

Numerical Linear Algebra Assignment 6

Exercise 1. (Shufang Xu, 10 points)

Consider the stationary iterative method

$$\mathbf{x}^{(m)} = \mathbf{R}\mathbf{x}^{(m-1)} + \mathbf{c}.$$

Assume $\mathbf{R} \in \mathbb{C}^{n \times n}$ and the spectral radius $\rho(\mathbf{R}) = 0$. For any given $\mathbf{c} \in \mathbb{R}^n$ and $\mathbf{x}^{(0)} \in \mathbb{C}^n$, prove that the n th iterate $\mathbf{x}^{(n)}$ is the solution of $(\mathbf{I} - \mathbf{R})\mathbf{x} = \mathbf{c}$.

Exercise 2. (Zhihao Cao, 10 points)

Consider the stationary iterative method

$$\mathbf{x}^{(m)} = \mathbf{R}\mathbf{x}^{(m-1)} + \mathbf{c}.$$

Assume that $\|\mathbf{R}\|_2 < 1$. Prove that

$$\|\mathbf{x}^{(m)} - \mathbf{x}_*\|_2 \leq \frac{\|\mathbf{R}\|_2^m}{1 - \|\mathbf{R}\|_2} \|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|_2,$$

where \mathbf{x}_* is the solution of $(\mathbf{I} - \mathbf{R})\mathbf{x} = \mathbf{c}$.

Exercise 3. (Shufang Xu, 10 points)

Consider the linear system $\mathbf{Ax} = \mathbf{b}$ with

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}.$$

Discuss the convergence of Jacobi's method and Gauss–Seidel method for this linear system.

Exercise 4. (10 points)

Prove that $0 < \omega < 2$ is required for the convergence of SOR(ω) for all starting vectors.

Exercise 5. (Programming, 10 points)

Construct a linear system $\mathbf{Ax} = \mathbf{b}$ with a strictly row diagonally dominant 15×15 matrix \mathbf{A} . Plot the convergence history of Jacobi's, Gauss–Seidel, SOR(0.5), and SSOR(0.5) methods in one figure. You must use MATLAB's `semilogy`: the x-axis is the iteration index m , and the y-axis is $\|\mathbf{x}^{(m)} - \mathbf{A}^{-1}\mathbf{b}\|_2$. For each method, stop at the 30th iteration.

Additional Exercise 1.

Let $\mathbf{A} \in \mathbb{C}^{m \times m}$. Prove that $\lim_{k \rightarrow \infty} \mathbf{A}^k = \mathbf{0} \Leftrightarrow \rho(\mathbf{A}) < 1 \Leftrightarrow \sum_{k=0}^{\infty} \mathbf{A}^k = (\mathbf{I} - \mathbf{A})^{-1}$.

Additional Exercise 2.

Let $\|\cdot\|$ be a given induced matrix norm. Let $\mathbf{A} \in \mathbb{C}^{m \times m}$. Prove that $\lim_{k \rightarrow \infty} \|\mathbf{A}^k\|^{\frac{1}{k}} = \rho(\mathbf{A})$.

Additional Exercise 3.

Let $\|\cdot\|$ be a given induced matrix norm. Assume that $\mathbf{A} \in \mathbb{C}^{m \times m}$ and $\|\mathbf{A}\| < 1$. Prove that $\mathbf{I} - \mathbf{A}$ is nonsingular and

$$\|(\mathbf{I} - \mathbf{A})^{-1}\| \leq \frac{1}{1 - \|\mathbf{A}\|}.$$