Numerical Linear Algebra Assignment 3

Exercise 1. (10 points)

(1) Let $\mathbf{P}$ be a projector. Given an explicit expression for the inverse of $\lambda\mathbf{I} - \mathbf{P}$, where $\lambda \neq 0, 1$.

(2) Suppose $\mathbf{A} \in \mathbb{C}^{m\times n}$ has a full SVD $\mathbf{A} = \mathbf{U\Sigma V}^*$, where

$$\mathbf{U} = [\mathbf{U}_r \quad \mathbf{U}_{\mathrm{c}}], \quad \mathbf{V} = [\mathbf{V}_r \quad \mathbf{V}_{\mathrm{c}}], \quad r = \mathrm{rank}(\mathbf{A}).$$

What are the orthogonal projections onto $\mathrm{null}(\mathbf{A})^\perp$, $\mathrm{null}(\mathbf{A})$, $\mathrm{range}(\mathbf{A})$ and $\mathrm{range}(\mathbf{A})^\perp$?

Exercise 2. (10 points)

Two subspaces $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathbb{C}^m$ are called *complementary subspaces* if they satisfy

$$\mathcal{S}_1 \cap \mathcal{S}_2 = \{\mathbf{0}\}, \qquad \mathcal{S}_1 + \mathcal{S}_2 = \mathbb{C}^m.$$

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be complementary subspaces. Prove that there exists a projector $\mathbf{P}$ with

$$\mathrm{range}(\mathbf{P}) = \mathcal{S}_1, \qquad \mathrm{null}(\mathbf{P}) = \mathcal{S}_2.$$

Exercise 3. (TreBau Exercise 6.1, 10 points)

If $\mathbf{P}$ is an orthogonal projector, then $\mathbf{I} - 2\mathbf{P}$ is unitary. Prove this algebraically, and give a geometric interpretation.

Exercise 4. (TreBau Exercise 6.5, 10 points)

Let $\mathbf{P} \in \mathbb{C}^{m\times m}$ be a nonzero projector. Show that $\|\mathbf{P}\|_2 \geq 1$, with equality if and only if $\mathbf{P}$ is an orthogonal projector.

Exercise 5. (10 points)

Compute a QR factorization of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ \sqrt{2} & 1+\sqrt{2} & 1 \\ 1 & 2 & 1 \end{bmatrix}$.

Exercise 6. (Programming, TreBau Exercise 8.2, 10 points)