# Numerical Linear Algebra Assignment 7

## Exercise 1. (10 points)

Prove that an eigenvalue is multiple if and only if it has a pair of orthogonal left and right eigenvectors.

## Exercise 2. (10 points)

Let $\lambda_1, \cdots \lambda_m$ be the $m$ eigenvalues of $\mathbf{A} \in \mathbb{C}^{m\times m}$. Let

$$\mathbf{M} = \frac{\mathbf{A}+\mathbf{A}^*}{2}, \quad \mathbf{N} = \frac{\mathbf{A}-\mathbf{A}^*}{2}.$$

Prove that

$$\sum_{i=1}^{m} |\lambda_i|^2 \leq \|\mathbf{A}\|_{\mathrm{F}}^2, \quad \sum_{i=1}^{m} |\mathrm{Re}\lambda_i|^2 \leq \|\mathbf{M}\|_{\mathrm{F}}^2, \quad \sum_{i=1}^{m} |\mathrm{Im}\lambda_i|^2 \leq \|\mathbf{N}\|_{\mathrm{F}}^2.$$

## Exercise 3. (TreBau Exercise 24.2, 10 points)

Gerschgorin's theorem: Every eigenvalue of $\mathbf{A} \in \mathbb{C}^{m\times m}$ lies in at least one of the $m$ circular disks in the complex plane with centers $a_{ii}$ and radii $\sum_{j\neq i} |a_{ij}|$. Moreover, if $n$ of these disks form a connected domain that is disjoint from the other $m-n$ disks, then there are precisely $n$ eigenvalues of $\mathbf{A}$ within this domain.

(a) Prove the first part of Gerschgorin's theorem. (Hint: Let $\lambda$ be any eigenvalue of $\mathbf{A}$, and $\mathbf{x}$ a corresponding eigenvector with largest entry 1.)

(b) Prove the second part. (Hint: Deform $\mathbf{A}$ to a diagonal matrix and use the fact that the eigenvalues of a matrix are continuous functions of its entries.)

(c) Give estimates based on Gerschgorin's theorem for the eigenvalues of

$$\mathbf{A} = \begin{bmatrix} 8 & 1 & 0 \\ 1 & 4 & \varepsilon \\ 0 & \varepsilon & 1 \end{bmatrix}, \qquad |\varepsilon| < 1.$$

(d) Find a way to establish the tighter bound $|\lambda_3 - 1| \leq \varepsilon^2$ on the smallest eigenvalue of $\mathbf{A}$. (Hint: Consider diagonal similarity tranformations.)

## Exercise 4. (TreBau Exercise 25.1, 10 points)

(a) Let $\mathbf{A} \in \mathbb{C}^{m\times m}$ be tridiagonal and Hermitian, with all its sub- and superdiagonal entries nonzero. Prove that the eigenvalues of $\mathbf{A}$ are distinct. (Hint: Show that for any $\lambda \in \mathbb{C}$, $\mathbf{A} - \lambda\mathbf{I}$ has rank at least $m-1$.)

(b) On the other hand, let $\mathbf{A}$ be upper-Hessenberg, with all its subdiagonal entries nonzero. Give an example that shows that the eigenvalues of $\mathbf{A}$ are not necessarily distinct.

## Exercise 5. (10 points)

A subspace $\mathbb{X}$ of $\mathbb{C}^n$ is an *invariant subspace* for $\mathbf{A}$ if $\mathbf{A}\mathbb{X} \subseteq \mathbb{X}$. Let the columns of the matrix $\mathbf{X} \in \mathbb{C}^{n\times p}$ form a basis for $\mathbb{X}$. Prove:

(a) $\mathbb{X}$ is an invariant subspace for $\mathbf{A}$ if and only if $\mathbf{AX} = \mathbf{XB}$ for some $\mathbf{B} \in \mathbb{C}^{p\times p}$.

(b) The $p$ eigenvalues of $\mathbf{B}$ are also eigenvalues of $\mathbf{A}$.

### Exercise 6. (10 points)

Suppose that $\mathbf{A}$ is normal and triangular. Show that $\mathbf{A}$ must be diagonal.

### Exercise 7. (10 points)

Let $\mathbf{A} \in \mathbb{C}^{m\times m}$, $\mathbf{B} \in \mathbb{C}^{n\times n}$ and $\mathbf{C} \in \mathbb{C}^{m\times n}$ be given. Prove that the Sylvester equation $\mathbf{AX}-\mathbf{XB} = \mathbf{C}$ has a unique solution $\mathbf{X} \in \mathbb{C}^{m\times n}$ if and only if $\Lambda(\mathbf{A}) \cap \Lambda(\mathbf{B}) = \emptyset$. (You MUST use Schur factorizations of $\mathbf{A}$ and $\mathbf{B}$ to prove the conclusion. Any other approach is NOT accepted.)

### Exercise 8. (10 points)

Let $\mathbf{H} \in \mathbb{C}^{m\times m}$ be an irreducible ($h_{i+1,i} \neq 0$ for $i = 1 : m-1$) upper Hessenberg matrix. Prove that any eigenvalue of $\mathbf{H}$ has geometric multiplicity 1.

**Compulsory requirement for programming: Use Matlab's `publish` to save all your code, comments, and results to a PDF file. You must use the programming format files: example_format.zip.**

### Programming 1. (TreBau Exercise 24.3, 10 points)

### Programming 2. (TreBau Exercise 26.2(a), 10 points)

### Programming 3. (10 points)

Design an algorithm to solve the Sylvester equation $\mathbf{AX}-\mathbf{XB} = \mathbf{C}$ (assume that $\Lambda(\mathbf{A})\cap\Lambda(\mathbf{B}) = \emptyset$). Test your code via a numerical experiment. Hint: Use the MATLAB command `schur`.