Numerical Linear Algebra Assignment 1

Exercise 1. (10 points)

Prove the Cauchy-Schwarz inequality: For any given inner product $\langle \cdot, \cdot \rangle$,

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} \sqrt{\langle \mathbf{y}, \mathbf{y} \rangle}.$$

The equality holds if and only if $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent.

Exercise 2. (10 points)

Prove that

$$\langle \mathbf{A}, \mathbf{B} \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \overline{b_{ij}}, \quad \forall \mathbf{A}, \mathbf{B} \in \mathbb{C}^{m \times n}$$

is an inner product on $\mathbb{C}^{m \times n}$. (The Frobenius norm on $\mathbb{C}^{m \times n}$ is induced by this inner product.)

Exercise 3. (10 points)

Let $\|\cdot\|$ denote any vector norm on $\mathbb{C}^m$ and $\mathbf{W} \in \mathbb{C}^{m \times m}$ be nonsingular. Prove that $\|\mathbf{x}\|_{\mathbf{W}} = \|\mathbf{W}\mathbf{x}\|$ is a vector norm on $\mathbb{C}^m$.

Exercise 4. (10 points)

Let $\mathbf{A} \in \mathbb{C}^{m \times n}$, $\mathbf{B} \in \mathbb{C}^{n \times r}$ and let $\|\cdot\|_\alpha$, $\|\cdot\|_\beta$, and $\|\cdot\|_\gamma$ be norms on $\mathbb{C}^m, \mathbb{C}^n$, and $\mathbb{C}^r$, respectively. Prove the induced matrix norms $\|\cdot\|_{\alpha,\gamma}$, $\|\cdot\|_{\alpha,\beta}$, and $\|\cdot\|_{\beta,\gamma}$ satisfy $\|\mathbf{AB}\|_{\alpha,\gamma} \leq \|\mathbf{A}\|_{\alpha,\beta} \|\mathbf{B}\|_{\beta,\gamma}$.

Exercise 5. (10 points)

Prove that $\|\mathbf{A}\|_{\infty,1} = \max_{i,j} |a_{ij}|$.

Exercise 6. (TreBau Exercise 3.4, 10 points)

Let $\mathbf{A}$ be an $m \times n$ matrix and let $\mathbf{B}$ be a submatrix of $\mathbf{A}$, that is, an $s \times t$ matrix ($s \leq m, t \leq n$) obtained by selecting certain rows and columns of $\mathbf{A}$.

(a) Explain how $\mathbf{B}$ can be obtained by multiplying $\mathbf{A}$ by certain row and column "deletion matices" as in step 7 of Exercise 1.1.

(b) Using this product, show that $\|\mathbf{B}\|_p \leq \|\mathbf{A}\|_p$ for any p with $1 \leq p \leq \infty$.