Lecture 7: Constrained optimization

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1. Convex optimization

- A convex optimization problem (or a convex problem)

$$\min_{\mathbf{x}\in\mathcal{C}} f(\mathbf{x}),$$

  where $\mathcal{C}$ is a convex set and f is a convex function.

- Convex optimization problems in functional form

$$\begin{array}{lll} \min & f(\mathbf{x}) & \\ \text{s.t.} & g_i(\mathbf{x}) \le 0, & i = 1, 2, \ldots, m, \\ & h_j(\mathbf{x}) = 0, & j = 1, 2, \ldots, p, \end{array}$$

  where $f, g_1, g_2, \ldots, g_m : \mathbb{R}^n \to \mathbb{R}$ are convex functions and $h_1, h_2, \ldots, h_p : \mathbb{R}^n \to \mathbb{R}$ are affine functions. The convex set $\mathcal{C}$ is

$$\mathcal{C} = \left(\bigcap_{i=1}^{m} \mathrm{Lev}(g_i, 0)\right) \bigcap \left(\bigcap_{j=1}^{p} \{\mathbf{x} : h_j(\mathbf{x}) = 0\}\right).$$

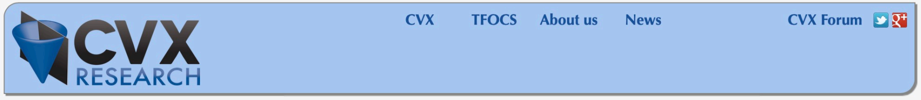
Theorem 1 (local = global in convex optimization)

Let $f : \mathcal{C} \to \mathbb{R}$ be a (strictly) convex function defined on the convex set $\mathcal{C}$. Let $\mathbf{x}_\star \in \mathcal{C}$ be a local minimizer of f over $\mathcal{C}$. Then $\mathbf{x}_\star$ is a (strict) global minimizer of f over $\mathcal{C}$.

Theorem 2

Let $f : \mathcal{C} \to \mathbb{R}$ be a convex function defined over the convex set $\mathcal{C} \subseteq \mathbb{R}^n$. Then the set of optimal solutions of the problem $\min\{f(\mathbf{x}) : \mathbf{x} \in \mathcal{C}\}$, which we denote by $\mathcal{X}_\star$, is convex. If, in addition, f is strictly convex over $\mathcal{C}$, then there exists at most one optimal solution.

CVX: a Matlab-based convex modeling framework



2. Optimization over a convex set

- Let f be a continuously differentiable function over a closed convex set $\mathcal{C}$. Then $\mathbf{x}_\star \in \mathcal{C}$ is called a *stationary point* of

$$(\mathrm{P}) \qquad \min f(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{x} \in \mathcal{C},$$

if $\nabla f(\mathbf{x}_\star)^\top (\mathbf{x} - \mathbf{x}_\star) \geq 0$ for any $\mathbf{x} \in \mathcal{C}$.

Theorem 3 (stationarity as a necessary optimality condition)

Let f be a continuously differentiable function over a closed convex set $\mathcal{C} \subseteq \mathbb{R}^n$, and let $\mathbf{x}_\star$ be a local minimizer of (P)*. Then $\mathbf{x}_\star$ is a stationary point of* (P)*.*

Theorem 4

Let f be a continuously differentiable convex function over a closed convex set $\mathcal{C} \subseteq \mathbb{R}^n$. Then $\mathbf{x}_\star \in \mathcal{C}$ is a stationary point of (P) *if and only if $\mathbf{x}_\star$ is an optimal solution of* (P)*.*

2.1 The gradient projection method

- The projection

$$\pi_{\mathcal{C}}(\mathbf{x}) = \operatorname*{argmin}_{\mathbf{y}\in\mathcal{C}} \|\mathbf{y}-\mathbf{x}\|_2.$$

Theorem 5

Let $\mathcal{C}$ be a nonempty closed convex set. Then for any $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$,

$$\|\pi_{\mathcal{C}}(\mathbf{v}) - \pi_{\mathcal{C}}(\mathbf{w})\|_2^2 \le (\pi_{\mathcal{C}}(\mathbf{v}) - \pi_{\mathcal{C}}(\mathbf{w}))^\top (\mathbf{v}-\mathbf{w}),$$

$$\|\pi_{\mathcal{C}}(\mathbf{v}) - \pi_{\mathcal{C}}(\mathbf{w})\|_2 \le \|\mathbf{v}-\mathbf{w}\|_2.$$

Theorem 6

Let f be a continuously differentiable function defined on the nonempty closed convex set $\mathcal{C}$, and let $s > 0$. Then $\mathbf{x}_\star \in \mathcal{C}$ is a stationary point of (P) if and only if

$$\mathbf{x}_\star = \pi_{\mathcal{C}}(\mathbf{x}_\star - s\nabla f(\mathbf{x}_\star)).$$

- The gradient projection method

$$\mathbf{x}_{k+1} = \pi_{\mathcal{C}}(\mathbf{x}_k - t_k \nabla f(\mathbf{x}_k)),$$

where $t_k > 0$ is obtained by using a line search procedure.

Lemma 7

Suppose that $f \in C_L^{1,1}(\mathcal{C})$, where $\mathcal{C}$ is a nonempty closed convex set. Then for any $\mathbf{x} \in \mathcal{C}$ and $t \in (0, 2/L)$ the following inequality holds:

$$f(\mathbf{x}) - f(\pi_{\mathcal{C}}(\mathbf{x} - t\nabla f(\mathbf{x}))) \geq (1/t - L/2)\|\mathbf{x} - \pi_{\mathcal{C}}(\mathbf{x} - t\nabla f(\mathbf{x}))\|_2^2$$

- Define the gradient mapping $G_M(\mathbf{x}) = M[\mathbf{x} - \pi_{\mathcal{C}}(\mathbf{x} - \nabla f(\mathbf{x})/M)]$

Lemma 8

Let f be a continuously differentiable function defined on a nonempty closed convex set $\mathcal{C}$. Suppose that $L_1 \geq L_2 > 0$. Then for any $\mathbf{x} \in \mathbb{R}^n$,

$$\|G_{L_1}(\mathbf{x})\|_2 \geq \|G_{L_2}(\mathbf{x})\|_2, \quad \|G_{L_1}(\mathbf{x})\|_2/L_1 \leq \|G_{L_2}(\mathbf{x})\|_2/L_2.$$

- **constant stepsize**:
$$t_k = \bar{t} \in \left(0, \frac{2}{L}\right).$$
- **backtracking**: $s > 0$, $\alpha \in (0,1)$, $\beta \in (0,1)$.

  First, set $t_k = s$. Then, while
$$f(\mathbf{x}_k) - f(\pi_{\mathcal{C}}(\mathbf{x}_k - t_k \nabla f(x_k))) < \alpha t_k \|G_{1/t_k}(\mathbf{x}_k)\|_2^2,$$
  set $t_k \leftarrow \beta t_k$. In other words, $t_k = s\beta^{i_k}$, where i_k is the smallest nonnegative integer satisfying (the sufficient decrease condition)
$$f(\mathbf{x}_k) - f(\pi_{\mathcal{C}}(\mathbf{x}_k - s\beta^{i_k} \nabla f(\mathbf{x}_k))) \geq \alpha s \beta^{i_k} \|G_{1/(s\beta^{i_k})}(\mathbf{x}_k)\|_2^2.$$
  If $f \in C_L^{1,1}(\mathcal{C})$, then the backtracking procedure ends when t_k is smaller than or equal to $2(1-\alpha)/L$. The chosen stepsize t_k satisfies
$$t_k \geq \min\left\{s, \frac{2(1-\alpha)\beta}{L}\right\}.$$

Theorem 9 (convergence of the gradient projection method)

Let $f \in C_L^{1,1}(\mathcal{C})$ and $\mathcal{C}$ be a nonempty closed convex set. Let $\{\mathbf{x}_k\}$ be the sequence generated by the gradient projection method for solving (P) *with either a constant stepsize $\bar{t} \in (0, 2/L)$ or with a stepsize chosen by the backtracking procedure with parameters $s > 0$, $\alpha \in (0,1)$, $\beta \in (0,1)$. Assume that f is bounded below. Then we have the following:*

(a) *The sequence $\{f(\mathbf{x}_k)\}$ is nonincreasing. In addition, for any $k > 0$, $f(\mathbf{x}_{k+1}) < f(\mathbf{x}_k)$ unless $\mathbf{x}_k$ is a stationary point of* (P).

(b) *$G_d(\mathbf{x}_k) \to \mathbf{0}$ as $k \to \infty$, and*

$$\min_{k=0,1,\ldots,n} \|G_d(\mathbf{x}_k)\|_2 \leq \sqrt{\frac{f(\mathbf{x}_0) - f_\star}{M(n+1)}},$$

where $f_\star = \lim_{k\to\infty} f(\mathbf{x}_k)$, and

$$d = \begin{cases} 1/\bar{t}, \\ 1/s, \end{cases} \quad M = \begin{cases} \bar{t}(1 - \bar{t}L/2), & \textit{constant stepsize}, \\ \alpha \min\{s, 2(1-\alpha)\beta/L\} & \textit{backtracking}. \end{cases}$$

Theorem 10

Let $f \in C_L^{1,1}(\mathcal{C})$ *be convex and* $\mathcal{C}$ *be a nonempty closed convex set. Let* $\{\mathbf{x}_k\}$ *be the sequence generated by the gradient projection method for solving* (P) *with a constant stepsize* $\bar{t} \in (0, 1/L]$. *Assume that the set of optimal solutions, denoted by* $\mathcal{X}_\star$, *is nonempty, and let* $f_\star$ *be the optimal value of* (P). *Then we have the following:*

(a) *for any* $k \geq 0$ *and* $\mathbf{x}_\star \in \mathcal{X}_\star$,

$$2\bar{t}(f(\mathbf{x}_{k+1}) - f(\mathbf{x}_\star)) \leq \|\mathbf{x}_k - \mathbf{x}_\star\|_2^2 - \|\mathbf{x}_{k+1} - \mathbf{x}_\star\|_2^2,$$

which implies

$$\|\mathbf{x}_{k+1} - \mathbf{x}_\star\|_2 \leq \|\mathbf{x}_k - \mathbf{x}_\star\|_2, \quad \text{(Fejér monotonicity)}$$

(b) *for any* $n \geq 0$,

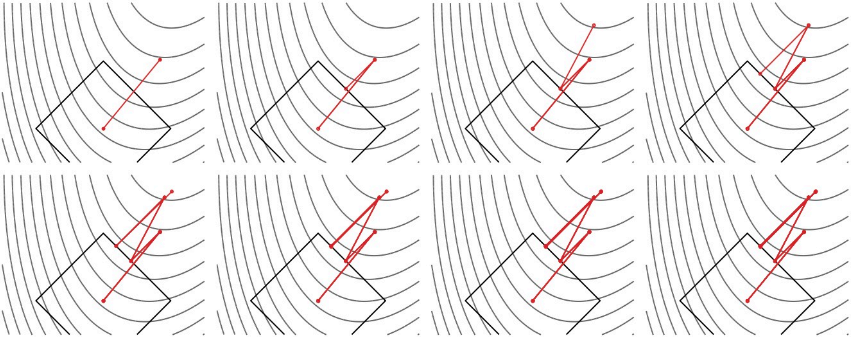
$$f(\mathbf{x}_n) - f_\star \leq \frac{\|\mathbf{x}_0 - \mathbf{x}_\star\|_2^2}{2\bar{t}n},$$

(c) *the sequence* $\{\mathbf{x}_k\}$ *converges to an optimal solution.*

2.2 Examples

- Example: LASSO or compressed sensing applications.

$$\min_{\mathbf{x}} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2 \quad \text{subject to} \quad \|\mathbf{x}\|_1 \leq 1$$



- Example: Suppose that $\mathcal{C} = \{\mathbf{x} : \|\mathbf{x}\|_p \le 1\}$ for $p = 1, 2, \infty$.

(1) $p = \infty$:

$$[\pi_{\mathcal{C}}(\mathbf{x})]_j = \min\{1, \max\{x_j, -1\}\},$$

that is, we simply truncate the coordinates of $\mathbf{x}$ to be in the range $[-1, 1]$.

(2) $p = 2$:

$$\pi_{\mathcal{C}}(\mathbf{x}) = \begin{cases} \mathbf{x} & \text{if } \|\mathbf{x}\|_2 \le 1 \\ \mathbf{x}/\|\mathbf{x}\|_2 & \text{otherwise.} \end{cases}$$

(3) $p = 1$: If $\|\mathbf{x}\|_1 \le 1$, then $\pi_{\mathcal{C}}(\mathbf{x}) = \mathbf{x}$. If $\|\mathbf{x}\|_1 > 1$, then

$$[\pi_{\mathcal{C}}(\mathbf{x})]_j = \mathrm{sign}(x_j)[|x_j| - t]_+,$$

where t is the unique $t \ge 0$ satisfying

$$\sum_{j=1}^{n} [|x_j| - t]_+ = 1.$$

- Example: Suppose that $\mathcal{C}$ is an affine set, represented by

$$\mathcal{C} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b}\},$$

  where $\mathbf{A} \in \mathbb{R}^{m \times n}$ with $m \le n$ is full rank. (So that $\mathbf{A}$ is a short and fat matrix and $\mathbf{A}\mathbf{A}^{\mathrm{T}} \succ \mathbf{0}$.) Then the projection of $\mathbf{x}$ onto $\mathcal{C}$ is

$$\pi_{\mathcal{C}}(\mathbf{x}) = (\mathbf{I} - \mathbf{A}^{\mathrm{T}}(\mathbf{A}\mathbf{A}^{\mathrm{T}})^{-1}\mathbf{A})\mathbf{x} + \mathbf{A}^{\mathrm{T}}(\mathbf{A}\mathbf{A}^{\mathrm{T}})^{-1}\mathbf{b}.$$

  If we begin the iterates from a point $\mathbf{x}^k \in \mathcal{C}$, i.e., with $\mathbf{A}\mathbf{x}^k = \mathbf{b}$, then

$$\mathbf{x}^{k+1} = \pi_{\mathcal{C}}(\mathbf{x}^k - \alpha_k \mathbf{g}^k) = \mathbf{x}^k - \alpha_k(\mathbf{I} - \mathbf{A}^{\mathrm{T}}(\mathbf{A}\mathbf{A}^{\mathrm{T}})^{-1}\mathbf{A})\mathbf{g}^k,$$

  that is, we simply project $\mathbf{g}^k$ onto the nullspace of $\mathbf{A}$ and iterate.
- For more examples and proofs, see FOMO §6.4.

Theorem 11 (KKT conditions for constrained problems)

Let $\mathbf{x}_\star$ *be a local minimizer of*

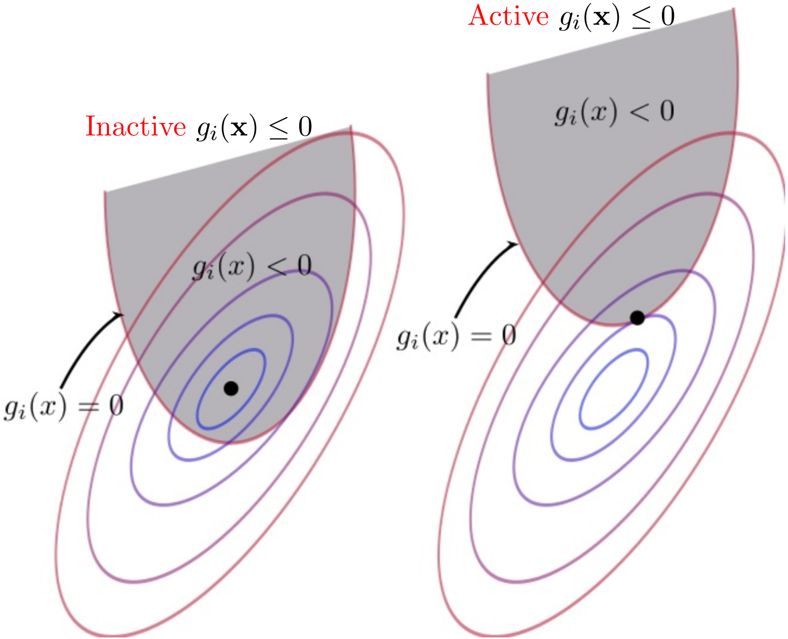
$$\min f(\mathbf{x}), \quad s.t. \quad g_i(\mathbf{x}) \le 0, \; h_j(\mathbf{x}) = 0, \; i = 1 : m, \; j = 1 : p,$$

where f, g_i, h_j *are continuously differentiable functions over* $\mathbb{R}^n$*. Suppose that the gradients of the active constraints and the equality constraints*

$$\{\nabla g_i(\mathbf{x}_\star) : i \in I(\mathbf{x}_\star)\} \cup \{\nabla h_j(\mathbf{x}_\star) : j = 1 : p\}$$

are linearly independent (where $I(\mathbf{x}_\star) = \{i : g_i(\mathbf{x}_\star) = 0\}$*). Then there exist multipliers* $\lambda_i \ge 0$ *and* $\mu_j \in \mathbb{R}$ *such that* $\lambda_i g_i(\mathbf{x}_\star) = 0, \; i = 1 : m,$

$$\nabla f(\mathbf{x}_\star) + \sum_{i=1}^{m} \lambda_i \nabla g_i(\mathbf{x}_\star) + \sum_{j=1}^{p} \mu_j \nabla h_j(\mathbf{x}_\star) = \mathbf{0}.$$


Inactive $g_i(\mathbf{x}) \leq 0$
$g_i(x) < 0$
$g_i(x) = 0$
Active $g_i(\mathbf{x}) \leq 0$
$g_i(x) < 0$
$g_i(x) = 0$

Theorem 12 (sufficiency of KKT conditions for convex problems)

Let $\mathbf{x}_\star$ *be a feasible solution of*

$$\min f(\mathbf{x}), \quad s.t. \quad g_i(\mathbf{x}) \leq 0, \ h_j(\mathbf{x}) = 0, \ i = 1:m, \ j = 1:p,$$

where f, g_i *are continuously differentiable convex functions over* $\mathbb{R}^n$ *and* h_j *are affine functions. Suppose that there exist multipliers* $\lambda_i \geq 0$ *and* $\mu_j \in \mathbb{R}$ *such that*

$$\lambda_i g_i(\mathbf{x}_\star) = 0, \ i = 1:m,$$

$$\nabla f(\mathbf{x}_\star) + \sum_{i=1}^{m} \lambda_i \nabla g_i(\mathbf{x}_\star) + \sum_{j=1}^{p} \mu_j \nabla h_j(\mathbf{x}_\star) = \mathbf{0}.$$

Then $\mathbf{x}_\star$ *is an optimal solution.*

Theorem 13 (necessity of KKT conditions under Slater's condition)

Let $\mathbf{x}_\star$ *be a local minimizer of* $\min f(\mathbf{x})$ *such that*

$$g_i(\mathbf{x}) \le 0,\ h_j(\mathbf{x}) \le 0,\ s_k(\mathbf{x}) = 0,\ i = 1:m,\ j = 1:p,\ k = 1:q,$$

where f, g_i *are continuously differentiable convex functions over* $\mathbb{R}^n$, *and* h_j, s_k *are affine functions. Suppose that there exists* $\widehat{\mathbf{x}}$ *such that*

$$g_i(\widehat{\mathbf{x}}) < 0,\ h_j(\widehat{\mathbf{x}}) \le 0,\ s_k(\widehat{\mathbf{x}}) = 0,\ i = 1:m,\ j = 1:p,\ k = 1:q.$$

Then there exist multipliers $\lambda_i \ge 0$, $\eta_j \ge 0$, *and* $\mu_j \in \mathbb{R}$ *such that*

$$\lambda_i g_i(\mathbf{x}_\star) = 0,\ i = 1:m,\ \eta_j h_j(\mathbf{x}_\star) = 0,\ j = 1:p,$$

$$\nabla f(\mathbf{x}_\star) + \sum_{i=1}^{m} \lambda_i \nabla g_i(\mathbf{x}_\star) + \sum_{j=1}^{p} \eta_j \nabla h_j(\mathbf{x}_\star) + \sum_{k=1}^{q} \mu_k \nabla s_k(\mathbf{x}_\star) = \mathbf{0}.$$

Then $\mathbf{x}_\star$ *is an optimal solution.*

4. Duality

- The *primal problem*: Consider the general model

$$
\begin{aligned}
f_\star = \min\ & f(\mathbf{x}) \\
\text{s.t.}\quad & g_i(\mathbf{x}) \le 0, \quad i = 1, 2, \ldots, m, \\
& h_j(\mathbf{x}) = 0, \quad j = 1, 2, \ldots, p, \\
& \mathbf{x} \in \mathcal{X},
\end{aligned}
$$

  where f, g_i, h_j are functions defined on the set $\mathcal{X} \subseteq \mathbb{R}^n$.

- The Lagrangian: $\mathbf{x} \in \mathcal{X}$, $\boldsymbol{\lambda} \in \mathbb{R}^m_+$, $\boldsymbol{\mu} \in \mathbb{R}^p$,

$$
\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i g_i(\mathbf{x}) + \sum_{j=1}^{p} \mu_j h_j(\mathbf{x}).
$$

- The *dual objective function* $q : \mathbb{R}^m_+ \times \mathbb{R}^p \to \mathbb{R} \cup \{-\infty\}$,

$$
q(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \min_{\mathbf{x} \in \mathcal{X}} \mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}).
$$

- The *dual problem*:

$$\begin{aligned} q_\star = \max\ & q(\boldsymbol{\lambda}, \boldsymbol{\mu}) \\ \text{s.t.}\quad & (\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \operatorname{dom}(q), \end{aligned}$$

where $\operatorname{dom}(q) = \{(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \mathbb{R}_+^m \times \mathbb{R}^p : q(\boldsymbol{\lambda}, \boldsymbol{\mu}) > -\infty\}$.

Theorem 14 (convexity of the dual problem)

The domain $\operatorname{dom}(q)$ *of the dual objective function is a convex set, and* q *is a concave (i.e.,* $-q$ *is convex) function over* $\operatorname{dom}(q)$.

Theorem 15 (weak duality theorem)

It holds that

$$q_\star \leq f_\star,$$

where $q_\star$ *and* $f_\star$ *are the optimal dual and primal values, respectively.*

4.1 Strong duality in the convex case

Theorem 16 (convex problems with inequality constraints)

Consider the optimization problem

$$f_\star = \min f(\mathbf{x}) \quad s.t. \quad g_i(\mathbf{x}) \le 0, \quad i = 1, 2, \ldots, m, \quad \mathbf{x} \in \mathcal{X},$$

where $\mathcal{X}$ is a convex set and f, g_i, are convex functions over $\mathcal{X}$. Suppose that there exists $\widehat{\mathbf{x}} \in \mathcal{X}$ for which $g_i(\widehat{\mathbf{x}}) < 0$ and the optimal value of the primal problem is finite. Then the optimal value of the dual problem

$$q_\star = \max\{q(\boldsymbol{\lambda}) : \boldsymbol{\lambda} \in \mathrm{dom}(q)\},$$

where $q(\boldsymbol{\lambda}) = \min_{\mathbf{x} \in \mathcal{X}} \mathcal{L}(\mathbf{x}, \boldsymbol{\lambda})$ is attained, and the optimal values of the primal and dual problems are the same:

$$f_\star = g_\star.$$

Theorem 17

Consider the optimization problem

$$f_\star = \min f(\mathbf{x}) \quad s.t.\ g_i(\mathbf{x}) \le 0,\ h_j(\mathbf{x}) \le 0,\ s_k(\mathbf{x}) = 0,\ \mathbf{x} \in \mathcal{X},$$

where $\mathcal{X}$ is a convex set and f, g_i, $i = 1 : m$, are convex functions over $\mathcal{X}$. The functions h_j, s_k, $j = 1 : p$, $k = 1 : q$, are affine functions. Suppose that there exists $\widehat{\mathbf{x}} \in \operatorname{int}(\mathcal{X})$ for which $g_i(\widehat{\mathbf{x}}) < 0$, $h_j(\widehat{\mathbf{x}}) \le 0$, $s_k(\widehat{\mathbf{x}}) = 0$. Then if the optimization problem has a finite optimal value, the optimal value of the dual problem

$$q_\star = \max\{q(\boldsymbol{\lambda}, \boldsymbol{\eta}, \boldsymbol{\mu}) : (\boldsymbol{\lambda}, \boldsymbol{\eta}, \boldsymbol{\mu}) \in \operatorname{dom}(q)\},$$

where $q : \mathbb{R}_+^m \times \mathbb{R}_+^p \times \mathbb{R}^q \to \mathbb{R} \cup \{-\infty\}$ is given by

$$q(\boldsymbol{\lambda}, \boldsymbol{\eta}, \boldsymbol{\mu}) = \min_{\mathbf{x} \in \mathcal{X}} f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i g_i(\mathbf{x}) + \sum_{j=1}^{p} \eta_j h_j(\mathbf{x}) + \sum_{k=1}^{q} \mu_k s_k(\mathbf{x}),$$

is attained, and $f_\star = g_\star$.