Improved TriCG and TriMR methods for symmetric quasi-definite linear systems

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joint work with Jia-Jun Fan and Ya-Lan Zhang

Symposium for Young Scholars in Numerical Algebra, Dalian, 2024

Outline

1. Symmetric quasi-definite (SQD) linear systems
2. TriCG and TriMR
3. Improved TriCG and TriMR
4. Numerical experiments
5. Summary

Symmetric quasi-definite (SQD) linear systems

- $\mathbf{M} \in \mathbb{R}^{m \times m}$ and $\mathbf{N} \in \mathbb{R}^{n \times n}$ are SPD, $\mathbf{A} \in \mathbb{R}^{m \times n}$ is nonzero, $\mathbf{b} \in \mathbb{R}^{m}$, and $\mathbf{c} \in \mathbb{R}^{n}$:

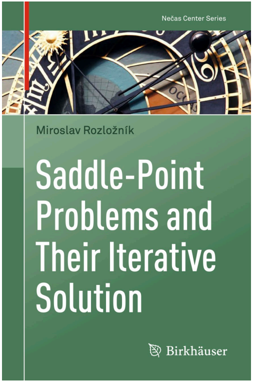
$$\begin{bmatrix} \mathbf{M} & \mathbf{A} \\ \mathbf{A}^\top & -\mathbf{N} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{c} \end{bmatrix}.$$

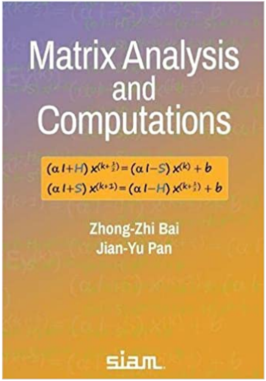
- Computational optimization and computational partial differential equations, etc.
- Symmetric, indefinite, nonsingular
- Monolithic methods: solving the system as a whole, for example, SYMMLQ, MINRES

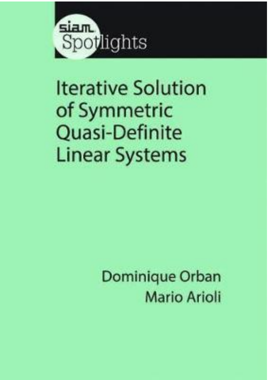
  Segregated methods: exploiting the block structure, excluding the preconditioning stage, for example, TriCG, TriMR

Review papers and books

- Michele Benzi, Gene H. Golub, and Jörg Liesen
  Numerical solution of saddle point problems.
  Acta Numerica (2005), pp. 1–137.







The generalized SSY tridiagonalization

Algorithm Generalized SSY tridiagonalization:

Require: $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{c} \in \mathbb{R}^n$,
subroutines for performing $\mathbf{M}^{-1}\mathbf{u}$ and $\mathbf{N}^{-1}\mathbf{v}$

1: $\mathbf{u}_0 = \mathbf{0}$, $\mathbf{v}_0 = \mathbf{0}$
2: $\beta_1 \mathbf{M}\mathbf{u}_1 = \mathbf{b}$
3: $\gamma_1 \mathbf{N}\mathbf{v}_1 = \mathbf{c}$
4: **for** $k = 1, 2, \ldots$ **do**
5: $\quad \mathbf{p} = \mathbf{A}\mathbf{v}_k - \gamma_k \mathbf{M}\mathbf{u}_{k-1}$
6: $\quad \alpha_k = \mathbf{u}_k^\top \mathbf{p}$
7: $\quad \beta_{k+1} \mathbf{M}\mathbf{u}_{k+1} = \mathbf{p} - \alpha_k \mathbf{M}\mathbf{u}_k$
8: $\quad \gamma_{k+1} \mathbf{N}\mathbf{v}_{k+1} = \mathbf{A}^\top \mathbf{u}_k - \beta_k \mathbf{N}\mathbf{v}_{k-1} - \alpha_k \mathbf{N}\mathbf{v}_k$
9: **end for**

The generalized SSY tridiagonalization

- The generalized Saunders–Simon–Yip tridiagonalization:

$$\begin{aligned}
\mathbf{A}\mathbf{V}_k &= \mathbf{M}\mathbf{U}_{k+1}\mathbf{T}_{k+1,k} = \mathbf{M}\mathbf{U}_k\mathbf{T}_k + \beta_{k+1}\mathbf{M}\mathbf{u}_{k+1}\mathbf{e}_k^\top,\\
\mathbf{A}^\top\mathbf{U}_k &= \mathbf{N}\mathbf{V}_{k+1}\mathbf{T}_{k,k+1}^\top = \mathbf{N}\mathbf{V}_k\mathbf{T}_k^\top + \gamma_{k+1}\mathbf{N}\mathbf{v}_{k+1}\mathbf{e}_k^\top,\\
\mathbf{U}_{k+1}^\top\mathbf{M}\mathbf{U}_{k+1} &= \mathbf{V}_{k+1}^\top\mathbf{N}\mathbf{V}_{k+1} = \mathbf{I}_{k+1},
\end{aligned}$$

where

$$\mathbf{U}_k = \begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k\end{bmatrix}, \quad \mathbf{V}_k = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k\end{bmatrix},$$

$$\mathbf{T}_k = \mathrm{tridiag}(\beta_i, \alpha_i, \gamma_{i+1}),$$

and

$$\mathbf{T}_{k+1,k} = \begin{bmatrix}\mathbf{T}_k \\ \beta_{k+1}\mathbf{e}_k^\top\end{bmatrix}, \quad \mathbf{T}_{k,k+1} = \begin{bmatrix}\mathbf{T}_k & \gamma_{k+1}\mathbf{e}_k\end{bmatrix}.$$

TriCG

- Assume that no breakdowns occur for the first k steps, i.e., $\mathbf{U}_k$, $\mathbf{V}_k$, and $\mathbf{T}_k$ are well defined. The kth TriCG iterate is

$$\begin{bmatrix} \mathbf{x}_k \\ \mathbf{y}_k \end{bmatrix} = \begin{bmatrix} \mathbf{U}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_k \end{bmatrix} \begin{bmatrix} \mathbf{I}_k & \mathbf{T}_k \\ \mathbf{T}_k^\top & -\mathbf{I}_k \end{bmatrix}^{-1} \begin{bmatrix} \beta_1 \mathbf{e}_1 \\ \gamma_1 \mathbf{e}_1 \end{bmatrix},$$

which satisfies the Galerkin condition

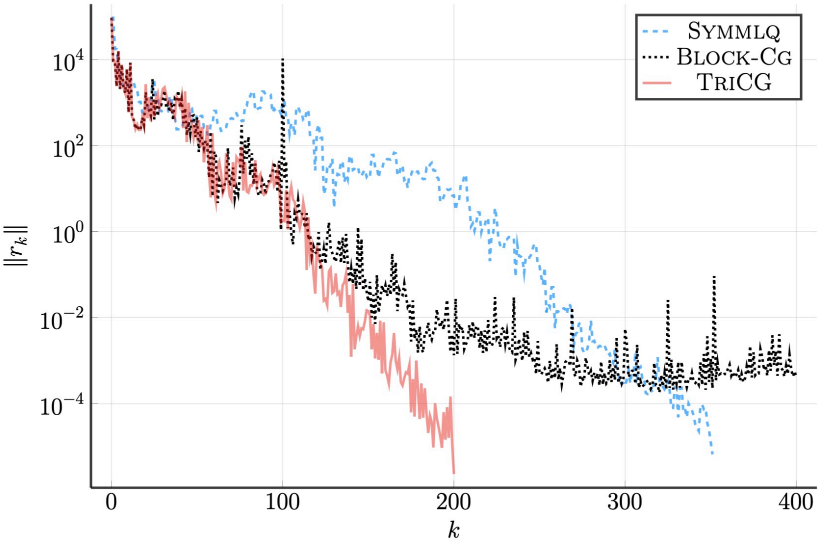
$$\begin{bmatrix} \mathbf{U}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_k \end{bmatrix}^\top \left(\begin{bmatrix} \mathbf{b} \\ \mathbf{c} \end{bmatrix} - \begin{bmatrix} \mathbf{M} & \mathbf{A} \\ \mathbf{A}^\top & -\mathbf{N} \end{bmatrix} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{y}_k \end{bmatrix} \right) = \mathbf{0}.$$

- Equivalent to preconditioned Block-CG:

$$\begin{bmatrix} \mathbf{M} & \mathbf{A} \\ \mathbf{A}^\top & -\mathbf{N} \end{bmatrix} \begin{bmatrix} \mathbf{x}^1 & \mathbf{x}^2 \\ \mathbf{y}^1 & \mathbf{y}^2 \end{bmatrix} = \begin{bmatrix} \mathbf{b} & \mathbf{0} \\ \mathbf{0} & \mathbf{c} \end{bmatrix}.$$

A. Montoison and D. Orban. *TriCG and TriMR: Two iterative methods for symmetric quasi-definite systems*. SISC, Vol. 43, Iss. 4 (2021)

Example: $\mathbf{M} = \mathbf{I}$, $\mathbf{N} = \mathbf{I}$, $\mathbf{A} = \texttt{lp\_osa\_07}$



TriMR

- The kth TriMR iterate is

$$\begin{bmatrix}\mathbf{x}_k \\ \mathbf{y}_k\end{bmatrix} = \underset{\mathbf{x}\in\mathrm{range}(\mathbf{U}_k),\ \mathbf{y}\in\mathrm{range}(\mathbf{V}_k)}{\mathrm{argmin}} \left\| \begin{bmatrix}\mathbf{b} \\ \mathbf{c}\end{bmatrix} - \begin{bmatrix}\mathbf{M} & \mathbf{A} \\ \mathbf{A}^\top & -\mathbf{N}\end{bmatrix} \begin{bmatrix}\mathbf{x} \\ \mathbf{y}\end{bmatrix} \right\|_{\mathbf{H}^{-1}},$$

  where

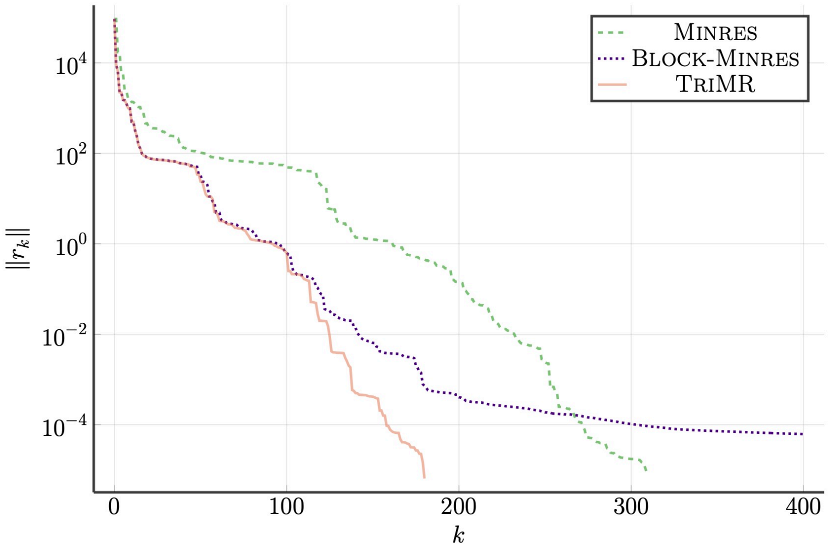
$$\mathbf{H} = \begin{bmatrix}\mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{N}\end{bmatrix}.$$

- Equivalent to preconditioned Block-MINRES:

$$\begin{bmatrix}\mathbf{M} & \mathbf{A} \\ \mathbf{A}^\top & -\mathbf{N}\end{bmatrix} \begin{bmatrix}\mathbf{x}^1 & \mathbf{x}^2 \\ \mathbf{y}^1 & \mathbf{y}^2\end{bmatrix} = \begin{bmatrix}\mathbf{b} & \mathbf{0} \\ \mathbf{0} & \mathbf{c}\end{bmatrix}.$$

A. Montoison and D. Orban. *TriCG and TriMR: Two iterative methods for symmetric quasi-definite systems*. SISC, Vol. 43, Iss. 4 (2021)

Example: $\mathbf{M} = \mathbf{I}$, $\mathbf{N} = \mathbf{I}$, $\mathbf{A}$ = `lp_osa_07`



The subproblem of TriMR

Let

$$\mathbf{W}_k = \begin{bmatrix} \mathbf{U}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_k \end{bmatrix} \Pi_{2k}, \quad \Pi_{2k} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_{k+1} & \cdots & \mathbf{e}_k & \mathbf{e}_{2k} \end{bmatrix},$$

and

$$\mathbf{S}_{k+1,k} = \Pi_{2k+2}^\top \begin{bmatrix} \begin{bmatrix} \mathbf{I}_k \\ \mathbf{0} \end{bmatrix} & \mathbf{T}_{k+1,k} \\ \mathbf{T}_{k,k+1}^\top & -\begin{bmatrix} \mathbf{I}_k \\ \mathbf{0} \end{bmatrix} \end{bmatrix} \Pi_{2k} = \begin{bmatrix} \Theta_1 & \Psi_2 & & \\ \Psi_2^\top & \Theta_2 & \ddots & \\ & \ddots & \ddots & \Psi_k \\ & & \ddots & \Theta_k \\ & & & \Psi_{k+1} \end{bmatrix},$$

where

$$\Theta_k = \begin{bmatrix} 1 & \alpha_k \\ \alpha_k & -1 \end{bmatrix} \quad \text{and} \quad \Psi_k = \begin{bmatrix} 0 & \gamma_k \\ \beta_k & 0 \end{bmatrix}.$$

The subproblem of TriMR

We have

$$\begin{bmatrix} \mathbf{M} & \mathbf{A} \\ \mathbf{A}^\top & -\mathbf{N} \end{bmatrix} \mathbf{W}_k = \mathbf{H}\mathbf{W}_{k+1}\mathbf{S}_{k+1,k}.$$

Then the kth TriMR iterate can be determined by

$$\begin{bmatrix} \mathbf{x}_k \\ \mathbf{y}_k \end{bmatrix} = \mathbf{W}_k \mathbf{z}_k$$

where $\mathbf{z}_k \in \mathbb{R}^{2k}$ solves

$$\min_{\mathbf{z}\in\mathbb{R}^{2k}} \|\mathbf{S}_{k+1,k}\mathbf{z} - (\beta_1 \mathbf{e}_1 + \gamma_1 \mathbf{e}_2)\|.$$

The vector $\mathbf{z}_k$ can be determined via the QR factorization

$$\mathbf{S}_{k+1,k} = \mathbf{Q}_k \begin{bmatrix} \mathbf{R}_k \\ \mathbf{0} \end{bmatrix},$$

The subproblem of TriMR

where

$$\mathbf{Q}_k \in \mathbb{R}^{(2k+2)\times(2k+2)}$$

is a product of reflections, and

$$\mathbf{R}_k = \begin{bmatrix} \delta_1 & \sigma_1 & \eta_1 & \lambda_1 & \mu_1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots & \mu_{2k-4} \\ & & & \ddots & \ddots & \ddots & \lambda_{2k-3} \\ & & & & \ddots & \ddots & \eta_{2k-2} \\ & & & & & \ddots & \sigma_{2k-1} \\ & & & & & & \delta_{2k} \end{bmatrix} \in \mathbb{R}^{(2k)\times(2k)}.$$

The subproblem of TriMR

Theorem ($\mathbf{R}_k$ has only three nonzero diagonals)

The upper triangular matrix $\mathbf{R}_k$ *of the QR factorization has the following form:*

$$\mathbf{R}_k = \begin{bmatrix} \delta_1 & 0 & \eta_1 & 0 & \mu_1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots & \mu_{2k-4} \\ & & & \ddots & \ddots & \ddots & 0 \\ & & & & \ddots & \ddots & \eta_{2k-2} \\ & & & & & \ddots & 0 \\ & & & & & & \delta_{2k} \end{bmatrix}.$$

Breakdowns of gSSY

Algorithm Generalized Saunders–Simon–Yip tridiagonalization: gSSY($\mathbf{M}, \mathbf{N}, \mathbf{A}, \mathbf{b}, \mathbf{c}$)

Require: $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{c} \in \mathbb{R}^n$, subroutines for performing $\mathbf{M}^{-1}\mathbf{u}$ and $\mathbf{N}^{-1}\mathbf{v}$

$\mathbf{u}_0 = \mathbf{0}$, $\mathbf{v}_0 = \mathbf{0}$

$\beta_1 \mathbf{M}\mathbf{u}_1 = \mathbf{b}$

$\gamma_1 \mathbf{N}\mathbf{v}_1 = \mathbf{c}$

for $k = 1, 2, \ldots$ **do**

  $\mathbf{p} = \mathbf{A}\mathbf{v}_k - \gamma_k \mathbf{M}\mathbf{u}_{k-1}$

  $\alpha_k = \mathbf{u}_k^\top \mathbf{p}$

  $\beta_{k+1}\mathbf{M}\mathbf{u}_{k+1} = \mathbf{p} - \alpha_k \mathbf{M}\mathbf{u}_k$

  $\gamma_{k+1}\mathbf{N}\mathbf{v}_{k+1} = \mathbf{A}^\top \mathbf{u}_k - \beta_k \mathbf{N}\mathbf{v}_{k-1} - \alpha_k \mathbf{N}\mathbf{v}_k$

end for

- gSSY must break down in $\ell \leq \min(m, n)$ steps in exact arithmetic, and either $\beta_{\ell+1} = 0$ or $\beta_{\ell+1} \neq 0$ and $\gamma_{\ell+1} = 0$.
- $\beta_{\ell+1} = \gamma_{\ell+1} = 0$ ensures a lucky breakdown.
- When $\beta_{\ell+1}$ and $\gamma_{\ell+1}$ are not simultaneous zero, unlucky breakdowns may occur.

Unlucky breakdowns of gSSY

Example (The case that $\beta_{\ell+1} = 0$ and $\gamma_{\ell+1} \neq 0$)

The solution to the SQD linear system with

$$\mathbf{M} = \mathbf{N} = \mathbf{I}_3, \quad \mathbf{A} = \begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \quad \mathbf{b} = \mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

is $\begin{bmatrix} 1 & 2 & 1 & -3 & 0 & 1 \end{bmatrix}^\top/4$. gSSY breaks down at step $\ell = 2$ with $\beta_{\ell+1} = 0$, and we have $\gamma_{\ell+1} = 1 \neq 0$ and $\mathbf{U}_\ell = \mathbf{V}_\ell = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{bmatrix}$. Obviously,

$$\begin{bmatrix} 1 & 2 & 1 & -3 & 0 & 1 \end{bmatrix}^\top \notin \operatorname{range}\left(\begin{bmatrix} \mathbf{U}_\ell & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_\ell \end{bmatrix} \right).$$

Unlucky breakdowns of gSSY

Example (The case that $\beta_{\ell+1} \neq 0$ and $\gamma_{\ell+1} = 0$)

The solution to the SQD linear system with

$$\mathbf{M} = \mathbf{N} = \mathbf{I}_3, \quad \mathbf{A} = \begin{bmatrix} -1 & 1 & 0 \\ 3 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \quad \mathbf{b} = \mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

is $\begin{bmatrix} 11 & 8 & -1 & -2 & 2 & 1 \end{bmatrix}^\top / 15$. gSSY breaks down at step $\ell = 2$ with $\beta_{\ell+1} = 1 \neq 0$ and $\gamma_{\ell+1} = 0$, and we have $\mathbf{U}_{\ell+1} = \mathbf{I}_3$, $\mathbf{V}_\ell = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{bmatrix}$. Obviously,

$$\begin{bmatrix} 11 & 8 & -1 & -2 & 2 & 1 \end{bmatrix}^\top \notin \operatorname{range}\left(\begin{bmatrix} \mathbf{U}_{\ell+1} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_\ell \end{bmatrix} \right).$$

Improved generalized SSY tridiagonalization

Algorithm Improved generalized Saunders–Simon–Yip tridiagonalization: igSSY($\mathbf{M}, \mathbf{N}, \mathbf{A}, \mathbf{b}, \mathbf{c}$)

Require: $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{c} \in \mathbb{R}^n$, subroutines for performing $\mathbf{M}^{-1}\mathbf{u}$ and $\mathbf{N}^{-1}\mathbf{v}$

$\mathbf{u}_0 = \mathbf{0}$, $\mathbf{v}_0 = \mathbf{0}$
$\mathbf{u} = \mathbf{M}^{-1}\mathbf{b}$, $\beta_1 = \sqrt{\mathbf{b}^\top \mathbf{u}}$; **if** $\beta_1 \neq 0$, **then** $\mathbf{u}_1 = \mathbf{u}/\beta_1$ **end if**
$\mathbf{v} = \mathbf{N}^{-1}\mathbf{c}$, $\gamma_1 = \sqrt{\mathbf{c}^\top \mathbf{v}}$; **if** $\gamma_1 \neq 0$, **then** $\mathbf{v}_1 = \mathbf{v}/\gamma_1$ **end if**
$k = 1$
while $\beta_k\gamma_k \neq 0$ **do**
    $\mathbf{p} = \mathbf{A}\mathbf{v}_k - \gamma_k \mathbf{M}\mathbf{u}_{k-1}$
    $\alpha_k = \mathbf{u}_k^\top \mathbf{p}$, $\mathbf{p} = \mathbf{p} - \alpha_k \mathbf{M}\mathbf{u}_k$
    $\mathbf{u} = \mathbf{M}^{-1}\mathbf{p}$, $\beta_{k+1} = \sqrt{\mathbf{p}^\top \mathbf{u}}$; **if** $\beta_{k+1} \neq 0$, **then** $\mathbf{u}_{k+1} = \mathbf{u}/\beta_{k+1}$ **end if**
    $\mathbf{q} = \mathbf{A}^\top \mathbf{u}_k - \beta_k \mathbf{N}\mathbf{v}_{k-1} - \alpha_k \mathbf{N}\mathbf{v}_k$
    $\mathbf{v} = \mathbf{N}^{-1}\mathbf{q}$, $\gamma_{k+1} = \sqrt{\mathbf{q}^\top \mathbf{v}}$; **if** $\gamma_{k+1} \neq 0$, **then** $\mathbf{v}_{k+1} = \mathbf{v}/\gamma_{k+1}$ **end if**
    $k = k + 1$
end while
$\ell = k - 1$
if $\beta_{\ell+1} = 0$ and $\gamma_{\ell+1} = 0$ **then**
    stop
end if
if $\beta_{\ell+1} = 0$ and $\gamma_{\ell+1} \neq 0$ **then**
    for $k = \ell+1, \ell+2, \dots$ **do**
        $\alpha_k \mathbf{M}\mathbf{u}_k = \mathbf{A}\mathbf{v}_k - \gamma_k \mathbf{M}\mathbf{u}_{k-1}$
        $\gamma_{k+1}\mathbf{N}\mathbf{v}_{k+1} = \mathbf{A}^\top \mathbf{u}_k - \alpha_k \mathbf{N}\mathbf{v}_k$
    end for
end if
if $\beta_{\ell+1} \neq 0$ and $\gamma_{\ell+1} = 0$ **then**
    for $k = \ell+1, \ell+2, \dots$ **do**
        $\alpha_k \mathbf{N}\mathbf{v}_k = \mathbf{A}^\top \mathbf{u}_k - \beta_k \mathbf{N}\mathbf{v}_{k-1}$
        $\beta_{k+1}\mathbf{M}\mathbf{u}_{k+1} = \mathbf{A}\mathbf{v}_k - \alpha_k \mathbf{M}\mathbf{u}_k$
    end for
end if

Breakdowns of igSSY

- Assume that gSSY breaks down at step ℓ, i.e., $\beta_{\ell+1} = 0$ or $\gamma_{\ell+1} = 0$.
- Assume that igSSY breaks down at step $L \geq \ell$. Five cases occur (see lines 15, 19, 20, 25, and 26): for $k = \ell, \ldots, L$,

  Case I: $\beta_{\ell+1} = \gamma_{\ell+1} = 0$;

  Case II: $\alpha_{L+1} = 0$, $\beta_{k+1} = 0$, $\gamma_{k+1} \neq 0$;

  Case III: $\alpha_{L+1} \neq 0$, $\beta_{k+1} = 0$, $\gamma_{k+1} \neq 0$, $\gamma_{L+2} = 0$;

  Case IV: $\alpha_{L+1} = 0$, $\beta_{k+1} \neq 0$, $\gamma_{k+1} = 0$;

  Case V: $\alpha_{L+1} \neq 0$, $\beta_{k+1} \neq 0$, $\gamma_{k+1} = 0$, $\beta_{L+2} = 0$.

  All are lucky breakdowns.

  The solution of the SQD linear system belongs to the final subspace generated by igSSY.

Elliptic singular value decomposition (ESVD)

- Given SPD $\mathbf{M}$ and $\mathbf{N}$, ESVD of $\mathbf{A}$ is

$$\mathbf{A} = \mathbf{M}\mathbf{P}\boldsymbol{\Sigma}\mathbf{Q}^\top\mathbf{N},$$

where $\boldsymbol{\Sigma} = \operatorname{diag}(\sigma_1, \ldots, \sigma_p)$, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0$, $p = \min(m, n)$, and $\mathbf{P}$ and $\mathbf{Q}$ satisfy

$$\mathbf{P}^\top\mathbf{M}\mathbf{P} = \mathbf{I}_m, \quad \mathbf{Q}^\top\mathbf{N}\mathbf{Q} = \mathbf{I}_n.$$

Theorem

Assume that igSSY breaks down at step L. If d is the number of distinct elliptic singular values of $\mathbf{A}$ and r is the rank of $\mathbf{A}$, then we have $L \leq \min(2d, r)$.

M. Arioli. *Generalized Golub–Kahan bidiagonalization and stopping criteria*. SIMAX, Vol. 34, Iss. 2 (2013)

Improved TriCG and TriMR

- Improved TriCG and TriMR solve SQD linear systems in the same fashion as TriCG and TriMR, but are based on igSSY instead of gSSY.

  The kth iTriCG iterate is

  $$\begin{bmatrix} \mathbf{x}_k \\ \mathbf{y}_k \end{bmatrix} = \begin{bmatrix} \mathbf{U}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_k \end{bmatrix} \begin{bmatrix} \mathbf{I}_k & \mathbf{T}_k \\ \mathbf{T}_k^\top & -\mathbf{I}_k \end{bmatrix}^{-1} \begin{bmatrix} \beta_1 \mathbf{e}_1 \\ \gamma_1 \mathbf{e}_1 \end{bmatrix}.$$

  The kth iTriMR iterate is

  $$\begin{bmatrix} \mathbf{x}_k \\ \mathbf{y}_k \end{bmatrix} = \underset{\mathbf{x} \in \mathrm{range}(\mathbf{U}_k),\ \mathbf{y} \in \mathrm{range}(\mathbf{V}_k)}{\operatorname{argmin}} \left\| \begin{bmatrix} \mathbf{b} \\ \mathbf{c} \end{bmatrix} - \begin{bmatrix} \mathbf{M} & \mathbf{A} \\ \mathbf{A}^\top & -\mathbf{N} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \right\|_{\mathbf{H}^{-1}},$$

  where $\mathbf{H} = \mathrm{blkdiag}(\mathbf{M}, \mathbf{N})$.
- The first ℓ iterates of iTriCG and iTriMR coincide with the first ℓ iterates of TriCG and TriMR, respectively.

Numerical examples

- Examples without unlucky breakdowns

  The `channel_domain` problem from IFISS (version 3.6)

- Examples with unlucky breakdowns

  Set $\mathbf{M} = \mathbf{I}_m$ and $\mathbf{N} = \mathbf{I}_n$, and $\mathbf{A}$ to be `lp_czprob` or `lp_osa_07` from the SuiteSparse Matrix Collection.

  Vectors $\mathbf{b}$ and $\mathbf{c}$ are generated as follows.

  Case I: `[P,S,Q] = svd(A);`

  `b = P(:,1:2)*ones(2,1); c = ones(n,1);`

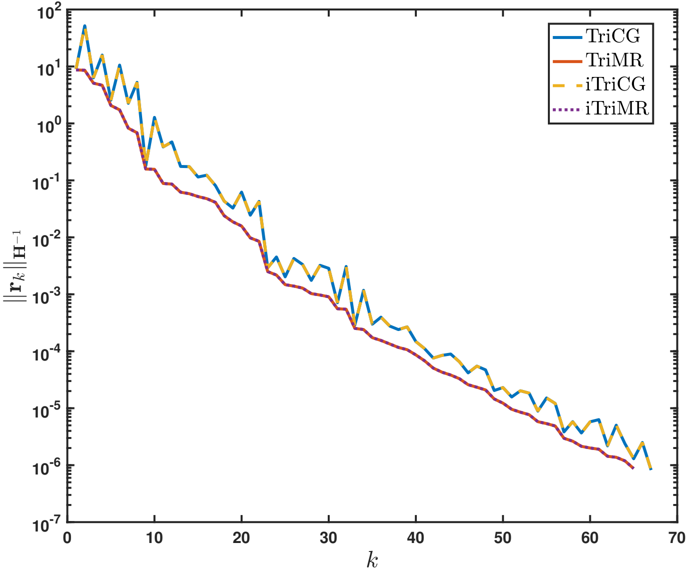
  In exact arithmetic, we have $\beta_5 = 0$ and $\gamma_5 \neq 0$.

  Case II: `[P,S,Q] = svd(A);`

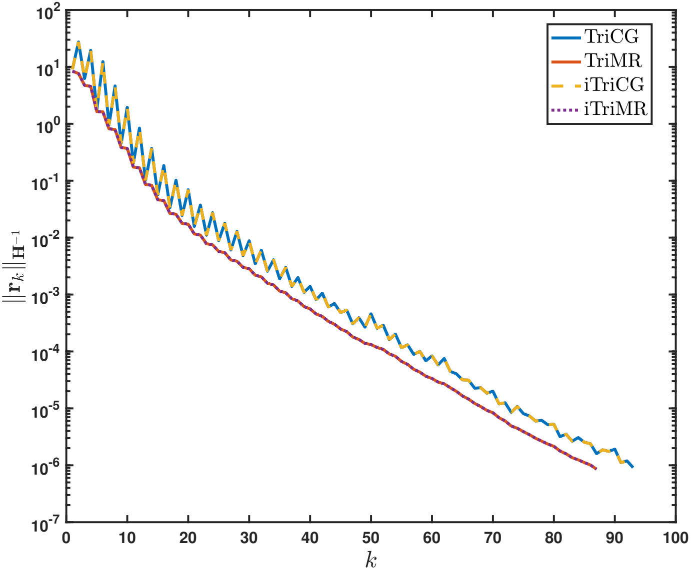
  `b = ones(m,1); c = Q(:,1:2)*ones(2,1);`

  In exact arithmetic, we have $\beta_5 \neq 0$ and $\gamma_5 = 0$.

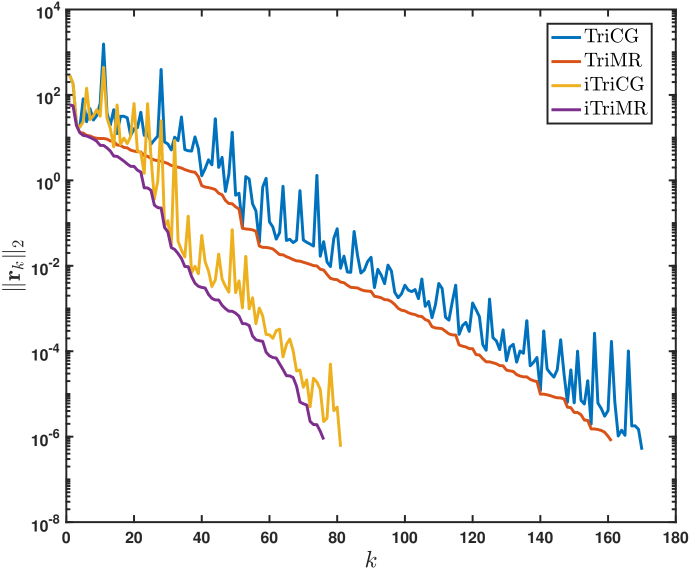
Example: `channel_domain`, Q_1–P_0 approximation



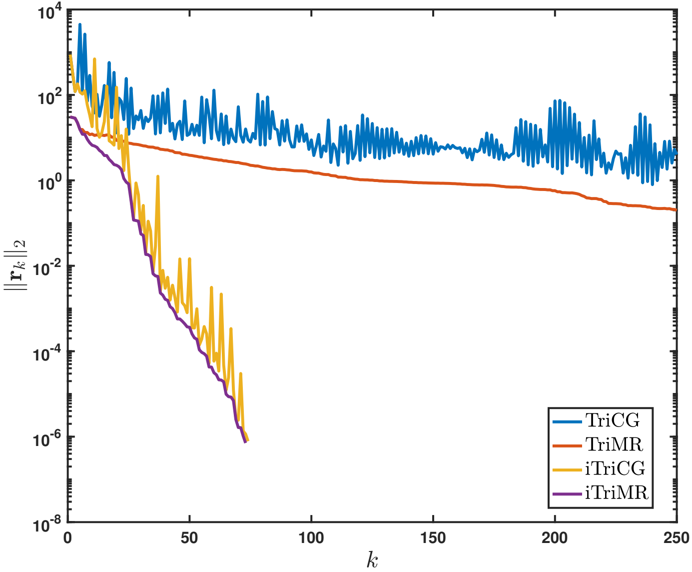
Example: `channel_domain`, Q_1–Q_1 approximation



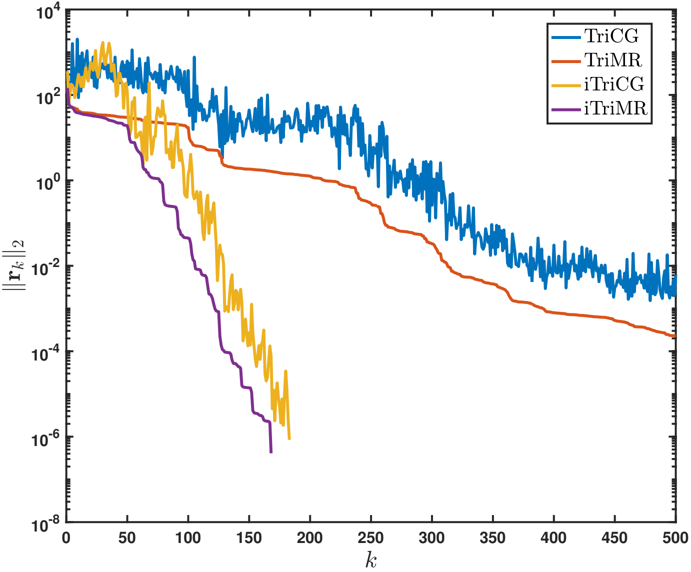
Example: `lp_czprob`, unlucky breakdown case I



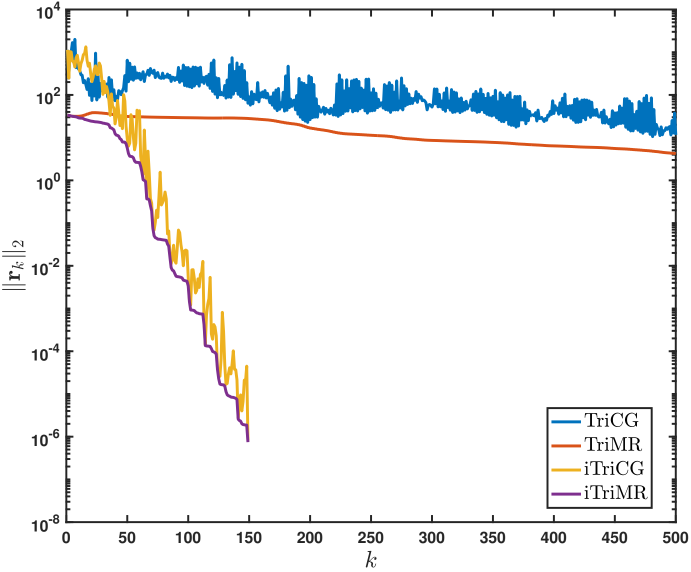
Example: `lp_czprob`, unlucky breakdown case II



Example: `lp_osa_07`, unlucky breakdown case I



Example: `lp_osa_07`, unlucky breakdown case II



Summary

- We proved that the upper triangular factor of the QR factorization used in TriMR only has three nonzero diagonals, and based on this fact we provided simplified short recurrences for TriMR, which reduce the work per iteration.
- We proposed an improved gSSY tridiagonalization process, which avoids unlucky breakdowns of the gSSY tridiagonalization process.
- We introduced two new iterative methods named iTriCG and iTriMR for solving SQD linear systems in the same fashion as TriCG and TriMR.
- iTriCG and iTriMR perform significantly better than TriCG and TriMR when unlucky breakdowns occur.

Thanks!