

Numerical Linear Algebra Assignment 11

Exercise 1. (10 points)

Let $z \in \mathbb{C}$, $\mathbf{A} \in \mathbb{C}^{m \times m}$, and $\mathbf{B} = \mathbf{A} + z\mathbf{I}$. Prove the translation-invariance of Krylov subspaces, i.e., $\forall j \in \mathbb{N}$,

$$\mathcal{K}_j(\mathbf{A}, \mathbf{r}) = \mathcal{K}_j(\mathbf{B}, \mathbf{r}).$$

Exercise 2. (10 points)

If the minimal polynomial of the nonsingular matrix \mathbf{A} has degree n , then the solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$ lies in the space $\mathcal{K}_n(\mathbf{A}, \mathbf{b})$. (Hint: Let $q(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_{n-1} z^{n-1} + z^n$ denote the minimal polynomial of \mathbf{A} . Then $\alpha_0 \neq 0$.)

Exercise 3. (10 points)

Suppose the minimal polynomial of the matrix $\mathbf{A} \in \mathbb{C}^{m \times m}$ has degree n and the Arnoldi process for \mathbf{A} and a nonzero \mathbf{r} breaks down at step k , i.e., $h_{k+1,k} = 0$ is encountered. Prove the following

- (i) $k \leq n$.
- (ii) $\mathcal{K}_k(\mathbf{A}, \mathbf{r}) = \mathcal{K}_{k+1}(\mathbf{A}, \mathbf{r}) = \mathcal{K}_{k+2}(\mathbf{A}, \mathbf{r}) = \cdots$.
- (iii) Each eigenvalue of \mathbf{H}_k is an eigenvalue of \mathbf{A} .
- (iv) If \mathbf{A} is nonsingular, then the solution \mathbf{x} of $\mathbf{A}\mathbf{x} = \mathbf{r}$ lies in $\mathcal{K}_k(\mathbf{A}, \mathbf{r})$.

Exercise 4. (10 points)

Assume $c_0 \neq 0$. Let $\mathbf{r}_0 = \mathbf{e}_1$ and

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_{m-1} \\ -c_0 & -\mathbf{c}_{m-1} \end{bmatrix} \in \mathbb{C}^{m \times m}, \quad \mathbf{c}_{m-1} = [c_1 \quad c_2 \quad \cdots \quad c_{m-1}].$$

Prove that

$$\|\mathbf{r}_0\|_2 = \|\mathbf{r}_1\|_2 = \cdots = \|\mathbf{r}_{m-1}\|_2, \quad \|\mathbf{r}_m\|_2 = 0.$$

The above example implies that GMRES can completely stagnate, i.e., the residual norm can be nondecreasing at the first $m - 1$ steps, and “convergence” occurs in the last step.

Exercise 5. (Programming, 10 points)

Write matlab code to plot the four pictures in Lecture 11.