Numerical Linear Algebra Assignment 5

Exercise 1. (TreBau Exercise 20.1, 10 points)

Let $\mathbf{A} \in \mathbb{C}^{m\times m}$ be nonsingular. Show that $\mathbf{A}$ has an LU factorization if and only if for each k with $1 \le k \le m$, the upper-left $k \times k$ block $\mathbf{A}_{1:k,1:k}$ is nonsingular. (Hints: The row operations of Gaussian elimination leave the determinants $\det(\mathbf{A}_{1:k,1:k})$ unchanged.) Prove that this LU factorization is unique.

Exercise 2. (TreBau Exercise 20.3, 10 points)

Suppose an $m \times m$ matrix $\mathbf{A}$ is written in the block form $\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$, where $\mathbf{A}_{11}$ is $n \times n$ and $\mathbf{A}_{22}$ is $(m-n)\times(m-n)$. Assume that $\mathbf{A}$ satisfies the condition of Exercise 1 (TreBau Exercise 20.1).

(a) Verify the formula

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12} \end{bmatrix}$$

for "elimination" of the block $\mathbf{A}_{21}$. The matrix $\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}$ is known as the *Schur complement* of $\mathbf{A}_{11}$ in $\mathbf{A}$.

(b) Suppose $\mathbf{A}_{21}$ is eliminated by means of n steps of Gaussian elimination. Show that the bottom-right $(m-n)\times(m-n)$ block of the result is again $\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}$.

Exercise 3. (10 points)

Compute the Cholesky factorization of the matrix $\mathbf{A} = \begin{bmatrix} 2 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 3 & 1+\sqrt{2} \\ \sqrt{2} & 1+\sqrt{2} & 4 \end{bmatrix}$.

Exercise 4. (Programming, TreBau Exercises 20.2, 10 points)

Answer the question in TreBau Exercises 20.2 and write MATLAB codes to provide an example with $p = 3$ for a 20×20 matrix $\mathbf{A}$. Plot the sparisity patterns of $\mathbf{A}, \mathbf{L}$ and $\mathbf{U}$ by using MATLAB's `spy`.

Exercise 5. (Programming, TreBau Exercises 20.4, 10 points)

Write two MATLAB functions, `[L,U]=gelu(A)` and `[L,U]=geoplu(A)`, to implement Algorithm 20.1 and the "outer product" form of Guassian elimination you have designed in Exercises 20.4, respectively. Compare the time required to run functions `gelu` and `geoplu` for a 500×500 matrix $\mathbf{A}$. Use MATLAB's `timeit` to measure time.

Further Reading

MathWorks Help Center: Performance and Memory
https://ww2.mathworks.cn/help/matlab/performance-and-memory.html
https://ww2.mathworks.cn/help/matlab/matlab_prog/vectorization.html

Exercise 6. (Programming, 10 points)

Write a MATLAB function, `[L,U,P]=gepp(A)`, to implement Algorithm 21.1 of TreBau's book. Test the 4×4 complex matrix ($\mathrm{i} = \sqrt{-1}$)

$$\mathbf{A} = \begin{bmatrix} 1+1\mathrm{i} & -1\mathrm{i} & 0 & 1\mathrm{i} \\ 1 & 1+1\mathrm{i} & 1-1\mathrm{i} & 1+3\mathrm{i} \\ 0 & 1\mathrm{i} & -1\mathrm{i} & -1\mathrm{i} \\ 2\mathrm{i} & 1 & 0 & 0 \end{bmatrix}.$$

Exercise 7. (Programming, 10 points)

Write a MATLAB function, `R=mychol(A)`, to implement Algorithm 23.1 of TreBau's book. Test the 4×4 Hermitian positive definite matrix ($\mathrm{i} = \sqrt{-1}$)

$$\mathbf{A} = \begin{bmatrix} 7 & -2\mathrm{i} & 1-1\mathrm{i} & 2+4\mathrm{i} \\ 2\mathrm{i} & 5 & -1-2\mathrm{i} & 2+2\mathrm{i} \\ 1+1\mathrm{i} & -1+2\mathrm{i} & 3 & -1+4\mathrm{i} \\ 2-4\mathrm{i} & 2-2\mathrm{i} & -1-4\mathrm{i} & 12 \end{bmatrix}.$$

Exercise 8. (Programming, 10 points)

Write a MATLAB function, `[Q,R,P]=hqrp(A)`, via Householder reflectors, to compute the so-called QR factorization with column pivoting: `AP=QR`, where `Q` is unitary, `R` is upper triangular, `P` is a permutation matrix, and `abs(diag(R))` is decreasing. Test the 4×4 matrix in Exercise 7.