Data Analysis and Matrix Computations Assignment 1

Exercise 1.

Prove the following: If $\mathbf{T}$ is any fixed $m \times n$ matrix, and $\mathbf{g} \in \mathbb{R}^n$ is a standard Gaussian random vector (i.e., $\mathbf{g} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$, which implies $\mathbf{g}$ has independent identically distributed (i.i.d.) random elements $g_i \sim \mathcal{N}(0, 1)$), then

$$\mathbb{E}(\|\mathbf{T}\mathbf{g}\|_2^2) = \|\mathbf{T}\|_{\mathrm{F}}^2.$$

Exercise 2.

Prove the expectation of a quadratic form: Let $\boldsymbol{X}$ be a random vector and $\mathbf{A}$ a fixed matrix. If $\mathbb{E}(\boldsymbol{X}) = \boldsymbol{\mu}$, then

$$\mathbb{E}(\boldsymbol{X}^\top \mathbf{A} \boldsymbol{X}) = \boldsymbol{\mu}^\top \mathbf{A} \boldsymbol{\mu} + \mathrm{tr}[\mathbf{A} \mathbb{V}\mathrm{ar}(\boldsymbol{X})],$$

where tr denotes the trace of the matrix.

Exercise 3.

Let $\mathbf{C}$ be a symmetric positive definite matrix. Assume $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{C})$ and $\mathbf{C} = \mathbf{R}^\top \mathbf{R}$ is a Cholesky factorization. Prove that the random vector

$$\boldsymbol{Z} = \mathbf{R}^{-\top}(\boldsymbol{X} - \boldsymbol{\mu})$$

is a standard normal random vector.