

Numerical Linear Algebra Assignment 9

Exercise 1. (10 points)

Consider the subspace iteration:

Algorithm 1: Subspace iteration

Pick $\mathbf{Q}_n^{(0)} \in \mathbb{C}^{m \times n}$ with orthonormal columns
for $k = 1, 2, 3, \dots$,
 $\mathbf{Q}_n^{(k)} \mathbf{R}_n^{(k)} = \mathbf{A} \mathbf{Q}_n^{(k-1)}$ (QR factorization)
end

Assume \mathbf{A} is diagonalizable with $\mathbf{A} = \mathbf{S} \mathbf{\Lambda} \mathbf{S}^{-1}$, $\mathbf{\Lambda} = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_m\}$, and

$$|\lambda_1| \geq \dots \geq |\lambda_n| > |\lambda_{n+1}| \geq \dots \geq |\lambda_m|.$$

Assume $\mathbf{X}_n := [\mathbf{I}_n \quad \mathbf{0}] \mathbf{S}^{-1} \mathbf{Q}_n^{(0)}$ has full rank. Prove that

$$\mathbf{Q}_n^{(k)} = \mathbf{A}^k \mathbf{Q}_n^{(0)} (\mathbf{R}_n^{(1)})^{-1} (\mathbf{R}_n^{(2)})^{-1} \dots (\mathbf{R}_n^{(k)})^{-1}.$$

Exercise 2. (Zhihao Cao, 10 points)

Let $\mathbf{A}_1 = \begin{bmatrix} a & b \\ \varepsilon & c \end{bmatrix} \in \mathbb{R}^{2 \times 2}$. Let $\mathbf{A}_2 = \mathbf{R}_1 \mathbf{Q}_1 + c \mathbf{I} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, where $\mathbf{Q}_1 \mathbf{R}_1 = \mathbf{A}_1 - c \mathbf{I}$ is a QR factorization of $\mathbf{A}_1 - c \mathbf{I}$. Prove:

- if a and c are not close (i.e., $\exists \delta > 0$, $|a - c| > \delta$), then $a_{21} = \mathcal{O}(\varepsilon^2)$;
- if further $b = \varepsilon$ (i.e., \mathbf{A}_1 is symmetric), then $a_{21} = \mathcal{O}(\varepsilon^3)$.

Exercise 3. (Programming, 10 points)

Write a function $[\mathbf{Q}, \mathbf{H}] = \text{myhess}(\mathbf{A})$ that reduces a complex square matrix to upper Hessenberg form by unitary similarity transformations, i.e., $\mathbf{H} = \mathbf{Q}^* \mathbf{A} \mathbf{Q}$. Your program should use only elementary MATLAB operations – not the function `hess`, for example. Apply your program to $\mathbf{A} = \text{randn}(5) + 1i * \text{randn}(5)$.

Exercise 4. (Programming, TreBau Exercise 29.1, 10 points)