Numerical Linear Algebra Assignment 3

Exercise 1. (10 points)

(1) Let $\mathbf{P}$ be a projector. Given an explicit expression for the inverse of $\lambda\mathbf{I} - \mathbf{P}$, where $\lambda \neq 0, 1$.

(2) Suppose $\mathbf{A} \in \mathbb{C}^{m\times n}$ has a full SVD $\mathbf{A} = \mathbf{U\Sigma V}^*$, where

$$\mathbf{U} = [\mathbf{U}_r \quad \mathbf{U}_\mathrm{c}], \quad \mathbf{V} = [\mathbf{V}_r \quad \mathbf{V}_\mathrm{c}], \quad r = \mathrm{rank}(\mathbf{A}).$$

What are the orthogonal projections onto $\mathrm{null}(\mathbf{A})^\perp$, $\mathrm{null}(\mathbf{A})$, $\mathrm{range}(\mathbf{A})$, and $\mathrm{range}(\mathbf{A})^\perp$?

Exercise 2. (Carl D. Meyer, 10 points)

Let $\mathbf{P}$ and $\mathbf{Q}$ be projectors (oblique or orthogonal).

(i) Prove that $\mathrm{range}(\mathbf{P}) = \mathrm{range}(\mathbf{Q})$ if and only if $\mathbf{PQ} = \mathbf{Q}$ and $\mathbf{QP} = \mathbf{P}$.

(ii) Prove that $\mathrm{null}(\mathbf{P}) = \mathrm{null}(\mathbf{Q})$ if and only if $\mathbf{PQ} = \mathbf{P}$ and $\mathbf{QP} = \mathbf{Q}$.

Exercise 3. (10 points)

Two subspaces $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathbb{C}^m$ are called *complementary subspaces* if they satisfy

$$\mathcal{S}_1 \cap \mathcal{S}_2 = \{\mathbf{0}\}, \qquad \mathcal{S}_1 + \mathcal{S}_2 = \mathbb{C}^m.$$

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be complementary subspaces. Prove that there exists a projector $\mathbf{P}$ with

$$\mathrm{range}(\mathbf{P}) = \mathcal{S}_1, \qquad \mathrm{null}(\mathbf{P}) = \mathcal{S}_2.$$

Exercise 4. (TreBau Exercise 6.1, 10 points)

If $\mathbf{P}$ is an orthogonal projector, then $\mathbf{I}-2\mathbf{P}$ is unitary. Prove this algebraically, and give a geometric interpretation.

Exercise 5. (TreBau Exercise 6.5, 10 points)

Let $\mathbf{P} \in \mathbb{C}^{m\times m}$ be a nonzero projector. Show that $\|\mathbf{P}\|_2 \geq 1$, with equality if and only if $\mathbf{P}$ is an orthogonal projector.

Exercise 6. (10 points)

Let $\mathcal{S} \subseteq \mathbb{C}^m$ and $\mathcal{T} \subseteq \mathbb{C}^m$. Let $\mathbf{P}_\mathcal{S}$ and $\mathbf{P}_\mathcal{T}$ be orthogonal projectors onto $\mathcal{S}$ and $\mathcal{T}$, respectively. Assume that $\mathcal{S} \subseteq \mathcal{T}$.

(i) Prove that $\mathbf{P}_\mathcal{S}\mathbf{P}_\mathcal{T} = \mathbf{P}_\mathcal{T}\mathbf{P}_\mathcal{S} = \mathbf{P}_\mathcal{S}$.

(ii) Prove that $\mathbf{P}_\mathcal{T} - \mathbf{P}_\mathcal{S}$ is also an orthogonal projection.

(iii) $\mathrm{range}(\mathbf{P}_\mathcal{T} - \mathbf{P}_\mathcal{S}) = ?$ $\quad \mathrm{null}(\mathbf{P}_\mathcal{T} - \mathbf{P}_\mathcal{S}) = ?$

Exercise 7. (TreBau Exercise 7.3, 10 points)

Let $\mathbf{A}$ be an $m \times m$ matrix, and let $\mathbf{a}_j$ be its jth column. Give an algebraic proof of *Hadamard's inequality*:

$$|\det(\mathbf{A})| \leq \prod_{j=1}^{m} \|\mathbf{a}_j\|_2.$$

Also give a geometric interpretation of this result, making use of the fact that the determinant equals the volume of a parallelepiped.

Exercise 8. (TreBau Exercise 7.5, 10 points)

Let $\mathbf{A}$ be an $m \times n$ matrix ($m \geq n$), and let $\mathbf{A} = \mathbf{Q}_n\mathbf{R}_n$ be a reduced QR factorization.

(a) Show that $\mathbf{A}$ has rank n if and only if all the diagonal entries of $\mathbf{R}_n$ are nonzero.

(b) Suppose $\mathbf{R}_n$ has k nonzero diagonal entries for some k with $0 \leq k \leq n$. What does this imply about the rank of $\mathbf{A}$? Exactly k? At least k? At most k? Give a precise answer, and prove it.

Exercise 9. (10 points)

Compute a QR factorization of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ \sqrt{2} & 1+\sqrt{2} & 1 \\ 1 & 2 & 1 \end{bmatrix}$.

Exercise 10. (Programming, TreBau Exercise 8.2, 10 points)

Additional Exercise 1.

Let $C[-1,1]$ denote the linear space of real-valued continuous functions on $[-1,1]$ with the inner product

$$\forall f, g \in C[-1,1], \qquad \langle f, g \rangle_w = \int_{-1}^{1} w(x)f(x)g(x)\mathrm{d}x,$$

where $w(x) \geq 0 (\not\equiv 0)$ is a weight function (continuous). For the case $w(x) = 1 + x^2$, complete the following:

(i) Write MATLAB code to compute the first six orthogonal (with respect to the inner product $\langle \cdot, \cdot \rangle_w$) polynomials ($P_j(x), j = 0, 1, 2, 3, 4, 5$, which are conventionally normalized so that $P_j(1) = 1$). Hint: you can use MATLAB's symbolic toolbox. For your reference, the polynomials are given by:

```
P =

                                           1
                                           x
                             (5*x^2)/3 - 2/3
                        (14*x^3)/5 - (9*x)/5
       (119*x^4)/24 - (161*x^2)/36 + 37/72
 (1221*x^5)/136 - (705*x^3)/68 + (325*x)/136
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(ii) Modify the code we used for discrete Legendre polynomials to plot the discrete polynomials corresponding to those obtained in (i).

Additional Exercise 2.

Assume that $\mathbf{B} \in \mathbb{R}^{k \times n}$ has full row rank and that $\mathbf{A} \in \mathbb{R}^{k \times r}$ has full column rank.

(1) Prove that $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\top \mathbf{B}\mathbf{B}^\top \mathbf{y}$ is an inner product.

(2) Give an orthonormal basis (with respect to the inner product in (1)) of range($\mathbf{A}$).