Problem Set 3

Due: November 30, 2023 (11:59 PM EST)

Exercise 1 (Hardcore Model on $K_{n,n}$). *Consider the hardcore model with fugacity $\lambda \geq 0$ on the complete bipartite graph $K_{n,n}$. Let μ denote the Gibbs distribution, and $Z(\lambda)$ denote the partition function.*

(a) Give a closed form expression for $Z(\lambda)$.

(b) Prove that there is a constant $C = C(\lambda) > 0$ such that for each $0 \leq k \leq 2n-2$, there exists $S \subseteq V$ with $|S| = k$ and a pinning $\tau : S \to \{\mathsf{in}, \mathsf{out}\}$ such that $\lambda_{\max}(\Psi_{\mu^\tau}) \geq C \cdot (n-k)$.

(If it makes life convenient, you may assume k is even.)

(c) Show that for every fixed $\lambda \geq 0$ independent of n, there is a constant $C = C(\lambda) > 0$ such that Glauber dynamics has spectral gap at most $\exp(-Cn)$. In particular, its worst-case mixing time is exponentially large.

*(d) Conclude that the "local-to-global theorem", i.e. that $O(1)$-spectral independence for all pinnings implies inverse polynomial spectral gap, **cannot** admit an "average-case" analog of the following form:*

"Let μ be a probability distribution on $\{\pm 1\}^n$. Suppose for every $\Omega(\log n) \leq k \leq n-2$, the conditional measure μ^τ is $O(1)$-spectrally independent with high probability (e.g. $1 - \frac{1}{\mathsf{poly}(n)}$) over a random pinning $\tau \sim \mu_S$ on a uniformly random set S of k coordinates. Then Glauber dynamics for μ has spectral gap at least $1/\mathsf{poly}(n)$, and polynomial mixing time."

Exercise 2 (Continuous to Discrete). *Let μ be a probability measure on $\{\pm 1\}^n$.*

(a) Suppose μ can be decomposed as

$$\mu(\sigma) = \int_{\mathbb{R}^n} (\mathcal{T}_x \nu)(\sigma)\, d\xi(x), \qquad \forall \sigma \in \{\pm 1\}^n,$$

for some distribution ν on $\{\pm 1\}^n$ and some distribution ξ on $\mathbb{R}^n$ satisfying the following properties:

- ***Poincaré Inequality for Mixture:*** *There is a constant $C > 0$ such that for every test function $\psi : \mathbb{R}^n \to \mathbb{R}$,*

$$\mathrm{Var}_{x \sim \xi}[\psi(x)] \leq C \cdot \int_{\mathbb{R}^n} \|\nabla \psi(x)\|_2^2\, d\xi(x). \tag{1}$$

- ***Spectral Independence for Components:*** *There is a constant $\eta > 0$ such that for every $x \in \mathbb{R}^n$, the exponential tilt $\mathcal{T}_x \nu$ of ν is η-spectrally independent.*

Prove that for every $x \in \mathbb{R}^n$, the exponential tilt $\mathcal{T}_x \mu$ of μ is $O(\alpha\eta^2)$-spectrally independent. (Note that since this holds for all exponential tilts, we also get $O(\alpha\eta^2)$-entropic independence for all exponential tilts.)

(b) Let A be a symmetric positive definite matrix. Show that

$$\exp\left(\frac{1}{2}\sigma^\top A \sigma\right) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\|x\|_2^2 + \left\langle A^{1/2} x, \sigma \right\rangle\right) dx$$

for every $\sigma \in \mathbb{R}^n$.[1]

[1] The "fancy name" for this is the *Hubbard–Stratonovich transform.* Up to scaling, the right-hand side is just a moment generating function.

(c) *Let* $\mu(\sigma) \propto \exp\left(\frac{1}{2}\sigma^\top A\sigma\right)$ *be the Gibbs distribution over* $\{\pm 1\}^n$ *of an Ising model with interaction matrix* $A \in \mathbb{R}^{n\times n}$ *satisfying* $\frac{\epsilon}{2} \cdot \mathsf{Id} \preceq A \preceq \left(1 - \frac{\epsilon}{2}\right) \cdot \mathsf{Id}$ *for some constant* $0 < \epsilon < 1$. *Note that we can always shift* A *by a diagonal matrix without affecting the distribution* μ, *and so we can assume* A *is positive (semi)definite without loss of generality; the key here is ensuring a bound on the spectral diameter* $\lambda_{\max}(A) - \lambda_{\min}(A) \leq 1 - \epsilon$. *Prove that* $\mathcal{T}_x\mu$ *is* $O(1/\epsilon)$*-spectrally independent for all* $x \in \mathbb{R}^n$.

Since the emphasis here is on the connection between the continuous and discrete, you may use the following fact from the continuous world without proof:

Theorem 0.1. *Let* $\xi(x) \propto \exp(-V(x))$ *be an* α*-strongly log-concave probability distribution on* $\mathbb{R}^n$, *i.e. for some constant* $\alpha > 0$, $\nabla^2 V(x) \succeq \alpha \cdot \mathsf{Id}$ *for all* $x \in \mathbb{R}^n$. *Then* ξ *satisfies the Poincaré Inequality*[2] *Eq. (1) with constant* $C = \frac{1}{\alpha}$.

(d) ***The Sherrington–Kirkpatrick Spin Glass:*** *Let* $G \sim \mathsf{GOE}(n)$, *i.e.* $G_{ij} \sim \mathcal{N}(0, 1/n)$ *i.i.d. for* $i < j$, $G_{ji} = G_{ij}$ *for* $i > j$, *and* $G_{ii} = 0$ *for* $i \in [n]$. *Prove that if* $\beta < \frac{1-\epsilon}{4}$ *for a constant* $0 < \epsilon < 1$, *then with high probability over the randomness of* G, *all exponential tilts of the Ising Gibbs measure with interaction matrix* $A = \beta G$ *are* $O(1/\epsilon)$*-spectrally independent.*[3] *Conclude that w.h.p. over* G, *Glauber dynamics mixes in polynomial-time.*[4] *On the other hand, what is the largest value of* β *for which you can make path coupling contract?*

For this problem, you may use the following standard fact from random matrix theory, which can be proved using the trace moment method.

Theorem 0.2. *With high probability over* $G \sim \mathsf{GOE}(n)$, *we have* $-2(1 + o(1)) \cdot \mathsf{Id} \preceq G \preceq 2(1 + o(1)) \cdot \mathsf{Id}$.

[2] Under this strong log-concavity assumption, a log-Sobolev inequality also holds. This is a consequence of the beautiful Bakry–Émery theory.

[3] It is known that for this distribution, we have $\|\Psi_\mu\|_{\ell_\infty \to \ell_\infty} \gtrsim \sqrt{n}$, roughly for the same reason that $\|G\|_{\ell_\infty \to \ell_\infty} \gtrsim \sqrt{n}$.

[4] The naïve bound one would get for the mixing time is $n^{1/\epsilon}$, although using concentration bounds for the norms of random submatrices, one can bring this down to $C^{1/\epsilon} n^c$ for (reasonable) universal constants $C, c > 0$. Using stochastic localization, one can actually establish the optimal $O(n \log n)$-mixing.