# Numerical Linear Algebra Assignment 6

**Exercise 1. (10 points)**

A splitting of $\mathbf{A} \in \mathbb{C}^{n \times n}$ is a decomposition $\mathbf{A} = \mathbf{M} - \mathbf{K}$, with $\mathbf{M}$ nonsingular. Prove that if $\mathbf{A}$ is singular, then $\rho(\mathbf{M}^{-1}\mathbf{K}) \geq 1$.

**Exercise 2. (Shufang Xu, 10 points)**

Consider the stationary iterative method

$$\mathbf{x}^{(m)} = \mathbf{R}\mathbf{x}^{(m-1)} + \mathbf{c}.$$

Assume $\mathbf{R} \in \mathbb{C}^{n \times n}$ and the spectral radius $\rho(\mathbf{R}) = 0$. For any given $\mathbf{c} \in \mathbb{R}^n$ and $\mathbf{x}^{(0)} \in \mathbb{C}^n$, prove that the $n$th iterate $\mathbf{x}^{(n)}$ is the solution of $(\mathbf{I} - \mathbf{R})\mathbf{x} = \mathbf{c}$.

**Exercise 3. (Zhihao Cao, 10 points)**

Consider the stationary iterative method

$$\mathbf{x}^{(m)} = \mathbf{R}\mathbf{x}^{(m-1)} + \mathbf{c}.$$

Assume that $\|\mathbf{R}\|_2 < 1$. Prove that

$$\|\mathbf{x}^{(m)} - \mathbf{x}_*\|_2 \leq \frac{\|\mathbf{R}\|_2^m}{1 - \|\mathbf{R}\|_2}\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|_2,$$

where $\mathbf{x}_*$ is the solution of $(\mathbf{I} - \mathbf{R})\mathbf{x} = \mathbf{c}$.

**Exercise 4. (Shufang Xu, 10 points)**

Consider the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ with

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}.$$

Discuss the convergence of Jacobi's method and Gauss–Seidel method for this linear system.

**Exercise 5. (10 points)**

Prove that $0 < \omega < 2$ is required for the convergence of $\mathrm{SOR}(\omega)$ for all starting vectors.

**<span style="color:red">Compulsory requirement for programming: Use Matlab's `publish` to save all your code, comments, and results to a PDF file. You must use the programming format files: example_format.zip.</span>**

**Programming 1. (10 points)**

Construct a linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ with a strictly row diagonally dominant $15 \times 15$ matrix $\mathbf{A}$. Plot the convergence history of Jacobi's, Gauss–Seidel, SOR(0.5), and SSOR(0.5) methods in one figure. You must use Matlab's `semilogy`: the x-axis is the iteration index $m$, and the y-axis is $\|\mathbf{x}^{(m)} - \mathbf{A}^{-1}\mathbf{b}\|_2$. For each method, stop at the 30th iteration.