Data Analysis and Matrix Computations Assignment 1

Exercise 1.

Prove the following: If $\mathbf{T}$ is any fixed $m \times n$ matrix, and $\mathbf{g} \in \mathbb{R}^n$ is a standard Gaussian random vector (i.e., $\mathbf{g} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$, which implies $\mathbf{g}$ has independent identically distributed (i.i.d.) random elements $g_i \sim \mathcal{N}(0,1)$), then

$$\mathbb{E}[\|\mathbf{T}\mathbf{g}\|_2^2] = \|\mathbf{T}\|_{\mathrm{F}}^2.$$

Exercise 2.

Prove the expectation of a quadratic form: Let $\boldsymbol{X}$ be a random vector and $\mathbf{A}$ a fixed matrix. If $\mathbb{E}(\boldsymbol{X}) = \boldsymbol{\mu}$, then

$$\mathbb{E}(\boldsymbol{X}^\top \mathbf{A} \boldsymbol{X}) = \boldsymbol{\mu}^\top \mathbf{A} \boldsymbol{\mu} + \mathrm{tr}[\mathbf{A} \mathbb{V}\mathrm{ar}(\boldsymbol{X})],$$

where tr denotes the trace of the matrix.

Exercise 3.

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{x}^0 \in \mathbb{R}^n$, and $\mathbf{b} \in \mathbb{R}^m$ be given. Solve the following problem by using Lagrange multiplier method.

$$\min_{\mathbf{x} \in \mathbb{R}^n,\ \mathbf{A}\mathbf{x} = \mathbf{b}} \frac{1}{2} \|\mathbf{x} - \mathbf{x}^0\|_2^2.$$

Exercise 4.

Consider the following relaxed RK algorithm.

Algorithm: Relaxed RK for $\mathbf{A}\mathbf{x} = \mathbf{b}$

Initialize $\mathbf{x}^0 \in \mathbb{R}^n$ and $0 < \alpha < 2$
for $k = 1, 2, \ldots,$ **do**
  Pick $i \in [m]$ with probability $\dfrac{\|\mathbf{A}_{i,:}\|_2^2}{\|\mathbf{A}\|_{\mathrm{F}}^2}$
  Set $\mathbf{x}^k = \mathbf{x}^{k-1} - \alpha \dfrac{\mathbf{A}_{i,:}\mathbf{x}^{k-1} - \mathbf{b}_i}{\|\mathbf{A}_{i,:}\|_2^2} (\mathbf{A}_{i,:})^\top$
end

Prove a convergence result of the relaxed RK for consistent linear systems.

Exercise 5.

Prove the convergence result of the REK algorithm:

$$\mathbb{E}\left[\|\mathbf{x}^k - \mathbf{x}_\star^0\|_2^2\right] \leq \rho^k \|\mathbf{x}^0 - \mathbf{x}_\star^0\|_2^2 + \frac{k\rho^k}{\|\mathbf{A}\|_{\mathrm{F}}^2} \|\mathbf{z}^0 - (\mathbf{I} - \mathbf{A}\mathbf{A}^\dagger)\mathbf{b}\|_2^2.$$