

# Numerical Linear Algebra Assignment 4

## Exercise 1. (10 points)

Let  $\mathbf{x} \in \mathbb{C}^m$  and  $x_1 = \mathbf{e}_1^\top \mathbf{x} \neq 0$ . Show that the matrix

$$\mathbf{H} = \mathbf{I} - 2 \frac{\mathbf{v}\mathbf{v}^*}{\mathbf{v}^*\mathbf{v}}, \quad \mathbf{v} = \pm \frac{x_1}{|x_1|} \|\mathbf{x}\|_2 \mathbf{e}_1 - \mathbf{x},$$

satisfies that

$$\mathbf{H}\mathbf{x} = \pm \frac{x_1}{|x_1|} \|\mathbf{x}\|_2 \mathbf{e}_1.$$

## Exercise 2. (10 points)

Prove that  $\mathbf{I} - \frac{\mathbf{v}\mathbf{v}^*}{\mathbf{v}^*\mathbf{v}}$  is the orthogonal projector which projects  $\mathbb{C}^m$  onto the *hyperplane*  $\text{span}\{\mathbf{v}\}^\perp$  along  $\text{span}\{\mathbf{v}\}$ .

## Exercise 3. (10 points)

Let  $\mathbf{x} = [3 \ 4 \ 0]^\top$  and  $\mathbf{e}_3 = [0 \ 0 \ 1]^\top$ . Try to find all possible numbers  $z \in \mathbb{C}$  and the corresponding Householder reflectors  $\mathbf{H}$  such that  $\mathbf{H}\mathbf{x} = z\mathbf{e}_3$ .

## Exercise 4. (10 points)

Let  $m > n$ ,  $\mathbf{A} \in \mathbb{C}^{m \times n}$  of rank  $n$ ,  $\mathbf{b} \in \mathbb{C}^m$ ,  $\mathbf{b} \notin \text{range}(\mathbf{A})$  and  $\mathbf{QR} = [\mathbf{A} \ \mathbf{b}]$  (i.e., full QR factorization of  $[\mathbf{A} \ \mathbf{b}]$ ). Show that (we use MATLAB's notation for convenience)

$$\min_{\mathbf{x} \in \mathbb{C}^n} \|\mathbf{b} - \mathbf{Ax}\|_2 = |\mathbf{R}(n+1, n+1)|,$$

and the least squares solution is given by

$$\mathbf{x} = \mathbf{R}(1:n, 1:n) \setminus \mathbf{R}(1:n, n+1).$$

## Exercise 5. (TreBau Exercise 19.1, 10 points)

Given  $\mathbf{A} \in \mathbb{C}^{m \times n}$  of rank  $n$  and  $\mathbf{b} \in \mathbb{C}^m$ , consider the block  $2 \times 2$  system of equations

$$\begin{bmatrix} \mathbf{I} & \mathbf{A} \\ \mathbf{A}^* & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix},$$

where  $\mathbf{I}$  is the  $m \times m$  identity. Show that this system has a unique solution  $\begin{bmatrix} \mathbf{r} \\ \mathbf{x} \end{bmatrix}$ , and that the vectors  $\mathbf{r}$  and  $\mathbf{x}$  are the residual and the least squares solution of the least squares problem: Given  $\mathbf{A} \in \mathbb{C}^{m \times n}$  of full rank,  $m \geq n$ ,  $\mathbf{b} \in \mathbb{C}^m$ , find  $\mathbf{x} \in \mathbb{C}^n$  such that  $\|\mathbf{b} - \mathbf{Ax}\|_2$  is minimized.

## Exercise 6. (Demmel Question 3.11, 10 points)

Let  $\mathbf{A} \in \mathbb{C}^{m \times n}$ . Show that  $\mathbf{X} = \mathbf{A}^\dagger$  (the Moore–Penrose pseudoinverse) minimizes  $\|\mathbf{AX} - \mathbf{I}\|_F$  over all  $n \times m$  matrices. What is the value (positive square root of some integer) of this minimum?

## Exercise 7. (10 points)

Let  $\mathbf{A} \in \mathbb{C}^{m \times n}$  and  $\mathbf{b} \in \mathbb{C}^m$ . Solve the *penalized* problem

$$\min_{\mathbf{x} \in \mathbb{C}^n} \left\{ \|\mathbf{b} - \mathbf{Ax}\|_2^2 + \lambda^2 \|\mathbf{x}\|_2^2 \right\},$$

where  $\lambda > 0$ . Hint: consider the LSP

$$\min_{\mathbf{x} \in \mathbb{C}^n} \left\| \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{A} \\ \lambda \mathbf{I} \end{bmatrix} \mathbf{x} \right\|_2.$$

### Exercise 8. (10 points)

Suppose a matrix  $\mathbf{A} \in \mathbb{R}^{m \times n}$  satisfies

$$\mathbf{A} = \mathbf{U}\mathbf{V}, \quad \mathbf{U} \in \mathbb{R}^{m \times l}, \quad \mathbf{V} \in \mathbb{R}^{l \times n}.$$

Prove the following statements:

- (i) If  $\text{rank}(\mathbf{U}) = \text{rank}(\mathbf{V}) = l$  (i.e.,  $\mathbf{A} = \mathbf{U}\mathbf{V}$  is a full-rank factorization), then  $\mathbf{A}^\dagger = \mathbf{V}^\dagger\mathbf{U}^\dagger$ .
- (ii) For all  $\mathbf{b} \in \text{range}(\mathbf{A})$ , if  $\text{rank}(\mathbf{U}) = l$  and  $\text{rank}(\mathbf{V}) = n$ , then  $\mathbf{A}^\dagger\mathbf{b} = \mathbf{V}^\dagger\mathbf{U}^\dagger\mathbf{b}$ .

### Exercise 9. (10 points)

For a given  $\mathbf{b} \in \mathbb{C}^m$  and any  $\mathbf{A} \in \mathbb{C}^{m \times n}$ , let  $\mathbf{y}$  be the closest point (we use  $\|\cdot\|_2$ ) to  $\mathbf{b}$  in  $\text{range}(\mathbf{A})$ . Prove that  $\mathbf{y}$  is located on the sphere of radius  $\|\mathbf{b}\|_2/2$  centered at  $\mathbf{b}/2$ .

### Exercise 10. (10 points)

Let  $\mathbf{A} \in \mathbb{C}^{m \times n}$ ,  $\mathbf{b} \in \mathbb{C}^n$ , and  $\mathbf{y} \in \mathbb{C}^m$  be given. Assume that  $\text{rank}(\mathbf{A}) < \min\{m, n\}$  and  $\mathbf{y} \in \text{range}(\mathbf{A})$ . Solve the following problem

$$\min_{\mathbf{x} \in \mathbb{C}^n, \mathbf{Ax}=\mathbf{y}} \|\mathbf{b} - \mathbf{x}\|_2.$$

### Exercise 11. (Programming, TreBau Exercises 10.2, 10 points)

### Exercise 12. (Programming, TreBau Exercises 10.3, 10 points)

### Additional Exercise 1. (Carl D. Meyer)

For a vector  $\mathbf{x} \in \mathbb{C}^m$  with  $\|\mathbf{x}\|_2 = 1$ , partition  $\mathbf{x}$  as  $\mathbf{x} = \begin{bmatrix} x_1 \\ \tilde{\mathbf{x}} \end{bmatrix}$ , where  $\tilde{\mathbf{x}} \in \mathbb{C}^{m-1}$ . Show that if  $|x_1| \neq 0, 1$ , and if

$$\alpha = \frac{1}{1 - |x_1|} \quad \text{and} \quad \beta = \frac{x_1}{|x_1|},$$

then for some  $\mathbf{v} \in \mathbb{C}^m$ ,

$$\mathbf{Q} = \begin{bmatrix} x_1 & \beta^2 \tilde{\mathbf{x}}^* \\ \tilde{\mathbf{x}} & \beta(\mathbf{I} - \alpha \tilde{\mathbf{x}} \tilde{\mathbf{x}}^*) \end{bmatrix} = \beta \left( \mathbf{I} - 2 \frac{\mathbf{v} \mathbf{v}^*}{\mathbf{v}^* \mathbf{v}} \right).$$

This result provides an easy way to extend a vector  $\mathbf{x}$  to a complete orthonormal set.