. (Programming, 10 points)

Design an algorithm to solve the Sylvester equation $\mathbf{AX}-\mathbf{XB} = \mathbf{C}$ (assume that $\Lambda(\mathbf{A})\cap\Lambda(\mathbf{B}) = \emptyset$). Test your code via a numerical experiment. Hint: Use the MATLAB command `schur`.