Lecture 6: Convex sets and convex functions

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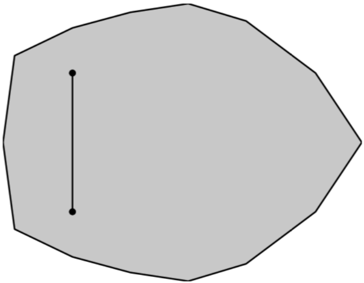
1. Convex sets

- A set $\mathcal{C} \in \mathbb{R}^n$ is a *convex set* if the straight line segment connecting any two points in $\mathcal{C}$ lies entirely inside $\mathcal{C}$. Formally,

$$\forall\, \mathbf{x}, \mathbf{y} \in \mathcal{C},\ \alpha \in [0,1] : \quad \alpha \mathbf{x} + (1-\alpha)\mathbf{y} \in \mathcal{C}.$$

Example: A convex set (left) and a non-convex set (right).



1.1 Basic properties of convex sets

- If $\alpha \in \mathbb{R}$ and $\mathcal{C}$ is convex, then
$$\alpha\mathcal{C} := \{\alpha\mathbf{x} : \mathbf{x} \in \mathcal{C}\}$$
is convex.
- If all $\mathcal{C}_i$, $i = 1 : m$, are convex. Then the Cartesian product
$$\mathcal{C}_1 \times \mathcal{C}_2 \times \cdots \times \mathcal{C}_m := \{(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_m) : \mathbf{x}_i \in \mathcal{C}_i\}$$
is convex.
- The convex hull of a set of points $\mathbf{x}_1, \cdots, \mathbf{x}_m \in \mathbb{R}^n$, defined by
$$\operatorname{conv}\{\mathbf{x}_1, \cdots, \mathbf{x}_m\} := \left\{\sum_{i=1}^{m} \lambda_i\mathbf{x}_i : \lambda_i \geq 0, \sum_{i=1}^{m} \lambda_i = 1\right\},$$
is convex. Let $\mathcal{S} \subseteq \mathbb{R}^n$. Then
$$\operatorname{conv}(\mathcal{S}) = \left\{\sum_{i=1}^{k} \lambda_i\mathbf{x}_i : \lambda_i \geq 0, \mathbf{x}_i \in \mathcal{S}, \sum_{i=1}^{k} \lambda_i = 1, k \in \mathbb{N}\right\}$$
is the "smallest" convex set containing $\mathcal{S}$.

- If $\alpha_i \in \mathbb{R}$ and all $\mathcal{C}_i$ are convex, then

$$\mathcal{C} = \sum_{i=1}^{m} \alpha_i \mathcal{C}_i := \left\{ \sum_{i=1}^{m} \alpha_i \mathbf{x}_i : \mathbf{x}_i \in \mathcal{C}_i \right\}$$

  is convex.

- Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a convex set and let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times m}$. Then the sets

$$\mathbf{A}(\mathcal{C}) := \{\mathbf{A}\mathbf{x} : \mathbf{x} \in \mathcal{C}\}, \quad \mathbf{B}^{-1}(\mathcal{C}) := \{\mathbf{y} \in \mathbb{R}^m : \mathbf{B}\mathbf{y} \in \mathcal{C}\}$$

  are both convex.

- If $\mathcal{C}_\alpha$ are convex sets for each $\alpha \in \mathcal{A}$, where $\mathcal{A}$ is an arbitrary index set (possibly infinite), then the intersection

$$\mathcal{C} = \bigcap_{\alpha \in \mathcal{A}} \mathcal{C}_\alpha$$

  is convex.

Theorem 1 (Projection onto closed convex sets)

Let $\mathcal{C}$ *be a closed convex set and* $\mathbf{x} \in \mathbb{R}^n$*. Then there is a unique point* $\pi_{\mathcal{C}}(\mathbf{x})$*, called the projection of* $\mathbf{x}$ *onto* $\mathcal{C}$*, such that*

$$\|\mathbf{x} - \pi_{\mathcal{C}}(\mathbf{x})\|_2 = \inf_{\mathbf{y} \in \mathcal{C}} \|\mathbf{y} - \mathbf{x}\|_2,$$

that is,

$$\pi_{\mathcal{C}}(\mathbf{x}) = \operatorname*{argmin}_{\mathbf{y} \in \mathcal{C}} \|\mathbf{y} - \mathbf{x}\|_2.$$

A point $\mathbf{z}$ *is the projection of* $\mathbf{x}$ *onto* $\mathcal{C}$*, i.e.,*

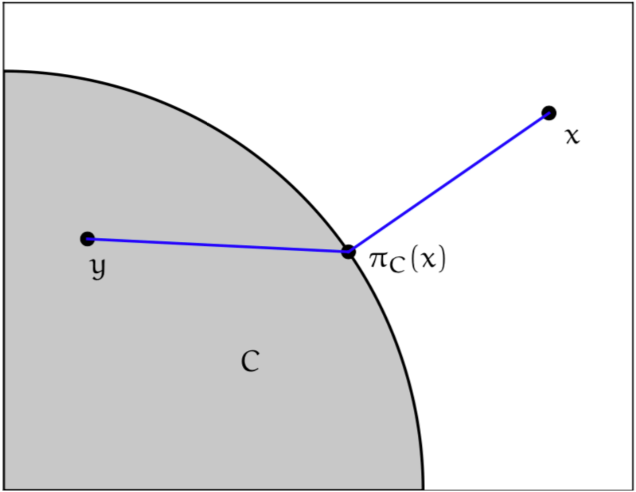
$$\mathbf{z} = \pi_{\mathcal{C}}(\mathbf{x}),$$

if and only if

$$\langle \mathbf{x} - \mathbf{z}, \mathbf{y} - \mathbf{z} \rangle \leq 0,$$

for all $\mathbf{y} \in \mathcal{C}$*.*

- Projection of the point $\mathbf{x}$ onto the set $\mathcal{C}$ (with projection $\pi_{\mathcal{C}}(\mathbf{x})$), exhibiting $\langle \mathbf{x} - \pi_{\mathcal{C}}(\mathbf{x}), \mathbf{y} - \pi_{\mathcal{C}}(\mathbf{x}) \rangle \leq 0$.



Corollary 2 (Nonexpansiveness)

Projections onto closed convex sets are nonexpansive, in particular,

$$\|\pi_{\mathcal{C}}(\mathbf{x}) - \mathbf{y}\|_2 \leq \|\mathbf{x} - \mathbf{y}\|_2$$

for any $\mathbf{x} \in \mathbb{R}^n$ *and* $\mathbf{y} \in \mathcal{C}$.

Theorem 3 (Strict separation of points)

Let $\mathcal{C}$ *be a closed convex set. For any* $\mathbf{x} \notin \mathcal{C}$, *the vector*

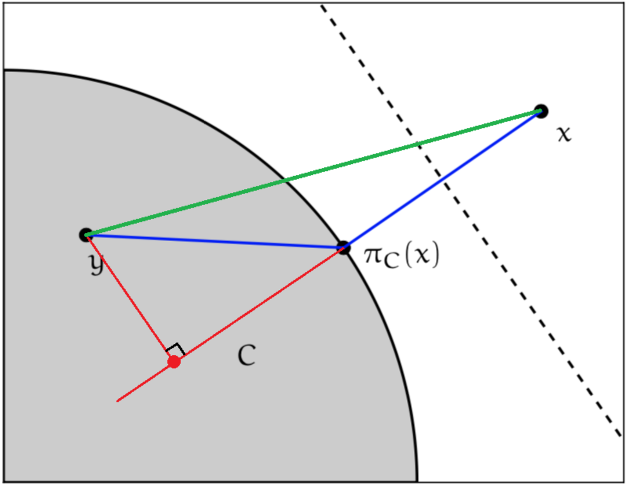
$$\mathbf{v} = \mathbf{x} - \pi_{\mathcal{C}}(\mathbf{x})$$

satisfies

$$\langle \mathbf{v}, \mathbf{x} \rangle \geq \sup_{\mathbf{y} \in \mathcal{C}} \langle \mathbf{v}, \mathbf{y} \rangle + \|\mathbf{v}\|_2^2 > \sup_{\mathbf{y} \in \mathcal{C}} \langle \mathbf{v}, \mathbf{y} \rangle.$$

This means the strict separation of the point $\mathbf{x} \notin \mathcal{C}$ *from the closed convex set* $\mathcal{C}$.

- Strict separation of $\mathbf{x}$ from $\mathcal{C}$ by the vector $\mathbf{v} = \mathbf{x} - \pi_{\mathcal{C}}(\mathbf{x})$.



- For nonempty sets $\mathcal{S}_1$ and $\mathcal{S}_2$ satisfying $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$, if there exist vector $\mathbf{v} \neq \mathbf{0}$ and scalar b such that

$$\langle \mathbf{v}, \mathbf{x} \rangle \geq b \quad \text{for all} \quad \mathbf{x} \in \mathcal{S}_1,$$

  and

$$\langle \mathbf{v}, \mathbf{x} \rangle \leq b \quad \text{for all} \quad \mathbf{x} \in \mathcal{S}_2,$$

  then

$$\{\mathbf{x} \in \mathbb{R}^n \mid \langle \mathbf{v}, \mathbf{x} \rangle = b\}$$

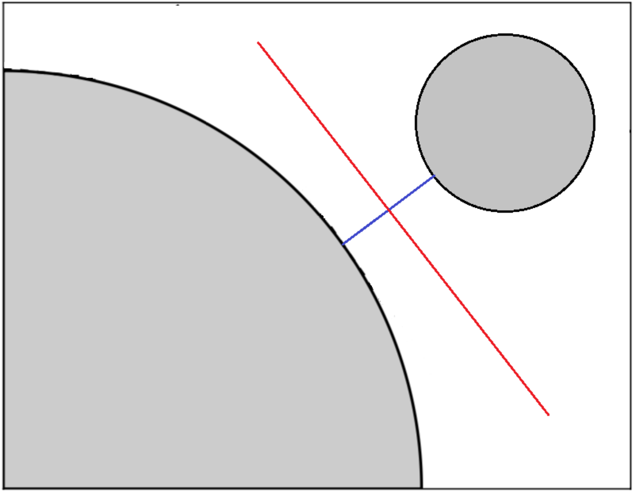
  is called a separating hyperplane for nonempty sets $\mathcal{S}_1$ and $\mathcal{S}_2$.

Theorem 4 (Strict separation of closed convex sets)

Let $\mathcal{C}_1, \mathcal{C}_2$ be closed convex sets, with $\mathcal{C}_2$ compact and $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$. Then there is a vector $\mathbf{v}$ such that

$$\inf_{\mathbf{x} \in \mathcal{C}_1} \langle \mathbf{v}, \mathbf{x} \rangle > \sup_{\mathbf{x} \in \mathcal{C}_2} \langle \mathbf{v}, \mathbf{x} \rangle.$$

- Strict separation of closed convex sets.



- For a set $\mathcal{S}$ and a boundary point $\mathbf{x}$, i.e.,

$$\mathbf{x} \in \mathrm{bd}\mathcal{S} := \mathrm{cl}\mathcal{S} \setminus \mathrm{int}\mathcal{S},$$

if vector $\mathbf{v} \neq \mathbf{0}$ satisfies

$$\langle \mathbf{v}, \mathbf{x} \rangle \geq \langle \mathbf{v}, \mathbf{y} \rangle \quad \text{for all} \quad \mathbf{y} \in \mathcal{S},$$

then

$$\{\mathbf{z} \in \mathbb{R}^n \mid \mathbf{v}^\top(\mathbf{z} - \mathbf{x}) = 0\}$$

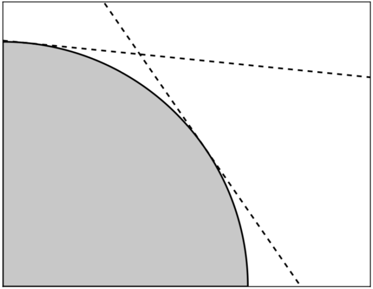
is called a supporting hyperplane supporting $\mathcal{S}$ at $\mathbf{x}$.

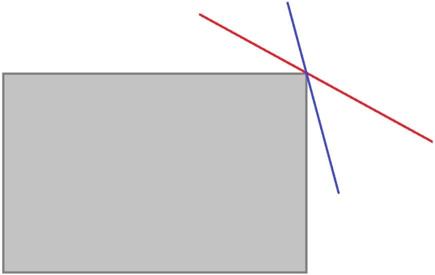
Theorem 5 (Supporting hyperplane theorem)

For convex set $\mathcal{C}$ and any $\mathbf{x} \in \mathrm{bd}\mathcal{C}$, theres exists a supporting hyperplane supporting $\mathcal{C}$ at $\mathbf{x}$, i.e., $\exists\ \mathbf{v} \neq \mathbf{0}$ satisfying

$$\langle \mathbf{v}, \mathbf{x} \rangle \geq \langle \mathbf{v}, \mathbf{y} \rangle \quad \textit{for all} \quad \mathbf{y} \in \mathcal{C}.$$

- Supporting hyperplanes to a convex set. (unique?)





Theorem 6 (Halfspace intersections)

Let $\mathcal{C} \subset \mathbb{R}^n$ be a closed convex set. Then $\mathcal{C}$ is the intersection of all the halfspaces containing it. Moreover, $\mathcal{C} = \bigcap_{\mathbf{x} \in \mathrm{bd}\,\mathcal{C}} \mathcal{H}_{\mathbf{x}}$, where $\mathcal{H}_{\mathbf{x}}$ denotes the intersection of the halfspaces contained in the hyperplanes supporting $\mathcal{C}$ at $\mathbf{x}$.

2. Convex functions

- A function $f : \mathcal{C} \to \mathbb{R}$ defined on a convex set $\mathcal{C} \subseteq \mathbb{R}^n$ is called *convex* (or *convex over* $\mathcal{C}$) if for any $\mathbf{x}, \mathbf{y} \in \mathcal{C}$, $\lambda \in [0, 1]$,

$$f(\lambda \mathbf{x} + (1 - \lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1 - \lambda) f(\mathbf{y}).$$

  It is called *strictly convex* if for any $\mathbf{x} \neq \mathbf{y}$, $\lambda \in (0, 1)$,

$$f(\lambda \mathbf{x} + (1 - \lambda)\mathbf{y}) < \lambda f(\mathbf{x}) + (1 - \lambda) f(\mathbf{y}).$$

  Examples of convex functions: afines functions, norms.

- Jensen's inequality.

  Let $f : \mathcal{C} \to \mathbb{R}$ be a convex function defined on the convex set $\mathcal{C} \subseteq \mathbb{R}^n$. Then for any $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \in \mathcal{C}$ and $\lambda_i \geq 0$, $\sum_{i=1}^k \lambda_i = 1$, the following inequality holds:

$$f\left(\sum_{i=1}^{k} \lambda_i \mathbf{x}_i\right) \leq \sum_{i=1}^{k} \lambda_i f(\mathbf{x}_i).$$

2.1 Characterizations of convex functions

Theorem 7 (the gradient inequality)

Let $f : \mathcal{C} \to \mathbb{R}$ be a continuously differentiable function defined on a nonempty convex set $\mathcal{C} \subseteq \mathbb{R}^n$. Then f is convex over $\mathcal{C}$ if and only if

$$f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x}) \le f(\mathbf{y}) \quad \textit{for any} \quad \mathbf{x}, \mathbf{y} \in \mathcal{C},$$

and f is strictly convex over $\mathcal{C}$ if and only if

$$f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x}) < f(\mathbf{y}) \ \textit{for any} \ \mathbf{x}, \mathbf{y} \in \mathcal{C} \ \textit{satisfying} \ \mathbf{x} \neq \mathbf{y}.$$

Theorem 8 (monotonicity of the gradient)

Suppose that f is a continuously differentiable function over a nonempty convex set $\mathcal{C} \subseteq \mathbb{R}^n$. Then f is convex over $\mathcal{C}$ if and only if

$$(\nabla f(\mathbf{x}) - \nabla f(\mathbf{y}))^\top (\mathbf{x} - \mathbf{y}) \ge 0 \quad \textit{for any} \quad \mathbf{x}, \mathbf{y} \in \mathcal{C}.$$

Proposition 9

Let f be a continuously differentiable function which is convex over a convex set $\mathcal{C} \subseteq \mathbb{R}^n$.

(1) *Suppose that $\nabla f(\mathbf{x}_\star) = \mathbf{0}$ for some $\mathbf{x}_\star \in \mathcal{C}$. Then $\mathbf{x}_\star$ is a global minimizer of f over $\mathcal{C}$.*

(2) *If $\mathcal{C} = \mathbb{R}^n$, then $\nabla f(\mathbf{x}_\star) = \mathbf{0}$ if and only if $\mathbf{x}_\star$ is a global minimizer of f over $\mathbb{R}^n$.*

Theorem 10 (second order characterization of convex functions)

Let f be a twice continuously differentiable function over a nonempty convex set $\mathcal{C} \subseteq \mathbb{R}^n$. Then

(1) *If $\nabla^2 f(\mathbf{x}) \succeq \mathbf{0}$ for any $\mathbf{x} \in \mathcal{C}$, then f is convex over $\mathcal{C}$.*

(2) *If $\nabla^2 f(\mathbf{x}) \succ \mathbf{0}$ for any $\mathbf{x} \in \mathcal{C}$, then f is strictly convex over $\mathcal{C}$.*

(3) *If $\mathcal{C}$ is open, then f is convex over $\mathcal{C}$ if and only if $\nabla^2 f(\mathbf{x}) \succeq \mathbf{0}$ for any $\mathbf{x} \in \mathcal{C}$.*

2.2 Operations preserving convexity

Theorem 11 (nonnegative scalar multiplication and summation)

(1) *Let f be a convex function defined over a convex set $\mathcal{C} \subseteq \mathbb{R}^n$ and let $\alpha \geq 0$. Then αf is a convex function over $\mathcal{C}$.*

(2) *Let $f_1, f_2, \ldots, f_p$ be convex functions over a convex set $\mathcal{C} \subseteq \mathbb{R}^n$. Then the sum function $f_1 + f_2 + \cdots + f_p$ is convex over $\mathcal{C}$.*

Theorem 12 (affine change of variables)

Let $f : \mathcal{C} \to \mathbb{R}$ be a convex function defined on a convex set $\mathcal{C} \subseteq \mathbb{R}^n$. Let $\mathbf{A} \in \mathbb{R}^{n \times m}$ and $\mathbf{b} \in \mathbb{R}^n$. Then the function g defined by

$$g(\mathbf{y}) = f(\mathbf{A}\mathbf{y} + \mathbf{b})$$

is convex over the convex set

$$\mathcal{D} = \{\mathbf{y} \in \mathbb{R}^m : \mathbf{A}\mathbf{y} + \mathbf{b} \in \mathcal{C}\}.$$

Theorem 13 (composition with a nondecreasing convex function)

Let $f : \mathcal{C} \to \mathbb{R}$ be a convex function over the convex set $\mathcal{C} \subseteq \mathbb{R}^n$. Let $g : \mathcal{I} \to \mathbb{R}$ be a one-dimensional nondecreasing convex function over the interval $\mathcal{I} \subseteq \mathbb{R}$. Assume that the image of $\mathcal{C}$ under f is contained in $\mathcal{I}$: $f(\mathcal{C}) \subseteq \mathcal{I}$. Then the composition of g with f defined by

$$h(\mathbf{x}) \equiv g(f(\mathbf{x})), \quad \mathbf{x} \in \mathcal{C},$$

is a convex function over $\mathcal{C}$.

Theorem 14 (pointwise maximum of convex functions)

Let $f_1, \ldots, f_p : \mathcal{C} \to \mathbb{R}$ be p convex functions over the convex set $\mathcal{C} \subseteq \mathbb{R}^n$. Then the maximum function

$$f(\mathbf{x}) = \max_{i=1,\ldots,p} f_i(\mathbf{x})$$

is a convex function over $\mathcal{C}$.

Theorem 15 (partial minimization)

Let $f : \mathcal{C} \times \mathcal{D} \to \mathbb{R}$ *be a convex function defined over the set* $\mathcal{C} \times \mathcal{D}$, *where* $\mathcal{C} \subseteq \mathbb{R}^m$ *and* $\mathcal{D} \subseteq \mathbb{R}^n$ *are convex sets. Let*

$$g(\mathbf{x}) = \min_{\mathbf{y} \in \mathcal{D}} f(\mathbf{x}, \mathbf{y}), \quad \mathbf{x} \in \mathcal{C},$$

where we assume that the minimal value (maybe not attained) in the above definition is finite. Then g *is convex over* $\mathcal{C}$.

- Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a nonempty convex set and $\|\cdot\|$ an arbitrary norm. The distance function defined by

$$d(\mathbf{x}, \mathcal{C}) = \min_{\mathbf{y} \in \mathcal{C}} \|\mathbf{x} - \mathbf{y}\|$$

is convex since the function $f(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ is convex over $\mathbb{R}^n \times \mathcal{C}$.

2.3 Level sets of convex functions

- Let $f : \mathcal{S} \to \mathbb{R}$ be a function defined over a set $\mathcal{S} \subseteq \mathbb{R}^n$. Then the *level set* of f with level $\alpha \in \mathbb{R}$ is given by

$$\mathrm{Lev}(f, \alpha) = \{\mathbf{x} \in \mathcal{S} : f(\mathbf{x}) \leq \alpha\}.$$

Theorem 16 (level sets of convex functions are convex)

Let $f : \mathcal{C} \to \mathbb{R}$ be a convex function defined over a convex set $\mathcal{C} \subseteq \mathbb{R}^n$. Then for any $\alpha \in \mathbb{R}$ the level set $\mathrm{Lev}(f, \alpha)$ is convex.

- A function $f : \mathcal{C} \to \mathbb{R}$ defined over the convex set $\mathcal{C} \subseteq \mathbb{R}^n$ is called *quasi-convex* if for any $\alpha \in \mathbb{R}$ the set $\mathrm{Lev}(f, \alpha)$ is convex.
- Quasi-convex functions may be nonconvex.

  For example, $f(x) = \sqrt{|x|}$ with level sets

$$\mathrm{Lev}(f, \alpha) = \begin{cases} [-\alpha^2, \alpha^2], & \alpha \geq 0, \\ \emptyset, & \alpha < 0. \end{cases}$$

2.4 Continuity and differentiability of convex functions

- Convex functions are always continuous at interior points of their domain. Thus, for example, functions which are convex over $\mathbb{R}^n$ are always continuous. A stronger result is given below.

Theorem 17 (local Lipschitz continuity at interior points)

Let $f : \mathcal{C} \to \mathbb{R}$ *be a convex function defined over a convex set* $\mathcal{C} \subseteq \mathbb{R}^n$*. Let* $\mathbf{x}_0 \in \operatorname{int}(\mathcal{C})$*. Then there exist* $\varepsilon > 0$ *and* $L > 0$ *such that* $\mathcal{B}[\mathbf{x}_0, \varepsilon] \subseteq \mathcal{C}$ *and*

$$|f(\mathbf{x}) - f(\mathbf{x}_0)| \leq L\|\mathbf{x} - \mathbf{x}_0\|$$

for all $\mathbf{x} \in \mathcal{B}[\mathbf{x}_0, \varepsilon]$.

Theorem 18 (existence of directional derivatives at interior points)

Let $f : \mathcal{C} \to \mathbb{R}$ *be a convex function defined over a convex set* $\mathcal{C} \subseteq \mathbb{R}^n$*. Let* $\mathbf{x} \in \operatorname{int}(\mathcal{C})$*. Then for any* $\mathbf{d} \neq \mathbf{0}$*, the directional derivative* $f'(\mathbf{x}; \mathbf{d})$ *exists.*

2.5 Extended real-valued function

- The *effective domain* of an *extended real-valued function* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined as

$$\mathrm{dom}(f) := \{\mathbf{x} \mid f(\mathbf{x}) < +\infty\}.$$

- An extended real-valued function is called *proper* if there exists at least one $\mathbf{x} \in \mathbb{R}^n$ such that $f(\mathbf{x}) < +\infty$, meaning that $\mathrm{dom}(f) \neq \emptyset$.
- An extended real-valued function f is convex if $\forall\ \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\alpha \in [0, 1]$ the following inequality holds:

$$f((1-\alpha)\mathbf{x} + \alpha\mathbf{y}) \leqslant (1-\alpha) f(\mathbf{x}) + \alpha f(\mathbf{y}),$$

where we use the arithmetic with $+\infty$:

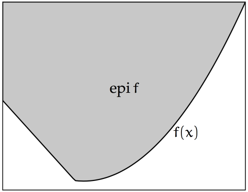
$$a + (+\infty) = +\infty \ (a \in \mathbb{R}), \quad b \cdot (+\infty) = +\infty \ (b > 0),$$

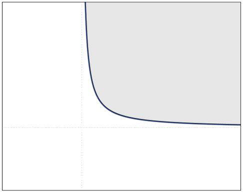
and

$$0 \cdot (+\infty) = 0.$$

- The definition of convexity of extended real-valued functions is equivalent to saying that $\mathrm{dom}(f)$ is a convex set and that the restriction of f to its effective domain $\mathrm{dom}(f)$ is a convex function.
- The *epigraph* of $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\mathrm{epi}(f) = \{(\mathbf{x}, y) : f(\mathbf{x}) \leq y, \mathbf{x} \in \mathbb{R}^n, y \in \mathbb{R}\}.$$





An extended real-valued function f convex "$\Leftrightarrow$" $\mathrm{epi}(f)$ convex.

Theorem 19

Let $f_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued convex function for any $i \in \mathcal{I}$ ($\mathcal{I}$ being an arbitrary index set). Then $f(\mathbf{x}) = \max_{i\in\mathcal{I}} f_i(\mathbf{x})$ is an extended real-valued convex function.

2.6 Maxima of convex functions

Theorem 20

Let $f : \mathcal{C} \to \mathbb{R}$ be a convex function which is not constant over the convex set $\mathcal{C}$. Then f does not attain a maximum at a point in $\operatorname{int}(\mathcal{C})$.

- Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a convex set. A point $\mathbf{x} \in \mathcal{C}$ is called an *extreme point* of $\mathcal{C}$ if there do not exist $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{C}, \mathbf{x}_1 \neq \mathbf{x}_2$, and $\lambda \in (0, 1)$ such that $\mathbf{x} = \lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2$. The set of extreme points is denoted by $\operatorname{ext}(\mathcal{C})$.

Theorem 21 (Krein–Milman)

Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a compact convex set. Then $\mathcal{C} = \operatorname{conv}(\operatorname{ext}(\mathcal{C}))$.

Theorem 22

Let $f : \mathcal{C} \to \mathbb{R}$ be a convex and continuous function over the nonempty convex and compact set $\mathcal{C} \subseteq \mathbb{R}^n$. Then there exists at least one maximizer of f over $\mathcal{C}$ that is an extreme point of $\mathcal{C}$.

2.7 Convexity and inequalities

- The arithmetic geometric mean inequality

  For any $x_1, \dots, x_n \geq 0$ and $\boldsymbol{\lambda} \in \Delta_n$ the following inequality holds:

$$\sum_{i=1}^{n} \lambda_i x_i \geq \prod_{i=1}^{n} x_i^{\lambda_i}.$$

- Young's inequality

  For any $s, t \geq 0$ and $p, q > 1$ satisfying $1/p + 1/q = 1$ it holds that

$$st \leq s^p/p + t^q/q.$$

- Hölder's inequality

  For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $p, q \in [1, \infty]$ satisfying $1/p + 1/q = 1$, it holds that

$$|\mathbf{x}^\top \mathbf{y}| \leq \|\mathbf{x}\|_p \|\mathbf{y}\|_q.$$

- Minkowski's inequality

  Let $p \geq 1$. For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\|\mathbf{x} + \mathbf{y}\|_p \leq \|\mathbf{x}\|_p + \|\mathbf{y}\|_p$.