

# Lecture 4: Householder reflector, Givens rotation, Least squares problem



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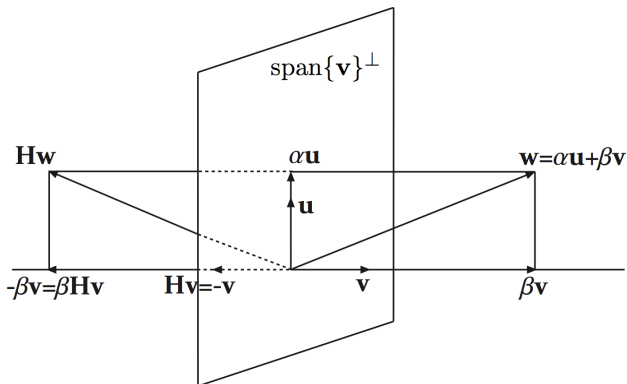
# 1. Householder reflector

- Let  $\mathbf{v} \in \mathbb{C}^m$  and  $\mathbf{v} \neq \mathbf{0}$ . Then the matrix

$$\mathbf{H} = \mathbf{I} - 2 \frac{\mathbf{v}\mathbf{v}^*}{\mathbf{v}^*\mathbf{v}},$$

is called a *Householder reflector*.

- Geometric interpretation



**Exercise:** Householder reflector  $\mathbf{H}$  satisfies the following properties:

- (1) It is *Hermitian*:  $\mathbf{H} = \mathbf{H}^*$
- (2) It is *unitary*:  $\mathbf{H}^* = \mathbf{H}^{-1}$
- (3) It is *involutory*:  $\mathbf{H}^2 = \mathbf{I}$

**Exercise:** What are the eigenvalues, the determinant, and the singular values of a Householder reflector  $\mathbf{H}$ ?

Hint: eigenvalues 1 with multiplicity  $m - 1$  and  $-1$  with multiplicity 1.

**Exercise:** Prove that  $\mathbf{I} - \frac{\mathbf{v}\mathbf{v}^*}{\mathbf{v}^*\mathbf{v}}$  is the orthogonal projector which projects  $\mathbb{C}^m$  onto the *hyperplane*  $\text{span}\{\mathbf{v}\}^\perp$  along  $\text{span}\{\mathbf{v}\}$ .

## Theorem 1

*For all  $\mathbf{x}, \mathbf{y} \in \mathbb{C}^m$  with  $\mathbf{x} \neq \mathbf{y}$ , there exists a Householder reflector  $\mathbf{H}$  such that  $\mathbf{H}\mathbf{x} = \mathbf{y}$  if and only if  $\|\mathbf{x}\|_2 = \|\mathbf{y}\|_2$  and  $\mathbf{x}^*\mathbf{y} \in \mathbb{R}$ .*

## Proof.

“ $\Rightarrow$ ” is easy. “ $\Leftarrow$ ”: let  $\mathbf{v} = \mathbf{y} - \mathbf{x}$ , verify  $\mathbf{H}\mathbf{x} = \mathbf{y}$ . □

## Corollary 2

*For all nonzero  $\mathbf{x}, \mathbf{y} \in \mathbb{C}^m$  with  $\mathbf{x} \neq \mathbf{y}$ , there exists a Householder reflector  $\mathbf{H}$  and  $z \in \mathbb{C}$  such that  $\mathbf{H}\mathbf{x} = z\mathbf{y}$ .*

## Proof.

Let

$$z = \frac{\|\mathbf{x}\|_2}{\|\mathbf{y}\|_2} \cdot c, \quad c = \begin{cases} \pm \mathbf{y}^*\mathbf{x} / |\mathbf{x}^*\mathbf{y}|, & \text{if } \mathbf{x}^*\mathbf{y} \neq 0, \\ e^{i\theta}, \theta \in [0, 2\pi), & \text{if } \mathbf{x}^*\mathbf{y} = 0, \end{cases}$$

and  $\mathbf{v} = z\mathbf{y} - \mathbf{x}$ . Verify  $\mathbf{H}\mathbf{x} = z\mathbf{y}$ . □

## 2. QR factorization via Householder reflectors

- Householder method:  $\mathbf{Q}_n \cdots \mathbf{Q}_2 \mathbf{Q}_1 \mathbf{A} = \mathbf{R}$  is upper-triangular.

$$\begin{array}{ccccc}
 \begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{bmatrix} & \xrightarrow{\mathbf{Q}_1} & \begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \end{bmatrix} & \xrightarrow{\mathbf{Q}_2} & \begin{bmatrix} \times & \times & \times \\ & \times & \times \\ & 0 & \times \\ & 0 & \times \\ & 0 & \times \end{bmatrix} & \xrightarrow{\mathbf{Q}_3} & \begin{bmatrix} \times & \times & \times \\ & \times & \times \\ & & \times \\ & & 0 \\ & & 0 \end{bmatrix} \\
 \mathbf{A} & & \mathbf{Q}_1 \mathbf{A} & & \mathbf{Q}_2 \mathbf{Q}_1 \mathbf{A} & & \mathbf{Q}_3 \mathbf{Q}_2 \mathbf{Q}_1 \mathbf{A}
 \end{array}$$

$\times$  denotes an entry not necessarily zero; “blank” are zeros

- At the  $k$ th step, the unitary matrix  $\mathbf{Q}_k$  has the form

$$\mathbf{Q}_k = \begin{bmatrix} \mathbf{I}_{k-1} & \\ & \mathbf{H}_k \end{bmatrix}.$$

Here  $\mathbf{H}_k$  is an  $(m - k + 1) \times (m - k + 1)$  Householder reflector, which maps an  $m - k + 1$ -vector to a scalar multiple of  $\mathbf{e}_1$ .

- The full QR factorization:  $\mathbf{A} = \mathbf{Q}_1^* \mathbf{Q}_2^* \cdots \mathbf{Q}_n^* \mathbf{R} = \mathbf{Q} \mathbf{R}$

- QR factorization with column pivoting:  $\mathbf{A}\mathbf{P} = \mathbf{Q}\mathbf{R}$ . Consider “qr”

## 2.1. Two possible Householder reflections in real case



- Choose the one that moves  $\mathbf{x}$  the larger distance, i.e.,  $\mathbf{v} = -\text{sign}(x_1)\|\mathbf{x}\|_2 \mathbf{e}_1 - \mathbf{x}$ , or  $\mathbf{v} = \text{sign}(x_1)\|\mathbf{x}\|_2 \mathbf{e}_1 + \mathbf{x}$
- Convention:  $\text{sign}(x_1) = 1$  if  $x_1 = 0$

## 2.2. Algorithms

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**Algorithm:** Householder QR factorization

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```
for  $k = 1$  to  $n$ 
     $\mathbf{x} = \mathbf{A}_{k:m,k}$ 
     $\mathbf{v}_k = \text{sign}(x_1) \|\mathbf{x}\|_2 \mathbf{e}_1 + \mathbf{x}$ 
     $\mathbf{v}_k = \mathbf{v}_k / \|\mathbf{v}_k\|_2$ 
     $\mathbf{A}_{k:m,k:n} = \mathbf{A}_{k:m,k:n} - 2\mathbf{v}_k(\mathbf{v}_k^* \mathbf{A}_{k:m,k:n})$ 
end
```

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**Algorithm:** Implicit calculations of  $\mathbf{Q}^* \mathbf{b}$  or  $\mathbf{Q} \mathbf{x}$

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for  $k = 1$  to  $n$ 
     $\mathbf{b}_{k:m} = \mathbf{b}_{k:m} - 2\mathbf{v}_k(\mathbf{v}_k^* \mathbf{b}_{k:m})$ 
end
for  $k = n$  downto  $1$ 
     $\mathbf{x}_{k:m} = \mathbf{x}_{k:m} - 2\mathbf{v}_k(\mathbf{v}_k^* \mathbf{x}_{k:m})$ 
end
```

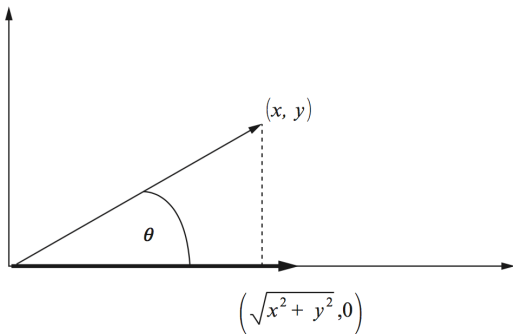
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### 3. Givens rotation (We mainly consider the real case.)

- The  $2 \times 2$  Givens rotation

$$\mathbf{G} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}, \quad \cos \theta = \frac{x}{\sqrt{x^2 + y^2}}, \quad \sin \theta = \frac{y}{\sqrt{x^2 + y^2}}$$

rotates vector  $\begin{bmatrix} x \\ y \end{bmatrix}$  in  $\mathbb{R}^2$  onto the  $x$ -axis, i.e., the second entry becomes zero.





- Givens rotation:  $\begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \tau \\ 0 \end{bmatrix}$  with  $\tau = \sqrt{x^2 + y^2}$

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**Algorithm:** Givens rotation zeroing the 2nd entry

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**function**  $[c, s, \tau] = \text{givens}(x, y)$

**if**  $y = 0$

$c = \text{sign}(x), \quad s = 0, \quad \tau = |x|$

**else**

$\tau = \sqrt{x^2 + y^2}, \quad c = \frac{x}{\tau}, \quad s = \frac{y}{\tau}$

**end**

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- Givens rotation:  $\begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ \tau \end{bmatrix}$  with  $\tau = \sqrt{x^2 + y^2}$

**Exercise:** Design a similar algorithm as above.

- Zeroing a particular entry in a vector using a Givens rotation.

Define the  $m \times m$  Givens rotation  $\mathbf{G}(i, j; \theta)$ ,  $i < j$ ,

$$\begin{aligned}\mathbf{G}(i, j; \theta) &= \mathbf{I} + [\mathbf{e}_i \quad \mathbf{e}_j] \begin{bmatrix} \cos \theta - 1 & \sin \theta \\ -\sin \theta & \cos \theta - 1 \end{bmatrix} \begin{bmatrix} \mathbf{e}_i^\top \\ \mathbf{e}_j^\top \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{I} & & & \\ & \cos \theta & & \sin \theta \\ & & \mathbf{I} & \\ & -\sin \theta & & \cos \theta \\ & & & & \mathbf{I} \end{bmatrix} \begin{matrix} \\ \text{row } i \\ \\ \text{row } j \\ \end{matrix}.\end{aligned}$$

**Exercise:** Prove that the matrix  $\mathbf{G}(i, j; \theta)$  is orthogonal.

- Creating a sequence of zeros in a vector using Givens rotations

$$\mathbf{G}_n \mathbf{G}_{n-1} \cdots \mathbf{G}_1 \mathbf{x}$$

- QR factorization via Givens rotations?

**Exercise:** Let  $\mathbf{A} \in \mathbb{R}^{n \times n}$  be given as

$$\mathbf{A} = \begin{bmatrix} \alpha_1 & \beta_2 & \beta_3 & \cdots & \beta_n \\ \gamma_2 & \alpha_2 & 0 & \cdots & 0 \\ \gamma_3 & 0 & \alpha_3 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \gamma_n & 0 & \cdots & 0 & \alpha_n \end{bmatrix}, \quad \begin{array}{ll} \alpha_i \neq 0, & i = 1:n, \\ \beta_i \neq 0, & i = 2:n, \\ \gamma_i \neq 0, & i = 2:n. \end{array}$$

Describe an algorithm for QR factorization of  $\mathbf{A}$  based on as few Givens rotations as possible.

- Complex case:

$$\mathbf{G} = \begin{bmatrix} c & \bar{s} \\ -s & c \end{bmatrix}, \quad c \in \mathbb{R}, \quad c^2 + |s|^2 = 1.$$

#### 4. The least squares problem (LSP)

- LSP: Given  $\mathbf{A} \in \mathbb{C}^{m \times n}$ ,  $\mathbf{b} \in \mathbb{C}^m$ ; find  $\mathbf{x}_{\text{ls}} \in \mathbb{C}^n$  such that

$$\|\mathbf{b} - \mathbf{A}\mathbf{x}_{\text{ls}}\|_2 = \min_{\mathbf{x} \in \mathbb{C}^n} \|\mathbf{b} - \mathbf{A}\mathbf{x}\|_2.$$

- Note that the 2-norm corresponds to Euclidean distance.

LSP means we seek a vector  $\mathbf{x}_{\text{ls}} \in \mathbb{C}^n$  such that the vector  $\mathbf{A}\mathbf{x}_{\text{ls}}$  is the closest point in  $\text{range}(\mathbf{A})$  to  $\mathbf{b}$ .

The *least squares solution*,  $\mathbf{x}_{\text{ls}}$ , maybe *not* unique. Why?

The *residual*,  $\mathbf{r}_{\text{ls}} = \mathbf{b} - \mathbf{A}\mathbf{x}_{\text{ls}}$ , is unique. Why?

- Assume that  $\mathbf{A}$  and  $\mathbf{b}$  are real. Define

$$f(\mathbf{x}) := \|\mathbf{b} - \mathbf{A}\mathbf{x}\|_2^2 = \mathbf{b}^\top \mathbf{b} - \mathbf{x}^\top \mathbf{A}^\top \mathbf{b} - \mathbf{b}^\top \mathbf{A}\mathbf{x} + \mathbf{x}^\top \mathbf{A}^\top \mathbf{A}\mathbf{x}.$$

Then the gradient of  $f(\mathbf{x})$  is

$$\nabla f(\mathbf{x}) = 2\mathbf{A}^\top \mathbf{A}\mathbf{x} - 2\mathbf{A}^\top \mathbf{b}.$$

## 4.1. Theory of the least squares problem

### Theorem 3

Let  $\mathbf{P}$  be the orthogonal projector onto  $\text{range}(\mathbf{A})$ . A vector  $\mathbf{x}$  is a least squares solution if and only if  $\mathbf{x}$  satisfies  $\mathbf{Ax} = \mathbf{Pb}$ .

### Hint:

$$\|\mathbf{b} - \mathbf{Ax}\|_2^2 = \|\mathbf{Pb} - \mathbf{Ax} + \mathbf{b} - \mathbf{Pb}\|_2^2 = \|\mathbf{Pb} - \mathbf{Ax}\|_2^2 + \|\mathbf{b} - \mathbf{Pb}\|_2^2. \quad \square$$

### Corollary 4

A vector  $\mathbf{x}$  is a least squares solution if and only if  $\mathbf{x}$  satisfies  $\mathbf{A}^*\mathbf{Ax} = \mathbf{A}^*\mathbf{b}$ , i.e.,  $\mathbf{A}^*\mathbf{r} = \mathbf{0}$ , or  $\mathbf{r} \perp \text{range}(\mathbf{A})$ , where  $\mathbf{r} := \mathbf{b} - \mathbf{Ax}$ .

### Proof.

$$\because \mathbf{A}^* = \mathbf{A}^*\mathbf{P}, \therefore \mathbf{A}^*\mathbf{r} = \mathbf{0} \Leftrightarrow \mathbf{A}^*(\mathbf{Pb} - \mathbf{Ax}) = \mathbf{0} \Leftrightarrow \mathbf{Ax} = \mathbf{Pb}. \quad \square$$

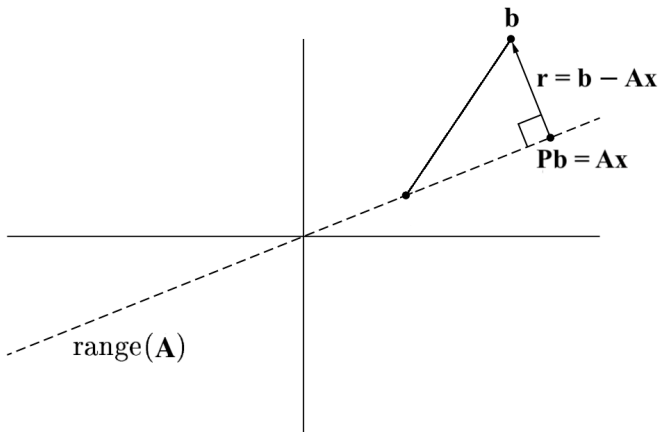
- The system  $\mathbf{A}^*\mathbf{Ax} = \mathbf{A}^*\mathbf{b}$  is called the *normal equations*.

## Corollary 5

*The least squares solution  $\mathbf{x}$  is unique if and only if  $\mathbf{A}^* \mathbf{A}$  has full rank, or equivalently,  $\mathbf{A}$  has full column rank, i.e.,  $\text{rank}(\mathbf{A}) = n$ .*

### 4.2. Geometric interpretation

- Let  $\mathbf{x}$  be a least squares solution. Obviously,  $\mathbf{r} = \mathbf{b} - \mathbf{Pb}$  is unique.



### 4.3. Moore–Penrose pseudoinverse solution $\mathbf{A}^\dagger \mathbf{b}$

- Let  $\mathbf{A} \in \mathbb{C}^{m \times n}$  have an SVD  $\mathbf{A} = \mathbf{U}_r \mathbf{\Sigma}_r \mathbf{V}_r^*$ . The matrix

$$\mathbf{A}^\dagger = \mathbf{V}_r \mathbf{\Sigma}_r^{-1} \mathbf{U}_r^* = \sum_{j=1}^r \frac{1}{\sigma_j} \mathbf{v}_j \mathbf{u}_j^* \in \mathbb{C}^{n \times m},$$

is called the *Moore–Penrose pseudoinverse* of  $\mathbf{A}$ . If  $\mathbf{A}$  has full column rank, then  $\mathbf{A}^\dagger = (\mathbf{A}^* \mathbf{A})^{-1} \mathbf{A}^*$ . (Full row rank case?)

#### Theorem 6

Let  $\mathbf{A} \in \mathbb{C}^{m \times n}$  have rank  $r < n$  and  $\mathbf{b} \in \mathbb{C}^m$ . Then the vector  $\mathbf{A}^\dagger \mathbf{b}$  is the unique least squares solution with minimum 2-norm.

#### Proof.

By SVD of  $\mathbf{A}$ , the least squares solutions can be expressed as

$$\mathbf{x}_{\text{ls}} = \mathbf{A}^\dagger \mathbf{b} + \mathbf{V}_c \mathbf{z}, \quad \mathbf{z} \in \mathbb{C}^{n-r}.$$

Then the statement follows from  $\mathbf{A}^\dagger \mathbf{b} \perp \mathbf{V}_c \mathbf{z}$ . □

#### 4.4. Full column rank LSP solvers: $\text{rank}(\mathbf{A}) = n$

- Normal equations: classical way to solve LSP, best for speed
- QR factorization: “modern classical” method to solve LSP, numerically stable. By

$$\mathbf{A} = \mathbf{Q}\mathbf{R} = [\mathbf{Q}_n \quad \mathbf{Q}_c] \begin{bmatrix} \mathbf{R}_n \\ \mathbf{0} \end{bmatrix},$$

we have

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{C}^n} \|\mathbf{b} - \mathbf{A}\mathbf{x}\|_2 &= \min_{\mathbf{x} \in \mathbb{C}^n} \|\mathbf{b} - \mathbf{Q}\mathbf{R}\mathbf{x}\|_2 = \min_{\mathbf{x} \in \mathbb{C}^n} \|\mathbf{Q}^*\mathbf{b} - \mathbf{R}\mathbf{x}\|_2 \\ &= \min_{\mathbf{x} \in \mathbb{C}^n} \left\| \begin{bmatrix} \mathbf{Q}_n^*\mathbf{b} - \mathbf{R}_n\mathbf{x} \\ \mathbf{Q}_c^*\mathbf{b} \end{bmatrix} \right\|_2 \end{aligned}$$

- SVD, numerically stable, for problems close to rank-deficient. By

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^* = \mathbf{U}_n\mathbf{\Sigma}_n\mathbf{V}^* = [\mathbf{U}_n \quad \mathbf{U}_c] \begin{bmatrix} \mathbf{\Sigma}_n \\ \mathbf{0} \end{bmatrix} \mathbf{V}^*,$$



we have

$$\begin{aligned}\min_{\mathbf{x} \in \mathbb{C}^n} \|\mathbf{b} - \mathbf{A}\mathbf{x}\|_2 &= \min_{\mathbf{x} \in \mathbb{C}^n} \|\mathbf{b} - \mathbf{U}\Sigma\mathbf{V}^*\mathbf{x}\|_2 \\ &= \min_{\mathbf{x} \in \mathbb{C}^n} \|\mathbf{U}^*\mathbf{b} - \Sigma\mathbf{V}^*\mathbf{x}\|_2 \\ &= \min_{\mathbf{x} \in \mathbb{C}^n} \left\| \begin{bmatrix} \mathbf{U}_n^*\mathbf{b} - \Sigma_n\mathbf{V}^*\mathbf{x} \\ \mathbf{U}_c^*\mathbf{b} \end{bmatrix} \right\|_2.\end{aligned}$$

**Exercise:** Given  $\mathbf{A} \in \mathbb{C}^{m \times n}$  of full column rank,  $m > n$ ,  $\mathbf{b} \in \mathbb{C}^m$ ,  $\mathbf{b} \notin \text{range}(\mathbf{A})$  and  $\mathbf{QR} = [\mathbf{A} \ \mathbf{b}]$  (i.e., full QR factorization of  $[\mathbf{A} \ \mathbf{b}]$ ). Show that

$$\min_{\mathbf{x} \in \mathbb{C}^n} \|\mathbf{b} - \mathbf{A}\mathbf{x}\|_2 = |\mathbf{R}(n+1, n+1)|,$$

and the least squares solution is given by

$$\mathbf{x} = \mathbf{R}(1:n, 1:n) \backslash \mathbf{R}(1:n, n+1).$$

#### 4.5. Rank-deficient LSP solvers: $\text{rank}(\mathbf{A}) = r < n$

- QR factorization with column pivoting:

$$\mathbf{AP} = \mathbf{QR} = \mathbf{Q} \begin{bmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

where  $\mathbf{P}$  is a permutation matrix,  $\mathbf{Q} \in \mathbb{C}^{m \times m}$  is unitary, and  $\mathbf{R}_{11} \in \mathbb{R}^{r \times r}$  is nonsingular upper triangular. Introduce the auxiliary vectors

$$\mathbf{Q}^* \mathbf{b} = \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{bmatrix} \quad \text{and} \quad \mathbf{P}^* \mathbf{x} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}.$$

The general least squares solution is

$$\mathbf{x}_{\text{ls}} = \mathbf{P} \begin{bmatrix} \mathbf{R}_{11}^{-1}(\mathbf{d}_1 - \mathbf{R}_{12}\mathbf{y}_2) \\ \mathbf{y}_2 \end{bmatrix}, \quad \mathbf{y}_2 = \text{arbitrary}.$$

The case  $\mathbf{y}_2 = \mathbf{0}$  yields a least squares solution with at least  $n - r$  zero components. Consider “\” in MATLAB.

- Complete orthogonal factorization (also called UTV factorization)

$$\mathbf{A} = \mathbf{U} \begin{bmatrix} \mathbf{R}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{V}^*,$$

where  $\mathbf{U} \in \mathbb{C}^{m \times m}$  is unitary,  $\mathbf{V} \in \mathbb{C}^{n \times n}$  is unitary, and  $\mathbf{R}_{11} \in \mathbb{R}^{r \times r}$  is nonsingular upper triangular. Introduce the auxiliary vectors

$$\mathbf{U}^* \mathbf{b} = \begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{bmatrix} \quad \text{and} \quad \mathbf{V}^* \mathbf{x} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}.$$

The general least squares solution is

$$\mathbf{x}_{\text{ls}} = \mathbf{V} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} = \mathbf{V} \begin{bmatrix} \mathbf{R}_{11}^{-1} \mathbf{g}_1 \\ \mathbf{y}_2 \end{bmatrix}, \quad \mathbf{y}_2 = \text{arbitrary}.$$

The case  $\mathbf{y}_2 = \mathbf{0}$  yields the minimum 2-norm least squares solution. Consider `lsqminnorm` in MATLAB.

<http://www.netlib.org/numeralgo/>

## 5. Solutions of $\mathbf{Ax} = \mathbf{b}$ with $\mathbf{A} \in \mathbb{C}^{m \times n}$ and $\mathbf{b} \in \mathbb{C}^m$

