

Numerical Linear Algebra Assignment 3

Exercise 1. (10 points)

- (1) Let \mathbf{P} be a projector. Given an explicit expression for the inverse of $\lambda\mathbf{I} - \mathbf{P}$, where $\lambda \neq 0, 1$.
- (2) Suppose $\mathbf{A} \in \mathbb{C}^{m \times n}$ has a full SVD $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^*$, where

$$\mathbf{U} = [\mathbf{U}_r \quad \mathbf{U}_c], \quad \mathbf{V} = [\mathbf{V}_r \quad \mathbf{V}_c], \quad r = \text{rank}(\mathbf{A}).$$

What are the orthogonal projections onto $\text{null}(\mathbf{A})^\perp$, $\text{null}(\mathbf{A})$, $\text{range}(\mathbf{A})$, and $\text{range}(\mathbf{A})^\perp$?

Exercise 2. (Carl D. Meyer, 10 points)

Let \mathbf{P} and \mathbf{Q} be projectors (oblique or orthogonal).

- (i) Prove that $\text{range}(\mathbf{P}) = \text{range}(\mathbf{Q})$ if and only if $\mathbf{PQ} = \mathbf{Q}$ and $\mathbf{QP} = \mathbf{P}$.
- (ii) Prove that $\text{null}(\mathbf{P}) = \text{null}(\mathbf{Q})$ if and only if $\mathbf{PQ} = \mathbf{P}$ and $\mathbf{QP} = \mathbf{Q}$.

Exercise 3. (10 points)

Two subspaces $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathbb{C}^m$ are called *complementary subspaces* if they satisfy

$$\mathcal{S}_1 \cap \mathcal{S}_2 = \{\mathbf{0}\}, \quad \mathcal{S}_1 + \mathcal{S}_2 = \mathbb{C}^m.$$

Let \mathcal{S}_1 and \mathcal{S}_2 be complementary subspaces. Prove that there exists a projector \mathbf{P} with

$$\text{range}(\mathbf{P}) = \mathcal{S}_1, \quad \text{null}(\mathbf{P}) = \mathcal{S}_2.$$

Exercise 4. (TreBau Exercise 6.1, 10 points)

If \mathbf{P} is an orthogonal projector, then $\mathbf{I} - 2\mathbf{P}$ is unitary. Prove this algebraically, and give a geometric interpretation.

Exercise 5. (TreBau Exercise 6.5, 10 points)

Let $\mathbf{P} \in \mathbb{C}^{m \times m}$ be a nonzero projector. Show that $\|\mathbf{P}\|_2 \geq 1$, with equality if and only if \mathbf{P} is an orthogonal projector.

Exercise 6. (10 points)

Let $\mathcal{S} \subseteq \mathbb{C}^m$ and $\mathcal{T} \subseteq \mathbb{C}^m$. Let $\mathbf{P}_{\mathcal{S}}$ and $\mathbf{P}_{\mathcal{T}}$ be orthogonal projectors onto \mathcal{S} and \mathcal{T} , respectively. Assume that $\mathcal{S} \subseteq \mathcal{T}$.

- (i) Prove that $\mathbf{P}_{\mathcal{S}}\mathbf{P}_{\mathcal{T}} = \mathbf{P}_{\mathcal{T}}\mathbf{P}_{\mathcal{S}} = \mathbf{P}_{\mathcal{S}}$.
- (ii) Prove that $\mathbf{P}_{\mathcal{T}} - \mathbf{P}_{\mathcal{S}}$ is also an orthogonal projection.
- (iii) $\text{range}(\mathbf{P}_{\mathcal{T}} - \mathbf{P}_{\mathcal{S}}) = ?$ $\text{null}(\mathbf{P}_{\mathcal{T}} - \mathbf{P}_{\mathcal{S}}) = ?$

Exercise 7. (TreBau Exercise 7.3, 10 points)

Let \mathbf{A} be an $m \times m$ matrix, and let \mathbf{a}_j be its j th column. Give an algebraic proof of *Hadamard's inequality*:

$$|\det(\mathbf{A})| \leq \prod_{j=1}^m \|\mathbf{a}_j\|_2.$$

Also give a geometric interpretation of this result, making use of the fact that the determinant equals the volume of a parallelepiped.

Exercise 8. (TreBau Exercise 7.5, 10 points)

Let \mathbf{A} be an $m \times n$ matrix ($m \geq n$), and let $\mathbf{A} = \mathbf{Q}_n \mathbf{R}_n$ be a reduced QR factorization.

- (a) Show that \mathbf{A} has rank n if and only if all the diagonal entries of \mathbf{R}_n are nonzero.
- (b) Suppose \mathbf{R}_n has k nonzero diagonal entries for some k with $0 \leq k \leq n$. What does this imply about the rank of \mathbf{A} ? Exactly k ? At least k ? At most k ? Give a precise answer, and prove it.

Exercise 9. (10 points)

Compute a QR factorization of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ \sqrt{2} & 1 + \sqrt{2} & 1 \\ 1 & 2 & 1 \end{bmatrix}$.

Exercise 10. (Programming, TreBau Exercise 8.2, 10 points)**Additional Exercise 1.**

Let $C[-1, 1]$ denote the linear space of real-valued continuous functions on $[-1, 1]$ with the inner product

$$\forall f, g \in C[-1, 1], \quad \langle f, g \rangle_w = \int_{-1}^1 w(x) f(x) g(x) dx,$$

where $w(x) \geq 0 (\neq 0)$ is a weight function (continuous). For the case $w(x) = 1 + x^2$, complete the following:

- (i) Write Matlab code to compute the first six orthogonal (with respect to the inner product $\langle \cdot, \cdot \rangle_w$) polynomials $(P_j(x), j = 0, 1, 2, 3, 4, 5)$, which are conventionally normalized so that $P_j(1) = 1$. Hint: you can use Matlab's symbolic toolbox. For your reference, the polynomials are given by:

P =

$$\begin{aligned} & 1 \\ & x \\ & (5x^2)/3 - 2/3 \\ & (14x^3)/5 - (9x)/5 \\ & (119x^4)/24 - (161x^2)/36 + 37/72 \\ & (1221x^5)/136 - (705x^3)/68 + (325x)/136 \end{aligned}$$

- (ii) Modify the code we used for discrete Legendre polynomials to plot the discrete polynomials corresponding to those obtained in (i).