Numerical Linear Algebra Assignment 4

Exercise 1. (10 points)

Let $\mathbf{x} = \begin{bmatrix} 3 & 4 & 0 \end{bmatrix}^\top$ and $\mathbf{e}_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top$. Try to find all possible numbers $z \in \mathbb{C}$ and the corresponding Householder reflectors $\mathbf{H}$ such that $\mathbf{H}\mathbf{x} = z\mathbf{e}_3$.

Exercise 2. (TreBau Exercise 19.1, 10 points)

Given $\mathbf{A} \in \mathbb{C}^{m\times n}$ of rank n and $\mathbf{b} \in \mathbb{C}^m$, consider the block 2×2 system of equations

$$\begin{bmatrix} \mathbf{I} & \mathbf{A} \\ \mathbf{A}^* & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix},$$

where $\mathbf{I}$ is the $m \times m$ identity. Show that this system has a unique solution $\begin{bmatrix} \mathbf{r} \\ \mathbf{x} \end{bmatrix}$, and that the vectors $\mathbf{r}$ and $\mathbf{x}$ are the residual and the least squares solution of the least squares problem: Given $\mathbf{A} \in \mathbb{C}^{m\times n}$ of full rank, $m \geq n$, $\mathbf{b} \in \mathbb{C}^m$, find $\mathbf{x} \in \mathbb{C}^n$ such that $\|\mathbf{b} - \mathbf{A}\mathbf{x}\|_2$ is minimized.

Exercise 3. (Demmel Question 3.11, 10 points)

Let $\mathbf{A} \in \mathbb{C}^{m\times n}$. Show that $\mathbf{X} = \mathbf{A}^\dagger$ (the Moore–Penrose pseudoinverse) minimizes $\|\mathbf{A}\mathbf{X} - \mathbf{I}\|_\mathrm{F}$ over all $n \times m$ matrices. What is the value (an integer) of this minimum?

Exercise 4. (10 points)

Let $\mathbf{A} \in \mathbb{C}^{m\times n}$ and $\mathbf{b} \in \mathbb{C}^m$. Solve the *penalized* problem

$$\min_{\mathbf{x}\in\mathbb{C}^n} \left\{ \|\mathbf{b} - \mathbf{A}\mathbf{x}\|_2^2 + \lambda^2 \|\mathbf{x}\|_2^2 \right\},$$

where $\lambda > 0$. Hint: consider the LSP

$$\min_{\mathbf{x}\in\mathbb{C}^n} \left\| \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{A} \\ \lambda\mathbf{I} \end{bmatrix} \mathbf{x} \right\|_2.$$

Exercise 5. (10 points)

Let $\mathbf{A} \in \mathbb{C}^{m\times n}$, $\mathbf{b} \in \mathbb{C}^n$, and $\mathbf{y} \in \mathbb{C}^m$ be given. Assume that $\operatorname{rank}(\mathbf{A}) < \min\{m, n\}$ and $\mathbf{y} \in \operatorname{range}(\mathbf{A})$. Solve the following problem

$$\min_{\mathbf{x}\in\mathbb{C}^n,\ \mathbf{A}\mathbf{x}=\mathbf{y}} \|\mathbf{b} - \mathbf{x}\|_2.$$

Exercise 6. (Programming, TreBau Exercises 10.2, 10 points)

Exercise 7. (Programming, TreBau Exercises 10.3, 10 points)