Data Analysis and Matrix Computations Assignment 3

Exercise 1.

Given a list $S \subseteq \{1, 2, 3, ..., n\}$ of column indices, the column Nyström approximation:

$$\mathbf{A}_{\langle S \rangle} := \mathbf{A}(:, S)\mathbf{A}(S, S)^{\dagger}\mathbf{A}(S, :).$$

Prove that the column Nyström approximation has the properties:

(1) $\text{range}(\mathbf{A}_{\langle S \rangle}) = \text{range}(\mathbf{A}(:, S))$;

(2) $\mathbf{0} \preccurlyeq \mathbf{A}_{\langle S \rangle} \preccurlyeq \mathbf{A}$.

Exercise 2.

Pivoted partial Cholesky:

- Set $\widehat{\mathbf{A}}_0 := \mathbf{0}$, $\mathbf{A}_0 := \mathbf{A}$, and $\mathbf{F}_0 := \mathbf{0}$.

  At each step $t = 1, 2, \ldots, s$, select $i_t \in \{1, 2, \ldots, n\}$, update

  $$\widehat{\mathbf{A}}_t := \widehat{\mathbf{A}}_{t-1} + \frac{\mathbf{A}_{t-1}(:, i_t)\mathbf{A}_{t-1}(i_t, :)}{\mathbf{A}_{t-1}(i_t, i_t)};$$

  $$\mathbf{A}_t := \mathbf{A}_{t-1} - \frac{\mathbf{A}_{t-1}(:, i_t)\mathbf{A}_{t-1}(i_t, :)}{\mathbf{A}_{t-1}(i_t, i_t)},$$

  set

  $$\mathbf{c}_t := \mathbf{A}(:, i_t) - \mathbf{F}_{t-1}(\mathbf{F}_{t-1}(i_t, :))^{\top},$$

  and update

  $$\mathbf{F}_t := \begin{bmatrix} \mathbf{F}_{t-1} & \mathbf{c}_t / \sqrt{\mathbf{c}_t(i_t)} \end{bmatrix}.$$

Prove that $\widehat{\mathbf{A}}_t = \mathbf{F}_t\mathbf{F}_t^{\top}$ for $t = 0, 1, 2, \ldots, s$. (Hint: $\mathbf{c}_t = \mathbf{A}(:, i_t) - \widehat{\mathbf{A}}_{t-1}(:, i_t)$.)