

## Lecture 2: Singular value decomposition (SVD)



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## 1. Singular value decomposition

- **Definition:** Let  $m$  and  $n$  be arbitrary positive integers ( $m \geq n$  or  $m < n$ ). Given  $\mathbf{A} \in \mathbb{C}^{m \times n}$ , not necessarily of full rank, a *singular value decomposition (SVD)* of  $\mathbf{A}$  is a factorization

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^*,$$

where  $\mathbf{U} \in \mathbb{C}^{m \times m}$  is unitary,  $\mathbf{V} \in \mathbb{C}^{n \times n}$  is unitary, and  $\mathbf{\Sigma} \in \mathbb{R}^{m \times n}$  is diagonal. In addition, it is assumed that the diagonal entries  $\sigma_i$  of  $\mathbf{\Sigma}$  are nonnegative and in nonincreasing order; that is

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0,$$

where  $p = \min\{m, n\}$ .

### Theorem 1 (Existence of SVD)

*Every matrix  $\mathbf{A} \in \mathbb{C}^{m \times n}$  has a singular value decomposition.*

**Proof.** Assume  $\mathbf{A} \neq \mathbf{0}$ ; otherwise we can take  $\mathbf{\Sigma} = \mathbf{0}$  and let  $\mathbf{U}$  and  $\mathbf{V}$  be arbitrary unitary matrices. Next, we use induction on  $m$  and  $n$  to prove the existence of SVD for the case  $m \geq n$  (consider  $\mathbf{A}^*$  if  $m < n$ ): Assume that an SVD exists for any  $(m-1) \times (n-1)$  matrix and prove it for any  $m \times n$  matrix.

(i) The basic step:  $m \geq n = 1$ .

Write  $\mathbf{A} = \mathbf{u}_1 \mathbf{\Sigma}_1 \mathbf{V}^*$  with  $\mathbf{u}_1 = \mathbf{A} / \|\mathbf{A}\|_2$ ,  $\mathbf{\Sigma}_1 = \|\mathbf{A}\|_2$  and  $\mathbf{V} = 1$ .

Choose  $\hat{\mathbf{U}}$  such that  $\mathbf{U} = \begin{bmatrix} \mathbf{u}_1 & \hat{\mathbf{U}} \end{bmatrix} \in \mathbb{C}^{m \times m}$  is unitary. Let

$\mathbf{\Sigma} = \begin{bmatrix} \mathbf{\Sigma}_1 & \mathbf{0} \end{bmatrix}^\top \in \mathbb{R}^{m \times 1}$ . Then  $\mathbf{A}$  has an SVD  $\mathbf{A} = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^*$ .

(ii) The induction step:  $m \geq n > 1$ .

Let  $\mathbf{v}_1 \in \mathbb{C}^n$  be a unit (i.e.,  $\|\mathbf{v}_1\|_2 = 1$ ) eigenvector corresponding to the eigenvalue  $\lambda_{\max}(\mathbf{A}^* \mathbf{A})$ . Then we have  $\|\mathbf{A} \mathbf{v}_1\|_2 = \|\mathbf{A}\|_2 > 0$ .

Let  $\mathbf{u}_1 = \mathbf{A} \mathbf{v}_1 / \|\mathbf{A} \mathbf{v}_1\|_2$ , which is a unit vector. Choose  $\hat{\mathbf{U}}$  and  $\hat{\mathbf{V}}$  such that  $\tilde{\mathbf{U}} = \begin{bmatrix} \mathbf{u}_1 & \hat{\mathbf{U}} \end{bmatrix} \in \mathbb{C}^{m \times m}$  and  $\tilde{\mathbf{V}} = \begin{bmatrix} \mathbf{v}_1 & \hat{\mathbf{V}} \end{bmatrix} \in \mathbb{C}^{n \times n}$  are unitary.

Now we have

$$\tilde{\mathbf{U}}^* \mathbf{A} \tilde{\mathbf{V}} = \begin{bmatrix} \mathbf{u}_1^* \\ \hat{\mathbf{U}}^* \end{bmatrix} \mathbf{A} \begin{bmatrix} \mathbf{v}_1 & \hat{\mathbf{V}} \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1^* \mathbf{A} \mathbf{v}_1 & \mathbf{u}_1^* \mathbf{A} \hat{\mathbf{V}} \\ \hat{\mathbf{U}}^* \mathbf{A} \mathbf{v}_1 & \hat{\mathbf{U}}^* \mathbf{A} \hat{\mathbf{V}} \end{bmatrix}.$$

We note that

$$\mathbf{u}_1^* \mathbf{A} \mathbf{v}_1 = \frac{(\mathbf{A} \mathbf{v}_1)^* (\mathbf{A} \mathbf{v}_1)}{\|\mathbf{A} \mathbf{v}_1\|_2} = \|\mathbf{A} \mathbf{v}_1\|_2 = \|\mathbf{A}\|_2,$$

and

$$\hat{\mathbf{U}}^* \mathbf{A} \mathbf{v}_1 = \hat{\mathbf{U}}^* \mathbf{u}_1 \|\mathbf{A} \mathbf{v}_1\|_2 = \mathbf{0}.$$

We claim  $\mathbf{u}_1^* \mathbf{A} \hat{\mathbf{V}} = \mathbf{0}$  too because otherwise

$$\begin{aligned} \sigma_1 &:= \|\mathbf{A}\|_2 = \|\tilde{\mathbf{U}}^* \mathbf{A} \tilde{\mathbf{V}}\|_2 \\ &= \left\| \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \right\|_2 \cdot \|\tilde{\mathbf{U}}^* \mathbf{A} \tilde{\mathbf{V}}\|_2 \\ &\geq \left\| \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \tilde{\mathbf{U}}^* \mathbf{A} \tilde{\mathbf{V}} \right\|_2 = \left\| [\sigma_1 \ \mathbf{u}_1^* \mathbf{A} \hat{\mathbf{V}}] \right\|_2 > \sigma_1, \end{aligned}$$

which is a contradiction.

Therefore,

$$\tilde{\mathbf{U}}^* \mathbf{A} \tilde{\mathbf{V}} = \begin{bmatrix} \sigma_1 & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{U}}^* \mathbf{A} \hat{\mathbf{V}} \end{bmatrix}.$$

By the induction hypothesis, we know that the  $(m-1) \times (n-1)$  matrix  $\hat{\mathbf{U}}^* \mathbf{A} \hat{\mathbf{V}}$  has an SVD:

$$\hat{\mathbf{U}}^* \mathbf{A} \hat{\mathbf{V}} = \mathbf{U}_0 \boldsymbol{\Sigma}_0 \mathbf{V}_0^*.$$

Then it follows from

$$\tilde{\mathbf{U}}^* \mathbf{A} \tilde{\mathbf{V}} = \begin{bmatrix} \sigma_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_0 \boldsymbol{\Sigma}_0 \mathbf{V}_0^* \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_0 \end{bmatrix} \begin{bmatrix} \sigma_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_0 \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_0 \end{bmatrix}^*$$

that

$$\mathbf{A} = \tilde{\mathbf{U}} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_0 \end{bmatrix} \begin{bmatrix} \sigma_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_0 \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_0 \end{bmatrix}^* \tilde{\mathbf{V}}^* =: \mathbf{U} \begin{bmatrix} \sigma_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_0 \end{bmatrix} \mathbf{V}^*.$$

This is an SVD of  $\mathbf{A}$  because  $\sigma_1 \geq \|\boldsymbol{\Sigma}_0\|_2$ .

□

- Full SVD:

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^*$$

- Reduced SVD (the case  $m \geq n$ ):

$$\mathbf{A} = \mathbf{U}_n\mathbf{\Sigma}_n\mathbf{V}^*$$

where

$$\mathbf{U}_n = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_n],$$

and

$$\mathbf{\Sigma}_n = \text{diag}\{\sigma_1, \sigma_2, \dots, \sigma_n\}.$$



- Rank SVD or compact SVD or condensed SVD:

$$\mathbf{A} = [\mathbf{U}_r \quad \mathbf{U}_c] \begin{bmatrix} \mathbf{\Sigma}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{V}_r^* \\ \mathbf{V}_c^* \end{bmatrix} = \mathbf{U}_r \mathbf{\Sigma}_r \mathbf{V}_r^* = \sum_{i=1}^r \sigma_i \mathbf{u}_i \mathbf{v}_i^*$$

where  $r = \text{rank}(\mathbf{A})$ ,

$$\mathbf{U}_r = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_r], \quad \mathbf{U}_c = [\mathbf{u}_{r+1} \quad \mathbf{u}_{r+2} \quad \cdots \quad \mathbf{u}_m],$$

$$\mathbf{V}_r = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_r], \quad \mathbf{V}_c = [\mathbf{v}_{r+1} \quad \mathbf{v}_{r+2} \quad \cdots \quad \mathbf{v}_n],$$

and

$$\mathbf{\Sigma}_r = \text{diag}\{\sigma_1, \sigma_2, \dots, \sigma_r\}.$$

- $\{\sigma_i^2, \mathbf{u}_i\}$  are eigenvalue-eigenvector pairs of  $\mathbf{A}\mathbf{A}^*$ , and  $\{\sigma_i^2, \mathbf{v}_i\}$  are eigenvalue-eigenvector pairs of  $\mathbf{A}^*\mathbf{A}$ :

$$\mathbf{A}\mathbf{A}^* \mathbf{u}_i = \sigma_i^2 \mathbf{u}_i, \quad \mathbf{A}^* \mathbf{A} \mathbf{v}_i = \sigma_i^2 \mathbf{v}_i, \quad i = 1, 2, \dots, p$$

- $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p$  are called the *singular values* of  $\mathbf{A}$ .

- $\mathbf{u}_i$  is called *left singular vector*, and  $\mathbf{v}_i$  is called *right singular vector*:  $\mathbf{u}_i^* \mathbf{A} = \sigma_i \mathbf{v}_i^*$ ,  $\mathbf{A} \mathbf{v}_i = \sigma_i \mathbf{u}_i$ ,  $i = 1, 2, \dots, p$

## Theorem 2

*The set of singular values  $\{\sigma_i\}$  is uniquely determined and invariant under unitary multiplication.*

## Theorem 3

*If  $\mathbf{A}$  is square and all the  $\sigma_i$  are distinct, the left and right singular vectors are uniquely determined up to complex signs (i.e., complex scalar factors of absolute value 1).*

**Hint:** There exists only one linearly independent eigenvector for each eigenvalue of  $\mathbf{A}^* \mathbf{A}$  or  $\mathbf{A} \mathbf{A}^*$ .

## Theorem 4 (Real SVD)

*Every matrix  $\mathbf{A} \in \mathbb{R}^{m \times n}$  has a real singular value decomposition.*



## 1.1. Geometric observation

- The image of the unit sphere (in the 2-norm) of  $\mathbb{C}^n$  under any  $m \times n$  matrix is a hyperellipse of  $\mathbb{C}^m$ .

For example,  $2 \times 2$  real matrix  $\mathbf{A}$



**SVD of a matrix can not be emphasized too much!**

## 2. Matrix properties via SVD: $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^*$

- 2-norm

$$\|\mathbf{A}\|_2 = \|\mathbf{A}^*\|_2 = \|\mathbf{A}^\top\|_2 = \|\overline{\mathbf{A}}\|_2 = \sigma_1$$

- F-norm

$$\|\mathbf{A}\|_F = \sqrt{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_r^2}$$

- $\text{range}(\mathbf{A})$ : *column space* of  $\mathbf{A}$ , spanned by the columns of  $\mathbf{A}$

$$\begin{aligned}\text{range}(\mathbf{A}) &:= \{\mathbf{y} \in \mathbb{C}^m \mid \exists \mathbf{x} \in \mathbb{C}^n \text{ s.t. } \mathbf{y} = \mathbf{A}\mathbf{x}\} \\ &= \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_r\}\end{aligned}$$

- $\text{null}(\mathbf{A})$ : *kernel* or *nullspace* of  $\mathbf{A}$

$$\begin{aligned}\text{null}(\mathbf{A}) &:= \{\mathbf{x} \in \mathbb{C}^n \mid \mathbf{A}\mathbf{x} = \mathbf{0}\} \\ &= \text{span}\{\mathbf{v}_{r+1}, \mathbf{v}_{r+2}, \dots, \mathbf{v}_n\}\end{aligned}$$

- Range and null space of  $\mathbf{A}^*$ :

$$\text{range}(\mathbf{A}^*) = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\} = \text{null}(\mathbf{A})^\perp$$

$$\text{null}(\mathbf{A}^*) = \text{span}\{\mathbf{u}_{r+1}, \mathbf{u}_{r+2}, \dots, \mathbf{u}_m\} = \text{range}(\mathbf{A})^\perp$$

- Relations between the four subspaces

$$\text{range}(\mathbf{A}^*) \perp \text{null}(\mathbf{A}), \quad \text{range}(\mathbf{A}^*) + \text{null}(\mathbf{A}) = \mathbb{C}^n$$

$$\text{range}(\mathbf{A}) \perp \text{null}(\mathbf{A}^*), \quad \text{range}(\mathbf{A}) + \text{null}(\mathbf{A}^*) = \mathbb{C}^m$$

- If  $\mathbf{A}$  is Hermitian, i.e.,  $\mathbf{A} = \mathbf{A}^*$

singular values are absolute values of eigenvalues

- Determinant of  $\mathbf{A} \in \mathbb{C}^{m \times m}$

$$|\det(\mathbf{A})| = \prod_{i=1}^m \sigma_i$$

## 2.1. Low-rank approximation (LRA)

### Theorem 5 (Eckart-Young-Mirski)

For any integer  $k$  with  $1 \leq k < r = \text{rank}(\mathbf{A})$ , define

$$\mathbf{A}_k = \sum_{i=1}^k \sigma_i \mathbf{u}_i \mathbf{v}_i^*.$$

Then

$$\|\mathbf{A} - \mathbf{A}_k\|_2 = \sigma_{k+1} = \min_{\substack{\mathbf{B} \in \mathbb{C}^{m \times n}, \\ \text{rank}(\mathbf{B}) \leq k}} \|\mathbf{A} - \mathbf{B}\|_2,$$

and

$$\|\mathbf{A} - \mathbf{A}_k\|_F = \sqrt{\sigma_{k+1}^2 + \cdots + \sigma_r^2} = \min_{\substack{\mathbf{B} \in \mathbb{C}^{m \times n}, \\ \text{rank}(\mathbf{B}) \leq k}} \|\mathbf{A} - \mathbf{B}\|_F.$$

- **Discussion:** Is the minimizer in Theorem 5 unique?

A random  $m \times m$  matrix is “always” nonsingular. Why?

## Proof of Theorem 5.

- Suppose there is some  $\mathbf{B} \in \mathbb{C}^{m \times n}$  with  $\text{rank}(\mathbf{B}) \leq k < r$  such that

$$\|\mathbf{A} - \mathbf{B}\|_2 < \sigma_{k+1}.$$

By  $\dim(\text{null}(\mathbf{B})^\perp) = \dim(\text{range}(\mathbf{B}^*)) = \text{rank}(\mathbf{B}^*) = \text{rank}(\mathbf{B}) \leq k$ , we have  $\dim(\text{null}(\mathbf{B})) = n - \dim(\text{null}(\mathbf{B})^\perp) \geq n - k$ . Then there exists an  $(n - k)$ -dimensional subspace  $\mathcal{W} \subseteq \text{null}(\mathbf{B})$ . For any nonzero  $\mathbf{x} \in \mathcal{W}$ , we have

$$\|\mathbf{A}\mathbf{x}\|_2 = \|(\mathbf{A} - \mathbf{B})\mathbf{x}\|_2 \leq \|\mathbf{A} - \mathbf{B}\|_2 \|\mathbf{x}\|_2 < \sigma_{k+1} \|\mathbf{x}\|_2.$$

Let  $\mathcal{V} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{k+1}\}$ . For any  $\mathbf{x} \in \mathcal{V}$ , we have

$$\|\mathbf{A}\mathbf{x}\|_2 = \|\mathbf{A}\mathbf{V}_{k+1}\mathbf{y}\|_2 = \|\mathbf{U}_{k+1}\mathbf{\Sigma}_{k+1}\mathbf{y}\|_2 = \|\mathbf{\Sigma}_{k+1}\mathbf{y}\|_2 \geq \sigma_{k+1} \|\mathbf{x}\|_2.$$

Since  $\dim \mathcal{W} + \dim \mathcal{V} = (n - k) + (k + 1) > n$ , there must be a nonzero vector lying in both, and this is a contradiction.

- Case  $\|\cdot\|_F$ : **Generalized Inverses: Theory and Applications, 2nd edition, Adi Ben-Israel and Thomas N.E. Greville, Page 213.** □

## Application of low-rank approximation: image compression

- An image can be represented as a matrix. For example, typical grayscale images consist of a rectangular array of pixels,  $m$  in the vertical direction,  $n$  in the horizontal direction. The color of each of those pixels is denoted by a single number, an integer between 0 (black) and 255 (white). (This gives  $2^8 = 256$  different shades of gray for each pixel. Color images are represented by three such matrices: one for red, one for green, and one for blue. Thus each pixel in a typical color image takes  $(2^8)^3 = 2^{24}$  shades.)
- The objective of image compression is to reduce irrelevance and redundancy of the image data in order to be able to store or transmit data in an efficient form.
- Low-rank SVD approximation is a good candidate. (Note: jpeg compression algorithm uses similar idea, on subimages)

### 3. Moore–Penrose pseudoinverse

- Let  $\mathbf{A} \in \mathbb{C}^{m \times n}$  have an SVD (rank form)  $\mathbf{A} = \mathbf{U}_r \mathbf{\Sigma}_r \mathbf{V}_r^*$ . The *Moore–Penrose pseudoinverse* of  $\mathbf{A}$ , denoted by  $\mathbf{A}^\dagger$ :

$$\mathbf{A}^\dagger := \mathbf{V}_r \mathbf{\Sigma}_r^{-1} \mathbf{U}_r^* = \sum_{i=1}^r \frac{1}{\sigma_i} \mathbf{v}_i \mathbf{u}_i^*.$$

- The matrix  $\mathbf{A}^\dagger$  is the *unique* matrix satisfying the four equations

$$\mathbf{A} \mathbf{X} \mathbf{A} = \mathbf{A}, \quad \mathbf{X} \mathbf{A} \mathbf{X} = \mathbf{X}, \quad (\mathbf{A} \mathbf{X})^* = \mathbf{A} \mathbf{X}, \quad (\mathbf{X} \mathbf{A})^* = \mathbf{X} \mathbf{A}.$$

For a proof, see Page 122 of [Numerical linear algebra \(in Chinese\) by Zhihao Cao](#).

- If  $\mathbf{A}$  has full column rank, then  $\mathbf{A}^\dagger = (\mathbf{A}^* \mathbf{A})^{-1} \mathbf{A}^*$ .  
If  $\mathbf{A}$  has full row rank, then  $\mathbf{A}^\dagger = \mathbf{A}^* (\mathbf{A} \mathbf{A}^*)^{-1}$ .

#### 4. A wonderful reference

- Zhihua Zhang

The singular value decomposition, applications and beyond  
arXiv:1510.08532

#### 5. Another proof of Theorem 5

- Holger Wendland

Numerical Linear Algebra An Introduction

Cambridge University Press, 2018.

See Page 295, Theorem 7.41.

#### 6. A computationally more feasible method for LRA

- Adaptive cross approximation (ACA)

See Page 297 of Numerical Linear Algebra An Introduction.