Numerical Linear Algebra Assignment 11

Exercise 1. (10 points)

Let $z \in \mathbb{C}$, $\mathbf{A} \in \mathbb{C}^{m \times m}$, and $\mathbf{B} = \mathbf{A} + z\mathbf{I}$. Prove the translation-invariance of Krylov subspaces, i.e., $\forall j \in \mathbb{N}$,

$$\mathcal{K}_j(\mathbf{A}, \mathbf{r}) = \mathcal{K}_j(\mathbf{B}, \mathbf{r}).$$

Exercise 2. (10 points)

If the minimal polynomial of the nonsingular matrix $\mathbf{A}$ has degree n, then the solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$ lies in the space $\mathcal{K}_n(\mathbf{A}, \mathbf{b})$. (Hint: Let $q(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_{n-1} z^{n-1} + z^n$ denote the minimal polynomial of $\mathbf{A}$. Then $\alpha_0 \neq 0$.)

Exercise 3. (10 points)

Suppose the minimal polynomial of the matrix $\mathbf{A} \in \mathbb{C}^{m \times m}$ has degree n and the Arnoldi process for $\mathbf{A}$ and a nonzero $\mathbf{r}$ breaks down at step k, i.e., $h_{k+1,k} = 0$ is encountered. Prove the following

(i) $k \leq n$.

(ii) $\mathcal{K}_k(\mathbf{A}, \mathbf{r}) = \mathcal{K}_{k+1}(\mathbf{A}, \mathbf{r}) = \mathcal{K}_{k+2}(\mathbf{A}, \mathbf{r}) = \cdots$.

(iii) Each eigenvalue of $\mathbf{H}_k$ is an eigenvalue of $\mathbf{A}$.

(iv) If $\mathbf{A}$ is nonsingular, then the solution $\mathbf{x}$ of $\mathbf{A}\mathbf{x} = \mathbf{r}$ lies in $\mathcal{K}_k(\mathbf{A}, \mathbf{r})$.

Exercise 4. (10 points)

Assume the Arnoldi process breaks down at step k, then $\mathbf{A}\mathbf{Q}_k = \mathbf{Q}_k\mathbf{H}_k$. Consider the QR factorization of the Krylov matrix

$$\mathbf{K}_j := \begin{bmatrix} \mathbf{r} & \mathbf{A}\mathbf{r} & \cdots & \mathbf{A}^{j-1}\mathbf{r} \end{bmatrix} = \mathbf{Q}_j\mathbf{R}_j, \qquad j \leq k.$$

Note that in the Arnoldi process, neither $\mathbf{K}_j$ nor $\mathbf{R}_j$ is formed explicitly. How to construct $\mathbf{R}_j$ by the Hessenberg matrices in the Arnoldi process, without computing QR factorizations of $\mathbf{K}_j$ explicitly?

Exercise 5. (10 points)

Assume that Arnoldi process breaks down at step k. For all $1 \leq j < k$, prove the following:

(a) The jth residual vector $\mathbf{r}_j$ of GMRES satisfies $\mathbf{r}_j \perp \mathbf{A}\mathcal{K}_j$.

(b) The jth residual vector $\mathbf{r}_j$ of GMRES can be *uniquely* expressed as

$$\mathbf{r}_j = p_j(\mathbf{A})\mathbf{r}_0, \qquad \deg(p_j) \leq j, \qquad p_j(0) = 1.$$

Exercise 6. (10 points)

Let the GMRES iteration be applied to a matrix $\mathbf{A} \in \mathbb{C}^{m \times m}$ and a vector $\mathbf{r}_0$. Prove the following invariance properties:

(a) Scale-invariance. If $\mathbf{A}$ is changed to $z\mathbf{A}$ for some $z \in \mathbb{C}$, and $\mathbf{r}_0$ is changed to $z\mathbf{r}_0$, the residuals $\{\mathbf{r}_j\}$ change to $\{z\mathbf{r}_j\}$.

(b) Invariance under unitary similarity transformations. If $\mathbf{A}$ is changed to $\mathbf{U}\mathbf{A}\mathbf{U}^*$ for some unitary matrix $\mathbf{U}$, and $\mathbf{r}_0$ is changed to $\mathbf{U}\mathbf{r}_0$, the residuals $\{\mathbf{r}_j\}$ change to $\{\mathbf{U}\mathbf{r}_j\}$.

Exercise 7. (10 points)

Assume $c_0 \neq 0$. Let $\mathbf{r}_0 = \mathbf{e}_1$ and

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_{m-1} \\ -c_0 & -\mathbf{c}_{m-1} \end{bmatrix} \in \mathbb{C}^{m \times m}, \quad \mathbf{c}_{m-1} = \begin{bmatrix} c_1 & c_2 & \cdots & c_{m-1} \end{bmatrix}.$$

Prove that

$$\|\mathbf{r}_0\|_2 = \|\mathbf{r}_1\|_2 = \cdots = \|\mathbf{r}_{m-1}\|_2, \qquad \|\mathbf{r}_m\|_2 = 0.$$

The above example implies that GMRES can completely stagnate, i.e., the residual norm can be nondecreasing at the first $m-1$ steps, and “convergence” occurs in the last step.

Compulsory requirement for programming: Use Matlab’s `publish` to save all your code, comments, and results to a PDF file. You must use the programming format files: example_format.zip.

Programming 1. (10 points)

Write matlab code to plot the four pictures in Lecture 11.