

# Numerical Linear Algebra Assignment 4

## Exercise 1. (10 points)

Let  $\mathbf{x} = [3 \ 4 \ 0]^\top$  and  $\mathbf{e}_3 = [0 \ 0 \ 1]^\top$ . Try to find all possible numbers  $z \in \mathbb{C}$  and the corresponding Householder reflectors  $\mathbf{H}$  such that  $\mathbf{H}\mathbf{x} = z\mathbf{e}_3$ .

## Exercise 2. (TreBau Exercise 19.1, 10 points)

Given  $\mathbf{A} \in \mathbb{C}^{m \times n}$  of rank  $n$  and  $\mathbf{b} \in \mathbb{C}^m$ , consider the block  $2 \times 2$  system of equations

$$\begin{bmatrix} \mathbf{I} & \mathbf{A} \\ \mathbf{A}^* & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix},$$

where  $\mathbf{I}$  is the  $m \times m$  identity. Show that this system has a unique solution  $\begin{bmatrix} \mathbf{r} \\ \mathbf{x} \end{bmatrix}$ , and that the vectors  $\mathbf{r}$  and  $\mathbf{x}$  are the residual and the least squares solution of the least squares problem: Given  $\mathbf{A} \in \mathbb{C}^{m \times n}$  of full rank,  $m \geq n$ ,  $\mathbf{b} \in \mathbb{C}^m$ , find  $\mathbf{x} \in \mathbb{C}^n$  such that  $\|\mathbf{b} - \mathbf{A}\mathbf{x}\|_2$  is minimized.

## Exercise 3. (Demmel Question 3.11, 10 points)

Let  $\mathbf{A} \in \mathbb{C}^{m \times n}$ . Show that  $\mathbf{X} = \mathbf{A}^\dagger$  (the Moore–Penrose pseudoinverse) minimizes  $\|\mathbf{A}\mathbf{X} - \mathbf{I}\|_F$  over all  $n \times m$  matrices. What is the value of this minimum?

## Exercise 4. (10 points)

Let  $\mathbf{A} \in \mathbb{C}^{m \times n}$  and  $\mathbf{b} \in \mathbb{C}^m$ . Solve the *penalized* problem

$$\min_{\mathbf{x} \in \mathbb{C}^n} \{ \|\mathbf{b} - \mathbf{A}\mathbf{x}\|_2^2 + \lambda^2 \|\mathbf{x}\|_2^2 \},$$

where  $\lambda > 0$ . Hint: consider the LSP

$$\min_{\mathbf{x} \in \mathbb{C}^n} \left\| \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{A} \\ \lambda \mathbf{I} \end{bmatrix} \mathbf{x} \right\|_2.$$

## Exercise 5. (10 points)

Let  $\mathbf{A} \in \mathbb{C}^{m \times n}$ ,  $\mathbf{b} \in \mathbb{C}^n$ , and  $\mathbf{y} \in \mathbb{C}^m$  be given. Assume that  $\text{rank}(\mathbf{A}) < \min\{m, n\}$  and  $\mathbf{y} \in \text{range}(\mathbf{A})$ . Solve the following problem

$$\min_{\mathbf{x} \in \mathbb{C}^n, \mathbf{A}\mathbf{x}=\mathbf{y}} \|\mathbf{b} - \mathbf{x}\|_2.$$

## Exercise 6. (Programming, TreBau Exercises 10.2, 10 points)

## Exercise 7. (Programming, TreBau Exercises 10.3, 10 points)