Numerical Linear Algebra Assignment 19

Exercise 1. (Yousef Saad, 10 points)

Let $\mathbf{A}$ be an $n \times n$ matrix where $n \geq 4$ and assume that

$$\|\mathbf{A}\|_2 = \frac{n-2}{2}, \qquad \|\mathbf{A}\|_{\mathrm{F}} = \frac{n}{2}, \qquad \operatorname{rank}(\mathbf{A}) = r \leq n.$$

Give the sharpest possible lower bound for the 2-norm condition number of $\mathbf{A}$.

Exercise 2. (Zhihao Cao, 10 points)

Let $\mathbf{R} \in \mathbb{C}^{m \times m}$ be a nonsingular upper triangular matrix. Show that the 2-norm condition number

$$\kappa_2(\mathbf{R}) \geq \frac{\max\limits_{i,j} |r_{ij}|}{\min\limits_{i} |r_{ii}|}.$$

Exercise 3. (10 points)

Let $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{A}$ be nonsingular. Let $\delta\mathbf{A}$ and $\delta\mathbf{b}$ be perturbations of the data $\mathbf{A}$ and $\mathbf{b}$, respectively. Let $\|\cdot\|$ denote a vector norm or the corresponding induced matrix norm. Assume that $\|\mathbf{A}^{-1}\|\|\delta\mathbf{A}\| < 1$. Prove the unique solution $\mathbf{x} + \delta\mathbf{x}$ of

$$(\mathbf{A} + \delta\mathbf{A})(\mathbf{x} + \delta\mathbf{x}) = \mathbf{b} + \delta\mathbf{b}$$

satisfies

$$\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \leq \frac{\kappa(\mathbf{A})}{1 - \|\mathbf{A}^{-1}\|\|\delta\mathbf{A}\|} \left(\frac{\|\delta\mathbf{A}\|}{\|\mathbf{A}\|} + \frac{\|\delta\mathbf{b}\|}{\|\mathbf{b}\|} \right).$$

Hint: you may use the following lemma: If $\mathbf{E} \in \mathbb{C}^{n \times n}$ and $\|\mathbf{E}\| < 1$, then $\mathbf{I} + \mathbf{E}$ is nonsingular and

$$(\mathbf{I} + \mathbf{E})^{-1} = \mathbf{I} - \mathbf{E} + \mathbf{E}^2 - \mathbf{E}^3 + \cdots, \qquad \|(\mathbf{I} + \mathbf{E})^{-1}\| \leq \frac{1}{1 - \|\mathbf{E}\|}.$$

Exercise 4. (10 points)

Prove $\widehat{\kappa}(f(x)) = \|\mathbf{J}(f(x))\|$ for all differentiable f.