Lecture 2: Singular value decomposition (SVD)

School of Mathematical Sciences, Xiamen University

1. Singular value decomposition

- Definition: Let m and n be arbitrary positive integers ($m \geq n$ or $m < n$). Given $\mathbf{A} \in \mathbb{C}^{m \times n}$, not necessarily of full rank, a *singular value decomposition* (SVD) of $\mathbf{A}$ is a factorization

$$\mathbf{A} = \mathbf{U\Sigma V}^*$$

where $\mathbf{U} \in \mathbb{C}^{m \times m}$ is unitary, $\mathbf{V} \in \mathbb{C}^{n \times n}$ is unitary, and $\mathbf{\Sigma} \in \mathbb{R}^{m \times n}$ is diagonal. In addition, it is assumed that the diagonal entries σ_i of $\mathbf{\Sigma}$ are nonnegative and in nonincreasing order; that is

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0,$$

where $p = \min\{m, n\}$.

Theorem 1 (Existence of SVD)

Every matrix $\mathbf{A} \in \mathbb{C}^{m \times n}$ has a singular value decomposition.

Proof. Assume $\mathbf{A} \neq \mathbf{0}$; otherwise we can take $\boldsymbol{\Sigma} = \mathbf{0}$ and let $\mathbf{U}$ and $\mathbf{V}$ be arbitrary unitary matrices. Next, we use induction on m and n to prove the existence of SVD for the case $m \geq n$ (consider $\mathbf{A}^*$ if $m < n$): Assume that an SVD exists for any $(m-1) \times (n-1)$ matrix and prove it for any $m \times n$ matrix.

(i) The basic step: $m \geq n = 1$.

Write $\mathbf{A} = \mathbf{u}_1 \boldsymbol{\Sigma}_1 \mathbf{V}^*$ with $\mathbf{u}_1 = \mathbf{A}/\|\mathbf{A}\|_2, \boldsymbol{\Sigma}_1 = \|\mathbf{A}\|_2$ and $\mathbf{V} = 1$. Choose $\widehat{\mathbf{U}}$ such that $\mathbf{U} = \begin{bmatrix} \mathbf{u}_1 & \widehat{\mathbf{U}} \end{bmatrix} \in \mathbb{C}^{m \times m}$ is unitary. Let $\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_1 & \mathbf{0} \end{bmatrix}^\top \in \mathbb{R}^{m \times 1}$. Then $\mathbf{A}$ has an SVD $\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^*$.

(ii) The induction step: $m \geq n > 1$.

Let $\mathbf{v}_1 \in \mathbb{C}^n$ be a unit (i.e., $\|\mathbf{v}_1\|_2 = 1$) eigenvector corresponding to the eigenvalue $\lambda_{\max}(\mathbf{A}^*\mathbf{A})$. Then we have $\|\mathbf{A}\mathbf{v}_1\|_2 = \|\mathbf{A}\|_2 > 0$. Let $\mathbf{u}_1 = \mathbf{A}\mathbf{v}_1/\|\mathbf{A}\mathbf{v}_1\|_2$, which is a unit vector. Choose $\widehat{\mathbf{U}}$ and $\widehat{\mathbf{V}}$ such that $\widetilde{\mathbf{U}} = \begin{bmatrix} \mathbf{u}_1 & \widehat{\mathbf{U}} \end{bmatrix} \in \mathbb{C}^{m \times m}$ and $\widetilde{\mathbf{V}} = \begin{bmatrix} \mathbf{v}_1 & \widehat{\mathbf{V}} \end{bmatrix} \in \mathbb{C}^{n \times n}$ are unitary.

Now we have

$$\widetilde{\mathbf{U}}^*\mathbf{A}\widetilde{\mathbf{V}} = \begin{bmatrix}\mathbf{u}_1^* \\ \widehat{\mathbf{U}}^*\end{bmatrix} \mathbf{A} \begin{bmatrix}\mathbf{v}_1 & \widehat{\mathbf{V}}\end{bmatrix} = \begin{bmatrix}\mathbf{u}_1^*\mathbf{A}\mathbf{v}_1 & \mathbf{u}_1^*\mathbf{A}\widehat{\mathbf{V}} \\ \widehat{\mathbf{U}}^*\mathbf{A}\mathbf{v}_1 & \widehat{\mathbf{U}}^*\mathbf{A}\widehat{\mathbf{V}}\end{bmatrix}.$$

We note that

$$\mathbf{u}_1^*\mathbf{A}\mathbf{v}_1 = \frac{(\mathbf{A}\mathbf{v}_1)^*(\mathbf{A}\mathbf{v}_1)}{\|\mathbf{A}\mathbf{v}_1\|_2} = \|\mathbf{A}\mathbf{v}_1\|_2 = \|\mathbf{A}\|_2,$$

and

$$\widehat{\mathbf{U}}^*\mathbf{A}\mathbf{v}_1 = \widehat{\mathbf{U}}^*\mathbf{u}_1\|\mathbf{A}\mathbf{v}_1\|_2 = \mathbf{0}.$$

We claim $\mathbf{u}_1^*\mathbf{A}\widehat{\mathbf{V}} = \mathbf{0}$ too because otherwise

$$\begin{aligned}
\sigma_1 &:= \|\mathbf{A}\|_2 = \|\widetilde{\mathbf{U}}^*\mathbf{A}\widetilde{\mathbf{V}}\|_2 \\
&= \|\begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}\|_2 \cdot \|\widetilde{\mathbf{U}}^*\mathbf{A}\widetilde{\mathbf{V}}\|_2 \\
&\geq \|\begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}\widetilde{\mathbf{U}}^*\mathbf{A}\widetilde{\mathbf{V}}\|_2 = \|[\sigma_1 \ \mathbf{u}_1^*\mathbf{A}\widehat{\mathbf{V}}]\|_2 > \sigma_1,
\end{aligned}$$

which is a contradiction.

Therefore,

$$\widetilde{\mathbf{U}}^*\mathbf{A}\widetilde{\mathbf{V}} = \begin{bmatrix} \sigma_1 & \mathbf{0} \\ \mathbf{0} & \widehat{\mathbf{U}}^*\mathbf{A}\widehat{\mathbf{V}} \end{bmatrix}.$$

The $(m-1)\times(n-1)$ matrix $\widehat{\mathbf{U}}^*\mathbf{A}\widehat{\mathbf{V}}$ has an SVD (by the induction hypothesis):

$$\widehat{\mathbf{U}}^*\mathbf{A}\widehat{\mathbf{V}} = \bar{\mathbf{U}}\bar{\mathbf{\Sigma}}\bar{\mathbf{V}}^*.$$

It follows from

$$\widetilde{\mathbf{U}}^*\mathbf{A}\widetilde{\mathbf{V}} = \begin{bmatrix} \sigma_1 & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{U}}\bar{\mathbf{\Sigma}}\bar{\mathbf{V}}^* \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{U}} \end{bmatrix} \begin{bmatrix} \sigma_1 & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{\Sigma}} \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{V}} \end{bmatrix}^*,$$

that

$$\mathbf{A} = \widetilde{\mathbf{U}} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{U}} \end{bmatrix} \begin{bmatrix} \sigma_1 & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{\Sigma}} \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{V}} \end{bmatrix}^* \widetilde{\mathbf{V}}^* =: \mathbf{U} \begin{bmatrix} \sigma_1 & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{\Sigma}} \end{bmatrix} \mathbf{V}^*.$$

This is an SVD of $\mathbf{A}$ because $\sigma_1 \geq \|\bar{\mathbf{\Sigma}}\|_2$. $\square$

- Full SVD:

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^*$$

- Reduced SVD (the case $m \geq n$):

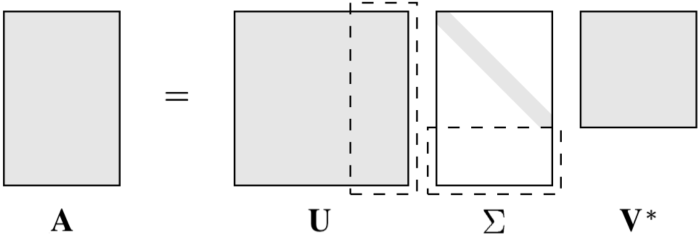
$$\mathbf{A} = \mathbf{U}_n\mathbf{\Sigma}_n\mathbf{V}^*$$

where

$$\mathbf{U}_n = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix},$$

and

$$\mathbf{\Sigma}_n = \mathrm{diag}\{\sigma_1, \sigma_2, \cdots, \sigma_n\}.$$



- Rank SVD or compact SVD or condensed SVD:

$$\mathbf{A} = \begin{bmatrix} \mathbf{U}_r & \mathbf{U}_\mathrm{c} \end{bmatrix} \begin{bmatrix} \mathbf{\Sigma}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{V}_r^* \\ \mathbf{V}_\mathrm{c}^* \end{bmatrix} = \mathbf{U}_r \mathbf{\Sigma}_r \mathbf{V}_r^* = \sum_{i=1}^r \sigma_i \mathbf{u}_i \mathbf{v}_i^*$$

  where $r = \mathrm{rank}(\mathbf{A})$,

$$\mathbf{U}_r = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_r \end{bmatrix}, \quad \mathbf{U}_\mathrm{c} = \begin{bmatrix} \mathbf{u}_{r+1} & \mathbf{u}_{r+2} & \cdots & \mathbf{u}_m \end{bmatrix},$$

$$\mathbf{V}_r = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_r \end{bmatrix}, \quad \mathbf{V}_\mathrm{c} = \begin{bmatrix} \mathbf{v}_{r+1} & \mathbf{v}_{r+2} & \cdots & \mathbf{v}_n \end{bmatrix},$$

  and

$$\mathbf{\Sigma}_r = \mathrm{diag}\{\sigma_1, \sigma_2, \cdots, \sigma_r\}.$$

- $\{\sigma_i^2, \mathbf{u}_i\}$ are eigenvalue-eigenvector pairs of $\mathbf{A}\mathbf{A}^*$, and $\{\sigma_i^2, \mathbf{v}_i\}$ are eigenvalue-eigenvector pairs of $\mathbf{A}^*\mathbf{A}$:

$$\mathbf{A}\mathbf{A}^*\mathbf{u}_i = \sigma_i^2 \mathbf{u}_i, \quad \mathbf{A}^*\mathbf{A}\mathbf{v}_i = \sigma_i^2 \mathbf{v}_i, \quad i = 1, 2, \ldots, p$$

- $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p$ are called the *singular values* of $\mathbf{A}$.

- $\mathbf{u}_i$ is called *left singular vector*, and $\mathbf{v}_i$ is called *right singular vector*: $\quad \mathbf{u}_i^* \mathbf{A} = \sigma_i \mathbf{v}_i^*, \quad \mathbf{A}\mathbf{v}_i = \sigma_i \mathbf{u}_i, \quad i = 1, 2, \ldots, p$

Theorem 2

The set of singular values $\{\sigma_i\}$ is uniquely determined and invariant under unitary multiplication.

Theorem 3

If $\mathbf{A}$ is square and all the σ_i are distinct, the left and right singular vectors are uniquely determined up to complex signs (i.e., complex scalar factors of absolute value 1).

Hint: There exists only one linearly independent eigenvector for each distinct eigenvalue of $\mathbf{A}^*\mathbf{A}$ and $\mathbf{A}^*\mathbf{A}$.

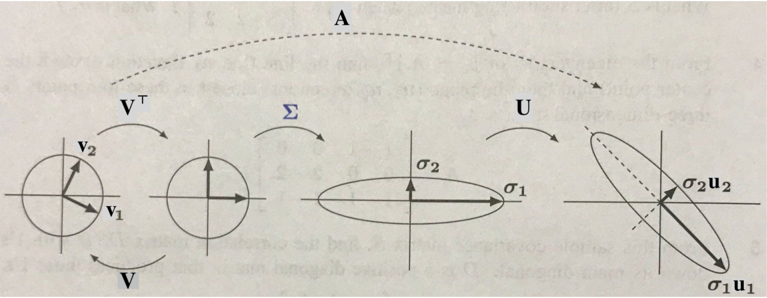
Theorem 4 (Real SVD)

Every matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ has a real singular value decomposition.

1.1. Geometric observation

- The image of the unit sphere (in the 2-norm) of $\mathbb{C}^n$ under any $m \times n$ matrix is a hyperellipse of $\mathbb{C}^m$.

  For example, 2×2 real matrix $\mathbf{A}$



SVD of a matrix can not be emphasized too much!

2. Matrix properties via SVD

- 2-norm

$$\|\mathbf{A}\|_2 = \sigma_1$$

- F-norm

$$\|\mathbf{A}\|_{\mathrm{F}} = \sqrt{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_r^2}$$

- range($\mathbf{A}$): *column space* of $\mathbf{A}$, spanned by the columns of $\mathbf{A}$

$$\begin{aligned} \mathrm{range}(\mathbf{A}) : &= \{\mathbf{y} \in \mathbb{C}^m \mid \exists \mathbf{x} \in \mathbb{C}^n \quad s.t. \quad \mathbf{y} = \mathbf{A}\mathbf{x}\} \\ &= \mathrm{span}\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_r\} \end{aligned}$$

- null($\mathbf{A}$): *kernel* or *null space* of $\mathbf{A}$

$$\begin{aligned} \mathrm{null}(\mathbf{A}) : &= \{\mathbf{x} \in \mathbb{C}^n \mid \mathbf{A}\mathbf{x} = \mathbf{0}\} \\ &= \mathrm{span}\{\mathbf{v}_{r+1}, \mathbf{v}_{r+2}, \cdots, \mathbf{v}_n\} \end{aligned}$$

- Range and null space of $\mathbf{A}^*$:

$$\text{range}(\mathbf{A}^*) = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r\} = \text{null}(\mathbf{A})^\perp$$

$$\text{null}(\mathbf{A}^*) = \text{span}\{\mathbf{u}_{r+1}, \mathbf{u}_{r+2}, \cdots, \mathbf{u}_m\} = \text{range}(\mathbf{A})^\perp$$

- Relations between the four subspaces

$$\text{range}(\mathbf{A}^*) \perp \text{null}(\mathbf{A}), \quad \text{range}(\mathbf{A}^*) + \text{null}(\mathbf{A}) = \mathbb{C}^n$$

$$\text{range}(\mathbf{A}) \perp \text{null}(\mathbf{A}^*), \quad \text{range}(\mathbf{A}) + \text{null}(\mathbf{A}^*) = \mathbb{C}^m$$

- If $\mathbf{A}$ is Hermitian, i.e., $\mathbf{A} = \mathbf{A}^*$

  singular values are absolute values of eigenvalues

- Determinant of $\mathbf{A} \in \mathbb{C}^{m\times m}$

$$|\det(\mathbf{A})| = \prod_{i=1}^{m} \sigma_i$$

2.1. Low-rank approximation (LRA)

Theorem 5 (Eckart-Young-Mirski)

For any integer k *with* $1 \le k < r = \operatorname{rank}(\mathbf{A})$, *define*

$$\mathbf{A}_k = \sum\nolimits_{i=1}^{k} \sigma_i \mathbf{u}_i \mathbf{v}_i^*.$$

Then

$$\|\mathbf{A} - \mathbf{A}_k\|_2 = \min_{\substack{\mathbf{B} \in \mathbb{C}^{m \times n}, \\ \operatorname{rank}(\mathbf{B}) \le k}} \|\mathbf{A} - \mathbf{B}\|_2 = \sigma_{k+1},$$

and

$$\|\mathbf{A} - \mathbf{A}_k\|_{\mathrm{F}} = \min_{\substack{\mathbf{B} \in \mathbb{C}^{m \times n}, \\ \operatorname{rank}(\mathbf{B}) \le k}} \|\mathbf{A} - \mathbf{B}\|_{\mathrm{F}} = \sqrt{\sigma_{k+1}^2 + \cdots + \sigma_r^2}.$$

- Discussion: Is the minimizer in Theorem 5 unique?

  A random $m \times m$ matrix is "always" nonsingular. Why?

Proof of Theorem 5.

- Suppose there is some $\mathbf{B} \in \mathbb{C}^{m\times n}$ with $\text{rank}(\mathbf{B}) \leq k < r$ such that

$$\|\mathbf{A} - \mathbf{B}\|_2 < \sigma_{k+1} = \|\mathbf{A} - \mathbf{A}_k\|_2.$$

  Then there exists an $(n-k)$-dimensional subspace $\mathcal{W} \subseteq \text{null}(\mathbf{B})$. For any nonzero $\mathbf{x} \in \mathcal{W}$, we have

$$\|\mathbf{A}\mathbf{x}\|_2 = \|(\mathbf{A} - \mathbf{B})\mathbf{x}\|_2 \leq \|\mathbf{A} - \mathbf{B}\|_2 \|\mathbf{x}\|_2 < \sigma_{k+1}\|\mathbf{x}\|_2.$$

  Let $\mathcal{V} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{k+1}\}$. For any $\mathbf{x} \in \mathcal{V}$, we have

$$\|\mathbf{A}\mathbf{x}\|_2 = \|\mathbf{A}\mathbf{V}_{k+1}\mathbf{y}\|_2 = \|\mathbf{U}_{k+1}\mathbf{\Sigma}_{k+1}\mathbf{y}\|_2 = \|\mathbf{\Sigma}_{k+1}\mathbf{y}\|_2 \geq \sigma_{k+1}\|\mathbf{x}\|_2.$$

  Since $\dim\mathcal{W} + \dim\mathcal{V} = (n-k) + (k+1) > n$, there must be a nonzero vector lying in both, and this is a contradiction.

- The case for $\|\cdot\|_\text{F}$, see Page 213 of Generalized Inverses: Theory and Applications, 2nd edition, Adi Ben-Israel and Thomas N.E. Greville. $\square$

Application of low-rank approximation: image compression

- An image can be represented as a matrix. For example, typical grayscale images consist of a rectangular array of pixels, m in the vertical direction, n in the horizontal direction. The color of each of those pixels is denoted by a single number, an integer between 0 (black) and 255 (white). (This gives $2^8 = 256$ different shades of gray for each pixel. Color images are represented by three such matrices: one for red, one for green, and one for blue. Thus each pixel in a typical color image takes $(2^8)^3 = 2^{24}$ shades.)
- The objective of image compression is to reduce irrelevance and redundancy of the image data in order to be able to store or transmit data in an efficient form.
- Low-rank SVD approximation is a good candidate. (Note: `jpeg` compression algorithm uses similar idea, on subimages)

3. Moore–Penrose pseudoinverse

- Let $\mathbf{A} \in \mathbb{C}^{m\times n}$ have an SVD (rank form) $\mathbf{A} = \mathbf{U}_r \mathbf{\Sigma}_r \mathbf{V}_r^*$. The *Moore–Penrose pseudoinverse* of $\mathbf{A}$, denoted by $\mathbf{A}^\dagger$:

$$\mathbf{A}^\dagger := \mathbf{V}_r \mathbf{\Sigma}_r^{-1} \mathbf{U}_r^* = \sum\nolimits_{i=1}^{r} \frac{1}{\sigma_i} \mathbf{v}_i \mathbf{u}_i^*.$$

- The matrix $\mathbf{A}^\dagger$ is the *unique* matrix satisfying the four equations

$$\mathbf{AXA} = \mathbf{A}, \quad \mathbf{XAX} = \mathbf{X}, \quad (\mathbf{AX})^* = \mathbf{AX}, \quad (\mathbf{XA})^* = \mathbf{XA}.$$

  For a proof, see Page 122 of Numerical linear algebra (in Chinese) by Zhihao Cao.

- If $\mathbf{A}$ has full column rank, then $\mathbf{A}^\dagger = (\mathbf{A}^*\mathbf{A})^{-1}\mathbf{A}^*$.
  If $\mathbf{A}$ has full row rank, then $\mathbf{A}^\dagger = \mathbf{A}^*(\mathbf{A}\mathbf{A}^*)^{-1}$.

4. A wonderful reference

- Zhihua Zhang
  The singular value decomposition, applications and beyond
  arXiv:1510.08532

5. An alternative proof of Theorem 5

- Holger Wendland
  Numerical Linear Algebra An Introduction
  Cambridge University Press, 2018.
  See Page 295, Theorem 7.41.

6. A computationally more feasible method for LRA

- Adaptive cross approximation (ACA)
  See Page 297 of Holger Wendland's book.