

Lecture 8: Power/Inverse iteration, Rayleigh quotient iteration



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1. Eigenvalue problem and polynomial rootfinding problem.

- The eigenvalues of a matrix $\mathbf{A} \in \mathbb{C}^{m \times m}$ are the m roots of its characteristic polynomial $p(z) = \det(z\mathbf{I} - \mathbf{A})$.
- Suppose we have the monic polynomial

$$p(z) = z^m + a_{m-1}z^{m-1} + \cdots + a_1z + a_0.$$

It is not hard to verify that $p(z) = \det(z\mathbf{I} - \mathbf{A})$, where the $m \times m$ matrix \mathbf{A} is

$$\mathbf{A} = \begin{bmatrix} 0 & & & & -a_0 \\ 1 & 0 & & & -a_1 \\ & 1 & 0 & & -a_2 \\ & & 1 & \ddots & \vdots \\ & & & \ddots & 0 & -a_{m-2} \\ & & & & 1 & -a_{m-1} \end{bmatrix}.$$

The matrix \mathbf{A} is called a *companion matrix* corresponding to $p(z)$.

- **Any eigenvalue solver must be iterative** because no explicit root expressing formula exists for polynomial of degree ≥ 5 . The goal of an eigenvalue solver is to produce sequences of numbers that converge rapidly towards eigenvalues.

- Convergence rate

Let e_1, e_2, e_3, \dots be a sequence of nonnegative numbers representing errors in some iterative process that converge to zero, and suppose there are a positive constant C and an exponent α such that for all sufficiently large k , $e_{k+1} \leq C(e_k)^\alpha$. Then,

- (1) $\alpha = 1$ and $C < 1$, *linear convergence* or *geometric convergence*;
- (2) $\alpha = 2$, *quadratic convergence*;
- (3) $\alpha = 3$, *cubic convergence*;

2. Rayleigh quotient

- The *Rayleigh quotient* of a nonzero vector $\mathbf{x} \in \mathbb{C}^m$ with respect to \mathbf{A} is the scalar

$$r(\mathbf{x}) = \frac{\mathbf{x}^* \mathbf{A} \mathbf{x}}{\mathbf{x}^* \mathbf{x}}.$$

Theorem 1

Let $\{\lambda, \mathbf{v}\}$ be an eigenpair of the matrix \mathbf{A} , i.e., $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$. We further assume that $\|\mathbf{v}\|_2 = 1$.

- (i) If \mathbf{A} is non-normal, then the Rayleigh quotient $r(\mathbf{x})$ is *generally* a linearly accurate estimate of the eigenvalue λ , i.e.,

$$|r(\mathbf{x}) - \lambda| = \mathcal{O}(\|\mathbf{x} - \mathbf{v}\|_2), \quad \text{as } \mathbf{x} \rightarrow \mathbf{v}.$$

- (ii) If \mathbf{A} is normal, then the Rayleigh quotient $r(\mathbf{x})$ is a quadratically accurate estimate of the eigenvalue λ , i.e.,

$$|r(\mathbf{x}) - \lambda| = \mathcal{O}(\|\mathbf{x} - \mathbf{v}\|_2^2), \quad \text{as } \mathbf{x} \rightarrow \mathbf{v}.$$

Proof.

Consider Schur form $\mathbf{T} = \mathbf{Q}^* \mathbf{A} \mathbf{Q}$ with $t_{11} = \lambda$ and $\mathbf{Q}\mathbf{e}_1 = \mathbf{v}$. □

3. Power iteration

Algorithm 1: Power iteration

$\mathbf{v}^{(0)}$ = some vector with $\|\mathbf{v}^{(0)}\|_2 = 1$

for $k = 1, 2, 3, \dots$,

$\mathbf{w} = \mathbf{A}\mathbf{v}^{(k-1)}$

$\mathbf{v}^{(k)} = \mathbf{w} / \|\mathbf{w}\|_2$

$\lambda^{(k)} = (\mathbf{v}^{(k)})^* \mathbf{A} \mathbf{v}^{(k)}$

end

- One application: Google's Pagerank.
- Power iteration can find only the eigenvector corresponding to the eigenvalue with the largest magnitude. The convergence is *linear*, which is very slow if the largest two eigenvalues are close in magnitude.

- Assume that $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$ is diagonalizable with $\|\mathbf{v}_1\|_2 = 1$ and

$$\mathbf{\Lambda} = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_m\}, \quad |\lambda_1| > |\lambda_2| \geq \dots \geq |\lambda_m|.$$

Let $\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} := \mathbf{V}^{-1}\mathbf{v}^{(0)}$. If $\alpha_1 \neq 0$, then we have

$$\mathbf{A}^k \mathbf{v}^{(0)} = \mathbf{V} \mathbf{\Lambda}^k \mathbf{V}^{-1} \mathbf{v}^{(0)} = \mathbf{V} \begin{bmatrix} \alpha_1 \lambda_1^k \\ \alpha_2 \lambda_2^k \\ \vdots \\ \alpha_m \lambda_m^k \end{bmatrix} = \alpha_1 \lambda_1^k \mathbf{V} \begin{bmatrix} 1 \\ \frac{\alpha_2}{\alpha_1} \frac{\lambda_2^k}{\lambda_1^k} \\ \vdots \\ \frac{\alpha_m}{\alpha_1} \frac{\lambda_m^k}{\lambda_1^k} \end{bmatrix}.$$

Then $\mathbf{v}^{(k)} = \frac{\mathbf{A}^k \mathbf{v}^{(0)}}{\|\mathbf{A}^k \mathbf{v}^{(0)}\|_2} \rightarrow e^{i\theta_k} \mathbf{v}_1$ and $\lambda^{(k)} \rightarrow \lambda_1$, where

$$\theta_k = k\theta + \theta_0, \quad e^{i\theta} = \lambda_1/|\lambda_1|, \quad e^{i\theta_0} = \alpha_1/|\alpha_1|.$$

Theorem 2

Suppose that \mathbf{A} is diagonalizable, i.e., $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$ with

$$\mathbf{\Lambda} = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_m\}.$$

Furthermore, suppose $\mathbf{e}_1^* \mathbf{V}^{-1} \mathbf{v}^{(0)} \neq 0$, $\|\mathbf{v}_1\|_2 = 1$, and

$$|\lambda_1| > |\lambda_2| \geq \dots \geq |\lambda_m|.$$

Then the iterates of power iteration satisfy, as $k \rightarrow \infty$,

$$\|\mathbf{v}^{(k)} - e^{i\theta_k} \mathbf{v}_1\|_2 = \mathcal{O}\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right),$$

and

$$|\lambda^{(k)} - \lambda_1| = \mathcal{O}\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right) \quad \text{or} \quad \mathcal{O}\left(\left|\frac{\lambda_2}{\lambda_1}\right|^{2k}\right).$$

4. Shifted inverse iteration

Proposition 3

For any μ that is not an eigenvalue, the eigenvectors of $(\mathbf{A} - \mu\mathbf{I})^{-1}$ are the same as the eigenvectors of \mathbf{A} , and the corresponding eigenvalues are $\{(\lambda_j - \mu)^{-1}\}$, where $\{\lambda_j\}$ are the eigenvalues of \mathbf{A} .

Algorithm 2: Shifted inverse iteration

$\mathbf{v}^{(0)}$ = some vector with $\|\mathbf{v}^{(0)}\|_2 = 1$

for $k = 1, 2, 3, \dots$,

 Solve $(\mathbf{A} - \mu\mathbf{I})\mathbf{w} = \mathbf{v}^{(k-1)}$ for \mathbf{w}

$\mathbf{v}^{(k)} = \mathbf{w} / \|\mathbf{w}\|_2$

$\lambda^{(k)} = (\mathbf{v}^{(k)})^* \mathbf{A} \mathbf{v}^{(k)}$

end

- Like power iteration, shifted inverse iteration exhibits only *linear* convergence. **Other important issue: Exercise 27.5**

Theorem 4

Suppose that \mathbf{A} is diagonalizable, i.e., $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$ with

$$\mathbf{\Lambda} = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_m\}.$$

Suppose λ_j is the closest eigenvalue to μ and λ_l is the second closest, that is,

$$|\lambda_j - \mu| < |\lambda_l - \mu| \leq |\lambda_i - \mu|$$

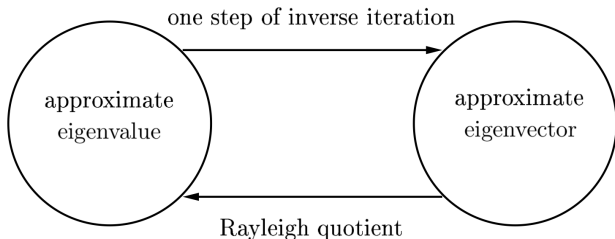
for each $i \neq j$. Furthermore, suppose $\mathbf{e}_j^* \mathbf{V}^{-1} \mathbf{v}^{(0)} \neq 0$ and $\|\mathbf{v}_j\|_2 = 1$. Then the iterates of shifted inverse iteration satisfy, as $k \rightarrow \infty$,

$$\|\mathbf{v}^{(k)} - e^{i\theta_k} \mathbf{v}_j\|_2 = \mathcal{O} \left(\left| \frac{\lambda_j - \mu}{\lambda_l - \mu} \right|^k \right),$$

and

$$|\lambda^{(k)} - \lambda_j| = \mathcal{O} \left(\left| \frac{\lambda_j - \mu}{\lambda_l - \mu} \right|^k \right) \quad \text{or} \quad \mathcal{O} \left(\left| \frac{\lambda_j - \mu}{\lambda_l - \mu} \right|^{2k} \right).$$

5. Rayleigh quotient iteration



Algorithm 3: Rayleigh quotient iteration

$\mathbf{v}^{(0)}$ = some vector with $\|\mathbf{v}^{(0)}\|_2 = 1$

$\lambda^{(0)} = (\mathbf{v}^{(0)})^* \mathbf{A} \mathbf{v}^{(0)}$

for $k = 1, 2, 3, \dots$,

 Solve $(\mathbf{A} - \lambda^{(k-1)} \mathbf{I}) \mathbf{w} = \mathbf{v}^{(k-1)}$ for \mathbf{w}

$\mathbf{v}^{(k)} = \mathbf{w} / \|\mathbf{w}\|_2$

$\lambda^{(k)} = (\mathbf{v}^{(k)})^* \mathbf{A} \mathbf{v}^{(k)}$

end

Theorem 5

Rayleigh quotient iteration converges to an eigenpair

$$\{\lambda, e^{i\theta_k} \mathbf{v}\}, \quad \|\mathbf{v}\|_2 = 1,$$

for all except a set of measure zero of starting vectors $\mathbf{v}^{(0)}$. When it converges, the convergence is ultimately quadratic ($\alpha = 2$) for non-normal case or cubic ($\alpha = 3$) for normal case in the sense that if $\mathbf{v}^{(k)}$ is sufficiently close to the eigenvector $e^{i\theta_k} \mathbf{v}$, then

$$\|\mathbf{v}^{(k+1)} - e^{i\theta_{k+1}} \mathbf{v}\|_2 = \mathcal{O}(\|\mathbf{v}^{(k)} - e^{i\theta_k} \mathbf{v}\|_2^\alpha)$$

and

$$|\lambda^{(k+1)} - \lambda| = \mathcal{O}(|\lambda^{(k)} - \lambda|^\alpha)$$

as $k \rightarrow \infty$.

Example:

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}, \quad \mathbf{v}^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} / \sqrt{3}.$$

The eigenvalue $\lambda = 5.214319743377$

Power iteration:

$$\begin{aligned} \lambda^{(0)} &= 5.1818\dots \\ \lambda^{(1)} &= 5.2081\dots \\ \lambda^{(2)} &= 5.2130\dots \end{aligned}$$

Rayleigh quotient iteration:

$$\begin{aligned} \lambda^{(0)} &= 5 \\ \lambda^{(1)} &= 5.2131\dots \\ \lambda^{(2)} &= 5.214319743184\dots \end{aligned}$$

The convergence of Rayleigh quotient iteration is spectacular:
each iteration triples the number of digits of accuracy.