Numerical Linear Algebra Assignment 2

Exercise 1. (10 points)

Show that if $\mathbf{A} \in \mathbb{C}^{m\times n}$, then the maximum singular value of $\mathbf{A}$ satisfies

$$\sigma_{\max}(\mathbf{A}) = \max_{\mathbf{0}\neq\mathbf{x}\in\mathbb{C}^n,\mathbf{0}\neq\mathbf{y}\in\mathbb{C}^m} \frac{|\mathbf{y}^*\mathbf{A}\mathbf{x}|}{\|\mathbf{x}\|_2\|\mathbf{y}\|_2}.$$

(Hint: Cauchy–Schwarz inequality and $\|\mathbf{A}\mathbf{x}\|_2 \leq \|\mathbf{A}\|_2\|\mathbf{x}\|_2$.)

Exercise 2. (10 points)

For any $\mathbf{A} \in \mathbb{C}^{m\times l}$ and $\mathbf{B} \in \mathbb{C}^{l\times n}$, show that $\|\mathbf{A}\mathbf{B}\|_{\mathrm{F}} \leq \|\mathbf{A}\|_2\|\mathbf{B}\|_{\mathrm{F}}$. (Hint: $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^*$, and F-norm is unitarily invariant.)

Exercise 3. (TreBau Exercise 4.5, 10 points)

Show that every $\mathbf{A} \in \mathbb{R}^{m\times n}$ has a real SVD $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$ with $\mathbf{U} \in \mathbb{R}^{m\times m}$ and $\mathbf{V} \in \mathbb{R}^{n\times n}$. (Hint: the proof is almost the same as that given in the lecture for the complex case except additional arguments should be given to illustrate all vectors and matrices involved are real.)

Exercise 4. (10 points)

Let $\mathbf{A} \in \mathbb{R}^{m\times n}$. Define

$$\|\mathbf{A}\|_* = \sup_{\mathbf{X}\in\mathbb{R}^{m\times n},\|\mathbf{X}\|_2\leq 1} \mathrm{tr}(\mathbf{A}^\top\mathbf{X}).$$

Prove that

$$\|\mathbf{A}\|_* = \sum_{i=1}^{r} \sigma_i(\mathbf{A}),$$

where $\{\sigma_i(\mathbf{A})\}_{i=1}^r$ are the nonzero singular values of $\mathbf{A}$ and $r = \mathrm{rank}(\mathbf{A})$.

Exercise 5. (10 points)

Let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ be the singular values of the matrix $\mathbf{A} \in \mathbb{C}^{m\times n}$ $(m \geq n)$. Assume that $z \in \mathbb{C}$. Compute the singular values of the $(m+n)\times(m+n)$ matrix

$$\begin{bmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{A}^* & z\mathbf{I} \end{bmatrix}.$$

Exercise 6. (Programming, TreBau Exercise 9.3, 10 points)