

Lecture 20: Backward stability of an algorithm



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1. Floating point number

- For given integers p and β , in the IEEE floating point standard (founded in 1985, updated in 2008, being undated again now), the elements of the floating point number system \mathbf{F} are the number 0 together with numbers of the form

$$x = \pm d_1.d_2 \cdots d_p \times \beta^e$$

where the integers d_i , e satisfy

$$0 \leq d_i \leq \beta - 1, \quad d_1 \neq 0, \quad e_{\min} \leq e \leq e_{\max}.$$

- One need to store sign bit (\pm), exponent (e), and mantissa ($d_1.d_2 \cdots d_p$); but not the *base* or *radix* ($\beta \geq 2$). Floating point number system usually uses $\beta = 2$ (10 sometimes, 16 historically).

| Precision | β | Bits | p | e_{\min} | e_{\max} | $\epsilon_{\text{machine}}$ |
|-------------|---------|---------|-----|------------|------------|-----------------------------|
| Single (32) | 2 | 1+8+23 | 24 | -126 | 127 | 2^{-24} |
| Double (64) | 2 | 1+11+52 | 53 | -1022 | 1023 | 2^{-53} |

[Reading](#). Higham and Mary, [Mixed precision algorithms in numerical linear algebra](#), Acta Numerica, 2022.

1.1. Limitations of digital representations

- Only a finite subset of the real numbers (or the complex numbers) can be represented. Therefore,
 - (i) the represented numbers cannot be arbitrarily large or small;
 - (ii) there are gaps between these numbers.

1.2. Floating point number machine accuracy

- In IEEE double precision arithmetic, the interval $[1, 2]$ is represented by the discrete subset

$$1, \quad 1 + 2^{-52}, \quad 1 + 2 \times 2^{-52}, \quad 1 + 3 \times 2^{-52}, \quad \dots, \quad 2.$$

The interval $[2, 4]$ is represented by the same numbers multiplied by 2,

$$2, \quad 2 + 2^{-51}, \quad 2 + 2 \times 2^{-51}, \quad 2 + 3 \times 2^{-51}, \quad \dots, \quad 4.$$

In general, the interval $[2^j, 2^{j+1}]$ is represented by the numbers for $[1, 2]$ times 2^j .

- For floating point number system, the machine accuracy, denoted by $\epsilon_{\text{machine}}$, is defined as: *half the distance between 1 and the next larger floating point number*. We have

$$\forall x \in [\theta, \Theta], \quad \exists x' \in \mathbf{F} \quad s.t., \quad |x - x'| \leq \epsilon_{\text{machine}}|x|.$$

In MATLAB, `eps` = $2\epsilon_{\text{machine}} = 2^{-52}$ in double precision.

- Let $\text{fl}: \mathbb{R} \rightarrow \mathbf{F}$ denote the function giving the closest floating point approximation. We have

$$\forall x \in [\theta, \Theta], \quad \exists \epsilon \in \mathbb{R} \quad s.t., \quad |\epsilon| \leq \epsilon_{\text{machine}} \quad and \quad \text{fl}(x) = x(1 + \epsilon).$$

Exercise. (James Demmel) Prove the following: If floating point numbers x and y satisfy $2y \geq x \geq y \geq 0$, then $\text{fl}(x - y) = x - y$, i.e., $x - y$ is an exact floating point number.

1.3. Floating point arithmetic

- $*$ ($+, -, \times, \div$) in \mathbb{R} ; \circledast ($\oplus, \ominus, \otimes, \oslash$) in \mathbf{F} ; $x \circledast y = \text{fl}(x * y)$.

Fundamental Axiom of Floating Point Arithmetic

For all $x, y \in \mathbf{F}$, there exists ϵ with $|\epsilon| \leq \epsilon_{\text{machine}}$ such that

$$x \circledast y = (x * y)(1 + \epsilon).$$

1.4. Programming exercise

- TreBau Exercise 13.3 (Horner's rule for polynomial evaluation).

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

Algorithm Horner's rule for $p(x) = \sum_{i=0}^n a_i x^i$.

```
p = a_n
for i = n - 1 : -1 : 0
    p = xp + a_i;
end
```

2. Algorithm

- Given a *problem* $f : \mathbb{X} \rightarrow \mathbb{Y}$. An *algorithm* for the problem f can be viewed as a map $\tilde{f} : \mathbb{X} \rightarrow \mathbb{Y}$.
- More precisely, assume that a problem f , a computer with floating point system, and a program for solving the problem are fixed:
 - (1) given $x \in \mathbb{X}$, let $\text{fl}(x)$ be the corresponding floating point representation;
 - (2) input $\text{fl}(x)$ to the program and run it in the computer;
 - (3) the output (computed result) of the program belongs to \mathbb{Y} and is called $\tilde{f}(x)$.
- A problem may have different algorithms (due to different programs). For example: the problem of sum of three numbers: $a + b + c$. Programs: $(a + b) + c$, $a + (b + c)$, and $(a + c) + b$.
- What can happen for an ill-conditioned problem? Since x is perturbed to $\text{fl}(x)$, then $\|\tilde{f}(x) - f(x)\|$ maybe large.

2.1. Accuracy

- An algorithm \tilde{f} for a problem f is *accurate* if for each $x \in \mathbb{X}$,

$$\frac{\|\tilde{f}(x) - f(x)\|}{\|f(x)\|} = \mathcal{O}(\epsilon_{\text{machine}}).$$

- The meaning of $\mathcal{O}(\cdot)$: $\phi(t) = \mathcal{O}(\psi(t))$ means there exists a positive constant C such that $|\phi(t)| \leq C\psi(t)$ for all t sufficiently close to an understood limit (e.g., $t \rightarrow 0$ or $t \rightarrow \infty$).

2.2. Stability

- An algorithm \tilde{f} for a problem f is *stable* if for each $x \in \mathbb{X}$, there exists $\tilde{x} \in \mathbb{X}$, such that

$$\frac{\|\tilde{x} - x\|}{\|x\|} = \mathcal{O}(\epsilon_{\text{machine}}), \quad \frac{\|\tilde{f}(x) - f(\tilde{x})\|}{\|f(\tilde{x})\|} = \mathcal{O}(\epsilon_{\text{machine}}).$$

Remark 1

A stable algorithm gives nearly the right answer to nearly the right question.

2.3. Backward stability

- An algorithm \tilde{f} for a problem f is *backward stable* if for each $x \in \mathbb{X}$, there exists $\tilde{x} \in \mathbb{X}$, such that

$$\frac{\|\tilde{x} - x\|}{\|x\|} = \mathcal{O}(\epsilon_{\text{machine}}), \quad \tilde{f}(x) = f(\tilde{x}).$$

Remark 2

A backward stable algorithm gives exactly the right answer to nearly the right question.

Remark 3

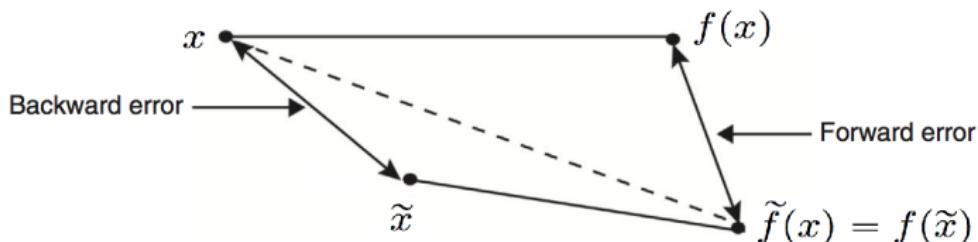
Backward stability obviously implies stability.

Remark 4

Backward stability is both *stronger and simpler* than stability. Many algorithms of NLA are backward stable.

3. Backward error analysis

- The first step is to investigate the conditioning of the problem.
The second step is to investigate the backward stability of the corresponding algorithm.
- Forward error \lesssim Condition number \times Backward error.



Theorem 5 (Accuracy of a backward stable algorithm)

Suppose \tilde{f} is backward stable for f . Let $\kappa(f(x))$ denote the condition number of the problem $f(x)$. Then the relative errors satisfy

$$\frac{\|\tilde{f}(x) - f(x)\|}{\|f(x)\|} = \mathcal{O}(\kappa(f(x))\epsilon_{\text{machine}}).$$

Proof.

By the definition of backward stability, we have there exists \tilde{x} such that

$$\frac{\|\tilde{x} - x\|}{\|x\|} = \mathcal{O}(\epsilon_{\text{machine}}), \quad \tilde{f}(x) = f(\tilde{x}).$$

By the definition of $\kappa(f(x))$,

$$\kappa(f(x)) = \lim_{\varepsilon \rightarrow 0^+} \sup_{\|\delta x\| \leqslant \varepsilon} \left(\frac{\|\delta f\|}{\|f(x)\|} / \frac{\|\delta x\|}{\|x\|} \right),$$

we have

$$\frac{\|\tilde{f}(x) - f(x)\|}{\|f(x)\|} \leqslant (\kappa(f(x)) + o(1)) \frac{\|\tilde{x} - x\|}{\|x\|},$$

where $o(1)$ denotes a quantity that converges to zero as $\epsilon_{\text{machine}} \rightarrow 0$.
Then the statement follows. □

4. Examples

- **Floating point arithmetic:** The floating point operations $\oplus, \ominus, \otimes, \oslash$ are all backward stable. Let $x_1, x_2 \in \mathbb{R}$. Consider the problem $f(x_1, x_2) = x_1 * x_2$ and the corresponding algorithm $\tilde{f}(x_1, x_2) = \text{fl}(x_1) \otimes \text{fl}(x_2)$. There exist $|\epsilon_1|, |\epsilon_2|, |\epsilon_3| \leq \epsilon_{\text{machine}}$ and $|\epsilon_4|, |\epsilon_5| \leq 2\epsilon_{\text{machine}} + \mathcal{O}(\epsilon_{\text{machine}}^2)$ such that (except \otimes and \oslash)

$$\begin{aligned}\tilde{f}(x_1, x_2) &= \text{fl}(x_1) \otimes \text{fl}(x_2) \\ &= ([x_1(1 + \epsilon_1)] * [x_2(1 + \epsilon_2)])(1 + \epsilon_3) \\ &= [x_1(1 + \epsilon_1)(1 + \epsilon_3)] * [x_2(1 + \epsilon_2)(1 + \epsilon_3)] \\ &= [x_1(1 + \epsilon_4)] * [x_2(1 + \epsilon_5)] \\ &= \tilde{x}_1 * \tilde{x}_2 = f(\tilde{x}_1, \tilde{x}_2).\end{aligned}$$

Backward stability follows from

$$\frac{|\tilde{x}_1 - x_1|}{|x_1|} = \mathcal{O}(\epsilon_{\text{machine}}), \quad \frac{|\tilde{x}_2 - x_2|}{|x_2|} = \mathcal{O}(\epsilon_{\text{machine}}).$$

- **Inner product**

$\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, $\alpha = \mathbf{x}^\top \mathbf{y}$. $\tilde{\alpha}$ by \otimes and \oplus . Backward stable.

- **Outer product**

$\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, $\mathbf{A} = \mathbf{x}\mathbf{y}^\top$. $\tilde{\mathbf{A}}$ by \otimes . Stable but not backward stable.

Explanation: the matrix $\tilde{\mathbf{A}}$ will be most unlikely to have rank exactly 1, i.e., cannot be written as $(\mathbf{x} + \delta\mathbf{x})(\mathbf{y} + \delta\mathbf{y})^*$.

As a rule, for problems where the dimension of the space \mathbb{Y} is greater than that of the space \mathbb{X} , backward stability is rare.

- **Compute** $f(x) = x + 1$

By \oplus , $\tilde{f}(x) = \text{fl}(x) \oplus 1$. Stable but not backward stable.

We have

$$\begin{aligned}\tilde{f}(x) &= \text{fl}(x) \oplus 1 = (x(1 + \epsilon_1) + 1)(1 + \epsilon_2) \\ &= x(1 + \epsilon_1 + \epsilon_2 + \epsilon_1\epsilon_2) + \epsilon_2 + 1.\end{aligned}$$

Obviously, $x(1 + \epsilon_1 + \epsilon_2 + \epsilon_1\epsilon_2) + \epsilon_2$ is not small compared with $x \rightarrow 0$, i.e., for $x \rightarrow 0$,

$$\frac{|x(1 + \epsilon_1 + \epsilon_2 + \epsilon_1\epsilon_2) + \epsilon_2|}{|x|} \neq \mathcal{O}(\epsilon_{\text{machine}}).$$

Explanation: For $x \approx 0$, \oplus introduces absolute errors of size $\mathcal{O}(\epsilon_{\text{machine}})$, which cannot be interpreted as caused by small relative perturbations in x . Therefore, not backward stable.

To show stability, for all x , let $\tilde{x} = x(1 + \epsilon_1)$. Note that

$$\frac{|\tilde{f}(x) - f(\tilde{x})|}{|f(\tilde{x})|} = \frac{|\epsilon_2(x(1 + \epsilon_1) + 1)|}{|x(1 + \epsilon_1) + 1|} = |\epsilon_2| = \mathcal{O}(\epsilon_{\text{machine}}).$$

Then stability follows.

Comparison: Let $x, y \in \mathbb{R}$. Consider $f(x, \tilde{y}) = x + y$ and the corresponding backward stable algorithm $\tilde{f}(x, y) = \text{fl}(x) \oplus \text{fl}(y)$.

4.1. Unitary matrix multiplication

- In the rest of this lecture, for simplicity, we always assume that the given data are floating point numbers already if not explicitly stated.

Theorem 6

Left and/or right unitary matrix multiplications are backward stable in the sense: Let \mathbf{Q} be a unitary matrix. The computed quantity $\tilde{\mathbf{B}}$ for $\mathbf{B} = \mathbf{Q}\mathbf{A}$ or $\mathbf{B} = \mathbf{A}\mathbf{Q}$ satisfies

$$\tilde{\mathbf{B}} = \mathbf{Q}(\mathbf{A} + \delta\mathbf{A}), \quad \text{or} \quad \tilde{\mathbf{B}} = (\mathbf{A} + \delta\mathbf{A})\mathbf{Q}, \quad \frac{\|\delta\mathbf{A}\|}{\|\mathbf{A}\|} = \mathcal{O}(\epsilon_{\text{machine}}).$$

Proof.

We only prove the real case. The complex case is similar. Consider the algorithm for the inner product $\mathbf{q}^\top \mathbf{a}$, then matrix-vector product $\mathbf{Q}\mathbf{a}$, and then matrix-matrix product $\mathbf{Q}\mathbf{A}$. □

TreBau Exercise 16.1

- Let unitary matrices $\mathbf{Q}_k \in \mathbb{C}^{m \times m}$ be fixed and consider the problem of computing, for $\mathbf{A} \in \mathbb{C}^{m \times n}$, the product $\mathbf{B} = \mathbf{Q}_k \mathbf{Q}_{k-1} \cdots \mathbf{Q}_1 \mathbf{A}$. Let the computation be carried out from right to left by straightforward floating point operations.
 - Show that this algorithm is backward stable. (Here \mathbf{A} is thought of as data that can be perturbed; the matrices \mathbf{Q}_j are fixed and not to be perturbed.) That is the computed $\tilde{\mathbf{B}}$ satisfies

$$\tilde{\mathbf{B}} = \mathbf{Q}_k \mathbf{Q}_{k-1} \cdots \mathbf{Q}_1 (\mathbf{A} + \delta \mathbf{A}), \quad \frac{\|\delta \mathbf{A}\|}{\|\mathbf{A}\|} = \mathcal{O}(\epsilon_{\text{machine}}).$$

- Give an example to show that this result no longer holds if the unitary matrices \mathbf{Q}_j are replaced by arbitrary matrices $\mathbf{X}_j \in \mathbb{C}^{m \times m}$. Hint: consider a singular matrix \mathbf{A} and nonsingular matrices \mathbf{X}_j .

4.2. Backward stability of back substitution

- The solution of the nonsingular upper-triangular system

$$\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1m} \\ r_{22} & \ddots & \vdots \\ \ddots & \vdots \\ r_{mm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$

can be obtained by the following back substitution algorithm

Algorithm: Back substitution

$$x_m = b_m / r_{mm}$$

$$x_{m-1} = (b_{m-1} - x_m r_{m-1,m}) / r_{m-1,m-1}$$

$$x_{m-2} = (b_{m-2} - x_{m-1} r_{m-2,m-1} - x_m r_{m-2,m}) / r_{m-2,m-2}$$

$$\vdots$$

$$x_j = (b_j - \sum_{k=j+1}^m x_k r_{jk}) / r_{jj}$$

Theorem 7

Back substitution is backward stable in the sense that the computed solution $\tilde{\mathbf{x}} \in \mathbb{C}^m$ satisfies

$$(\mathbf{R} + \delta\mathbf{R})\tilde{\mathbf{x}} = \mathbf{b},$$

for some upper-triangular $\delta\mathbf{R} \in \mathbb{C}^{m \times m}$ with

$$\frac{\|\delta\mathbf{R}\|}{\|\mathbf{R}\|} = \mathcal{O}(\epsilon_{\text{machine}}).$$

Specifically, for each i, j ,

$$\frac{|\delta r_{ij}|}{|r_{ij}|} \leq m\epsilon_{\text{machine}} + \mathcal{O}(\epsilon_{\text{machine}}^2).$$

- Our task is to express every floating point error as a perturbation of the input.

(i) The case $m = 1$:

$$\tilde{x}_1 = b_1 \oslash r_{11} = \frac{b_1(1 + \epsilon_1)}{r_{11}}, \quad |\epsilon_1| \leq \epsilon_{\text{machine}}$$

Set $1 + \epsilon'_1 = 1/(1 + \epsilon_1)$. We have

$$\epsilon'_1 = -\frac{\epsilon_1}{1 + \epsilon_1} \Rightarrow \tilde{x}_1 = \frac{b_1}{r_{11}(1 + \epsilon'_1)}, \quad |\epsilon'_1| \leq \epsilon_{\text{machine}} + \mathcal{O}(\epsilon_{\text{machine}}^2).$$

Therefore

$$(r_{11} + \delta r_{11})\tilde{x}_1 = b_1; \quad \delta r_{11} = \epsilon'_1 r_{11}; \quad \frac{|\delta r_{11}|}{|r_{11}|} \leq \epsilon_{\text{machine}} + \mathcal{O}(\epsilon_{\text{machine}}^2).$$

(ii) The case $m = 2$. The first step is the same as in $m = 1$ case,

$$\tilde{x}_2 = b_2 \oslash r_{22} = \frac{b_2}{r_{22}(1 + \epsilon_1)}, \quad |\epsilon_1| \leq \epsilon_{\text{machine}} + \mathcal{O}(\epsilon_{\text{machine}}^2).$$

The second step: there exist $|\epsilon_2|, |\epsilon_3|, |\epsilon_4| \leq \epsilon_{\text{machine}}$,

$$\begin{aligned}
\tilde{x}_1 &= (b_1 \ominus (\tilde{x}_2 \otimes r_{12})) \oplus r_{11} = (b_1 \ominus \tilde{x}_2 r_{12}(1 + \epsilon_2)) \oplus r_{11} \\
&= (b_1 - \tilde{x}_2 r_{12}(1 + \epsilon_2))(1 + \epsilon_3) \oplus r_{11} \\
&= \frac{(b_1 - \tilde{x}_2 r_{12}(1 + \epsilon_2))(1 + \epsilon_3)}{r_{11}}(1 + \epsilon_4).
\end{aligned}$$

Shift ϵ_3 and ϵ_4 to the denominator

$$\tilde{x}_1 = \frac{b_1 - \tilde{x}_2 r_{12}(1 + \epsilon_2)}{r_{11}(1 + \epsilon'_3)(1 + \epsilon'_4)},$$

or equivalently,

$$\tilde{x}_1 = \frac{b_1 - \tilde{x}_2 r_{12}(1 + \epsilon_2)}{r_{11}(1 + 2\epsilon_5)}, \quad |\epsilon'_3|, |\epsilon'_4|, |\epsilon_5| \leq \epsilon_{\text{machine}} + \mathcal{O}(\epsilon_{\text{machine}}^2).$$

Obviously, \tilde{x}_1 is exactly correct if r_{22} , r_{12} and r_{11} perturbed by factors $(1 + \epsilon_1)$, $(1 + \epsilon_2)$ and $(1 + 2\epsilon_5)$, respectively. Thus,

$$(\mathbf{R} + \delta\mathbf{R})\tilde{\mathbf{x}} = \mathbf{b},$$

where the entries δr_{ij} of $\delta\mathbf{R}$ satisfy

$$\begin{bmatrix} |\delta r_{11}| & |\delta r_{12}| \\ \frac{|\delta r_{11}|}{|r_{11}|} & \frac{|\delta r_{12}|}{|r_{12}|} \\ \frac{|\delta r_{22}|}{|r_{22}|} & \end{bmatrix} = \begin{bmatrix} 2|\epsilon_5| & |\epsilon_2| \\ & |\epsilon_1| \end{bmatrix} \leq \begin{bmatrix} 2 & 1 \\ & 1 \end{bmatrix} \epsilon_{\text{machine}} + \mathcal{O}(\epsilon_{\text{machine}}^2).$$

The last formula guarantees $\|\delta\mathbf{R}\|/\|\mathbf{R}\| = \mathcal{O}(\epsilon_{\text{machine}})$ in any norm.

(iii) The case $m = 3$. The first two steps are the same as before:

$$\tilde{x}_3 = b_3 \oplus r_{33} = \frac{b_3}{r_{33}(1 + \epsilon_1)},$$

$$\tilde{x}_2 = (b_2 \ominus (\tilde{x}_3 \otimes r_{23})) \oplus r_{22} = \frac{b_2 - \tilde{x}_3 r_{23}(1 + \epsilon_2)}{r_{22}(1 + 2\epsilon_3)},$$

where

$$\begin{bmatrix} 2|\epsilon_3| & |\epsilon_2| \\ & |\epsilon_1| \end{bmatrix} \leq \begin{bmatrix} 2 & 1 \\ & 1 \end{bmatrix} \epsilon_1 + \mathcal{O}(\epsilon_{\text{machine}}^2)$$

The third step:

$$\begin{aligned}
 \tilde{x}_1 &= [(b_1 \ominus (\tilde{x}_2 \otimes r_{12})) \ominus (\tilde{x}_3 \otimes r_{13})] \oplus r_{11} \\
 &= [(b_1 - \tilde{x}_2 r_{12}(1 + \epsilon_4))(1 + \epsilon_6) - \tilde{x}_3 r_{13}(1 + \epsilon_5)](1 + \epsilon_7) \oplus r_{11} \\
 &= \frac{[(b_1 - \tilde{x}_2 r_{12}(1 + \epsilon_4))(1 + \epsilon_6) - \tilde{x}_3 r_{13}(1 + \epsilon_5)](1 + \epsilon_7)}{r_{11}(1 + \epsilon'_8)} \\
 &= \frac{b_1 - \tilde{x}_2 r_{12}(1 + \epsilon_4) - \tilde{x}_3 r_{13}(1 + \epsilon_5)(1 + \epsilon'_6)}{r_{11}(1 + \epsilon'_6)(1 + \epsilon'_7)(1 + \epsilon'_8)},
 \end{aligned}$$

r_{13} has two perturbations of size at most $\epsilon_{\text{machine}}$, r_{11} has three. Then we have $(\mathbf{R} + \delta\mathbf{R})\tilde{\mathbf{x}} = \mathbf{b}$ with the entries δr_{ij} satisfying

$$\left[\begin{array}{ccc} \frac{|\delta r_{11}|}{|r_{11}|} & \frac{|\delta r_{12}|}{|r_{12}|} & \frac{|\delta r_{13}|}{|r_{13}|} \\ & \frac{|\delta r_{22}|}{|r_{22}|} & \frac{|\delta r_{23}|}{|r_{23}|} \\ & & \frac{|\delta r_{33}|}{|r_{33}|} \end{array} \right] \leq \left[\begin{array}{ccc} 3 & 1 & 2 \\ & 2 & 1 \\ & & 1 \end{array} \right] \epsilon_{\text{machine}} + \mathcal{O}(\epsilon_{\text{machine}}^2).$$

(iv) General m : Higher-dimensional cases are similar. For example,
 5×5 case:

$$\frac{|\delta \mathbf{R}|}{|\mathbf{R}|} \leqslant \begin{bmatrix} 5 & 1 & 2 & 3 & 4 \\ & 4 & 1 & 2 & 3 \\ & & 3 & 1 & 2 \\ & & & 2 & 1 \\ & & & & 1 \end{bmatrix} \epsilon_{\text{machine}} + \mathcal{O}(\epsilon_{\text{machine}}^2).$$

The entries of the matrix in this formula are obtained from three components. The multiplications $\tilde{x}_k r_{jk}$ introduce $\epsilon_{\text{machine}}$ perturbations in the pattern

$$\otimes : \tilde{x}_k r_{jk} \quad \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ & 0 & 1 & 1 & 1 \\ & & 0 & 1 & 1 \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}. \quad (\text{inner level})$$

The division by r_{kk} introduce perturbations in the pattern

$$\ominus : \text{divisions by } r_{kk} \quad \left[\begin{array}{ccccc} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{array} \right]. \quad (\text{outer level})$$

Finally, the subtractions also occur in the pattern for \otimes , and, due to the decision to compute from left to right, each one introduces a perturbation on the diagonal and at each position to the right. This adds up to the pattern

$$\ominus : \left[\begin{array}{ccccc} 4 & 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 & \\ 2 & 0 & 1 & & \\ 1 & 0 & & & \\ 0 & & & & \end{array} \right].$$

Remark 8

Perturbations of order $\epsilon_{\text{machine}}$ are composed additively and moved freely between numerators and denominators since the difference is of order $\epsilon_{\text{machine}}^2$.

Remark 9

More than one error bound can be derived for a given algorithm. In the present case, we could have perturbed b_j as well as r_{ij} , avoiding the need for the trickery represented pattern for \ominus . On the other hand, a final result in which only \mathbf{R} is perturbed is appealing clean.

Remark 10

We have done *componentwise* backward error bound. If $r_{ij} = 0$, this entry undergoes no perturbation at all: $\delta\mathbf{R}$ has the same sparsity pattern as \mathbf{R} .