Lecture : Domain decomposition

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1. Motivations for domain decomposition

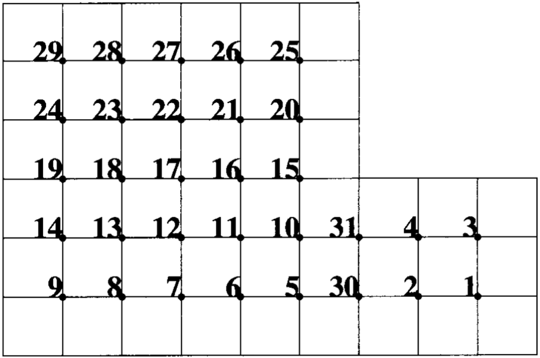
- The region of solution of real problems is irregular. Even more complicated problems such as that with different equations in different regions.
- The problem is too large to fit in the computer memory and may have to be solved "in pieces".
- We may want to break the problem into subproblems that can be solved in parallel on a parallel computer.

2. The idea of domain decomposition

- Domain decomposition breaks a large problem into subproblems, then uses simpler methods solve the individual subproblems, and finally combines the solutions together to get the overall solution.
- These subproblems can be solved one at a time if the whole problem does not fit into memory, or in parallel on a parallel computer.

3. Nonoverlapping methods

- This method is also called *substructuring* or a *Schur complement method* in the literature.
- Example: Poisson's equation with Dirichlet boundary conditions on an L-shaped region discretized with a 5-point stencil.



- The discretized matrix

$$
\left[\begin{array}{cc|cc|ccccc|ccccc|ccccc|ccccc|ccccc|cc}
4&-1 & -1& & &&&& & &&&& & &&&& & &&&& & &&&& & & \\
-1&4 & &-1 & &&&& & &&&& & &&&& & &&&& & &&&& & -1& \\
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&-1 & -1&4 & &&&& & &&&& & &&&& & &&&& & &&&& & &-1 \\
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& & & & 4&-1&&& & -1&&&& & &&&& & &&&& & &&&& & -1& \\
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& & & & &&-1&4&-1 & &&&-1& & &&&& & &&&& & &&&& & & \\
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& & & & &&-1&& & &-1&4&-1& & &&-1&& & &&&& & &&&& & & \\
& & & & &&&-1& & &&-1&4&-1 & &&&-1& & &&&& & &&&& & & \\
& & & & &&&&-1 & &&&-1&4 & &&&&-1 & &&&& & &&&& & & \\
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& & & & &&&& & &&-1&& & &-1&4&-1& & &&-1&& & &&&& & & \\
& & & & &&&& & &&&-1& & &&-1&4&-1 & &&&-1& & &&&& & & \\
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& & & & &&&& & &&&& & &&&& & &&&-1& & &&-1&4&-1 & & \\
& & & & &&&& & &&&& & &&&& & &&&&-1 & &&&-1&4 & & \\
\hline
&-1 & & & -1&&&& & &&&& & &&&& & &&&& & &&&& & 4&-1 \\
& & &-1 & &&&& & -1&&&& & &&&& & &&&& & &&&& & -1&4
\end{array}\right]
$$

- The discretized matrix is a block 3×3 matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} & \mathbf{A}_{13} \\ \mathbf{0} & \mathbf{A}_{22} & \mathbf{A}_{23} \\ \mathbf{A}_{13}^\top & \mathbf{A}_{23}^\top & \mathbf{A}_{33} \end{bmatrix}.$$

  Note that $\mathbf{A}_{12} = \mathbf{0}$, since there is no direct coupling between the interior grid points of the two subdomains.

- The block LDU decomposition of $\mathbf{A}$ is

$$\mathbf{A} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{A}_{13}^\top \mathbf{A}_{11}^{-1} & \mathbf{A}_{23}^\top \mathbf{A}_{22}^{-1} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{S} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} & \mathbf{A}_{13} \\ \mathbf{0} & \mathbf{A}_{22} & \mathbf{A}_{23} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix},$$

  where

$$\mathbf{S} = \mathbf{A}_{33} - \mathbf{A}_{13}^\top \mathbf{A}_{11}^{-1} \mathbf{A}_{13} - \mathbf{A}_{23}^\top \mathbf{A}_{22}^{-1} \mathbf{A}_{23}$$

  is called the Schur complement of the leading principal submatrix containing $\mathbf{A}_{11}$ and $\mathbf{A}_{22}$.

- Then we have $\mathbf{A}^{-1} =$

$$\begin{bmatrix} \mathbf{A}_{11}^{-1} & \mathbf{0} & -\mathbf{A}_{11}^{-1}\mathbf{A}_{13} \\ \mathbf{0} & \mathbf{A}_{22}^{-1} & -\mathbf{A}_{22}^{-1}\mathbf{A}_{23} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{S}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ -\mathbf{A}_{13}^{\top}\mathbf{A}_{11}^{-1} & -\mathbf{A}_{23}^{\top}\mathbf{A}_{22}^{-1} & \mathbf{I} \end{bmatrix}.$$

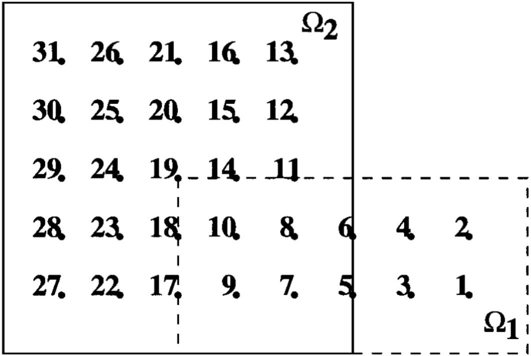
  To multiply a vector by $\mathbf{A}^{-1}$ we need to multiply by the blocks in the entries of this factored form of $\mathbf{A}^{-1}$, namely, $\mathbf{A}_{13}$ and $\mathbf{A}_{23}$ (and their transposes), $\mathbf{A}_{11}^{-1}$, $\mathbf{A}_{22}^{-1}$, and $\mathbf{S}^{-1}$. Multiplying by $\mathbf{A}_{13}$ and $\mathbf{A}_{23}$ is cheap, and multiplying by $\mathbf{A}_{11}^{-1}$ and $\mathbf{A}_{22}^{-1}$ is also cheap. It remains to explain how to multiply by $\mathbf{S}^{-1}$. Preconditioned Krylov iterative methods are preferred.

- The case for k subdomains is a block $(k+1)\times(k+1)$ matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \cdots & \mathbf{0} & \mathbf{A}_{1,k+1} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \cdots & \mathbf{A}_{kk} & \mathbf{A}_{k,k+1} \\ \mathbf{A}_{1,k+1}^{\top} & \cdots & \mathbf{A}_{k,k+1}^{\top} & \mathbf{A}_{k+1,k+1} \end{bmatrix}.$$

4. Overlapping methods (Schwarz domain decomposition)

- In nonoverlapping methods, the domains corresponding to the nodes in $\mathbf{A}_{ii}$ were disjoint. In overlapping methods, we use overlapping domains.
- Example: Poisson's equation with Dirichlet boundary conditions on an L-shaped region discretized with a 5-point stencil.



- The discretized matrix

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&&&&&&&&&&&&&&&&&&&&&&&&& -1 &&&& -1 & 4 \\
\hline
\end{array}\right|
$$

- Partition $\mathbf{A}$ as follows:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{\Omega_1,\Omega_1} & \mathbf{A}_{\Omega_1,\Omega\backslash\Omega_1} \\ \mathbf{A}_{\Omega\backslash\Omega_1,\Omega_1} & \mathbf{A}_{\Omega\backslash\Omega_1,\Omega\backslash\Omega_1} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{\Omega\backslash\Omega_2,\Omega\backslash\Omega_2} & \mathbf{A}_{\Omega\backslash\Omega_2,\Omega_2} \\ \mathbf{A}_{\Omega_2,\Omega\backslash\Omega_2} & \mathbf{A}_{\Omega_2,\Omega_2} \end{bmatrix}.$$

- Partition $\mathbf{x}$ conformally:

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_{\Omega_1} \\ \mathbf{x}_{\Omega\backslash\Omega_1} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{1:10} \\ \mathbf{x}_{11:31} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{\Omega\backslash\Omega_2} \\ \mathbf{x}_{\Omega_2} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{1:6} \\ \mathbf{x}_{7:31} \end{bmatrix}.$$

4.1. Additive Schwarz

Algorithm: Additive Schwarz or overlapping block Jacobi

$\mathbf{r} = \mathbf{b} - \mathbf{A}\mathbf{x}^{(i)};$ `compute residual`

$\mathbf{x}^{(i+1)} = \mathbf{x}^{(i)};$

$\mathbf{x}_{\Omega_1}^{(i+1)} = \mathbf{x}_{\Omega_1}^{(i)} + \mathbf{A}_{\Omega_1,\Omega_1}^{-1}\mathbf{r}_{\Omega_1};$ `update solution on` Ω_1

$\mathbf{x}_{\Omega_2}^{(i+1)} = \mathbf{x}_{\Omega_2}^{(i+1)} + \mathbf{A}_{\Omega_2,\Omega_2}^{-1}\mathbf{r}_{\Omega_2};$ `update solution on` Ω_2

- This algorithm also can be written in one line as

$$\mathbf{x}^{(i+1)} = \mathbf{x}^{(i)} + \begin{bmatrix} \mathbf{A}_{\Omega_1,\Omega_1}^{-1} \mathbf{r}_{\Omega_1} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{A}_{\Omega_2,\Omega_2}^{-1} \mathbf{r}_{\Omega_2} \end{bmatrix}.$$

- Since the additive Schwarz method is iterative, it is not necessary to solve the problems on Ω_i exactly, i.e., $\mathbf{A}_{\Omega_1,\Omega_1}^{-1} \mathbf{r}_{\Omega_1}$ and $\mathbf{A}_{\Omega_2,\Omega_2}^{-1} \mathbf{r}_{\Omega_2}$ can be computed inexactly.
- Indeed, the additive Schwarz method is typically used as a preconditioner for a Krylov subspace method. The preconditioner $\mathbf{M}$ is given by

$$\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{A}_{\Omega_1,\Omega_1}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{\Omega_2,\Omega_2}^{-1} \end{bmatrix}.$$

If Ω_1 and Ω_2 do not overlap, then $\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{A}_{\Omega_1,\Omega_1}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{\Omega_2,\Omega_2}^{-1} \end{bmatrix}$.

4.2. Multiplicative Schwarz

Algorithm: Multiplicative Schwarz or overlapping block G–S

$\mathbf{r}_{\Omega_1} = (\mathbf{b} - \mathbf{A}\mathbf{x}^{(i)})_{\Omega_1};$	`compute residual on` Ω_1
$\mathbf{x}^{(i+1/2)} = \mathbf{x}^{(i)};$	
$\mathbf{x}^{(i+1/2)}_{\Omega_1} = \mathbf{x}^{(i)}_{\Omega_1} + \mathbf{A}^{-1}_{\Omega_1,\Omega_1}\mathbf{r}_{\Omega_1};$	`update solution on` Ω_1
$\mathbf{r}_{\Omega_2} = (\mathbf{b} - \mathbf{A}\mathbf{x}^{(i+1/2)})_{\Omega_2};$	`compute residual on` Ω_2
$\mathbf{x}^{(i+1)} = \mathbf{x}^{(i+1/2)};$	
$\mathbf{x}^{(i+1)}_{\Omega_2} = \mathbf{x}^{(i+1/2)}_{\Omega_2} + \mathbf{A}^{-1}_{\Omega_2,\Omega_2}\mathbf{r}_{\Omega_2};$	`update solution on` Ω_2

- This algorithm first solves Poisson's equation on Ω_1 using boundary data from $\mathbf{x}^{(i)}$. It then solves Poisson's equation on Ω_2, but using boundary data that has just been updated.

- Exercise: The multiplicative Schwarz method also can be used as a preconditioner for a Krylov subspace method. What is the corresponding preconditioner?

- In practice more domains than just two (Ω_1 and Ω_2) are used. This is done if the domain of solution is more complicated or if there are many independent parallel processors available to solve independent problems $\mathbf{A}_{\Omega_i,\Omega_i}^{-1}\mathbf{r}_{\Omega_i}$, or just to keep the subproblems $\mathbf{A}_{\Omega_i,\Omega_i}^{-1}\mathbf{r}_{\Omega_i}$ small and inexpensive to solve.

- Two-level correction: $\mathbf{x}^{(i+1)} = \mathbf{x}^{(i+1)} + \mathbf{R}^\top \mathbf{A}_H^{-1} \mathbf{R}(\mathbf{b} - \mathbf{A}\mathbf{x}^{(i+1)})$

5. Further reading

- Iterative Methods and Preconditioners for Systems of Linear Equations

  Gabriele Ciaramella and Martin J. Gander

  SIAM, 2022