

# Lecture 13: Biorthogonalization methods



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## 1. Tridiagonal biorthogonalization

- Assume that  $\mathbf{A}$  is Hermitian, i.e.,  $\mathbf{A} = \mathbf{A}^*$ . The Lanczos process is a process of tridiagonal orthogonalization. Actually, there exists a unitary  $\mathbf{Q}$  and a tridiagonal  $\mathbf{T}$  such that

$$\mathbf{A} = \mathbf{Q}\mathbf{T}\mathbf{Q}^*.$$

Note that “unitary transformations” and “symmetric tridiagonal”.

- For non-Hermitian  $\mathbf{A}$ :

(1) The Arnoldi process gives up the symmetric tridiagonal structure:

$$\mathbf{A} = \mathbf{Q}\mathbf{H}\mathbf{Q}^*, \quad \mathbf{H} \text{ is upper-Hessenberg.}$$

(2) Tridiagonal biorthogonalization methods give up the unitary transformations and symmetry:

$$\mathbf{A} = \mathbf{V}\mathbf{T}\mathbf{V}^{-1}, \quad \mathbf{T} \text{ is tridiagonal but generally not symmetric,}$$

and  $\mathbf{V}$  is nonsingular but generally not unitary.

- The term “biorthogonal” refers to the fact that although the columns of  $\mathbf{V}$  are not orthogonal to each other, they are orthogonal to the columns of  $\mathbf{V}^{-*}$ , as follows trivially from the identity

$$(\mathbf{V}^{-*})^* \mathbf{V} = \mathbf{V}^{-1} \mathbf{V} = \mathbf{I}.$$

- Define  $\mathbf{W} = \mathbf{V}^{-*}$ . Let  $\mathbf{v}_j$  and  $\mathbf{w}_j$  denote the  $j$ th columns of  $\mathbf{V}$  and  $\mathbf{W}$ , respectively. These vectors are biorthogonal in the sense that

$$\mathbf{w}_i^* \mathbf{v}_j = \delta_{ij},$$

where  $\delta_{ij}$  is the Kronecker delta function. Define

$$\mathbf{V}_j = [\mathbf{v}_1 \quad \cdots \quad \mathbf{v}_j], \quad \mathbf{W}_j = [\mathbf{w}_1 \quad \cdots \quad \mathbf{w}_j].$$

In matrix form, the biorthogonality condition can be written

$$\mathbf{W}_j^* \mathbf{V}_j = \mathbf{V}_j^* \mathbf{W}_j = \mathbf{I}_j.$$

- The Lanczos relation

$$\mathbf{A}\mathbf{Q}_j = \mathbf{Q}_{j+1}\tilde{\mathbf{T}}_j, \quad \mathbf{T}_j = \mathbf{Q}_j^*\mathbf{A}\mathbf{Q}_j.$$

- The Arnoldi relation

$$\mathbf{A}\mathbf{Q}_j = \mathbf{Q}_{j+1}\tilde{\mathbf{H}}_j, \quad \mathbf{H}_j = \mathbf{Q}_j^*\mathbf{A}\mathbf{Q}_j.$$

- Corresponding formulas for biorthogonalization methods:

$$\mathbf{A}\mathbf{V}_j = \mathbf{V}_{j+1}\tilde{\mathbf{T}}_j, \quad \mathbf{A}^*\mathbf{W}_j = \mathbf{W}_{j+1}\tilde{\mathbf{S}}_j,$$

and

$$\mathbf{T}_j = \mathbf{S}_j^* = \mathbf{W}_j^*\mathbf{A}\mathbf{V}_j.$$

$\mathbf{T}_j$  is tridiagonal and is obtained by deleting the last row of  $\tilde{\mathbf{T}}_j$ .  
 $\mathbf{S}_j$  is tridiagonal and is obtained by deleting the last row of  $\tilde{\mathbf{S}}_j$ .

- Note that

$$\mathbf{A}\mathbf{V}_j = \mathbf{V}_{j+1}\tilde{\mathbf{T}}_j$$

takes the form

$$\mathbf{A} \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_j \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_{j+1} \end{bmatrix} \begin{bmatrix} \alpha_1 & \gamma_1 & & & \\ \beta_1 & \alpha_2 & \gamma_2 & & \\ & \beta_2 & \alpha_3 & \ddots & \\ & & \ddots & \ddots & \gamma_{j-1} \\ & & & \beta_{j-1} & \alpha_j \\ & & & & \beta_j \end{bmatrix},$$

which corresponds to the three-term recurrence relation

$$\mathbf{A}\mathbf{v}_j = \gamma_{j-1}\mathbf{v}_{j-1} + \alpha_j\mathbf{v}_j + \beta_j\mathbf{v}_{j+1}.$$

- Similarly,

$$\mathbf{A}^* \mathbf{W}_j = \mathbf{W}_{j+1} \tilde{\mathbf{S}}_j$$

takes the form

$$\mathbf{A}^* \begin{bmatrix} \mathbf{w}_1 & \cdots & \mathbf{w}_j \end{bmatrix} = \begin{bmatrix} \mathbf{w}_1 & \cdots & \mathbf{w}_{j+1} \end{bmatrix} \begin{bmatrix} \bar{\alpha}_1 & \bar{\beta}_1 & & & \\ \bar{\gamma}_1 & \bar{\alpha}_2 & \bar{\beta}_2 & & \\ & \bar{\gamma}_2 & \bar{\alpha}_3 & \ddots & \\ & & \ddots & \ddots & \bar{\beta}_{j-1} \\ & & & \bar{\gamma}_{j-1} & \bar{\alpha}_j \\ & & & & \bar{\gamma}_j \end{bmatrix},$$

which corresponds to the three-term recurrence relation

$$\mathbf{A}^* \mathbf{w}_j = \bar{\beta}_{j-1} \mathbf{w}_{j-1} + \bar{\alpha}_j \mathbf{w}_j + \bar{\gamma}_j \mathbf{w}_{j+1}.$$

- These recurrence relations suggest an algorithm. Begin with vectors  $\mathbf{v}_1$  and  $\mathbf{w}_1$  that are arbitrary except for satisfying  $\mathbf{v}_1^* \mathbf{w}_1 = 1$ , and set  $\beta_0 = \gamma_0 = 0$  and  $\mathbf{v}_0 = \mathbf{w}_0 = \mathbf{0}$ . Now, for each  $j = 1, 2, \dots$ , set

$$\alpha_j = \mathbf{w}_j^* \mathbf{A} \mathbf{v}_j.$$

The vectors  $\mathbf{v}_{j+1}$  and  $\mathbf{w}_{j+1}$  are then determined by the recurrence relations up to scalar factors. These factors may be chosen arbitrarily, subject to the normalization

$$\mathbf{v}_{j+1}^* \mathbf{w}_{j+1} = 1.$$

- The vectors generated by the procedure just described satisfy

$$\mathcal{K}_j(\mathbf{A}, \mathbf{v}_1) = \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_j\}, \quad \mathcal{K}_j(\mathbf{A}^*, \mathbf{w}_1) = \text{span}\{\mathbf{w}_1, \dots, \mathbf{w}_j\}.$$

- Breakdown at step  $k$ ? Case (1): invariant subspaces

$$\mathbf{A} \mathbf{v}_k - \gamma_{k-1} \mathbf{v}_{k-1} - \alpha_k \mathbf{v}_k = \mathbf{0} \text{ or } \mathbf{A}^* \mathbf{w}_k - \bar{\beta}_{k-1} \mathbf{w}_{k-1} - \bar{\alpha}_k \mathbf{w}_k = \mathbf{0};$$

$$\text{Case (2): } (\beta_k \mathbf{v}_{k+1})^* (\bar{\gamma}_k \mathbf{w}_{k+1}) = 0 \text{ with } \beta_k \mathbf{v}_{k+1} \neq \mathbf{0}, \bar{\gamma}_k \mathbf{w}_{k+1} \neq \mathbf{0}.$$

## 2. Bi-conjugate gradients (Bi-CG)

- The principle of Bi-CG is to pick

$$\mathbf{x}_j \in \mathbf{x}_0 + \mathcal{K}_j(\mathbf{A}, \mathbf{r}_0)$$

subject to the orthogonality condition

$$\mathbf{r}_j \perp \mathcal{K}_j(\mathbf{A}^*, \mathbf{w}_1),$$

where  $\mathbf{w}_1 \in \mathbb{C}^n$  is an arbitrary vector satisfying ( $\mathbf{v}_1 = \mathbf{r}_0$ )

$$\mathbf{w}_1^* \mathbf{r}_0 = 1.$$

- Proceeding in the same manner as for the derivation of CG, we can derive the scheme for Bi-CG.

See Saad's book – *Iterative methods for sparse linear systems* (Chapters 6 and 7) for details.



- Here we provide an alternative based on preconditioned CG.  
Consider the following linear system

$$\begin{bmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{A}^\top & \mathbf{0} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{x}} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \tilde{\mathbf{b}} \end{bmatrix}.$$

Apply the PCG scheme with the preconditioner

$$\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}.$$

Note that the corresponding inner product is replaced by the bilinear form

$$\left\langle \begin{bmatrix} \tilde{\mathbf{x}} \\ \mathbf{x} \end{bmatrix}, \begin{bmatrix} \tilde{\mathbf{y}} \\ \mathbf{y} \end{bmatrix} \right\rangle_{\text{Bi-CG}} = \begin{bmatrix} \tilde{\mathbf{y}} \\ \mathbf{y} \end{bmatrix}^\top \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{x}} \\ \mathbf{x} \end{bmatrix}.$$

We obtain the Bi-CG scheme for  $\mathbf{Ax} = \mathbf{b}$ :

$$\begin{aligned}\mathbf{r}_0 &= \mathbf{p}_0 = \mathbf{b} - \mathbf{Ax}_0; & \tilde{\mathbf{r}}_0 &= \tilde{\mathbf{p}}_0 = \tilde{\mathbf{b}} - \mathbf{A}^\top \tilde{\mathbf{x}}_0; \\ \mathbf{x}_j &= \mathbf{x}_{j-1} + \alpha_j \mathbf{p}_{j-1}; & \tilde{\mathbf{x}}_j &= \tilde{\mathbf{x}}_{j-1} + \alpha_j \tilde{\mathbf{p}}_{j-1}; \\ \mathbf{r}_j &= \mathbf{r}_{j-1} - \alpha_j \mathbf{Ap}_{j-1}; & \tilde{\mathbf{r}}_j &= \tilde{\mathbf{r}}_{j-1} - \alpha_j \mathbf{A}^\top \tilde{\mathbf{p}}_{j-1}; \\ \mathbf{p}_j &= \mathbf{r}_j + \beta_j \mathbf{p}_{j-1}; & \tilde{\mathbf{p}}_j &= \tilde{\mathbf{r}}_j + \beta_j \tilde{\mathbf{p}}_{j-1};\end{aligned}$$

where

$$\alpha_j = \frac{\tilde{\mathbf{r}}_{j-1}^\top \mathbf{r}_{j-1} + \mathbf{r}_{j-1}^\top \tilde{\mathbf{r}}_{j-1}}{\mathbf{p}_{j-1}^\top \mathbf{A}^\top \tilde{\mathbf{p}}_{j-1} + \tilde{\mathbf{p}}_{j-1}^\top \mathbf{Ap}_{j-1}} = \frac{\tilde{\mathbf{r}}_{j-1}^\top \mathbf{r}_{j-1}}{\tilde{\mathbf{p}}_{j-1}^\top \mathbf{Ap}_{j-1}};$$

and

$$\beta_j = \frac{\tilde{\mathbf{r}}_j^\top \mathbf{r}_j + \mathbf{r}_j^\top \tilde{\mathbf{r}}_j}{\tilde{\mathbf{r}}_{j-1}^\top \mathbf{r}_{j-1} + \mathbf{r}_{j-1}^\top \tilde{\mathbf{r}}_{j-1}} = \frac{\tilde{\mathbf{r}}_j^\top \mathbf{r}_j}{\tilde{\mathbf{r}}_{j-1}^\top \mathbf{r}_{j-1}}.$$

Obviously, by writing  $\mathbf{s}_j = \widetilde{\mathbf{r}}_j$ ,  $\mathbf{q}_j = \widetilde{\mathbf{p}}_j$ ,  $\widehat{\mathbf{b}} = \widetilde{\mathbf{b}}$ , and  $\widehat{\mathbf{x}}_j = \widetilde{\mathbf{x}}_j$ , we have

$$\begin{aligned} \mathbf{r}_0 &= \mathbf{p}_0 = \mathbf{b} - \mathbf{A}\mathbf{x}_0; & \mathbf{s}_0 &= \mathbf{q}_0 = \widehat{\mathbf{b}} - \mathbf{A}^*\widehat{\mathbf{x}}_0; \\ \mathbf{x}_j &= \mathbf{x}_{j-1} + \alpha_j \mathbf{p}_{j-1}; & \widehat{\mathbf{x}}_j &= \widehat{\mathbf{x}}_{j-1} + \overline{\alpha}_j \mathbf{q}_{j-1}; \\ \mathbf{r}_j &= \mathbf{r}_{j-1} - \alpha_j \mathbf{A}\mathbf{p}_{j-1}; & \mathbf{s}_j &= \mathbf{s}_{j-1} - \overline{\alpha}_j \mathbf{A}^* \mathbf{q}_{j-1}; \\ \mathbf{p}_j &= \mathbf{r}_j + \beta_j \mathbf{p}_{j-1}; & \mathbf{q}_j &= \mathbf{s}_j + \overline{\beta}_j \mathbf{q}_{j-1}; \end{aligned}$$

where

$$\alpha_j = \frac{\mathbf{s}_{j-1}^* \mathbf{r}_{j-1}}{\mathbf{q}_{j-1}^* \mathbf{A} \mathbf{p}_{j-1}}; \quad \beta_j = \frac{\mathbf{s}_j^* \mathbf{r}_j}{\mathbf{s}_{j-1}^* \mathbf{r}_{j-1}}.$$

As the case in CG, it is readily shown that  $\mathbf{s}_j^* \mathbf{r}_i = 0$  and  $\mathbf{q}_j^* \mathbf{A} \mathbf{p}_i = 0$  for  $i < j$ .

### • Discussion:

What happens if Bi-CG is applied to a linear system  $\mathbf{A}\mathbf{x} = \mathbf{b}$  with Hermitian positive definite  $\mathbf{A}$ ?

- Comparison of GMRES and Bi-CG for a  $500 \times 500$  matrix



### 3. Conjugate orthogonal conjugate gradients (COCG)

- If the system matrix  $\mathbf{A}$  is complex symmetric, i.e.,

$$\mathbf{A} \in \mathbb{C}^{m \times m}, \quad \mathbf{A} = \mathbf{A}^\top,$$

with the choice  $\mathbf{s}_0 = \bar{\mathbf{r}}_0$ , where  $\mathbf{s}_0$  is the initial residual of the dual system

$$\mathbf{A}^* \hat{\mathbf{x}} = \hat{\mathbf{b}}$$

arising in Bi-CG and  $\bar{\mathbf{r}}_0$  denotes the complex conjugate of  $\mathbf{r}_0$ , Bi-CG reduces to COCG because

$$\mathbf{s}_j = \bar{\mathbf{r}}_j, \quad \mathbf{q}_j = \bar{\mathbf{p}}_j.$$

- The details of COCG and its preconditioned version (PCOCG) are given below.

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### Algorithm COCG: $\mathbf{Ax} = \mathbf{b}$

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Choose arbitrary  $\mathbf{x}_0$ ; set  $\mathbf{r}_0 = \mathbf{b} - \mathbf{Ax}_0$  and  $\mathbf{p}_0 = \mathbf{r}_0$ ;  
**for**  $k = 1, 2, \dots$ , **do** until convergence or breakdown:

$$\mathbf{x}_j = \mathbf{x}_{j-1} + \alpha_j \mathbf{p}_{j-1};$$

$$\mathbf{r}_j = \mathbf{r}_{j-1} - \alpha_j \mathbf{Ap}_{j-1};$$

$$\mathbf{p}_j = \mathbf{r}_j + \beta_j \mathbf{p}_{j-1};$$

where

$$\alpha_j = \frac{\mathbf{r}_{j-1}^\top \mathbf{r}_{j-1}}{\mathbf{p}_{j-1}^\top \mathbf{Ap}_{j-1}}; \quad \beta_j = \frac{\mathbf{r}_j^\top \mathbf{r}_j}{\mathbf{r}_{j-1}^\top \mathbf{r}_{j-1}}.$$

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- The only essential change of COCG with respect to CG is the replacement of the standard inner product  $\mathbf{y}^* \mathbf{x}$  by the bilinear form  $\mathbf{y}^\top \mathbf{x}$ .
- Discussion:

What happens if COCG is applied to a linear system  $\mathbf{Ax} = \mathbf{b}$  with Hermitian positive definite  $\mathbf{A}$ ?

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**Algorithm PCOCG:  $\mathbf{A}\mathbf{M}^{-1}\mathbf{z} = \mathbf{b}$ ,  $\mathbf{x} = \mathbf{M}^{-1}\mathbf{z}$** 

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Choose  $\mathbf{x} = \mathbf{x}_0$ ;

set  $\mathbf{r}_0 = \mathbf{b} - \mathbf{A}\mathbf{x}_0$  and  $\mathbf{p}_0 = \mathbf{M}^{-1}\mathbf{r}_0$ ;

for  $k = 1, 2, \dots$ , do until convergence or breakdown:

$$\mathbf{x}_j = \mathbf{x}_{j-1} + \alpha_j \mathbf{p}_{j-1};$$

$$\mathbf{r}_j = \mathbf{r}_{j-1} - \alpha_j \mathbf{A}\mathbf{p}_{j-1};$$

$$\mathbf{p}_j = \mathbf{M}^{-1}\mathbf{r}_j + \beta_j \mathbf{p}_{j-1};$$

where

$$\alpha_j = \frac{\mathbf{r}_{j-1}^\top \mathbf{M}^{-1} \mathbf{r}_{j-1}}{\mathbf{p}_{j-1}^\top \mathbf{A} \mathbf{p}_{j-1}}; \quad \beta_j = \frac{\mathbf{r}_j^\top \mathbf{M}^{-1} \mathbf{r}_j}{\mathbf{r}_{j-1}^\top \mathbf{M}^{-1} \mathbf{r}_{j-1}}.$$

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- The preconditioner  $\mathbf{M}$  in PCOCG has to be complex symmetric. The only essential change of PCOCG with respect to COCG is the replacement of the bilinear form  $\mathbf{y}^\top \mathbf{x}$  by the bilinear form  $\mathbf{y}^\top \mathbf{M}^{-1} \mathbf{x}$ .

## 4. Advantage and disadvantages of Bi-CG

- One great advantage over GMRES:

It involves three-term recurrences, enabling the work per step and the storage requirements to remain under control even when many steps are needed.

- Two disadvantages:

(1) Its convergence is slower and often erratic. It may have the consequence of reducing the ultimately attainable accuracy because of rounding error. In the extreme, it becomes the phenomenon of breakdown of the iteration, where an inner product becomes zero and no further progress is possible.

(2) It requires multiplication by  $\mathbf{A}^*$  as well as  $\mathbf{A}$ . Depending on how these products are implemented both mathematically and in terms of computer architecture, this may be anything from a minor addition burden to effectively impossible.



## 5. Other variants

- Desired properties

Smoothed convergence curves + breakdown free + transpose free

- Look-ahead Lanczos

avoid breakdowns

- CGS = conjugate gradients squares

transpose free, but is twice as erratic

- QMR = quasi-minimal residuals

Pronounced effect on the smoothness of convergence

- Bi-CGSTAB = stabilized Bi-CG (via stabilizing CGS)

Significantly smooths the convergence of Bi-CG, transpose free

- TFQMR = transpose-free QMR

transpose free and smooth convergence

## 6. Overview of iterative methods

### 6.1. Why iterate?

- Direct methods for general matrices requires  $\mathcal{O}(m^3)$  work. It is too much both in the absolute sense that  $m^3$  is huge when  $m$  is large, and in the relative sense that since the input to most matrix problems involves only  $\mathcal{O}(m^2)$  numbers, it seems unreasonable that  $\mathcal{O}(m^3)$  work must be expended in solving them.
- Use iterative methods when direct methods are too slow, or use too much memory (from fill-in during factorization), or you don't need as much accuracy.
- A good iterative method exploits knowledge of the underlying physical or mathematical model that gives rise to matrix.

### 6.2. Matrix-vector product

- Krylov subspace methods require nothing more than the ability to determine  $\mathbf{Ax}$  for any  $\mathbf{x}$ . Some iterative algorithms also require the computation of  $\mathbf{A}^*\mathbf{x}$ .

- Black box:



- Sparsity: obtain  $\mathbf{Ax}$  in only  $\mathcal{O}(\text{nnz})$  rather than  $\mathcal{O}(m^2)$  operations.
- Other structure: fast matrix-vector product

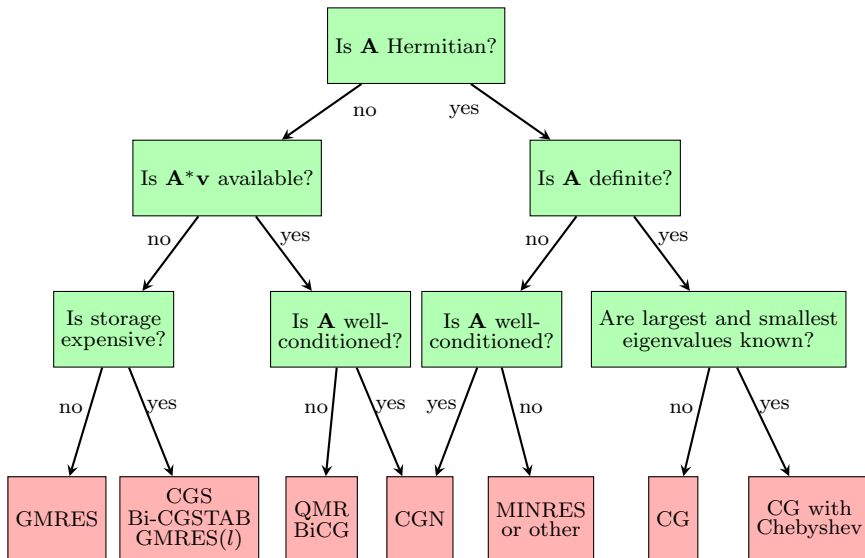
### 6.3. Computational cost

- The work per step in a Krylov subspace method depends mainly on the structure of the matrix and on what advantage is taken of this structure in the  $\mathbf{x} \mapsto \mathbf{Ax}$  black box.
- The ideal iterative method in linear algebra reduces the number of steps from  $\mathcal{O}(m)$  to  $\mathcal{O}(1)$  and the work per step from  $\mathcal{O}(m^2)$  to  $\mathcal{O}(m)$ , reducing the total work from  $\mathcal{O}(m^3)$  to  $\mathcal{O}(m)$ . Such extraordinary speedups do occur in practical problems, but a more typical improvement is perhaps from  $\mathcal{O}(m^3)$  to  $\mathcal{O}(m^2)$ .

## 6.4. Preconditioning for linear systems

- Diagonal scaling or Jacobi.
- Incomplete Cholesky or LU factorization.
- Coarse-grid approximation.
- Local approximation.
- Block preconditioners and domain decomposition.
- Low-order discretization.
- Constant-coefficient or symmetric approximation.
- Splitting of a multi-term operator.
- Dimensional splitting or ADI.
- One step of a classical iterative method, a single multigrid cycle.
- Periodic or convolution approximation.
- Unstable direct method.
- Polynomial preconditioners.
- Randomized techniques, Machine learning, AI . . . . .

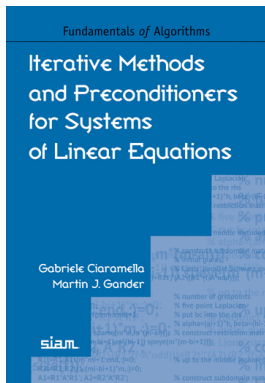
## 6.5. Decision tree for choosing an iterative solver for $Ax = b$



## 6.6. A Reference Book

### ● Iterative Methods and Preconditioners for Systems of Linear Equations

Authors: Gabriele Ciaramella and Martin J. Gander  
SIAM, 2022





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## Iterative Methods for Linear Systems

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One of the most important and common applications of numerical linear algebra is the solution of linear systems that can be expressed in the form  $A*x = b$ . When  $A$  is a large sparse matrix, you can solve the linear system using iterative methods, which enable you to trade-off between the run time of the calculation and the precision of the solution. This topic describes the iterative methods available in MATLAB® to solve the equation  $A*x = b$ .

### Direct vs. Iterative Methods

There are two types of methods for solving linear equations  $A*x = b$ :

- **Direct methods** are variants of Gaussian elimination. These methods involve the individual matrix elements directly, through matrix operations such as LU, QR, or Cholesky factorization. You can use direct methods to solve linear equations with a high level of precision, but these methods can be slow when operating on large sparse matrices. The speed of solving a linear system with a direct method strongly depends on the size of the coefficient matrix.

For example, this code solves a small linear system.

```
A = magic(5);  
b = sum(A,2);  
x = A\b;  
norm(A*x-b)
```

