Lecture 9: Monomer-Dimer Systems and the Heilmann–Lieb Theorem

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In this lecture, we see another method for establishing zero-free regions based on self-avoiding walks, similar to the computation tree gadget used in the correlation decay algorithm. We'll use this to prove the Heilmann–Lieb Theorem on real-rootedness of the matching polynomial. We'll then apply Barvinok's polynomial interpolation algorithm to obtain an FPTAS for matchings. In the setting of matchings, we'll further illustrate how one can improve the algorithm from quasipolynomial running time to polynomial, assuming the graph has bounded maximum degree. The main algorithmic result of this lecture is the following.

Theorem 0.1 (Patel–Regts; [PR17]). *There exists a deterministic algorithm such that given a graph $G = (V, E)$ of maximum degree Δ and an error parameter $\epsilon > 0$, the algorithm outputs a $(1 \pm \epsilon)$-multiplicative approximation to the total number of matchings in G in time $(n/\epsilon)^C$ for some constant $C = C(\Delta)$ depending only on Δ.*

1 Zero-Freeness for Matchings

Recall that for a graph $G = (V, E)$, a matching $M \subseteq E$ is a subset of edges such that no vertex is incident to more than one edge in M. For a parameter $\lambda \geq 0$, we define the *monomer-dimer model* as the following distribution over matchings:

$$\mu(M) \propto \lambda^{|M|}, \qquad \forall \text{ matchings } M \subseteq E.$$

The associated partition function

$$\mathcal{M}_G(\lambda) = \sum_{M \subseteq E \text{ matching}} \lambda^{|M|} = \sum_{k=0}^{\lfloor n/2 \rfloor} m_k \lambda^k,$$

is the matching polynomial of G, where $m_k = m_k(G)$ gives the number of size-k matchings in G. We prove the following theorem.

Theorem 1.1 (Heilmann–Lieb; [HL72]). *For every graph $G = (V, E)$, the roots of $\mathcal{M}_G$ all lie on the negative real axis. Furthermore, if G has maximum degree Δ, then every such root z satisfies $z \leq -\frac{1}{4(\Delta-1)}$.*

Remark 1. Note that the monomer-dimer model on G can be viewed as the hardcore model on the *line graph* $L(G)$ of G, where each vertex of $L(G)$ corresponds to an edge in G, and two such vertices in $L(G)$ are directly connected if their corresponding edges in G share a common vertex. In contrast, for the hardcore model on arbitrary graphs of maximum degree Δ, the partition function is zero-free only in some strip around the interval $[0, \lambda_c(\Delta))$. Hence, Theorem 1.1 says that the class of line graphs have much better behaved independence polynomials.

We will prove Theorem 1.1 in two steps, similar to the proof of correlation decay for the hardcore model. We will (inductively) construct a tree T, with the same maximum degree as G, such that $\mathcal{M}_G$ divides $\mathcal{M}_T$ as polynomials. This in particular says that the zeros of $\mathcal{M}_G$ are a subset of the zeros of $\mathcal{M}_T$. We will then directly analyze the zeros of $\mathcal{M}_T$ for all bounded-degree trees.

1.1 Reduction to Trees: Return of the Self-Avoiding Walks

Fix a vertex $r \in G$, and define the *tree of self-avoiding walks* $T_{\mathsf{SAW}}(G,r)$ as follows:

- Each vertex of $T_{\mathsf{SAW}}(G,r)$ corresponds to a walk $r = u_0 \to \cdots \to u_k = v$ in G started at r, subject to the constraint that no vertex is visited more than once. Such walks are called *self-avoiding*.
- Two such self-avoiding walks are adjacent in $T_{\mathsf{SAW}}(G,r)$ if one walk extends the other.

Note that the root vertex of $T_{\mathsf{SAW}}(G,r)$ corresponds to the trivial walk $\{r\}$ which hasn't moved anywhere. Furthermore, each vertex of $T_{\mathsf{SAW}}(G,r)$ can be identified with a vertex in G corresponding to the last visited vertex. Finally, if the maximum degree of G is Δ, then the maximum degree of $T_{\mathsf{SAW}}(G,r)$ is also (at most) Δ.

We prove the following divisibility result, which is well-known in algebraic combinatorics. It immediately implies that the set of roots of $\mathcal{M}_G$ is contained in the set of roots of $\mathcal{M}_T$ for $T = T_{\mathsf{SAW}}(G,r)$.

Theorem 1.2 (Godsil–Gutman; see e.g. [God93])**.** *For every graph $G = (V,E)$ and every vertex $r \in V$, there exists a polynomial $q(\lambda)$ such that $\mathcal{M}_T(\lambda) = \mathcal{M}_G(\lambda) \cdot q(\lambda)$ for $T = T_{\mathsf{SAW}}(G,r)$.*

The first step in the reduction is to decompose the matching polynomial $\mathcal{M}_G$ in a manner similar to the tree recursions we used in the context of correlation decay. By Remark 1, we could view $\mathcal{M}_G$ as the independence polynomial of the line graph, and then leverage the tree recursion for the hardcore model. However, deducing real-rootedness seems difficult, since at some point we need to take advantage of the special structure of matchings. We derive a different decomposition instead.

Lemma 1.3 (Recursion for Matching Polynomial)**.** *For every graph $G = (V,E)$ and every vertex $r \in V$,*

$$\mathcal{M}_G(\lambda) = \mathcal{M}_{G-r}(\lambda) + \lambda \sum_{v \sim r} \mathcal{M}_{G-r-v}(\lambda).$$

Proof. We can partition the collection of matchings in G into $1 + \deg_G(r)$ groups:

- One group consists of all matchings which do not contain any edge incident to r.
- For each $v \sim r$, we have a group consisting of all matchings containing the edge $\{r,v\}$.

The first group contributes $\mathcal{M}_{G-r}(\lambda)$, while each of the other groups contributes $\lambda \cdot \mathcal{M}_{G-r-v}(\lambda)$. $\square$

Proof of Theorem 1.2. We use Lemma 1.3 to inductively prove the following identity:

$$\frac{\mathcal{M}_G(\lambda)}{\mathcal{M}_{G-r}(\lambda)} = \frac{\mathcal{M}_T(\lambda)}{\mathcal{M}_{T-r}(\lambda)}. \tag{1}$$

Note if we interpret both sides probabilistically, this identity is exactly saying the marginal probability of r being saturated by an edge in a random matching is the same in both G and $T = T_{\mathsf{SAW}}(G,r)$. This is precisely the same kind of relationship we needed from the computation tree when we studied correlation decay and the hardcore model. Here, at least for the moment, we are just using the tree T as a mode of analysis, rather than as an algorithm directly. This is a fairly generic technique for establishing zero-free regions, assuming you can control the zeros in the special case of trees.

Note that the base case of Eq. (1), where G is a tree (e.g. two vertices connected by a single edge), is immediate, since in that case $G = T_{\mathsf{SAW}}(G,r)$. For the inductive step, observe that

$$\begin{aligned} \frac{\mathcal{M}_G(\lambda)}{\mathcal{M}_{G-r}(\lambda)} &= 1 + \lambda \sum_{v \sim r} \frac{\mathcal{M}_{(G-r)-v}(\lambda)}{\mathcal{M}_{G-r}(\lambda)} && \text{(Lemma 1.3)} \\ &= 1 + \lambda \sum_{v \sim r} \frac{\mathcal{M}_{T_{\mathsf{SAW}}(G-r,v)-v}(\lambda)}{\mathcal{M}_{T_{\mathsf{SAW}}(G-r,v)}(\lambda)}. && \text{(Inductive Hypothesis)} \end{aligned}$$

Here is the crucial observation: If we delete r from $T_{\mathsf{SAW}}(G,r)$, we obtain $\deg_G(r)$ many disjoint subtrees, each rooted at a neighbor $v \sim r$. Furthermore, the subtree corresponding to v is *precisely* $T_{\mathsf{SAW}}(G-r,v)$, since for any self-avoiding walk in $G-r$ started at v, we can append r at the beginning to produce a self-avoiding walk in G started at r which moves to v in the first step. It follows that for every $v \sim r$,

$$\frac{\mathcal{M}_{T_{\mathsf{SAW}}(G-r,v)-v}(\lambda)}{\mathcal{M}_{T_{\mathsf{SAW}}(G-r,v)}(\lambda)} = \frac{\mathcal{M}_{T_{\mathsf{SAW}}(G,r)-r-v}(\lambda)}{\mathcal{M}_{T_{\mathsf{SAW}}(G,r)-r}(\lambda)}.$$

There is a slight subtlety here, in that $T_{\mathsf{SAW}}(G,r)-r$ also has subtrees corresponding to other neighbors $u \sim r$ besides v. However, since $T_{\mathsf{SAW}}(G,r)-r$ is a disjoint union of trees which are not connected to one another, its matching polynomial factorizes as a product of matching polynomials of its components. In particular, the terms coming from the other neighbors $u \neq v$ of r cancel in both the numerator and denominator.

Once we have this, applying Lemma 1.3 again, we obtain

$$\begin{aligned} 1+\lambda\sum_{v\sim r}\frac{\mathcal{M}_{T_{\mathsf{SAW}}(G-r,v)-v}(\lambda)}{\mathcal{M}_{T_{\mathsf{SAW}}(G-r,v)}(\lambda)} &= 1+\lambda\sum_{v\sim r}\frac{\mathcal{M}_{T_{\mathsf{SAW}}(G,r)-r-v}(\lambda)}{\mathcal{M}_{T_{\mathsf{SAW}}(G,r)-r}(\lambda)} \\ &= \frac{\mathcal{M}_T(\lambda)}{\mathcal{M}_{T-r}(\lambda)}. \end{aligned}$$

This completes the proof of Eq. (1). Observe that Eq. (1) implies

$$\mathcal{M}_T(\lambda) = \mathcal{M}_G(\lambda)\cdot\frac{\mathcal{M}_{T-r}(\lambda)}{\mathcal{M}_{G-r}(\lambda)}.$$

We can use this with an inductive argument to prove divisibility. In particular, since $\mathcal{M}_{G-r}(\lambda)$ divides $\mathcal{M}_{T-r}(\lambda)$ by induction, the ratio $\frac{\mathcal{M}_{T-r}(\lambda)}{\mathcal{M}_{G-r}(\lambda)}$ is itself a polynomial, and so it follows that $\mathcal{M}_G(\lambda)$ divides $\mathcal{M}_T(\lambda)$. $\square$

1.2 The Matching Polynomial on Trees

Theorem 1.2 allowed us to reduce our task to studying the matching polynomial on trees. For this analysis, it will be convenient to perform a slight change of variables. Define

$$\widetilde{\mathcal{M}}_G(z) \overset{\text{def}}{=} \sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k m_k z^{n-2k} = z^n\mathcal{M}_G(-1/z^2).$$

For trees, we have the following relationship between this (transformed) matching polynomial and the characteristic polynomial of the adjacency matrix.

Lemma 1.4. *For every tree T, we have $\widetilde{\mathcal{M}}_T(z) = \det(zI - A_T)$. In particular, the roots of $\widetilde{\mathcal{M}}_T(z)$ are precisely the eigenvalues of A_T.*

Proof. We expand out the determinant as

$$\begin{aligned} \det(zI - A_T) &= \sum_{\sigma:V\to V}(-1)^{\mathrm{sign}(\sigma)}\prod_{v\in V}(zI-A)_{v,\sigma(v)} \\ &= \sum_{k=0}^{n}(-1)^k z^{n-k}\sum_{S\in\binom{V}{k}}\det(A_{S,S}), \end{aligned}$$

where $A_{S,S}$ denotes the $S\times S$ principal submatrix of A. The final step follows from first summing over the possible sets of fixed points $S\subseteq V$, and then summing over the possible permutations such that $S=\{v:\sigma(v)=v\}$. We can compactly write this inner summation as $\det(A_{S,S})$ because A has zero diagonal.

To prove the claim, it suffices to show that for every $k=0,\ldots,n$ and every $S\in\binom{V}{k}$, the number $(-1)^{k/2}\det(A_{S,S})$ counts the number of perfect matchings in the subgraph of T induced by S. If we can show this, then $(-1)^k m_k = \sum_{S\in\binom{V}{2k}}\det(A_{S,S})$ and $\widetilde{\mathcal{M}}_T(z) = \det(zI - A_T)$ follows by comparing coefficients.

For simplicity, we'll just prove that $(-1)^{n/2}\det(A)$ gives the number of perfect matchings in T. The argument for the induced subgraphs is nearly identical. We have

$$\det(A) = \sum_{\sigma:V\to V} (-1)^{\text{sign}(\sigma)} \prod_{v\in V} A_{v,\sigma(v)}.$$

All permutations $\sigma : V \to V$ with nonzero contribution must satisfy $\{v, \sigma(v)\} \in E$ for all $v \in V$. All permutations also admit a decomposition into a disjoint union of cycles, each formed by starting with some $v \in V$ and collecting the successive iterates $\{v, \sigma(v), \sigma(\sigma(v)), \dots\}$. Since T does not have any (nontrivial) cycles, we only get a nonzero contribution from σ whose cycle decomposition consists of inversions, i.e. $\sigma(\sigma(v)) = v$ for all $v \in V$. Clearly, such permutations are in one-to-one correspondence with perfect matchings on T (i.e. $M = \{(v, \sigma(v)) : v \in V\} \subseteq E$ and vice versa) Furthermore, since $\text{sign}(\sigma)$ counts the number of inversions in σ, we get $(-1)^{\text{sign}(\sigma)} = (-1)^{n/2}$ for such permutations. $\square$

Lemma 1.5. *For every tree T of maximum degree Δ, the eigenvalues of A_T are at most $2\sqrt{\Delta - 1}$.*

Remark 2. Note that for any graph G of maximum degree Δ, we always have the trivial bound of $\lambda_{\max}(A_G) \le \Delta$ by virtue of the maximum absolute row sum. While this trivial bound is attained by Δ-regular graphs, Lemma 1.5 says that for trees, we get a square root improvement. Note that since T is bipartite, the eigenvalues of A_T are symmetric about 0, and so Lemma 1.5 also says that the eigenvalues of A_T are fully contained in the interval $\left[-2\sqrt{\Delta-1}, 2\sqrt{\Delta-1}\right]$.

Proof. We use the trace moment method. In particular, since the trace is the sum of eigenvalues, and the eigenvalues of A_T^k are the eigenvalues of A_T raised to the kth power, it is immediate that[1]

$$\lambda_{\max}(A_T)^{2k} \le \operatorname{Tr}\left(A_T^{2k}\right), \qquad \forall k \in \mathbb{N}.$$

We claim that $\left(A_T^{2k}\right)(r, r) \le 2^{2k}(\Delta - 1)^k$ for every $r \in V$ and $k \in \mathbb{N}$ via combinatorial argument. If we can show this, then $\lambda_{\max}(A_T)^{2k} \le n \cdot 2^{2k}(\Delta - 1)^k$ for all $k \in \mathbb{N}$, and so the claim follows by taking $2k$th roots and sending $k \to \infty$.

Recall that $\left(A_T^{2k}\right)(r, r)$ counts the number of walks in T starting and ending at r. We classify each step of each walk as either up or down, depending on if the step increases or decreases the distance from r. The crucial observation is that since T is a tree, the number of up steps must equal the number of down steps in order to return to r; in particular, they must both be equal to k. Hence, there are at most $\binom{2k}{k}$ ways of choosing which steps of the length-$2k$ walk are up or down. Furthermore, each down step has at most $\Delta - 1$ choices for the destination vertex, while each up step only has one choice. It follows that there are at most $\binom{2k}{k}(\Delta - 1)^k \le 2^{2k}(\Delta - 1)^k$ such walks, as desired. $\square$

With all of these ingredients in hand, we complete the proof of the Heilmann–Lieb Theorem.

Proof of Theorem 1.1. Since $\mathcal{M}_G$ divides $\mathcal{M}_T$ by Theorem 1.2, where $T = T_{\mathsf{SAW}}(G, r)$ for an arbitrarily chosen vertex $r \in G$, all roots of $\mathcal{M}_G$ are roots of $\mathcal{M}_T$ as well. Furthermore, if G has maximum degree Δ, then so does T. Hence, it suffices to prove the claims for $\mathcal{M}_T$. By Lemma 1.4, $\widetilde{\mathcal{M}}_T(z)$ is the characteristic polynomial of a symmetric matrix, namely A_T, so its roots are real. This immediately implies the roots of $\mathcal{M}_T$ are real. Furthermore, since $\mathcal{M}_T$ has nonnegative coefficients and $m_0 = 1$ (so that $\mathcal{M}_T(0) \ne 0$), its roots must be strictly negative. Finally, if T has maximum degree Δ, then the roots of $\widetilde{\mathcal{M}}_T(z)$ are at most $2\sqrt{\Delta - 1}$ by Lemma 1.5. It follows that the roots of $\mathcal{M}_T$ are at at least $\frac{1}{4(\Delta-1)}$ in absolute value. $\square$

2 Improved Running Time for Polynomial Interpolation

In this section, we prove Theorem 0.1. We prove the following more general result.

Theorem 2.1 (Patel–Regts; [PR17]). *There exists a deterministic algorithm such that given a graph $G = (V, E)$ of maximum degree Δ, a fugacity $\lambda \ge 0$, and an error parameter $\epsilon > 0$, the algorithm outputs a $(1 \pm \epsilon)$-multiplicative approximation to $\mathcal{M}_G(\lambda)$ in time $(n/\epsilon)^C$ for some constant $C = C(\lambda, \Delta)$ depending only on λ, Δ.*

[1]In fact, we have $\lambda_{\max}(A_T) = \|A_T\|_{\mathsf{op}} = \lim_{k\to\infty} \operatorname{Tr}\left(A_T^{2k}\right)^{1/2k}$.

For simplicity, we begin by explaining the idea when $\lambda < \frac{1}{4(\Delta-1)}$, since in this setting, we have a zero-free disk around the origin and we can directly apply Barvinok's polynomial interpolation algorithm. For the general case, we'll need the extension to strips, along with a small additional trick.

Recall that to implement the vanilla version of Barvinok's algorithm when we have a zero-free disk around the origin, we need to compute the numbers $\{f^{(k)}(0)\}_{k=0}^{m}$ for $m \leq O(\log(n/\epsilon))$, where $f(\lambda) = \log \mathcal{M}_G(\lambda)$ and $f^{(k)}(\lambda) = \frac{d^k}{d\lambda^k} f(\lambda)$ denotes the kth-order derivative. Furthermore, if $\zeta_1, \ldots, \zeta_{\lfloor n/2 \rfloor}$ are the roots of $\mathcal{M}_G$, then

$$\frac{f^{(k)}(0)}{k!} = -\frac{1}{k} \sum_{i=1}^{\lfloor n/2 \rfloor} \zeta_i^{-k}, \qquad \forall k \in \mathbb{N}.$$

In the previous lecture, we computed the $f^{(k)}(0)$ by first computing the $\mathcal{M}_G^{(k)}(0) = k! \cdot m_k$ and then solving an appropriate linear system. Now, we can of course compute each m_k in $n^{O(k)}$-time by brute force, but this will lead to quasipolynomial running time once k is of logarithmic order.

The next insight here is to compute the inverse power sums $\sum_{i=1}^{\lfloor n/2 \rfloor} |\zeta_i|^{-k}$ of (the absolute values of) the roots by interpreting them combinatorially, again using walks. Since the graph has bounded maximum degree, in the end, this will bring the cost of estimating $f^{(k)}(0)$ down to $\Delta^{O(k)}$.

Proposition 2.2 (Moments of Roots)**.** *For every $r \in V$, let $T(r) = T_{\mathsf{SAW}}(G, r)$. Then*

$$\sum_{i=1}^{\lfloor n/2 \rfloor} |\zeta_i|^{-k} = \frac{1}{2} \sum_{r \in V} \left(A_{T(r)}^{2k} \right)(r, r).$$

Proof of Theorem 2.1 (Special Case). We illustrate the main ideas in the special case $|\lambda| < \frac{1}{4(\Delta-1)}$; the general case is treated in Section 2.2. We use Barvinok's polynomial interpolation algorithm. Since $\mathcal{M}_G$ is zero-free in a disk of radius $R = \frac{1}{4(\Delta-1)}$ around the origin, we can estimate $f(\lambda) = \log \mathcal{M}_G(\lambda)$ to ϵ-additive error for any $\lambda \in \mathbb{C}$ with $|\lambda| < R$ using its degree-m Taylor series with $m \leq O_{\lambda,\Delta}(\log(n/\epsilon))$. All that remains is to exactly compute the coefficients of this Taylor approximation, i.e. compute each $\sum_{i=1}^{\lfloor n/2 \rfloor} |\zeta_i|^{-k}$ for all $0 \leq k \leq m$.

By Proposition 2.2, $\sum_{i=1}^{\lfloor n/2 \rfloor} |\zeta_i|^{-k} = \frac{1}{2} \sum_{r \in V} \left(A_{T(r)}^{2k} \right)(r, r)$ for all k. Since $T(r)$ has maximum degree Δ, the number $\left(A_{T(r)}^{2k} \right)(r, r)$ can be computed exactly in time $\Delta^{O(k)}$ via brute force enumeration; again, this is because this quantity exactly counts the number of length-$2k$ walks in $T(r)$ which start and end at the root r. This gives us an $n\Delta^{O(k)}$-time algorithm for exactly computing $\sum_{i=1}^{\lfloor n/2 \rfloor} |\zeta_i|^{-k}$, for all k. Hence, the total running time of the algorithm is $\Delta^{O(m)} = \Delta^{O_\lambda(\log(n/\epsilon))} = (n/\epsilon)^C$ for a constant $C = C(\lambda, \Delta)$ depending on Δ and how far λ is from the boundary of the radius-$\frac{1}{4(\Delta-1)}$ disk. □

2.1 On Moments of Roots of $\mathcal{M}_G$: Proof of Proposition 2.2

To see this connection between moments of roots and walks, it is again easier to work with $\widetilde{\mathcal{M}}_G$ instead of $\mathcal{M}_G$, in light of Lemma 1.4. By Theorem 1.1 (and $\mathcal{M}_G(0) = m_0 = 1$), we may express

$$\mathcal{M}_G(\lambda) = \prod_{i=1}^{\lfloor n/2 \rfloor} \left(1 - \frac{\lambda}{\zeta_i}\right),$$

and so

$$\widetilde{\mathcal{M}}_G(z) = z^n \mathcal{M}_G(-1/z^2) = z^{n-2\cdot\lfloor n/2 \rfloor} \cdot \prod_{i=1}^{\lfloor n/2 \rfloor} \left(z + |\zeta_i|^{-1/2}\right)\left(z - |\zeta_i|^{-1/2}\right), \tag{2}$$

using the fact that the ζ_i are all strictly negative. Hence, the kth inverse power sums of roots of $\mathcal{M}_G$ are equal to the $2k$th inverse power sums of roots of $\widetilde{\mathcal{M}}_G$ (up to a factor of 1/2). We need the following two lemmas.

Lemma 2.3. *For every tree T and every vertex $r \in T$, we have*

$$z^{-1}\frac{\widetilde{\mathcal{M}}_{T-r}(z^{-1})}{\widetilde{\mathcal{M}}_{T}(z^{-1})} = \sum_{k=0}^{\infty}(A_T^{2k})(r,r)\cdot z^{2k}.$$

Lemma 2.4. *For every graph $G=(V,E)$, we have*

$$\widetilde{\mathcal{M}}'_G(z) = \sum_{r\in V}\widetilde{\mathcal{M}}_{G-r}(z).$$

Before we prove these technical lemmas, we show how to use them to relate moments of roots to closed walks in the self-avoiding walk trees.

Proof of Proposition 2.2. The idea is to encode $\sum_{i=1}^{\lfloor n/2\rfloor}|\zeta_i|^{-k}$ in a power series. We'll then express this power series in two different ways, one based on moments of roots, and the other based on walks. The claim will then follow by comparing coefficients.

We use the logarithmic derivative $\left(\log\widetilde{\mathcal{M}}_G(z)\right)'$. On the one hand, we have

$$\begin{aligned}\left(\log\widetilde{\mathcal{M}}_G(z)\right)' &= (n-2\cdot\lfloor n/2\rfloor)\,(\log z)' + \sum_{i=1}^{\lfloor n/2\rfloor}\left(\log\left(z+|\zeta_i|^{-1/2}\right)' + \log\left(z-|\zeta_i|^{-1/2}\right)'\right) && \text{(Using Eq. (2))}\\ &= (n-2\cdot\lfloor n/2\rfloor)\cdot z^{-1} + 2z^{-1}\sum_{i=1}^{\lfloor n/2\rfloor}\frac{1}{1-\frac{1}{z^2|\zeta_i|}} && \text{(Calculation)}\\ &= z^{-1}\cdot\left(n+2\sum_{k=1}^{\infty}z^{-2k}\sum_{i=1}^{\lfloor n/2\rfloor}|\zeta_i|^{-k}\right). && \text{(Taylor series for } \tfrac{1}{1-x}\text{)}\end{aligned}$$

On the other hand,

$$\begin{aligned}\left(\log\widetilde{\mathcal{M}}_G(z)\right)' &= \frac{\widetilde{\mathcal{M}}'_G(z)}{\widetilde{\mathcal{M}}_G(z)}\\ &= \sum_{r\in V}\frac{\widetilde{\mathcal{M}}_{G-r}(z)}{\widetilde{\mathcal{M}}_G(z)} && \text{(Lemma 2.4)}\\ &= \sum_{r\in V}\frac{\widetilde{\mathcal{M}}_{T(r)-r}(z)}{\widetilde{\mathcal{M}}_{T(r)}(z)} && \text{(Theorem 1.2)}\\ &= z^{-1}\sum_{k=0}^{\infty}z^{-2k}\sum_{r\in V}\left(A_{T(r)}^{2k}\right)(r,r). && \text{(Lemma 2.3)}\end{aligned}$$

Comparing coefficients yields the claim. ☐

Proof of Lemma 2.3. By Lemma 1.4, the left-hand side is precisely $z^{-1}\frac{\det(z^{-1}I-A_{T-r})}{\det(z^{-1}I-A_T)}$. On the other hand, the right-hand side is precisely the diagonal entry

$$\sum_{k=0}^{\infty}z^kA_T^k(r,r) = (I-zA_T)^{-1}(r,r) = z^{-1}\cdot(z^{-1}I-A_T)^{-1}(r,r),$$

again using the fact that the odd powers of A_T have zero diagonal. Hence, the claim is equivalent to showing that

$$\det(z^{-1}I-A_T)\cdot\left(z^{-1}I-A_T\right)^{-1}(r,r) = \det(z^{-1}I-A_{T-r}).$$

Observe that $z^{-1}I-A_{T-r}$ is precisely the submatrix of $z^{-1}I-A_T$ corresponding to vertices not equal to u. Hence, if we write $B=z^{-1}I-A_T$, then we're asking for $\det(B)\cdot B^{-1}(r,r)=\det(B_{-r})$ where B_{-r} is the submatrix of B obtained by deleting the row and column corresponding to r. For this, define the *adjugate* of B by $\mathsf{adj}(B)=\det(B)\cdot B^{-1}$. The claim is equivalent to saying that $\mathsf{adj}(B)_{r,r}=\det(B_{-r})$. This follows immediately from the relation $B\cdot\mathsf{adj}(B)=\det(B)\cdot I$ and the fact that $\det(B)$ admits a decomposition into determinants of submatrices of B. ☐

Proof of Lemma 2.4. The proof is essentially the same idea as the one used to show Lemma 1.3. We have

$$\widetilde{\mathcal{M}}'_G(z) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k z^{(n-1)-2k} \cdot (n-2k) \cdot m_k(G).$$

On the other hand,

$$\begin{aligned}\sum_{r \in V} \widetilde{\mathcal{M}}_{G-r}(z) &= \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k z^{(n-1)-2k} \cdot \sum_{r \in V} m_k(G-r) \\ &= \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k z^{(n-1)-2k} \cdot (n-2k) \cdot m_k(G).\end{aligned}$$

Here, in the final step, we used the fact that every size-k matching in $G-r$ is also a size-k matching in G. Furthermore, each such matching gets counted $n-2k$ times, once for each unmatched vertex. $\square$

2.2 Estimating $\mathcal{M}_G(\lambda)$ on All of $\mathbb{R}_{\geq 0}$

Now suppose we wish to compute $\mathcal{M}_G(\lambda)$ for some $\lambda \in \mathbb{R}_{\geq 0}$. In this more general setting, if $\lambda > \frac{1}{4(\Delta-1)}$, we no longer have a zero-free disk containing 0 and λ. However, Theorem 1.1 gives us a zero-free strip around the interval $[0, \lambda]$. So, we let $\varphi_\lambda : \mathbb{C} \to \mathbb{C}$ be some low-degree polynomial mapping the unit disk to a strip around $[0, \lambda]$ (e.g. a modification of the one constructed in the previous lecture). Then the composition $\mathcal{M}_G(\varphi_\lambda(z))$ is a polynomial which is zero-free within a disk around 0 by Theorem 1.1 and the properties imposed on φ_λ. We can then apply Barvinok's vanilla algorithm to this polynomial.

For this, we need to compute the coefficients $\{F^{(k)}(0)\}_{k=0}^m$ where $F(z) = \log \mathcal{M}_G(\varphi_\lambda(z))$. We could view this as the composition $(f \circ \varphi_\lambda)(z)$ where $f = \log \mathcal{M}_G$ as above, and apply something like Faà di Bruno's Formula to write $\{F^{(k)}(0)\}_{k=0}^m$ in terms of $\{f^{(k)}(0)\}_{k=0}^m$ and $\{\varphi_\lambda^{(k)}(0)\}_{k=0}^m$. However, this would naïvely cost at least quasipolynomial running time again, which we want to avoid. Instead, we use the moments $\sum_{i=1}^{\lfloor n/2 \rfloor} |\zeta_i|^{-k}$ for $k = 0, \ldots, m$ to compute the low-degree coefficients $\left\{\mathcal{M}_G^{(k)}(0)\right\}_{k=0}^m = \{k! \cdot m_k\}_{k=0}^m$ of the matching polynomial. This allows us to efficiently compute the low-degree coefficients of the polynomial $\mathcal{M}_G \circ \varphi_\lambda$, from which we can obtain the low-degree Taylor coefficients of $\log \mathcal{M}_G(\varphi_\lambda(z))$ as we did in the previous lecture.

This first step is made possible with the help of the following lemma. Once we have it, the proof of Theorem 2.1 in the general case is immediate.

Lemma 2.5 (Newton Identities)**.** *Let $q(x) = \sum_{k=0}^n a_k x^k$ be a univariate polynomial with roots $r_1, \ldots, r_n$. Then for every $k \in \mathbb{N}$,*

$$k \cdot a_k = -\sum_{j=0}^{k-1} a_j \cdot \sum_{i=1}^n r_i^{-(k-j)}.$$

(Here, we take $a_k = 0$ if $k > n$.)

Proof. On the one hand, since $q(x) = a_0 \cdot \prod_{i=1}^n \left(1 - \frac{x}{r_i}\right)$,

$$(\log q(x))' = \left(\log(a_0) + \sum_{i=1}^n \log\left(1 - \frac{x}{r_i}\right)\right)' = \sum_{i=1}^n \frac{-1/r_i}{1 - \frac{x}{r_i}} = -\sum_{k=1}^\infty x^{k-1} \sum_{i=1}^n r_i^{-k}.$$

In particular, using $(\log q(x))' = \frac{q'(x)}{q(x)}$,

$$q'(x) = -q(x) \cdot \sum_{k=1}^\infty x^{k-1} \sum_{i=1}^n r_i^{-k}.$$

On the other hand,

$$q'(x) = \sum_{k=1}^{n} k a_k \cdot x^{k-1}.$$

Thus,

$$\sum_{k=1}^{n} k a_k \cdot x^{k-1} = -\sum_{j=0}^{n} \sum_{k=1}^{\infty} a_j x^{j+k-1} \sum_{i=1}^{n} r_i^{-k}.$$

Comparing coefficients yields the claim. □

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