Lecture 7: Preliminaries III. Optimization

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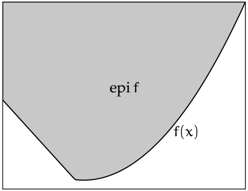
1. Basic definitions

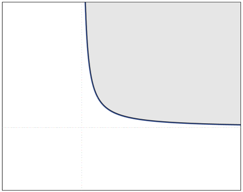
- The *effective domain* of $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined as

$$\mathrm{dom}(f) := \{\mathbf{x} \mid f(\mathbf{x}) < +\infty\}.$$

- A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is called *proper* if there exists at least one $\mathbf{x} \in \mathbb{R}^n$ such that $f(\mathbf{x}) < +\infty$, meaning that $\mathrm{dom}(f) \neq \emptyset$.
- The *epigraph* of $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\mathrm{epi}(f) = \{(\mathbf{x}, y) : f(\mathbf{x}) \leq y, \mathbf{x} \in \mathbb{R}^n, y \in \mathbb{R}\}.$$





- A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is called *closed* if $\mathrm{epi}(f)$ is closed.

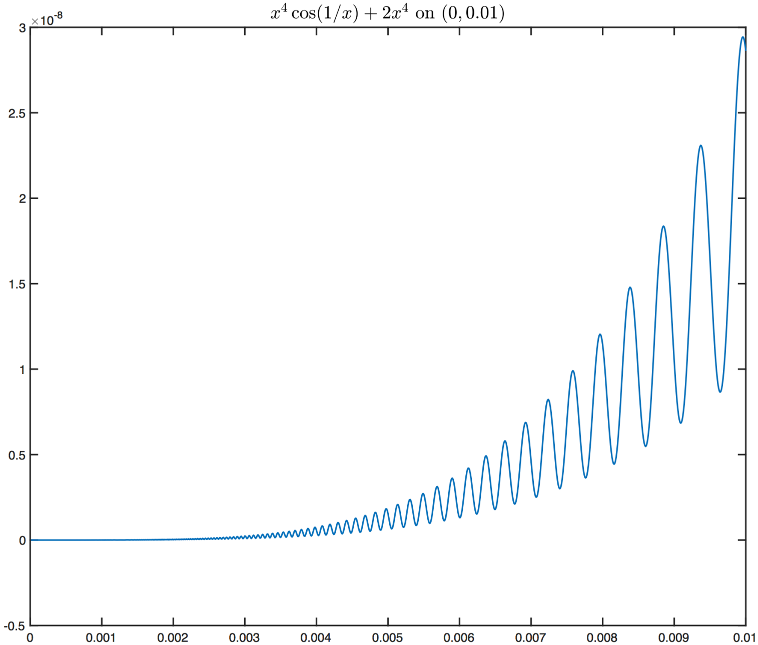
2. Solutions of $\min_{\mathbf{x}} f(\mathbf{x})$

- $\mathbf{x}_\star$ is a *local minimizer* of f if there is a neighborhood $\mathcal{N}$ of $\mathbf{x}_\star$ such that $f(\mathbf{x}) \geq f(\mathbf{x}_\star)$ for all $\mathbf{x} \in \mathcal{N}$.
- $\mathbf{x}_\star$ is a *strict local minimizer* if it is a local minimizer on some neighborhood $\mathcal{N}$ and in addition $f(\mathbf{x}) > f(\mathbf{x}_\star)$ for all $\mathbf{x} \in \mathcal{N}$ with $\mathbf{x} \neq \mathbf{x}_\star$.
- $\mathbf{x}_\star$ is an *isolated local minimizer* if there is a neighborhood $\mathcal{N}$ of $\mathbf{x}_\star$ such that $f(\mathbf{x}) \geq f(\mathbf{x}_\star)$ for all $\mathbf{x} \in \mathcal{N}$ and in addition, $\mathcal{N}$ contains no local minimizers other than $\mathbf{x}_\star$.

  Strict local minimizers are not always isolated: for example,

$$f(x) = x^4 \cos(1/x) + 2x^4, \;\; f(0) = 0.$$

  All isolated local minimizers are strict.
- $\mathbf{x}_\star$ is a *global minimizer* of f if $f(\mathbf{x}) \geq f(\mathbf{x}_\star)$ for all $\mathbf{x} \in \mathbb{R}^n$.


$x^4\cos(1/x) + 2x^4$ on $(0, 0.01)$
$\times 10^{-8}$
3
2.5
2
1.5
1
0.5
0
-0.5
0
0.001
0.002
0.003
0.004
0.005
0.006
0.007
0.008
0.009
0.01

3. Convexity

- A set $\Omega \subseteq \mathbb{R}^n$ is called *convex* if it has the property that

$$\forall\ \mathbf{x}, \mathbf{y} \in \Omega \Rightarrow (1-\alpha)\mathbf{x} + \alpha\mathbf{y} \in \Omega \quad \forall\ \alpha \in [0,1].$$

  We usually deal with closed convex sets.

- For a convex set $\Omega \subseteq \mathbb{R}^n$ we define the *indicator function* I_Ω as follows

$$\mathrm{I}_\Omega(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \in \Omega \\ +\infty & \text{otherwise.} \end{cases}$$

  The constrained optimization problem

$$\min_{\mathbf{x} \in \Omega} f(\mathbf{x})$$

  can be restated equivalently as follows:

$$\min_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x}) + \mathrm{I}_\Omega(\mathbf{x}).$$

- A convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ has the following defining property: $\mathrm{dom}(f)$ is convex, and $\forall\ \mathbf{x}, \mathbf{y} \in \mathrm{dom}(f)$, $\forall\ \alpha \in [0, 1]$,

$$f((1-\alpha)\mathbf{x} + \alpha\mathbf{y}) \leqslant (1-\alpha) f(\mathbf{x}) + \alpha f(\mathbf{y}).$$

Theorem 1 (First-order convexity condition)

Differentiable f is convex if and only if $\mathrm{dom}(f)$ is convex and

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x}), \quad \forall\ \mathbf{x}, \mathbf{y} \in \mathrm{dom}(f).$$

Theorem 2 (Second-order convexity conditions)

Assume f is twice continuously differentiable. Then f is convex if and only if $\mathrm{dom}(f)$ is convex and

$$\nabla^2 f(\mathbf{x}) \succeq \mathbf{0}, \quad \forall\ \mathbf{x} \in \mathrm{dom}(f)$$

that is, $\nabla^2 f(\mathbf{x})$ is positive semidefinite.

- Important properties for convex objective functions:
  - ⋆ Any local minimizer is also a global minimizer (see Theorem 12).
  - ⋆ The set of global minimizers is a convex set. (easy to prove)
- If there exists a value $\gamma > 0$ such that

$$f((1-\alpha)\mathbf{x} + \alpha\mathbf{y}) \leqslant (1-\alpha)f(\mathbf{x}) + \alpha f(\mathbf{y}) - \frac{\gamma}{2}\alpha(1-\alpha)\|\mathbf{x} - \mathbf{y}\|^2$$

  for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\alpha \in [0, 1]$, we say that f is *strongly convex with modulus of convexity* γ.

  Exercise: If f is strongly convex with modulus of convexity γ, then $f(\mathbf{x}) - \frac{\gamma}{2}\|\mathbf{x}\|^2$ is convex.

- For differentiable f: Equivalent definition of *strongly convex with modulus of convexity* γ

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^\top(\mathbf{y} - \mathbf{x}) + \frac{\gamma}{2}\|\mathbf{y} - \mathbf{x}\|^2.$$

4. Subgradient and subdifferential

- Definition: A vector $\mathbf{v} \in \mathbb{R}^n$ is a *subgradient* of $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ at a point $\mathbf{x}$ if for all $\mathbf{y} \in \mathbb{R}^n$, it holds

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \mathbf{v}^\top(\mathbf{y} - \mathbf{x}).$$

The *subdifferential*, denoted $\partial f(\mathbf{x})$, is the set of all subgradients of f at $\mathbf{x}$. (see FOMO for concrete examples)

Lemma 3 (Monotonicity of subdifferentials)

For all $\mathbf{a} \in \partial f(\mathbf{x})$ *and* $\mathbf{b} \in \partial f(\mathbf{y})$, *we have* $(\mathbf{a} - \mathbf{b})^\top(\mathbf{x} - \mathbf{y}) \geq 0$.

Proof. By the definition of subgradient, we have

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \mathbf{a}^\top(\mathbf{y} - \mathbf{x}) \quad \text{and} \quad f(\mathbf{x}) \geq f(\mathbf{y}) + \mathbf{b}^\top(\mathbf{x} - \mathbf{y}).$$

Adding these two inequalities yields the statement. □

Theorem 4 (Fermat's lemma: generalization in convex functions)

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be a proper convex function. Then the point* $\mathbf{x}_\star$ *is a minimizer of* $f(\mathbf{x})$ *if and only if*

$$\mathbf{0} \in \partial f(\mathbf{x}_\star).$$

Theorem 5

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be proper convex, and let* $\mathbf{x} \in \operatorname{int}(\operatorname{dom}(f))$.

(i) *If* f *is differentiable at* $\mathbf{x}$, *then* $\partial f(\mathbf{x}) = \{\nabla f(\mathbf{x})\}$.

(ii) *If* $\partial f(\mathbf{x})$ *is a singleton (a set containing a single vector), then* f *is differentiable at* $\mathbf{x}$ *with gradient equal to the unique subgradient.*

- If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable and convex, then

$$\partial f(\mathbf{x}) = \{\nabla f(\mathbf{x})\} \quad \text{and} \quad f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x})$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

5. Taylor's theorem

- Taylor's theorem shows how smooth functions can be locally approximated by low-order (e.g., linear or quadratic) functions.

定理 12.3.1(Taylor 公式) 设 $f(x,y)$ 在点 (x_0,y_0) 的邻域 $U=O((x_0,y_0),r)$ 上具有 $k+1$ 阶连续偏导数,那么对于 U 内每一点 $(x_0+\Delta x, y_0+\Delta y)$ 都成立

$$
\begin{aligned}
& f(x_0+\Delta x, y_0+\Delta y) \\
= & f(x_0,y_0) + \left(\Delta x\frac{\partial}{\partial x} + \Delta y\frac{\partial}{\partial y}\right)f(x_0,y_0) \\
& + \frac{1}{2!}\left(\Delta x\frac{\partial}{\partial x} + \Delta y\frac{\partial}{\partial y}\right)^2 f(x_0,y_0) + \cdots \\
& + \frac{1}{k!}\left(\Delta x\frac{\partial}{\partial x} + \Delta y\frac{\partial}{\partial y}\right)^k f(x_0,y_0) + R_k,
\end{aligned}
$$

其中 $R_k = \dfrac{1}{(k+1)!}\left(\Delta x\dfrac{\partial}{\partial x} + \Delta y\dfrac{\partial}{\partial y}\right)^{k+1} f(x_0+\theta\Delta x, y_0+\theta\Delta y)(0<\theta<1)$ 称为 **Lagrange 余项**.

Theorem 6 (Taylor's theorem)

Given a continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, and given $\mathbf{x}, \mathbf{p} \in \mathbb{R}^n$, we have that

$$f(\mathbf{x}+\mathbf{p}) = f(\mathbf{x}) + \int_0^1 \nabla f(\mathbf{x}+t\mathbf{p})^\top \mathbf{p} \mathrm{d}t,$$

$$f(\mathbf{x}+\mathbf{p}) = f(\mathbf{x}) + \nabla f(\mathbf{x}+\xi\mathbf{p})^\top \mathbf{p}, \ \textit{for some } \xi \in (0,1).$$

If f is twice continuously differentiable, we have

$$\nabla f(\mathbf{x}+\mathbf{p}) = \nabla f(\mathbf{x}) + \int_0^1 \nabla^2 f(\mathbf{x}+t\mathbf{p})\mathbf{p} \mathrm{d}t,$$

$$f(\mathbf{x}+\mathbf{p}) = f(\mathbf{x}) + \nabla f(\mathbf{x})^\top \mathbf{p} + \frac{1}{2}\mathbf{p}^\top \nabla^2 f(\mathbf{x}+\xi\mathbf{p})\mathbf{p},$$

for some $\xi \in (0,1)$.

- Lipschitz continuously differentiable with constant L:

$$\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\| \leq L\|\mathbf{x} - \mathbf{y}\|, \text{ for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

  By Taylor's theorem, we have

$$f(\mathbf{y}) - f(\mathbf{x}) - \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x}) \leq \frac{L}{2}\|\mathbf{y} - \mathbf{x}\|^2$$

  for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Lemma 7

Suppose f is strongly convex with modulus of convexity γ and Lipschitz continuously differentiable with constant L. We have $\forall\ \mathbf{x}, \mathbf{y}$ that

$$\frac{\gamma}{2}\|\mathbf{y} - \mathbf{x}\|^2 \leq f(\mathbf{y}) - f(\mathbf{x}) - \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x}) \leq \frac{L}{2}\|\mathbf{y} - \mathbf{x}\|^2.$$

- If further f is twice continuously differentiable, then for all $\mathbf{x}$, the inequalities in Lemma 7 is equivalent to $\gamma \mathbf{I} \preceq \nabla^2 f(\mathbf{x}) \preceq L\mathbf{I}$.

Theorem 8

Let f be differentiable and strongly convex with modulus of convexity $\gamma > 0$. Then the minimizer $\mathbf{x}_\star$ of f exists and is unique.

Proof. (i) Compactness of level set: Show that for any point $\mathbf{x}^0$, the level set

$$\{\mathbf{x} \mid f(\mathbf{x}) \leq f(\mathbf{x}^0)\}$$

is closed and bounded, and hence compact.

(ii) Existence: Since f is continuous, it attains its minimum on the compact level set, which is also the solution of $\min_{\mathbf{x}} f(\mathbf{x})$.

(iii) Uniqueness: Suppose for contradiction that the minimizer is not unique, so that we have two points $\mathbf{x}_\star^1$ and $\mathbf{x}_\star^2$ that minimize f. By using the strongly convex property, we can prove

$$f\left(\frac{\mathbf{x}_\star^1 + \mathbf{x}_\star^2}{2}\right) < f(\mathbf{x}_\star^1) = f(\mathbf{x}_\star^2).$$

This is a contradiction. □

6. Optimality conditions for smooth functions

Theorem 9 (First-order necessary condition)

If $\mathbf{x}_\star$ is a local minimizer of f and f is continuously differentiable in an open neighborhood of $\mathbf{x}_\star$, then $\nabla f(\mathbf{x}_\star) = \mathbf{0}$.

Proof. Suppose for contradiction that $\nabla f(\mathbf{x}_\star) \neq \mathbf{0}$. Define the vector $\mathbf{p} = -\nabla f(\mathbf{x}_\star)$ and note that $\mathbf{p}^\top \nabla f(\mathbf{x}_\star) = -\|\nabla f(\mathbf{x}_\star)\|^2 < 0$. Because ∇f is continuous near $\mathbf{x}_\star$, there is a scalar $T > 0$ such that

$$\mathbf{p}^\top \nabla f(\mathbf{x}_\star + t\mathbf{p}) < 0, \quad \text{for all} \quad t \in [0, T].$$

For any $s \in (0, T]$, we have by Taylor's theorem that

$$f(\mathbf{x}_\star + s\mathbf{p}) = f(\mathbf{x}_\star) + s\mathbf{p}^\top \nabla f(\mathbf{x}_\star + \xi s\mathbf{p}) \quad \text{for some} \quad \xi \in (0, 1).$$

Therefore, $f(\mathbf{x}_\star + s\mathbf{p}) < f(\mathbf{x}_\star)$ for all $s \in (0, T]$. We have found a direction leading away from $\mathbf{x}_\star$ along which f decreases, so $\mathbf{x}_\star$ is not a local minimizer, and we have a contradiction. □

Theorem 10 (Second-order necessary conditions)

If $\mathbf{x}_\star$ is a local minimizer of f and $\nabla^2 f$ is continuous in an open neighborhood of $\mathbf{x}_\star$, then $\nabla f(\mathbf{x}_\star) = \mathbf{0}$ and $\nabla^2 f(\mathbf{x}_\star)$ is positive semidefinite.

Proof. We know from Theorem 9 that $\nabla f(\mathbf{x}_\star) = \mathbf{0}$. Assume that $\nabla^2 f(\mathbf{x}_\star)$ is not positive semidefinite. Then we can choose a vector $\mathbf{p}$ such that $\mathbf{p}^\top \nabla^2 f(\mathbf{x}_\star)\mathbf{p} < 0$, and because $\nabla^2 f$ is continuous near $\mathbf{x}_\star$, there is a scalar $T > 0$ such that

$$\mathbf{p}^\top \nabla^2 f(\mathbf{x}_\star + t\mathbf{p})\mathbf{p} < 0, \quad \text{for all} \quad t \in [0, T].$$

By doing a Taylor series expansion around $\mathbf{x}_\star$, we have for all $s \in (0, T]$ and some $\xi \in (0, 1)$ that

$$f(\mathbf{x}_\star + s\mathbf{p}) = f(\mathbf{x}_\star) + s\mathbf{p}^\top \nabla f(\mathbf{x}_\star) + \frac{1}{2} s^2 \mathbf{p}^\top \nabla^2 f(\mathbf{x}_\star + \xi s\mathbf{p})\mathbf{p} < f(\mathbf{x}_\star).$$

As in Theorem 9, we have found a direction from $\mathbf{x}_\star$ along which f is decreasing, and so again, $\mathbf{x}_\star$ is not a local minimizer. $\square$

Theorem 11 (Second-order sufficient conditions)

Suppose that $\nabla^2 f$ is continuous in an open neighborhood of $\mathbf{x}_\star$ and that $\nabla f(\mathbf{x}_\star) = \mathbf{0}$ and $\nabla^2 f(\mathbf{x}_\star)$ is positive definite. Then $\mathbf{x}_\star$ is a strict local minimizer of f.

Proof. Because the Hessian $\nabla^2 f$ is continuous and positive definite at $\mathbf{x}_\star$, we can choose a radius $r > 0$ so that $\nabla^2 f(\mathbf{x})$ remains positive definite for all $\mathbf{x}$ in the open ball $\mathcal{B} = \{\mathbf{z} \mid \|\mathbf{z} - \mathbf{x}_\star\| < r\}$. Taking any nonzero vector $\mathbf{p}$ with $\|\mathbf{p}\| < r$, we have $\mathbf{x}_\star + \mathbf{p} \in \mathcal{B}$ and

$$\begin{aligned} f(\mathbf{x}_\star + \mathbf{p}) &= f(\mathbf{x}_\star) + \mathbf{p}^\top \nabla f(\mathbf{x}_\star) + \frac{1}{2}\mathbf{p}^\top \nabla^2 f(\mathbf{x}_\star + \xi\mathbf{p})\mathbf{p} \\ &= f(\mathbf{x}_\star) + \frac{1}{2}\mathbf{p}^\top \nabla^2 f(\mathbf{x}_\star + \xi\mathbf{p})\mathbf{p}, \end{aligned}$$

for some $\xi \in (0, 1)$. Since $\mathbf{x}_\star + \xi\mathbf{p} \in \mathcal{B}$, we have $\mathbf{p}^\top \nabla^2 f(\mathbf{x}_\star + \xi\mathbf{p})\mathbf{p} > 0$, and therefore $f(\mathbf{x}_\star + \mathbf{p}) > f(\mathbf{x}_\star)$, giving the result. □

- A point $\mathbf{x}$ is called a *stationary point* if

$$\nabla f(\mathbf{x}) = \mathbf{0}.$$

- A stationary point $\mathbf{x}$ is called a *saddle point* if there exist $\mathbf{u}$ and $\mathbf{v}$ such that

$$f(\mathbf{x} + \alpha\mathbf{u}) < f(\mathbf{x}) \quad \text{and} \quad f(\mathbf{x} + \alpha\mathbf{v}) > f(\mathbf{x})$$

  for all sufficiently small $\alpha > 0$.
- Stationary points are not necessarily local minimizers. Stationary points can be *local maximizers* or *saddle points*.
- If $\nabla f(\mathbf{x}) = \mathbf{0}$, and $\nabla^2 f(\mathbf{x})$ has both strictly positive and strictly negative eigenvalues, then $\mathbf{x}$ is a saddle point.
- If $\nabla^2 f(\mathbf{x})$ is positive semidefinite or negative semidefinite, then $\nabla^2 f(\mathbf{x})$ alone is insufficient to classify $\mathbf{x}$.

Theorem 12

(i) $\forall$ *convex* f*, any local minimizer* $\mathbf{x}_\star$ *is a global minimizer of* f*.*

(ii) *If* f *is convex and differentiable, then any stationary point* $\mathbf{x}_\star$ *is a global minimizer of* f*.*

Proof. (i) Suppose that $\mathbf{x}_\star$ is a local but not a global minimizer. Then we can find a point $\mathbf{z} \in \mathbb{R}^n$ with $f(\mathbf{z}) < f(\mathbf{x}_\star)$. Consider the line segment that joins $\mathbf{x}_\star$ to $\mathbf{z}$, that is,

$$\mathbf{x} = \lambda \mathbf{z} + (1-\lambda)\mathbf{x}_\star, \quad \text{for some} \quad \lambda \in (0,1].$$

By the convexity property for f, we have

$$f(\mathbf{x}) \leq \lambda f(\mathbf{z}) + (1-\lambda) f(\mathbf{x}_\star) < f(\mathbf{x}_\star).$$

Any neighborhood $\mathcal{N}$ of $\mathbf{x}_\star$ contains a piece of the line segment, so there will always be points $\mathbf{x} \in \mathcal{N}$ at which the last inequality is satisfied. Hence, $\mathbf{x}_\star$ is not a local minimizer.

(ii) Suppose that $\mathbf{x}_\star$ is not a global minimizer and choose $\mathbf{z}$ as above. Then, from convexity, we have

$$\begin{aligned}
\nabla f(\mathbf{x}_\star)^\top(\mathbf{z}-\mathbf{x}_\star) &= \frac{\mathrm{d}}{\mathrm{d}\lambda} f(\mathbf{x}_\star + \lambda(\mathbf{z}-\mathbf{x}_\star))|_{\lambda=0} \\
&= \lim_{\lambda\to 0^+} \frac{f(\mathbf{x}_\star + \lambda(\mathbf{z}-\mathbf{x}_\star)) - f(\mathbf{x}_\star)}{\lambda} \\
&\leq \lim_{\lambda\to 0^+} \frac{\lambda f(\mathbf{z}) + (1-\lambda) f(\mathbf{x}_\star) - f(\mathbf{x}_\star)}{\lambda} \\
&= f(\mathbf{z}) - f(\mathbf{x}_\star) < 0.
\end{aligned}$$

Therefore, $\nabla f(\mathbf{x}_\star) \neq \mathbf{0}$, and so $\mathbf{x}_\star$ is not a stationary point. □

- *Remark*: Theorems 9-12 provide the foundations for unconstrained optimization algorithms.
- Numerical algorithms try to seek a point where ∇f vanishes.

7. Karush–Kuhn–Tucker conditions

Theorem 13 (KKT conditions)

Consider the minimization problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x}) \quad s.t. \quad g_i(\mathbf{x}) \le 0, \quad i = 1:m,$$

where $f : \mathbb{R}^n \to \mathbb{R}$ *and all* $g_i : \mathbb{R}^n \to \mathbb{R}$ *are convex functions.*

- *Let* $\mathbf{x}_\star$ *be an optimal solution and assume Slater's condition*

  $$\exists\ \mathbf{x} \in \mathbb{R}^n, \quad s.t. \quad g_i(\mathbf{x}) < 0, \quad i = 1:m,$$

  hold. Then there exist $\lambda_1, \cdots, \lambda_m \ge 0$ *satisfying*

  $$\mathbf{0} \in \partial f(\mathbf{x}_\star) + \sum_{i=1}^{m} \lambda_i \partial g_i(\mathbf{x}_\star), \quad \lambda_i g_i(\mathbf{x}_\star) = 0, \quad i = 1:m.$$

- *If* $\mathbf{x}_\star$ *satisfies the above conditions, called* KKT conditions, *then it is an optimal solution of the optimization problem.*

8. Proximal operator

- For a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the *proximal operator* of f is

$$\operatorname{prox}_f(\mathbf{x}) := \operatorname*{argmin}_{\mathbf{u}} \left\{ f(\mathbf{u}) + \frac{1}{2}\|\mathbf{u} - \mathbf{x}\|^2 \right\}.$$

- For a closed proper convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, from the optimality condition (see Theorem 4), we have

$$\mathbf{0} \in \partial f(\operatorname{prox}_f(\mathbf{x})) + (\operatorname{prox}_f(\mathbf{x}) - \mathbf{x}).$$

- For a closed proper convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the point $\mathbf{x}_\star$ is a minimizer of f if and only if

$$\mathbf{x}_\star = \operatorname{prox}_f(\mathbf{x}_\star).$$

Lemma 14 (Nonexpansivity of proximal operator)

For a closed proper convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, *we have for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$\|\mathrm{prox}_f(\mathbf{x}) - \mathrm{prox}_f(\mathbf{y})\| \le \|\mathbf{x} - \mathbf{y}\|.$$

Proof. From the optimality conditions at two points $\mathbf{x}$ and $\mathbf{y}$, we have

$$\mathbf{x} - \mathrm{prox}_f(\mathbf{x}) \in \partial f(\mathrm{prox}_f(\mathbf{x})) \quad \text{and} \quad \mathbf{y} - \mathrm{prox}_f(\mathbf{y}) \in \partial f(\mathrm{prox}_f(\mathbf{y})).$$

By applying monotonicity (see Lemma 3), we have

$$((\mathbf{x} - \mathrm{prox}_f(\mathbf{x})) - (\mathbf{y} - \mathrm{prox}_f(\mathbf{y})))^\top (\mathrm{prox}_f(\mathbf{x}) - \mathrm{prox}_f(\mathbf{y})) \ge 0.$$

Rearranging this and applying the Cauchy–Schwartz inequality yields

$$\begin{aligned}\|\mathrm{prox}_f(\mathbf{x}) - \mathrm{prox}_f(\mathbf{y})\|^2 &\le (\mathbf{x} - \mathbf{y})^\top (\mathrm{prox}_f(\mathbf{x}) - \mathrm{prox}_f(\mathbf{y})) \\ &\le \|\mathbf{x} - \mathbf{y}\| \|\mathrm{prox}_f(\mathbf{x}) - \mathrm{prox}_f(\mathbf{y})\|. \qquad \square\end{aligned}$$

- Examples of several proximal operators

(1) $f(\mathbf{x}) = 0$:

$$\operatorname{prox}_f(\mathbf{x}) = \mathbf{x}.$$

(2) $f(\mathbf{x}) = \lambda\|\mathbf{x}\|_1$ with $\lambda > 0$:

$$\begin{aligned}
\left[\operatorname{prox}_{\lambda\|\cdot\|_1}(\mathbf{x})\right]_i &= \operatorname*{argmin}_{u\in\mathbb{R}} \left\{\lambda|u| + \frac{1}{2}(u - x_i)^2\right\} \\
&= \begin{cases} x_i - \lambda & \text{if } x_i > \lambda, \\ 0 & \text{if } x_i \in [-\lambda, \lambda], \\ x_i + \lambda & \text{if } x_i < -\lambda, \end{cases}
\end{aligned}$$

which is known as *soft-thresholding*.

(3) $f(\mathbf{x}) = \lambda\|\mathbf{x}\|_0$: the number of nonzero components, non-convex

$$\left[\mathrm{prox}_{\lambda\|\cdot\|_0}(\mathbf{x})\right]_i = \begin{cases} x_i & \text{if } |x_i| > \sqrt{2\lambda}, \\ \{0, x_i\} & \text{if } |x_i| = \sqrt{2\lambda}, \\ 0 & \text{if } |x_i| < \sqrt{2\lambda}, \end{cases}$$

which is known as *hard-thresholding.*

(4) $f(\mathbf{x}) = \mathrm{I}_\Omega(\mathbf{x})$:

$$\mathrm{prox}_{\mathrm{I}_\Omega}(\mathbf{x}) = \underset{\mathbf{u}}{\mathrm{argmin}} \left\{ \mathrm{I}_\Omega(\mathbf{u}) + \frac{1}{2}\|\mathbf{u} - \mathbf{x}\|^2 \right\} = \underset{\mathbf{u}\in\Omega}{\mathrm{argmin}}\, \|\mathbf{u} - \mathbf{x}\|,$$

which is simply the projection of $\mathbf{x}$ onto the set Ω. This shows the nonexpansivity of projection operator.