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The Optional Stopping Theorem

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1 How to Stop a Stochastic Process

This lecture is all about *stopping times* for martingales. These are particularly useful for analyzing *hitting times* for stochastic processes, e.g. given a sample trajectory of such a process $\{X_n\}_{n\geq 0}$ viewed as an evolution over time, when is the first time n such that the state of X_n comes to possess some property of interest.

Definition 1 (Stopping Time)**.** *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables. We say a random variable T taking values in $\mathbb{N}\cup\{+\infty\}$ is a* stopping time *with respect to $\{X_n\}_{n\in\mathbb{N}}$ if for every $n \in \mathbb{N}$, the event $\{T = n\}$ is completely determined by $X_0, \dots, X_n$.*

As with how we defined martingales, the proper level of generality in which a stopping time should be defined is with respect to a filtration. Since we won't need this in the bulk of this lecture, we relegate a formal discussion to Appendix A.

One should imagine there being a random object which is evolving over time according to $\{X_n\}_{n\in\mathbb{N}}$. The essential feature behind a stopping time is that to decide whether or not to stop the evolution, you only have to look at the past and present. For example, if you are at a casino, then all of the following are valid stopping times:

- "the first round of blackjack you lose"
- "773 rounds after your first loss"
- "the 13th round you win"
- "the third time you triple your money"

However, "the last time you lose" is not a valid stopping time, since you would need to see the future in order to decide whether or not to leave the casino right now.

Fact 1.1. *If T is a stopping time with respect to $\{X_n\}_{n\in\mathbb{N}}$, then so is $T \wedge n \overset{\mathsf{def}}{=} \min\{T, n\}$ for any fixed deterministic $n \in \mathbb{N}$.*

As we argued in the previous lecture, for a martingale $\{Y_n\}_{n\geq 0}$ with respect to $\{X_n\}_{n\geq 0}$, we always have $\mathbb{E}[Y_n] = \mathbb{E}[Y_0]$ for every fixed deterministic $n \in \mathbb{N}$. One might naturally expect that the analogous statement, when we replace n by a random stopping time T, also holds, i.e. $\mathbb{E}[Y_T] = \mathbb{E}[Y_0]$. Unfortunately, this definitely cannot be true in full generality. For example, suppose $\{X_n\}_{n\geq 0}$ is a sequence of independent $\mathsf{Unif}\{\pm 1\}$ random variables, $Y_n = \sum_{k=0}^n X_k$ for each $n \geq 0$ (initialized at $Y_0 = 0$), and $T = \inf\{n \in \mathbb{N} : S_n = +1\}$. Then by definition, we have $\mathbb{E}[Y_T] = +1 \neq 0 = \mathbb{E}[Y_0]$.

The following theorem formalizes several generic conditions under which we do get the "expected conclusion" $\mathbb{E}[Y_T] = \mathbb{E}[Y_0]$. It is among the most fundamental results for stopping times, and also goes by the name of *Doob's Optional Sampling Theorem.*

Theorem 1.2 (Optional Stopping Theorem)**.** *Let $\{Y_n\}_{n\in\mathbb{N}}$ be a martingale and T be a stopping time, both with respect to another sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$. Assume at least one of the following three conditions hold:*

- *There exists a finite constant $L \geq 0$ such that $\Pr[T \leq L] = 1$.*
- *There exists a finite constant $B \geq 0$ such that $|Y_{T\wedge n}| \leq B$ almost surely for all $n \in \mathbb{N}$.*
- *$\mathbb{E}[T] < \infty$ and there exists a finite constant $C \geq 0$ such that*

$$\mathbb{E}\left[|Y_n - Y_{n-1}| \mid X_0, \ldots, X_{n-1}\right] \leq C$$

on the event $\{T \geq n\}$,[1] *for all $n \in \mathbb{N}$.*

Then the random variable Y_T satisfies $\mathbb{E}[Y_T] = \mathbb{E}[Y_0]$.

Remark 1. Regarding the third condition, oftentimes it is not obvious that $\mathbb{E}[T] < \infty$ holds, even if T is finite almost surely. However, as we will see in the proofs for Lemmas 2.2 and 3.2, a generic way to establish $\mathbb{E}[T] < \infty$ is to again consider the truncated stopping times $T \wedge n$. If one can show that $\mathbb{E}[T \wedge n]$ is uniformly bounded in n by a finite constant, then by the Monotone Convergence Theorem, we have $\mathbb{E}[T] < \infty$.

To prove Theorem 1.2, we will need one key lemma.

Lemma 1.3. *Let $\{Y_n\}_{n\in\mathbb{N}}$ be a martingale and T be a stopping time, both with respect to another sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$. For each $n \in \mathbb{N}$, define a new random variable $Z_n \stackrel{\text{def}}{=} Y_{T\wedge n}$. Then $\{Z_n\}_{n\in\mathbb{N}}$ is a martingale with respect to $\{X_n\}_{n\in\mathbb{N}}$.*

Proof. By inserting indicator random variables, we can write

$$Z_n = \sum_{k=0}^{n-1} Y_k \cdot \mathbf{1}_{T=k} + Y_n \cdot \mathbf{1}_{T\geq n}.$$

Now, observe that the random variables $\mathbf{1}_{T=0}, \ldots, \mathbf{1}_{T=n-1}, Y_0, \ldots, Y_{n-1}$ and $\mathbf{1}_{T\geq n} = 1 - \mathbf{1}_{T\leq n-1}$ are all completely determined by $X_0, \ldots, X_{n-1}$; only Y_n isn't. Hence,

$$\begin{aligned}
\mathbb{E}\left[Z_n \mid X_0, \ldots, X_{n-1}\right] &= \sum_{k=0}^{n-1} Y_k \cdot \mathbf{1}_{T=k} + \mathbb{E}\left[Y_n \mid X_0, \ldots, X_{n-1}\right] \cdot \mathbf{1}_{T\geq n} \\
&= \sum_{k=0}^{n-2} Y_k \cdot \mathbf{1}_{T=k} + \underbrace{Y_{n-1} \cdot \mathbf{1}_{T=n-1} + Y_{n-1} \cdot \mathbf{1}_{T\geq n}}_{=Y_{n-1}\cdot\mathbf{1}_{T\geq n-1}} \\
&\qquad\qquad (\{Y_n\}_{n\in\mathbb{N}} \text{ is a martingale w.r.t. } \{X_n\}_{n\in\mathbb{N}}) \\
&= Z_{n-1}.
\end{aligned}$$

□

Proof of Theorem 1.2. Lemma 1.3 and its proof already gives the conclusion of the theorem in the case where T is almost surely bounded. Indeed, if there is a finite $L \geq 0$ such that $\Pr[T \leq L] = 1$, then $Y_T = Z_L$ for $\{Z_n\}_{n\in\mathbb{N}}$ as in Lemma 1.3. Hence, $\mathbb{E}[Y_T] = \mathbb{E}[Z_L] = \mathbb{E}[Z_0] = \mathbb{E}[Y_0]$, where in the middle equality, we used the fact the $\{Z_n\}_{n\in\mathbb{N}}$ is a martingale.

Now, we consider the other two conditions of the theorem. In either case, we know that $Y_{T\wedge n}$ converges to Y_T pointwise. We have already established $\mathbb{E}[Y_{T\wedge n}] = \mathbb{E}[Y_0]$ for all $n \in \mathbb{N}$, and so all that remains is to establish $\mathbb{E}[Y_{T\wedge n}] \to \mathbb{E}[Y_T]$. This is an immediate consequence of the Bounded Convergence Theorem when we assume the second condition. For the third condition, we use the Dominated Convergence Theorem. To do this, we just need to construct a random variable W such that $|Y_{T\wedge n}| \leq W$ almost surely for all $n \in \mathbb{N}$, and $\mathbb{E}[W] < \infty$.

[1] Since T is a stopping time and the event $\{T \geq n\}$ is the complement of $\bigcup_{k=0}^{n-1}\{T = k\}$, it is completely determined by the values of $X_0, \ldots, X_{n-1}$. Hence, we just mean that the expected increments are bounded for all values of $X_0, \ldots, X_{n-1}$ for which the event $\{T \geq n\}$ occurs.

Define

$$W \stackrel{\text{def}}{=} |Y_0| + \sum_{k=0}^{\infty} |Y_{k+1} - Y_k| \cdot \mathbf{1}_{T \geq k+1}.$$

Then

$$\begin{aligned}
|Y_{T\wedge n}| &= \left| \sum_{k=0}^{n-1} Y_k \cdot \mathbf{1}_{T=k} + Y_n \cdot \mathbf{1}_{T\geq n} \right| \\
&= \left| \sum_{k=0}^{n-2} Y_k \cdot \mathbf{1}_{T=k} + Y_{n-1} \cdot \mathbf{1}_{T\geq n-1} + (Y_n - Y_{n-1}) \cdot \mathbf{1}_{T\geq n} \right| \\
&= \cdots && \text{(Induction/Telescoping)} \\
&= \left| Y_0 + \sum_{k=0}^{n-1} (Y_{k+1} - Y_k) \cdot \mathbf{1}_{T\geq k+1} \right| \\
&\leq W. && \text{(Triangle Inequality)}
\end{aligned}$$

Finally, we show $\mathbb{E}[W]$ is finite using our assumptions.

$$\begin{aligned}
\mathbb{E}[W] &= \mathbb{E}\left[|Y_0|\right] + \sum_{k=0}^{\infty} \mathbb{E}\left[|Y_{k+1} - Y_k| \cdot \mathbf{1}_{T\geq k+1}\right] \\
&= \mathbb{E}\left[|Y_0|\right] + \sum_{k=0}^{\infty} \mathbb{E}\left[\mathbb{E}\left[|Y_{k+1} - Y_k| \cdot \mathbf{1}_{T\geq k+1} \mid X_0, \ldots, X_k\right]\right] \\
&= \mathbb{E}\left[|Y_0|\right] + \sum_{k=0}^{\infty} \mathbb{E}\left[\mathbb{E}\left[|Y_{k+1} - Y_k| \mid X_0, \ldots, X_k\right] \cdot \mathbf{1}_{T\geq k+1}\right] \\
& && (\mathbf{1}_{T\geq k+1} = 1 - \mathbf{1}_{T\leq k} \text{ is determined by } X_0, \ldots, X_k) \\
&\leq \mathbb{E}\left[|Y_0|\right] + C \cdot \sum_{k=0}^{\infty} \Pr\left[T \geq k+1\right] && \text{(Second assumption)} \\
&= \mathbb{E}\left[|Y_0|\right] + C \cdot \mathbb{E}[T] && \text{(Layered cake representation for an expectation)} \\
&< \infty. && \text{(First assumption)}
\end{aligned}$$

Applying the Dominated Convergence Theorem yields $\mathbb{E}[Y_0] = \lim_{n\to\infty} \mathbb{E}[Y_{T\wedge n}] = \mathbb{E}[Y_T]$ as desired. $\square$

2 Hitting Times for Random Walk on $\mathbb{Z}$

Consider a p-biased random walk on $\mathbb{Z}$. More precisely, let $\{X_n\}_{n\geq 0}$ is a sequence of i.i.d. random variables where $X_i = +1$ with probability p, and $X_i = -1$ with probability $1-p$. Consider the biased random walk $S_n = \sum_{k=0}^{n} X_k$ with initialization $S_0 = 0$; note that $\{S_n\}_{n\in\mathbb{N}}$ is a valid martingale with respect to $\{X_n\}_{n\in\mathbb{N}}$ if and only if $p = 1/2$. For each $x \in \mathbb{Z}$, let $T_x = \inf\{n : S_n = x\}$, which is a valid stopping time.

The gambling interpretation for this set up is the following: Suppose you enter a casino with A dollars in your pocket for some $A > 0$. You will leave the casino if either you go broke (i.e. you lose all A dollars), or you win at least B dollars for some $B > 0$ (e.g. you double your money by setting $B = A$). If we let S_n denote your "winnings", then $S_n = B$ if and only if you leave after gaining B dollars, and $S_n = -A$ if and only if you lose all your money. Each X_n denotes the outcome of one round of play, with you either winning or losing a dollar. The probability $0 \leq p \leq 1/2$ represents how "fair" the casino is, with $p = 1/2$ representing perfect fairness. Naturally, you are interested in the probability of losing all your money (i.e. $\Pr[T_{-A} < T_B]$), and also the amount of time you are expected to stay at the casino.

We begin with the simplest case where $p = 1/2$ and the casino is fair.

Lemma 2.1. *Suppose $p = 1/2$ and $A, B > 0$. Then*

$$\Pr\left[T_{-A} < T_B\right] = \frac{B}{A+B} \qquad \textit{and} \qquad \mathbb{E}\left[\min\left\{T_{-A}, T_B\right\}\right] = A \cdot B.$$

Proof. Observe that by the definition of T, we have $|S_{T\wedge n}| \le \max\{A, B\}$ almost surely for all $n \in \mathbb{N}$. Hence, we may invoke the Optional Stopping Theorem (with the second condition satisfied) to deduce that

$$0 = \mathbb{E}[S_0] = \mathbb{E}[S_T] = \Pr[T = T_B] \cdot B - \Pr[T = T_{-A}] \cdot A.$$

Rearranging and noting that $\Pr[T_{-A} < T_B] = \Pr[T = T_{-A}]$ then establishes the first claim. For the second, observe that the new sequence of random variables $Y_n \overset{\mathsf{def}}{=} S_n^2 - n$ is also a martingale w.r.t. $\{X_n\}_{n\ge 0}$; this is the "second moment martingale" we saw in the previous lecture. Moreover, $|Y_{T\wedge n}| \le \max\left\{A^2 - A, B^2 - B\right\}$ almost surely for all $n \in \mathbb{N}$. Hence, we may again use the Optional Stopping Theorem to obtain

$$0 = \mathbb{E}[Y_0] = \mathbb{E}[Y_T] = \mathbb{E}[S_T^2] - \mathbb{E}[T].$$

Rearranging yields

$$\mathbb{E}[T] = \mathbb{E}[S_T^2] = \Pr[T = T_{-A}] \cdot A^2 + \Pr[T = T_B] \cdot B^2 = \frac{B}{A+B} \cdot A^2 + \frac{A}{A+B} \cdot B^2 = A \cdot B$$

as desired. □

2.1 Asymmetric Gambler's Ruin

Now let's consider the unfair case, where $p < 1/2$. The proof is conceptually the same, although requires a little more calculation.

Lemma 2.2. *Suppose* $0 \le p < 1/2$ *and* $A, B > 0$. *Then*

$$\begin{aligned}
\Pr\left[T_{-A} < T_B\right] &= \frac{1 - \left(\frac{p}{1-p}\right)^B}{1 - \left(\frac{p}{1-p}\right)^{A+B}} \\
\Pr\left[T_B < \infty\right] &= \left(\frac{p}{1-p}\right)^B \\
\Pr\left[T_{-A} < \infty\right] &= 1 \\
\mathbb{E}\left[T_{-A}\right] &= \frac{A}{1-2p}.
\end{aligned}$$

Proof. Consider the function $\varphi(x) \overset{\mathsf{def}}{=} \left(\frac{1-p}{p}\right)^x$. We first claim that the random variables $\{\varphi(S_n)\}_{n\ge 0}$ form a martingale with respect to $\{X_n\}_{n\ge 0}$. Indeed,

$$\begin{aligned}
\mathbb{E}\left[\varphi(S_n) \mid X_0, \ldots, X_{n-1}\right] &= \mathbb{E}\left[\varphi(S_{n-1}) \cdot \left(\frac{1-p}{p}\right)^{X_n} \,\middle|\, X_0, \ldots, X_{n-1}\right] \\
&= \varphi(S_{n-1}) \cdot \mathbb{E}\left[\left(\frac{1-p}{p}\right)^{X_n} \,\middle|\, X_0, \ldots, X_{n-1}\right] \\
&\qquad (S_{n-1} \text{ is determined by } X_0, \ldots, X_{n-1}) \\
&= \varphi(S_{n-1}) \cdot \left(p \cdot \left(\frac{1-p}{p}\right) + (1-p) \cdot \left(\frac{1-p}{p}\right)^{-1}\right) \\
&= \varphi(S_{n-1}).
\end{aligned}$$

Now let $T = \min\{T_{-A}, T_B\}$ denote the time you leave casino. Then

$$0 \le \varphi\left(S_{T\wedge n}\right) \le \max\left\{\left(\frac{1-p}{p}\right)^A, \left(\frac{1-p}{p}\right)^B\right\}, \qquad \forall n \in \mathbb{N},$$

and so we may invoke the Optional Stopping Theorem with stopping T using the second condition. It follows that

$$1 = \mathbb{E}[\varphi(S_0)] = \mathbb{E}[\varphi(S_T)] = \Pr[T = T_{-A}] \cdot \varphi(-A) + \Pr[T = T_B] \cdot \varphi(B).$$

Rearranging and noting that $\Pr[T_{-A} < T_B] = \Pr[T = T_{-A}]$ then establishes the first identity. For the second, note that

$$\{T_B < \infty\} = \bigcup_{A=1}^{\infty} \{T_B < T_{-A}\} \qquad \{T_{-A} < \infty\} = \bigcup_{B=1}^{\infty} \{T_{-A} < T_B\}.$$

Hence, using the previously obtained formula for $\Pr[T_{-A} < T_B]$, we have

$$\Pr[T_B < \infty] = \lim_{A\to\infty} \Pr[T_B < T_{-A}] = \left(\frac{p}{1-p}\right)^B$$
$$\Pr[T_A < \infty] = \lim_{A\to\infty} \Pr[T_{-A} < T_B] = 1.$$

Finally, to compute $\mathbb{E}[T_{-A}]$, observe that while $\{S_n\}_{n\in\mathbb{N}}$ isn't a martingale with respect to $\{X_n\}_{n\in\mathbb{N}}$ (since $p \neq 1/2$), it is easy to see that $S_n - (2p-1)n$ is. Hence, if we could apply the Optional Stopping Theorem to $S_n - (2p-1)n$, then we would obtain

$$0 = \mathbb{E}[S_0] = \mathbb{E}[S_{T_{-A}} - (2p-1)T_{-A}] = (1-2p)\cdot\mathbb{E}[T_{-A}] - A,$$

which upon rearranging, would yield the claim. Hence, we must verify one of the three conditions of the Optional Stopping Theorem for the martingale $\{S_n - (2p-1)n\}_{n\in\mathbb{N}}$ and the stopping time T_{-A}.

Unfortunately, while we proved T_{-A} to be almost surely finite, it is not almost surely bounded. Hence, $S_{T_{-A}\wedge n} - (2p-1)\cdot(T_{-A}\wedge n)$ also fails to be almost surely bounded uniformly in n. So, we verify the third condition. Boundedness of the expected increments is immediate, and so it suffices to prove that $\mathbb{E}[T_{-A}] < \infty$. To do this, we use the idea outlined in Remark 1. Let us consider again the stopping times $T_{-A}\wedge n$, which are almost surely bounded by n. By the Optional Stopping Theorem,

$$0 = \mathbb{E}[S_0] = \mathbb{E}[S_{T_{-A}\wedge n} - (2p-1)\cdot(T_{-A}\wedge n)] = (1-2p)\cdot\mathbb{E}[T_{-A}\wedge n] + \mathbb{E}[S_{T_{-A}\wedge n}].$$

Of course, $\mathbb{E}[S_{T_{-A}\wedge n}] \geq -A$ since $S_{T_{-A}} = -A$, and $S_n \geq -A$ if $T_{-A} \geq n$. Hence, we see that $\mathbb{E}[T_{-A}\wedge n] \leq \frac{A}{1-2p}$ for all $n \in \mathbb{N}$. By the Monotone Convergence Theorem, the limit $\lim_{n\to\infty}\mathbb{E}[T_{-A}\wedge n]$ exists and equals $\mathbb{E}[T_{-A}]$ since $T_{-A}\wedge n \to T_{-A}$ pointwise. It follows that $\mathbb{E}[T_{-A}] \leq \frac{A}{1-2p} < \infty$. Hence, the above use of the Optional Stopping Theorem is valid, and we conclude $\mathbb{E}[T_{-A}] = \frac{A}{1-2p}$. □

3 Supermartingales and Submartingales

The application of martingales to simple random walk on $\mathbb{Z}$ described in Section 2 is nice and slick, but arguably isn't the most convincing, since Lemmas 2.1 and 2.2 are also obtainable through bare-handed calculations without too much effort. This is made possible by the *Markovian* nature of those stochastic processes; the random variable S_n was independent of $S_0, \ldots, S_{n-2}$ given S_{n-1}.[2]

To better appreciate the full power of martingale theory, let us now move to a setting where there is no Markovian structure whatsoever. We begin by substantially generalizing the theory of martingales.

Definition 2 (Supermartingale/Submartingale)**.** *We say a (possibly finite) sequence of random variables $\{Y_n\}_{n\geq 0}$ is a* supermartingale *(resp.* submartingale*) with respect to another sequence of random variables $\{X_n\}_{n\geq 0}$ if for every n,*

- $\mathbb{E}[|Y_n|] < \infty$,
- *Y_n is a function of $X_0, \ldots, X_n$, and*
- $\mathbb{E}[Y_{n+1} \mid X_0, \ldots, X_n] \leq Y_n$ *(resp.* $\mathbb{E}[Y_{n+1} \mid X_0, \ldots, X_n] \geq Y_n$*).*

[2] To get a sense of how such an argument could work, consider the setting of Lemma 2.2. We can write a recurrence like $\Pr[T_B < \infty] = (1-p)\cdot\Pr[T_{B+1} < \infty] + p\cdot\Pr[T_{B-1} < \infty]$. Rearranging then yields a fixed-point equation of the form $x = \frac{x-p}{x(1-p)}$ for the ratios of consecutive terms, which has roots 1 and $\frac{p}{1-p}$. This at least explains why $\Pr[T_B < \infty]$ has the form $\left(\frac{p}{1-p}\right)^B$. Such calculations are tractable only if the process has a Markovian structure.

Roughly speaking, supermartingales (resp. submartingales) tend to *decay* (resp. *grow*) faster than a standard martingale.[3] We can construct them from standard martingales extremely easily. The following lemma is an immediate consequence of Jensen's Inequality.

Lemma 3.1. *Let $\{Y_n\}_{n\geq 0}$ be a martingale with respect to $\{X_n\}_{n\geq 0}$.*

- *If $f : \mathbb{R} \to \mathbb{R}$ is a convex function satisfying $\mathbb{E}[|f(Y_n)|] < \infty$ for all n, then $\{f(Y_n)\}_{n\geq 0}$ is a* submartingale *with respect to $\{X_n\}_{n\geq 0}$.*
- *If $f : \mathbb{R} \to \mathbb{R}$ is a concave function satisfying $\mathbb{E}[|f(Y_n)|] < \infty$ for all n, then $\{f(Y_n)\}_{n\geq 0}$ is a* supermartingale *with respect to $\{X_n\}_{n\geq 0}$.*

Naturally, supermartingales (resp. submartingales) with bounded increments satisfy the upper (resp. lower) tail version of the Azuma–Hoeffding concentration bound. The Optional Stopping Theorem also extends in a straightforward manner to these processes:

$$\mathbb{E}[Y_T] \leq \mathbb{E}[Y_0] \qquad \text{(for supermartingales)}$$
$$\mathbb{E}[Y_T] \geq \mathbb{E}[Y_0] \qquad \text{(for submartingales)}$$

Using this, we can derive an upper bound for the hitting time of a martingale on $\mathbb{Z}$ satisfying extremely mild conditions.

Lemma 3.2. *Let $\{Y_t\}_{t\geq 0}$ be a supermartingale with respect to $\{X_t\}_{t\geq 0}$, and assume $\{Y_t\}_{t\geq 0}$ takes values in $\{0, \ldots, n\}$. Let T denote the stopping time $\min\{t \geq 0 : Y_t = 0\}$, and suppose there exists $\sigma^2 > 0$ such that for all $t < T$,*

$$\mathbb{E}\left[(Y_t - Y_{t-1})^2 \mid X_0, \ldots, X_{t-1}\right] \geq \sigma^2$$

almost surely. Then $\mathbb{E}[T] \leq n^2/\sigma^2$.

Proof. For parameters α, β to be determined later, consider the random variables $Z_t \stackrel{\text{def}}{=} Y_t^2 + \alpha Y_t + \beta t$. Our goal is to pick α, β so that $\{Z_t\}_{t\geq 0}$ is a submartingale w.r.t. $\{X_t\}_{t\geq 0}$. For this, observe that

$$\begin{aligned}
&\mathbb{E}[Z_{t+1} \mid X_0, \ldots, X_t] \\
&= \mathbb{E}\left[(Y_t + (Y_{t+1} - Y_t))^2 + \alpha(Y_t + (Y_{t+1} - Y_t)) + \beta(t+1) \mid X_0, \ldots, X_t\right] \\
&= Z_t + (2Y_t + \alpha) \cdot \underbrace{\mathbb{E}[Y_{t+1} - Y_t \mid X_0, \ldots, X_t]}_{\leq 0 \text{ by supermartingale assumption}} + \underbrace{\mathbb{E}\left[(Y_{t+1} - Y_t)^2 \mid X_0, \ldots, X_t\right]}_{\geq \sigma^2 \text{ by assumption}} + \beta.
\end{aligned}$$

To guarantee that this is at least Z_t, it suffices to set $\alpha = -2n$ and $\beta = -\sigma^2$ by using our assumptions on the process. Applying the Optional Stopping Theorem to the submartingale $\{Z_t\}_{t\geq 0}$ with the stopping time $T \wedge t$, we obtain

$$\begin{aligned}
\max_{s\in\{0,\ldots,n\}} \left\{s^2 - 2ns\right\} &\leq \mathbb{E}[Z_0] \\
&\leq \mathbb{E}[Z_{T\wedge t}] \\
&= \mathbb{E}[Y_{T\wedge t}^2] - 2n \cdot \mathbb{E}[Y_{T\wedge t}] - \sigma^2 \cdot \mathbb{E}[T \wedge t] \\
&= -\sigma^2 \cdot \mathbb{E}[T \wedge t],
\end{aligned}$$

where in the final inequality, we used the fact that $Y_{T\wedge t}^2 - 2n \cdot Y_{T\wedge t} \leq 0$ almost surely. Rearranging and using the fact that the left-hand side is equal to n^2 (by setting $s = n$), we see that $\mathbb{E}[T \wedge t] \leq n^2/\sigma^2$, which is independent of t. Applying the Monotone Convergence Theorem, we have $\lim_{t\to\infty} \mathbb{E}[T \wedge t] = \mathbb{E}[T]$, and so $\mathbb{E}[T] \leq n^2/\sigma^2$ as well. □

[3] The terminology for supermartingales/submartingales are almost surely confusing to me. Perhaps Lemma 3.1 suggests we should switch "supermartingale/submartingale/martingale" to "concave/convex/linear" martingale?

3.1 Stochastic Local Search for 2-SAT

We use Lemma 3.2 to show that the following randomized local search algorithm solves 2-SAT[4] in polynomial-time: Fix an arbitrary CNF-formula $\Phi = (\mathcal{V}, \mathcal{C})$ such that every clause contains at most two literals.

1. Select an arbitrary initial assignment $\boldsymbol{x}_0 : \mathcal{V} \mapsto \{\mathsf{T}, \mathsf{F}\}$.
2. If $\boldsymbol{x}_t$ doesn't satisfy Φ, pick an arbitrary unsatisfied clause $C \in \mathcal{C}$. We then pick a uniformly random literal $v \in C$, and resample its assignment to obtain a new assignment $\boldsymbol{x}_{t+1}$, i.e. $\boldsymbol{x}_{t+1}(u) = \boldsymbol{x}_t(u)$ for all $u \neq v$ and $\boldsymbol{x}_{t+1}(v) \sim \mathsf{Unif}\{\mathsf{T}, \mathsf{F}\}$.[5]

Theorem 3.3 ([Pap91; McD93])**.** *For any satisfiable* 2-SAT *instance* Φ *with* n *variables and* m *clauses, the above algorithm finds a satisfying assignment using* $O(n^2)$*-rounds in expectation. In particular, the expected running time is* $O(n^2 m)$*.*

Remark 2. A simple variant of the above algorithm, sometimes referred as the FIX algorithm or the *Moser–Tardos algorithm*, works as follows: We initialize to a *uniformly random assignment* $\boldsymbol{x}_0$. Then in each step, we pick an arbitrary unsatisfied clause, and resample *all* of its variables according to $\mathsf{Unif}\{\mathsf{T}, \mathsf{F}\}$ independently. As we previously mentioned, in a major breakthrough, Moser–Tardos [MT10] showed that in the Lovász Local Lemma regime, this algorithm successfully finds a satisfying assignment using a nearly-linear number of rounds.

Proof. Let $\boldsymbol{x}^*$ be some arbitrary satisfying assignment, and consider the "potential function" $f(\boldsymbol{x}) = d_H(\boldsymbol{x}, \boldsymbol{x}^*)$. Let $Y_t \stackrel{\mathsf{def}}{=} f(\boldsymbol{x}_t)$. If $\boldsymbol{x}_t$ isn't yet a satisfying assignment, then in each unsatisfied clause C, $\boldsymbol{x}_t$ must differ from $\boldsymbol{x}^*$ in the assignment of at least one of the (at most) two literals in C. Since we picked a uniformly random such literal and rerandomized its assignment from $\mathsf{Unif}\{\mathsf{T}, \mathsf{F}\}$, it follows that

$$\Pr[Y_t - Y_{t-1} = -1 \mid \boldsymbol{x}_0, \ldots, \boldsymbol{x}_{t-1}] \geq \Pr[Y_t - Y_{t-1} = +1 \mid \boldsymbol{x}_0, \ldots, \boldsymbol{x}_{t-1}]$$

almost surely. In particular, $\{Y_t\}_{t \geq 0}$ is a supermartingale with respect to $\{\boldsymbol{x}_t\}_{t \geq 0}$. Moreover,

$$\mathbb{E}\left[(Y_t - Y_{t-1})^2 \mid \boldsymbol{x}_0, \ldots, \boldsymbol{x}_{t-1}\right] \geq \sigma^2$$

for some constant $\sigma^2 > 0$ because we are resampling assignments from $\mathsf{Unif}\{\mathsf{T}, \mathsf{F}\}$. The claim then follows by Lemma 3.2. □

References

[McD93] Colin McDiarmid. "A Random Recolouring Method for Graphs and Hypergraphs". In: *Combinatorics, Probability and Computing* 2 (3 1993), pp. 363–365 (cit. on p. 7).

[MT10] Robin A. Moser and Gábor Tardos. "A Constructive Proof of the General Lovász Local Lemma". In: *J. ACM* 57.2 (Feb. 2010). ISSN: 0004-5411. DOI: 10.1145/1667053.1667060 (cit. on p. 7).

[Pap91] Christos H. Papadimitriou. "On selecting a satisfying truth assignment (extended abstract)". In: *Proceedings of the 32nd Annual Symposium on Foundations of Computer Science*. SFCS '91. San Juan, Puerto Rico: IEEE Computer Society, 1991, pp. 163–169. ISBN: 0818624450. DOI: 10.1109/SFCS.1991.185365 (cit. on p. 7).

A The Measure-Theoretic Definition of Stopping Times

Following the measure-theoretic definition of filtrations and martingales from the Appendix of the previous lecture, we can formally define stopping times as follows.

Definition 3. *Fix a probability space* $(\Omega, \mathcal{F}, \mu)$ *and a filtration* $\{\mathcal{F}_n\}_{n \geq 0}$*. We say a random variable (i.e. a* $\mathcal{F}$*-measurable function)* $T : \Omega \to \mathbb{N} \cup \{+\infty\}$ *is a* stopping time *with respect to* $\{\mathcal{F}_n\}_{n \geq 0}$ *if the events* $\{T = n\} \stackrel{\mathsf{def}}{=} T^{-1}(n)$ *satisfy* $\{T = n\} \in \mathcal{F}_n$ *for all* $n \in \mathbb{N} \cup \{+\infty\}$*.*

[4] Note that there is also a simple deterministic algorithm for this.

[5] One can also just "flip" the variable, i.e. set $\boldsymbol{x}_{t+1}(v) = \neg \boldsymbol{x}_t(v)$.

Example 1. Fix a probability space $(\Omega, \mathcal{F}, \mu)$. For a measurable space $(\Sigma, \mathcal{G})$, let $\{X_t\}_{t\geq 0}$ be a sequence of $(\Sigma, \mathcal{G})$-valued random variables adapted to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Then for any measurable set $\mathcal{P} \in \mathcal{G}$, the function

$$T(\omega) \stackrel{\text{def}}{=} \inf\{t \geq 0 : X_t(\omega) \in \mathcal{P}\}, \qquad \forall \omega \in \Omega,$$

is a stopping time. For instance, imagine Ω is given by the set of all possible trajectories of an infinite random walk on $\mathbb{Z}$, and each $X_t \in \mathbb{Z}$ denotes the location of the random walk at time-t. Then ω is a complete trajectory, and $T(\omega)$ denotes the first time step t that the random walk hits some set $\mathcal{P} \subseteq \mathbb{Z}$.