

## 6.7720/18.619/15.070 Lecture 3

# The First & Second Moment Methods: Statistical Inference for the Broadcast Process

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## 1 Broadcasting on Trees

A fundamental problem in statistical inference is that of *hypothesis testing*. We are given a *sample* (often called the “observation”)  $X$  from some mystery probability distribution, and there are two possibilities (or *hypotheses*)  $\mu, \nu$  for what that mystery distribution can be. Our goal is to distinguish whether  $X \sim \mu$  or  $X \sim \nu$ . Of course, one can also study the setting in which we receive multiple independent samples  $X_1, \dots, X_n$ .

In this lecture, we consider a special kind of hypothesis testing problem called the *reconstruction problem*, which was devised as basic statistical model for evolutionary genetics; see e.g. [Dur08]. To state the problem, we begin by defining a new type of correlated stochastic process called the *broadcast process*.

**Definition 1** ((Binary Symmetric) Broadcast Process on  $\hat{\mathbb{T}}_d$ ). *Fix  $d \in \mathbb{N}$  and an error parameter  $0 \leq \epsilon \leq 1/2$ . The associated (binary symmetric) broadcast process on the infinite  $d$ -ary tree  $\hat{\mathbb{T}}_d$  rooted at  $r$  is a random assignment  $\sigma_v \in \{\pm 1\}$  to each vertex  $v$  of  $\hat{\mathbb{T}}_d$  generated as follows:*

- *We initialize the process by sampling  $\sigma_r \sim \text{Unif}\{\pm 1\}$  for the root vertex.*
- *Independently for each child  $v$  of  $r$ , we set  $\sigma_v = \sigma_r$  with probability  $1 - \epsilon$  and  $\sigma_v = -\sigma_r$  with probability  $\epsilon$ .*
- *This process continues recursively in the subtrees rooted at each of the children  $v$  of  $r$ .*

*Remark 1.* This process can be vastly generalized to allow for larger state spaces, as well as arbitrary Markov chains for the channels on the edges.

Informally, the vertex  $r$  is “broadcasting” its random assignment to its descendants through channels corrupted by an  $\epsilon$  amount of noise. The basic question is whether or not we can *infer* the assignment of the root vertex given only information about far away vertices. More precisely, if we let  $L(n)$  denote the set of vertices at distance exactly  $n$  from the root, then our goal is to design an estimator  $\hat{\sigma}_{r,n}$  for  $\sigma_r$  based on only the assignments  $\sigma_{L(n)}$  of the vertices in  $L(n)$  for  $n$  large. For instance,  $\hat{\sigma}_{r,n}$  could be a deterministic function mapping  $\{\pm 1\}^{L(n)}$  to  $\{\pm 1\}$  which tells us what to guess upon seeing some specific  $\sigma_{L(n)}$ . At the highest level of generality, our goal is to design a function  $\hat{p} : \{\pm 1\}^{L(n)} \rightarrow [0, 1]$  which outputs the probability that we should set our estimator  $\hat{\sigma}_{r,n} \in \{\pm 1\}$  to, say,  $+1$ .

**Definition 2** (Reconstruction Problem). *Fix  $d \in \mathbb{N}$  and  $0 \leq \epsilon \leq 1/2$ . For an estimator  $\hat{\sigma}_{r,n}$  given access to only  $\sigma_{L(n)}$ , write  $\frac{1+b(\hat{\sigma}_{r,n})}{2} = \Pr[\hat{\sigma}_{r,n} = \sigma_r]$ , where the probability is calculated with respect to a random draw of the broadcast process  $\sigma$  and the randomness of the estimator, and*

$b(\hat{\sigma}_{r,n})$  denotes the “advantage” of the estimator over random guessing.<sup>1</sup> We also write  $\frac{1+b^*}{2}$  for the optimal success probability, where  $b^* = b^*(n, \hat{\mathbb{T}}_d, \epsilon) \geq 0$  is given by maximizing  $b(\hat{\sigma}_{r,n})$  over all possible estimators  $\hat{\sigma}_{r,n}$ .

**Theorem 1.1** (Kesten–Stigum’1966 [KS66], Bleher–Ruiz–Zagreb’1995 [BRZ95]). *Fix  $d \in \mathbb{N}$ , and let  $\theta = 1 - 2\epsilon$  for  $0 \leq \epsilon \leq 1/2$  (so that  $\epsilon = \frac{1-\theta}{2}$ ). Let  $\theta_c$  satisfy  $d \cdot \theta_c^2 = 1$ , and correspondingly define  $\epsilon_c = \frac{1}{2} - \frac{1}{2\sqrt{d}}$ . Then we have the following phase transition for the reconstruction problem:*

$$\lim_{n \rightarrow \infty} b^*(n, \hat{\mathbb{T}}_d, \epsilon) \begin{cases} = 0, & \text{if } \epsilon > \epsilon_c \text{ (or equivalently } d \cdot \theta^2 < 1) \\ > 0, & \text{if } \epsilon < \epsilon_c \text{ (or equivalently } d \cdot \theta^2 > 1) \end{cases}.$$

*Remark 2.* The reconstruction problem has also been studied on more general classes of trees; see e.g. [Eva+00].

*Remark 3.* It turns out that in the regime  $\epsilon > \epsilon_c$ , we have that there is some  $\delta = \delta(\epsilon) > 0$  such that we have an exponential decay rate

$$b^*(n, \hat{\mathbb{T}}_d, \epsilon) \lesssim (1 - \delta)^n.$$

At criticality, when  $\epsilon = \epsilon_c$ , a closer inspection of the proof reveals that  $b^*(n, \hat{\mathbb{T}}_d, \epsilon) \leq O(1/n)$ . This decay rate is much slower than the decay in the regime  $\epsilon > \epsilon_c$ , but it still yields  $b^*(n, \hat{\mathbb{T}}_d, \epsilon) \rightarrow 0$ .

Very roughly speaking, this phase transition occurs due to two competing effects: There is the exponential-in- $n$  growth of the number of vertices which could receive the “signal”  $\sigma_r$ , but there’s also the exponential decay of information due to the addition of noise in each step as we increase the distance  $n$ .

**A Brief Word on Notation** Throughout the remainder of the course, for a random variable  $X$ , we write  $\text{Law}(X)$  for the associated probability measure  $X$  according to which  $X$  is distributed. We also write  $\text{supp}(X)$  for the *support* of the distribution  $\text{Law}(X)$ , i.e. the set of values which occur with positive probability.

## 2 The Branching Process Perspective

] [Theorem 1.1](#) establishes a phase transition phenomenon for the reconstruction problem on  $\hat{\mathbb{T}}_d$ . Rather than diving head-first into bare-handed calculations, let us first try to relate this to the only phase transition phenomenon we’ve already studied, namely percolation on  $\hat{\mathbb{T}}_d$  (or more generally, Galton–Watson branching processes). Because the broadcast process is about noisy transmission of information, it is natural to track the number of vertices of  $\hat{\mathbb{T}}_d$  which “receive uncorrupted information from the root”. One clean way to formalize this is to reframe the broadcast process as follows:

- We initialize the process by sampling  $\sigma_r \sim \text{Unif}\{\pm 1\}$  for the root vertex.
- Independently for each child  $v$  of  $r$ , we set  $\sigma_v = \sigma_r$  with probability  $\theta = 1 - 2\epsilon$ , and otherwise sample  $\sigma_v \sim \text{Unif}\{\pm 1\}$ .
- This process continues recursively in the subtrees rooted at each of the children  $v$  of  $r$ .

In this way of looking at the broadcast process, we see that independently to each child  $v$ , the parent  $u$  “perfectly transmits” its assignment with probability  $\theta = 1 - 2\epsilon$ . Otherwise, with probability  $2\epsilon$ , the child completely ignores its parent’s assignment and samples a fresh bit in  $\{\pm 1\}$ . In the former case, let us mark the corresponding edge  $\{u, v\}$  with a 1, indicating the edge is “open”; otherwise, we mark the edge  $\{u, v\}$  with a 0, indicating the edge is “closed”. The open edges in  $\hat{\mathbb{T}}_d$ , i.e. the edges which receive a 1, are collectively distributed as bond percolation on  $\hat{\mathbb{T}}_d$  with edge probability  $\theta = 1 - 2\epsilon$ .

<sup>1</sup>A trivial estimator is to just output a uniformly random element of  $\{\pm 1\}$ . This estimator clearly guesses the correct answer with probability  $\frac{1}{2}$ .

For each  $n \in \mathbb{N}$ , let  $Z_n$  denote the number of vertices in  $L(n)$  which are connected to the root  $r$  by a path of open edges (i.e. edges marked by a 1); for each of these vertices, its assignment, as well as the assignments of its ancestors, all perfectly copy the assignment of the root. A key observation is that the collection of random variables  $\mathcal{Z} = \{Z_n\}_{n \in \mathbb{N}}$  is a Galton–Watson branching process with binomial offspring distribution  $\text{Bin}(d, \theta)$ , where recall that  $\theta = 1 - 2\epsilon$ . This branching process is critical if and only if  $d\theta = 1$  and so we expect a phase transition at this point.

In a future lecture, we will discuss the phase transition at  $d\theta = 1$  in the context of the famous (ferromagnetic) *Ising model* from statistical physics, which is intimately related to the broadcast process we’re studying here. However, unfortunately, this transition point doesn’t match the one in [Theorem 1.1](#).

The reason is that even if  $\theta > \frac{1+\epsilon}{d}$ , it could be that the number of vertices  $Z_n$  copying the root is too small relative to the total number of vertices  $d^n$  in  $L(n)$ . In other words, even if there is a “signal” (whose magnitude is  $Z_n$ ) received by  $L(n)$ , it is “drowned out” by the sea of noise from the remaining  $d^n - Z_n$  vertices who receive random bits independent of the root. To get a better sense of “how large” the signal needs to be, imagine an alternative world where all vertices of  $L(n)$  receive independent random bits drawn from  $\text{Unif}\{\pm 1\}$ , and there is simply no way to estimate the root  $\sigma_r$  better than a coin flip. In this “pure-noise” world (sometimes called the *null hypothesis*), the expected number of vertices which have same assignment as the root is  $\frac{1}{2}d^n$  while the standard deviation is  $\frac{1}{2}d^{n/2}$ . In order to “rule out” this alternative universe, we would need  $\mathbb{E}[Z_n] > \frac{1}{2}d^{n/2}$  for large  $n$ . Since  $\mathbb{E}[Z_n] = d^n \theta^n$ , this inequality is equivalent to  $d\theta^2 > 1$ , which is the threshold described by [Theorem 1.1](#). This is a fairly generic phenomenon: The difference in expectations between two distributions needs to exceed the standard deviation of one of them in order to distinguish them. We formalize this in [Section 3.2](#).

### 3 Reinterpreting $b^*$ as a Metric

The problem of recovering  $\sigma_r$  becomes easier if  $\epsilon < \epsilon_c$  is small because the value of  $\sigma_r$  has a noticeable effect on the distribution of  $\sigma_{L(n)}$ . If  $\epsilon < \epsilon_c$  is small and  $\sigma_r = +1$  (resp.  $\sigma_r = -1$ ), then we expect a disproportionate fraction of the vertices in  $L(n)$  to be assigned  $+1$  (resp.  $-1$ ). It turns out we can actually interpret  $b^*$  as measuring the size of this effect. More formally, we can express it as the *total variation distance* between the distribution of  $\sigma_{L(n)}$  conditioned on  $\sigma_r = +1$  and the distribution of  $\sigma_{L(n)}$  conditioned on  $\sigma_r = -1$ . To see this, observe that

$$\begin{aligned} b^* &= 2 \cdot \sup_{\hat{p}} \Pr[\hat{\sigma}_{r,n} = \sigma_r] - 1 \\ &= 2 \cdot \sup_{\hat{p}} \left\{ \frac{1}{2} \Pr[\hat{\sigma}_{r,n} = \sigma_r \mid \sigma_r = +1] + \frac{1}{2} \Pr[\hat{\sigma}_{r,n} = \sigma_r \mid \sigma_r = -1] \right\} - 1 \\ &\quad \text{(Law of Total Probability)} \\ &= \sup_{\hat{p}} \{ \Pr[\hat{\sigma}_{r,n} = +1 \mid \sigma_r = +1] - \Pr[\hat{\sigma}_{r,n} = +1 \mid \sigma_r = -1] \} \\ &= \sup_{\hat{p}} \{ \mathbb{E}_{\sigma} [\hat{p}(\sigma_{L(n)}) \mid \sigma_r = +1] - \mathbb{E}_{\sigma} [\hat{p}(\sigma_{L(n)}) \mid \sigma_r = -1] \} \\ &= \mathcal{D}_{\text{TV}} (\text{Law}(\sigma_{L(n)} \mid \sigma_r = +1), \text{Law}(\sigma_{L(n)} \mid \sigma_r = -1)), \end{aligned} \quad \text{(Lemma 3.1)}$$

where the supremum is over all possible functions  $\hat{p} : \{\pm 1\}^{L(n)} \rightarrow [0, 1]$  and the probability is with respect to  $\sigma$  drawn from the broadcast process and  $\hat{\sigma}_{r,n}$  drawn independently from  $\text{Ber}(\hat{p}(\sigma_{L(n)}))$  given  $\sigma$ . We justify the final step using the following lemma.

**Lemma 3.1** (Total Variation Distance and Test Functions). *Let  $\mu, \nu$  be two probability measures over a common state space  $\Omega$ . Then the total variation distance  $\mathcal{D}_{\text{TV}}(\mu, \nu) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{\omega \in \Omega} |\mu(\omega) - \nu(\omega)|$  may be equivalently written as*

$$\mathcal{D}_{\text{TV}}(\mu, \nu) = \sup_{A \subseteq \Omega} |\mu(A) - \nu(A)| = \sup_f |\mathbb{E}_{\mu}[f] - \mathbb{E}_{\nu}[f]|,$$

where the supremum in the right-most expression is over all 1-bounded functions  $f : \Omega \rightarrow [0, 1]$ .

*Proof.* Since  $\Omega$  is finite, we can view  $f : \Omega \rightarrow [0, 1]$  and  $\mu, \nu$  as big vectors in  $[0, 1]^{\Omega}$ . Then  $|\mathbb{E}_{\mu}[f] - \mathbb{E}_{\nu}[f]| = |\langle \mu - \nu, f \rangle|$ , which is convex in  $f$ . This convexity immediately implies the second equality via the natural correspondence between  $A \subseteq \Omega$  and its  $\{0, 1\}$ -indicator function.

For the first equality, observe that since  $|\mu(A) - \nu(A)| = |\mu(\Omega \setminus A) - \nu(\Omega \setminus A)|$ ,

$$|\mu(A) - \nu(A)| = \frac{1}{2} \left| \sum_{\omega \in A} (\mu(\omega) - \nu(\omega)) \right| + \frac{1}{2} \left| \sum_{\omega \in \Omega \setminus A} (\mu(\omega) - \nu(\omega)) \right| \leq \|\mu - \nu\|_{\text{TV}} \quad \forall A \subseteq \Omega$$

by the Triangle Inequality. On the other hand, this inequality is saturated by the set  $A = \{\omega \in \Omega : \mu(\omega) > \nu(\omega)\}$ , and so we must have equality.  $\square$

### 3.1 Designing Estimators

Given the interpretation of  $b^*$  as the total variation distance between  $\mu_n^+ \stackrel{\text{def}}{=} \text{Law}(\sigma_{L(n)} \mid \sigma_r = +1)$  and  $\mu_n^- \stackrel{\text{def}}{=} \text{Law}(\sigma_{L(n)} \mid \sigma_r = -1)$ , [Lemma 3.1](#) suggests an estimator  $\hat{\sigma}_{r,n}^{\text{MLE}}$  known as the *maximum likelihood estimator*: If  $\sigma_{L(n)} = \tau$  for some  $\tau \in \{\pm 1\}^{L(n)}$ , then

$$\hat{\sigma}_{r,n}^{\text{MLE}} = \begin{cases} +1, & \text{if } \mu_n^+(\tau) > \mu_n^-(\tau) \\ -1, & \text{if } \mu_n^+(\tau) < \mu_n^-(\tau) \end{cases}.$$

This is the optimal estimator whose success probability is precisely  $\frac{1+b^*}{2}$ .

*Remark 4.* Given  $\tau \in \{\pm 1\}^{L(n)}$ , one can of course compute the probabilities  $\mu_n^+(\tau), \mu_n^-(\tau)$  through brute force enumeration of all possible assignments for the vertices in  $L(1), \dots, L(n-1)$ . However, if the goal is to just evaluate whether or not  $\mu_n^+(\tau) > \mu_n^-(\tau)$ , then we can actually do this in time polynomial in the size of the input  $\tau$ . The idea is that by Bayes' Rule,  $\mu_n^+(\tau) > \mu_n^-(\tau)$  holds if and only if

$$\Pr[\sigma_r = +1 \mid \sigma_{L(n)} = \tau] > \Pr[\sigma_r = -1 \mid \sigma_{L(n)} = \tau].$$

These two quantities can be computed via recursion. The specific algorithm is a *message-passing algorithm* known in the literature as *belief propagation*, the *tree recursion*, or the *sum-product algorithm*, and is extremely well-studied. We will discuss this in greater depth in [Section 5](#); see also [Appendix B](#) for a derivation of this recursion.

Another natural estimator to consider is simply the one given by majority vote:

$$\hat{\sigma}_{r,n}^{\text{MAJ}} \stackrel{\text{def}}{=} \text{sign}(S_n) \quad \text{where} \quad S_n \stackrel{\text{def}}{=} \sum_{v \in L(n)} \sigma_v.$$

We note that the maximum likelihood and majority estimators are not equal.

**Exercise 1.** Prove that  $\hat{\sigma}_{r,n}^{\text{MLE}} \neq \hat{\sigma}_{r,n}^{\text{MAJ}}$ .

While these two estimators have different advantages, i.e.  $b(\hat{\sigma}_{r,n}^{\text{MLE}}) \neq b(\hat{\sigma}_{r,n}^{\text{MAJ}})$ , fortunately for us, it will turn out that  $\lim_{n \rightarrow \infty} b(\hat{\sigma}_{r,n}^{\text{MAJ}}) > 0$  if and only if  $\lim_{n \rightarrow \infty} b(\hat{\sigma}_{r,n}^{\text{MLE}}) > 0$ . This will be a consequence of the proof of [Theorem 1.1](#). We note that there are many other natural broadcast processes for which these two regimes *do not coincide* [[Mos01](#)].

### 3.2 Second Moment Bounds on Total Variation Distance

Before we move to the proof of [Theorem 1.1](#), we will need a technical lemma for bounding the total variation distance. This formalizes the intuition we mentioned in [Section 2](#), where the “signal strength” (i.e. difference in expectations between two hypotheses) needs to exceed the “noise level” (i.e. the typical fluctuations). To state it, for a probability measure  $\mu$  on a state space  $\Omega$  and a function  $f : \Omega \rightarrow \mathbb{R}$ , we abbreviate  $\mathbb{E}_\mu[f] \stackrel{\text{def}}{=} \mathbb{E}_{\omega \sim \mu}[f(\omega)]$  and  $\text{Var}_\mu(f) \stackrel{\text{def}}{=} \text{Var}_{\omega \sim \mu}(f(\omega))$ .

**Lemma 3.2.** *Let  $X, Y$  be two real-valued random variables, and let  $W$  have law  $\frac{1}{2}\text{Law}(X) + \frac{1}{2}\text{Law}(Y)$ .<sup>2</sup> Then we have the lower bound*

$$\mathcal{D}_{\text{TV}}(\text{Law}(X), \text{Law}(Y)) \geq \frac{1}{4} \cdot \frac{(\mathbb{E}[X] - \mathbb{E}[Y])^2}{\text{Var}(W)}.$$

<sup>2</sup>Note that  $W$  can be sampled by first tossing a fair coin, and then setting  $W = X$  (resp.  $W = Y$ ) if the coin comes up heads (resp. tails).

In particular, if  $\mu, \nu$  are two probability measures on a common state space  $\Omega$ , then

$$\mathcal{D}_{\text{TV}}(\mu, \nu) \geq \frac{1}{4} \cdot \sup_{f: \Omega \rightarrow \mathbb{R}} \frac{(\mathbb{E}_\mu[f] - \mathbb{E}_\nu[f])^2}{\text{Var}_{\frac{\mu+\nu}{2}}(f)}.$$

For intuition, we also provide a kind of converse to the above.

**Lemma 3.3.** *Let  $\mu, \nu$  be two probability measures on a common state space  $\Omega$ . Then*

$$\mathcal{D}_{\text{TV}}(\mu, \nu)^2 \leq \frac{1}{4} \cdot \sup_{f: \Omega \rightarrow \mathbb{R}} \frac{(\mathbb{E}_\mu[f] - \mathbb{E}_\nu[f])^2}{\text{Var}_\mu(f)}.$$

*Remark 5.* For comparison, note that by the Law of Total Variance,

$$\text{Var}_{\frac{\mu+\nu}{2}}(f) = \frac{1}{2} \text{Var}_\mu(f) + \frac{1}{2} \text{Var}_\nu(f) + \frac{1}{4} (\mathbb{E}_\mu[f] - \mathbb{E}_\nu[f])^2.$$

Note that the third term is necessary to ensure the lower bound never exceeds 1, since  $\mathcal{D}_{\text{TV}}(\mu, \nu) \in [0, 1]$  for any  $\mu, \nu$ . So, if  $(\mathbb{E}_\mu[f] - \mathbb{E}_\nu[f])^2 \gg \text{Var}_\mu(f), \text{Var}_\nu(f)$ , then Lemma 3.2 yields  $\mathcal{D}_{\text{TV}}(\mu, \nu) \geq 1 - o(1)$ . On the other hand, if  $(\mathbb{E}_\mu[f] - \mathbb{E}_\nu[f])^2 \ll \text{Var}_\mu(f), \text{Var}_\nu(f)$ , then Lemma 3.3 yields  $\mathcal{D}_{\text{TV}}(\mu, \nu) \leq o(1)$ .

Proofs for these lemmas are largely based on Cauchy–Schwarz, and provided in Appendix A.

## 4 Proof of Theorem 1.1 in the Case $\epsilon < \epsilon_c$

In this section, we work in the regime  $\epsilon < \epsilon_c$ , or equivalently,  $d\theta^2 > 1$ . From our previous analysis, we have that

$$\begin{aligned} b^* &\geq \mathcal{D}_{\text{TV}}(\text{Law}(\hat{\sigma}_{r,n}^{\text{MAJ}} \mid \sigma_r = +1), \text{Law}(\hat{\sigma}_{r,n}^{\text{MAJ}} \mid \sigma_r = -1)) \\ &= \mathcal{D}_{\text{TV}}(\text{Law}(S_n \mid \sigma_r = +1), \text{Law}(S_n \mid \sigma_r = -1)). \end{aligned}$$

Note the final equality holds due to symmetry between  $\pm 1$ . More concretely, the random variables  $S_n \mid \sigma_r = -1$  and  $-S_n \mid \sigma_r = +1$  have the same law, and so

$$\{s \in \mathbb{Z} : \Pr[S_n = s \mid \sigma_r = +1] > \Pr[S_n = s \mid \sigma_r = -1]\} = \{s \in \mathbb{Z} : s > 0\}$$

and we may apply Lemma 3.1.

We will show that when  $\epsilon < \epsilon_c$ , the right-hand side is lower bounded by a universal constant even in the large  $n$  limit. We will achieve this by combining the second moment method with Lemma 3.2, where  $X_n = S_n \mid \sigma_r = +1$  and  $Y_n = S_n \mid \sigma_r = -1$ ; note that  $W_n = S_n$  without any conditioning. We now compute the quantities involved in Lemma 3.2. If we consider again the Galton–Watson branching process  $\mathcal{Z} = \{Z_n\}_{n \in \mathbb{N}}$  we discussed previously, which has offspring distribution  $\text{Bin}(d, \theta)$  with mean  $d\theta$ , then observe that  $X_n$  is equal to  $Z_n$  plus a sum of  $d^n - Z_n$  many (dependent)  $\text{Unif}\{\pm 1\}$  random variables. Hence,

$$\mathbb{E}[X_n] = \mathbb{E}[Z_n] = d^n \theta^n \quad \text{and} \quad \mathbb{E}[Y_n] = -\mathbb{E}[Z_n] = -d^n \theta^n.$$

We now calculate the variance of  $S_n$ .

$$\begin{aligned} \text{Var}(S_n) &= \mathbb{E}[S_n^2] && \text{(Using } \mathbb{E}[S_n] = 0\text{)} \\ &= \sum_{u, v \in L(n)} \mathbb{E}[\sigma_u \sigma_v] \\ &= \sum_{u \in L(n)} \mathbb{E}[\sigma_u^2] + \sum_{u \in L(n)} \sum_{\ell=0}^{n-1} \sum_{\substack{v \neq u \\ \text{LCA}(u, v) \in L(\ell)}} \mathbb{E}[\sigma_u \sigma_v]. \end{aligned}$$

Fix a pair of distinct vertices  $u, v \in L(n)$  such that their least common ancestor  $w = \text{LCA}(u, v)$  lies in  $L(\ell)$ . Then by the Law of Total Expectation and the fact that an independently sampled  $\text{Unif}\{\pm 1\}$  is assigned when a child fails to copy its parent’s assignment, we have

$$\begin{aligned} \mathbb{E}[\sigma_u \sigma_v] &= \Pr[\sigma_u, \sigma_v \text{ both copy } \sigma_w] \\ &= \Pr[\sigma_u \text{ copies } \sigma_w] \cdot \Pr[\sigma_v \text{ copies } \sigma_w] \\ &= \theta^{2(n-\ell)}. \end{aligned}$$

Combining with the fact that  $\#\{v \neq u : \text{LCA}(u, v) \in L(\ell)\} = d^{n-\ell-1}(d-1)$  for every  $u \in L(n)$ , the above simplifies to

$$\text{Var}(S_n) = d^n + d^{n-1}(d-1) \sum_{k=1}^n (d\theta^2)^k = d^n + d^n \left( (d\theta^2)^n - 1 \right) \cdot \frac{(d-1)\theta^2}{d\theta^2 - 1}.$$

With this in hand, we find that

$$\begin{aligned} b^* &\geq \mathcal{D}_{\text{TV}}(\text{Law}(S_n \mid \sigma_r = +1), \text{Law}(S_n \mid \sigma_r = -1)) && \text{(Previous display)} \\ &\geq \frac{1}{4} \cdot \frac{(\mathbb{E}[X_n] - \mathbb{E}[Y_n])^2}{\text{Var}(S_n)} && \text{(Lemma 3.2)} \\ &= \frac{d^{2n}\theta^{2n}}{d^n + d^n((d\theta^2)^n - 1) \cdot \frac{(d-1)\theta^2}{d\theta^2 - 1}} \\ &= \frac{1}{(d\theta^2)^{-n} + \left(1 - (d\theta^2)^{-n}\right) \cdot \frac{(d-1)\theta^2}{d\theta^2 - 1}} \\ &\rightarrow \frac{d\theta^2 - 1}{(d-1)\theta^2} && \text{(In the } n \rightarrow \infty \text{ limit, using } d\theta^2 > 1\text{)} \\ &= \frac{d}{d-1} \left(1 - (d\theta^2)^{-1}\right). \end{aligned}$$

As a sanity check, this is at most 1 since  $\theta \leq 1$ . It is positive since  $d\theta^2 > 1$  by assumption, which completes the proof of [Theorem 1.1](#).

## 5 Bonus Material: Proof of [Theorem 1.1](#) in the Case $\epsilon > \epsilon_c$

Now suppose  $\epsilon < \epsilon_c$ , i.e.  $d\theta^2 < 1$ . Our goal is to show that  $b^*(n, \widehat{\mathbb{T}}_d, \epsilon) \rightarrow 0$  as  $n \rightarrow \infty$ . Towards this, observe that we can apply Bayes' Rule to obtain that

$$\begin{aligned} &\mathcal{D}_{\text{TV}}(\text{Law}(\sigma_{L(n)} \mid \sigma_r = +1), \text{Law}(\sigma_{L(n)} \mid \sigma_r = -1)) \\ &= \frac{1}{2} \sum_{\tau \in \{\pm 1\}^{L(n)}} |\Pr[\sigma_{L(n)} = \tau \mid \sigma_r = +1] - \Pr[\sigma_{L(n)} = \tau \mid \sigma_r = -1]| \\ &= \frac{1}{2} \sum_{\tau \in \{\pm 1\}^{L(n)}} \left| \Pr[\sigma_r = +1 \mid \sigma_{L(n)} = \tau] \frac{\Pr[\sigma_{L(n)} = \tau]}{\Pr[\sigma_r = +1]} - \Pr[\sigma_{L(n)} = \tau \mid \sigma_r = -1] \frac{\Pr[\sigma_{L(n)} = \tau]}{\Pr[\sigma_r = -1]} \right| \\ &= \mathbb{E} |\Pr[\sigma_r = +1 \mid \sigma_{L(n)}] - \Pr[\sigma_r = -1 \mid \sigma_{L(n)}]|. \end{aligned}$$

This suggests that we look at the quantity

$$\begin{aligned} \mathcal{M}_n(\tau) &\stackrel{\text{def}}{=} \Pr[\sigma_r = +1 \mid \sigma_{L(n)} = \tau] - \Pr[\sigma_r = -1 \mid \sigma_{L(n)} = \tau] \\ &= \mathbb{E}[\sigma_r \mid \sigma_{L(n)} = \tau], \quad \forall \tau \in \{\pm 1\}^{L(n)}. \end{aligned}$$

We write  $\mathcal{M}_n$  for the corresponding random variable, where the randomness comes from drawing  $\sigma_{L(n)}$  according to the broadcast process. Due to connections with the *(ferromagnetic) Ising model*, this quantity is sometimes referred to as the *magnetization* of the root vertex. Our goal will be to show that  $\mathbb{E}|\mathcal{M}_n|$  decays as  $n \rightarrow \infty$ . Rather than study  $|\mathcal{M}_n|$ , we will instead show step-wise decay for the second moment, which is more tractable.

**Lemma 5.1.** *We have the inequality  $\mathbb{E}|\mathcal{M}_n| \leq \sqrt{\mathbb{E}[\mathcal{M}_n^2]}$ .*

*Proof.* The claim is equivalent to nonnegativity of the variance of  $|\mathcal{M}_n|$ . □

Our goal will be to show that

$$\mathbb{E}[\mathcal{M}_n^2] \leq d\theta^2 \cdot \mathbb{E}[\mathcal{M}_{n-1}^2], \quad (1)$$

since our assumption that  $d\theta^2 < 1$  implies that  $\mathbb{E}|\mathcal{M}_n| \leq \sqrt{\mathbb{E}[\mathcal{M}_n^2]}$  is decaying to zero exponentially fast as  $n \rightarrow \infty$ .

The benefit of switching to the viewpoint of  $\mathcal{M}_n$  is that we can use recursion to understand the probabilities  $\Pr[\sigma_r = +1 \mid \sigma_{L(n)} = \tau]$  and  $\Pr[\sigma_r = -1 \mid \sigma_{L(n)} = \tau]$ . To state this precisely, let  $u_1, \dots, u_d$  be the children of the root  $r$ , and for each  $i = 1, \dots, d$ , let  $L_i(n-1) \subseteq L(n)$  denote the vertices in the subtree rooted at  $u_i$  which are at distance  $n-1$  from  $u_i$ . For  $\tau \in \{\pm 1\}^{L(n)}$ , we write  $\tau_i \in \{\pm 1\}^{L_i(n-1)}$  for the restriction of  $\tau$  to  $L_i(n-1)$ . Now let  $\mathcal{M}_{n-1}^{(i)}(\tau_i)$  be given by

$$\mathcal{M}_{n-1}^{(i)}(\tau_i) \stackrel{\text{def}}{=} \Pr[\sigma_{u_i} = +1 \mid \sigma_{L_i(n-1)} = \tau_i] - \Pr[\sigma_{u_i} = -1 \mid \sigma_{L_i(n-1)} = \tau_i].$$

Note the corresponding random variables  $\mathcal{M}_{n-1}^{(1)}, \dots, \mathcal{M}_{n-1}^{(d)}$  are correlated through the random root assignment  $\sigma_r$  if  $(\tau_1, \dots, \tau_d) \sim \mu_n$ . However, they become independent if we fix the value of  $\sigma_r$ , i.e. if  $(\tau_1, \dots, \tau_d) \sim \mu_n^{\pm 1}$ . What will be important for us is that  $\text{Law}(\mathcal{M}_{n-1}^{(i)}) = \text{Law}(\mathcal{M}_{n-1})$  individually for each  $i = 1, \dots, d$ . This is because marginally,  $\sigma_{u_i}$  is distributed as  $\text{Unif}\{\pm 1\}$  and the assignment  $\sigma_{L_i(n-1)}$  is independent of  $\sigma_r$  given  $\sigma_{u_i}$ , so  $\text{Law}(\sigma_{L_i(n-1)}) = \text{Law}(\sigma_{L(n-1)})$ .

Finally, let  $f(x) \stackrel{\text{def}}{=} \frac{1-x}{1+x}$ , and for  $d \in \mathbb{N}$ , define

$$g_d(x_1, \dots, x_d) \stackrel{\text{def}}{=} f\left(\prod_{i=1}^d f(x_i)\right).$$

We have the following theorem.

**Theorem 5.2.** *For any  $\tau \in \{\pm 1\}^{L(n)}$ ,*

$$\mathcal{M}_n(\tau) = g_d\left(\theta \cdot \mathcal{M}_{n-1}^{(1)}(\tau_1), \dots, \theta \cdot \mathcal{M}_{n-1}^{(d)}(\tau_d)\right). \quad (2)$$

*In particular, we have the distributional recursion*

$$\mathcal{M}_n \stackrel{d}{=} g_d\left(\theta \cdot \mathcal{M}_{n-1}^{(1)}, \dots, \theta \cdot \mathcal{M}_{n-1}^{(d)}\right). \quad (3)$$

Let us content ourselves for the moment with an informal explanation of how this recursion is derived; we formally prove this in [Appendix B](#). The idea is that because  $\sigma_{L_i(n-1)}$  is independent of  $\sigma_r$  given the value of  $\sigma_{u_i}$ , we can express  $\Pr[\sigma_r = +1 \mid \sigma_{L(n)} = \tau]$  as a function of the probabilities  $\Pr[\sigma_{u_i} = \pm 1 \mid \sigma_{L_i(n-1)} = \tau_i]$ , for any  $\tau \in \{\pm 1\}^{L(n)}$ . The specific function is known in the literature as *belief propagation*, the *tree recursion*, or the *sum-product algorithm*, and is extremely well-studied. Typically, it is discussed in the context of spin systems like the Ising model; we derive it using Bayes' Rule in [Appendix B](#) and postpone a more in-depth treatment of spin systems to a future lecture.

Taking [Theorem 5.2](#) as a given, to establish an inequality like [Eq. \(1\)](#), our goal is now to bound the rather complicated right-hand side of [Eq. \(3\)](#) by something like

$$\mathbb{E}\left[f\left(\prod_{i=1}^d f\left(\theta \cdot \mathcal{M}_{n-1}^{(i)}\right)\right)\right]^2 \leq \theta^2 \sum_{i=1}^d \mathbb{E}\left[\left(\mathcal{M}_{n-1}^{(i)}\right)^2\right] = d\theta^2 \cdot \mathbb{E}[\mathcal{M}_{n-1}^2]. \quad (4)$$

With this as our aim, let us begin by massaging [Eq. \(3\)](#). We build up  $g_d$  inductively.

**Lemma 5.3.** *For every  $d \in \mathbb{N}$ , we have that*

$$g_d(x_1, \dots, x_d) = g_2(g_{d-1}(x_1, \dots, x_{d-1}), x_d),$$

*where  $g_2$  is explicitly given by the expression*

$$g_2(x, y) = \frac{1 - \frac{1-x}{1+x} \cdot \frac{1-y}{1+y}}{1 + \frac{1-x}{1+x} \cdot \frac{1-y}{1+y}} = \frac{x+y}{1+xy}.$$

*Proof.* This is a straightforward calculation, taking advantage of the fact that  $f(f(x)) = x$ .  $\square$

The combinatorial interpretation of the conjunction of [Theorem 5.2](#) and [Lemma 5.3](#) is the following. Imagine we inductively build up our infinite  $d$ -ary tree  $\widehat{\mathbb{T}}_d$  rooted at  $r$  by first building  $d$  trees  $T_1, \dots, T_d$ , where each tree  $T_i$  is an infinite  $d$ -ary tree rooted at  $u_i$  along with a single edge



joining  $u_i$  to a special leaf vertex  $r_i$ . Independently in each  $T_i$ , we can run the same broadcast process initialized from  $r_i$ . A direct calculation reveals that  $\theta \cdot \mathbb{E} \left[ \mathcal{M}_{n-1}^{(i)} \right]$  is precisely the total variation distance/optimal advantage  $b^*(n, T_i, \epsilon)$  with respect to  $T_i$ . If we merge the vertices  $r_1, \dots, r_d$  into a single vertex, then we recover  $\mathbb{T}_d$ . The purpose of the function  $g_d$  is to compute the effect this merge operation induces on the reconstruction probability for the root  $r$ . [Lemma 5.3](#) captures the step-by-step effect of merging  $r_1, r_2$ , then merging  $r_1, r_2, r_3$ , etc.

Towards an inequality analogous to [Eq. \(4\)](#), we would like to establish that the factor  $\frac{1}{1+xy}$  is at most 1 in absolute value, at least in expectation when we plug in our random variables. This expression looks unwieldy, so let us attempt to “linearize”  $g_2$ . Using the identity  $\frac{1}{1+r} = 1 - r + \frac{r^2}{1+r}$ , we have that

$$\begin{aligned} g_2(x, y) &= (x + y) \left( 1 - xy + \frac{x^2 y^2}{1 + xy} \right) = (x + y) - x^2 y - xy^2 + x^2 y^2 \cdot g_2(x, y) \\ &\leq (x + y) - x^2 y - xy^2 + x^2 y^2, \end{aligned} \quad (\text{For } x, y \geq -1)$$

where in the final step, we used the fact that  $f([-1, +\infty]) = \mathbb{R}_{\geq 0} \cup \{+\infty\}$ ; in particular,  $-1$  is not in the range of  $f$  in our context. Our goal will be to ensure that the terms  $-x^2 y - xy^2 + x^2 y^2$  are negative, at least in expectation, when we plug in our random variables. For this, we need one final rather curious set of identities.

**Lemma 5.4.** *Recalling the notation  $\mu_n^+ = \text{Law}(\sigma_{L(n)} \mid \sigma_r = +1)$  and  $\mu_n = \text{Law}(\sigma_{L(n)})$ , we have the identities*

$$\begin{aligned} \mathbb{E}_{\sigma_{L(n)} \sim \mu_n^+} [\mathcal{M}_n] &= \mathbb{E}_{\sigma_{L(n)} \sim \mu_n} [\mathcal{M}_n^2] \\ \mathbb{E}_{\sigma_{L(n)} \sim \mu_n^+} [\mathcal{M}_{n-1}^{(i)}] &= \theta \cdot \mathbb{E}_{\sigma_{L(n-1)} \sim \mu_{n-1}} [\mathcal{M}_{n-1}^2] \\ \mathbb{E}_{\sigma_{L(n)} \sim \mu_n^+} \left[ \left( \mathcal{M}_{n-1}^{(i)} \right)^2 \right] &= \mathbb{E}_{\sigma_{L(n-1)} \sim \mu_{n-1}} [\mathcal{M}_{n-1}^2], \end{aligned}$$

for any  $i = 1, \dots, d$ .

We prove [Lemma 5.4](#) at the end of this section. With these tools in hand, we prove our final contraction inequality. We do the case  $d = 2$  here (with suppressed notation); the general case follows by a straightforward induction, using [Lemma 5.3](#) to add one child subtree at a time. We have

$$\begin{aligned} \mathbb{E}_{\mu_n} [\mathcal{M}_n^2] &= \mathbb{E}_{\mu_n^+} [\mathcal{M}_n] && (\text{Lemma 5.4}) \\ &= \mathbb{E}_{\mu_n^+} \left[ g_2 \left( \theta \cdot \mathcal{M}_{n-1}^{(1)}, \theta \cdot \mathcal{M}_{n-1}^{(2)} \right) \right] && (\text{Using Eq. (2)}) \\ &\leq \theta \cdot \mathbb{E}_{\mu_n^+} [\mathcal{M}_{n-1}^{(1)}] + \theta \cdot \mathbb{E}_{\mu_n^+} [\mathcal{M}_{n-1}^{(2)}] \\ &\quad - \theta^3 \cdot \mathbb{E}_{\mu_n^+} \left[ \left( \mathcal{M}_{n-1}^{(1)} \right)^2 \right] \cdot \mathbb{E}_{\mu_n^+} [\mathcal{M}_{n-1}^{(2)}] - \theta^3 \cdot \mathbb{E}_{\mu_n^+} [\mathcal{M}_{n-1}^{(1)}] \cdot \mathbb{E}_{\mu_n^+} \left[ \left( \mathcal{M}_{n-1}^{(2)} \right)^2 \right] \\ &\quad + \theta^4 \cdot \mathbb{E}_{\mu_n^+} \left[ \left( \mathcal{M}_{n-1}^{(1)} \right)^2 \right] \cdot \mathbb{E}_{\mu_n^+} \left[ \left( \mathcal{M}_{n-1}^{(2)} \right)^2 \right] && (\text{Independence under } \mu_n^+) \\ &= 2\theta^2 \cdot \mathbb{E}_{\mu_{n-1}} [\mathcal{M}_{n-1}^2] - \theta^4 \cdot \mathbb{E}_{\mu_{n-1}} [\mathcal{M}_{n-1}^2]^2 && (\text{Lemma 5.4}) \\ &\leq 2\theta^2 \cdot \mathbb{E}_{\mu_{n-1}} [\mathcal{M}_{n-1}^2]. \end{aligned}$$

All that remains to complete the proof of [Theorem 1.1](#) in the  $\epsilon > \epsilon_c$  regime is to prove [Lemma 5.4](#) and [Theorem 5.2](#). The former is proved here, while the latter is relegated to [Appendix B](#).

*Proof of Lemma 5.4.* For this first identity, observe that

$$\begin{aligned} \mathbb{E}_{\sigma_{L(n)} \sim \mu_n^+} [\mathcal{M}_n] &= \mathbb{E}_{\sigma_{L(n)} \sim \mu_n} \left[ \frac{\Pr[\sigma_{L(n)} \mid \sigma_r = +1]}{\Pr[\sigma_{L(n)}]} (\Pr[\sigma_{L(n)} \mid \sigma_r = +1] - \Pr[\sigma_{L(n)} \mid \sigma_r = -1]) \right] \\ &\quad (\text{Change of densities}) \\ &= \mathbb{E}_{\sigma_{L(n)} \sim \mu_n} [2 \Pr[\sigma_r = +1 \mid \sigma_{L(n)}] (\Pr[\sigma_{L(n)} \mid \sigma_r = +1] - \Pr[\sigma_{L(n)} \mid \sigma_r = -1])] \\ &\quad (\text{Bayes' Rule}) \\ &= \mathbb{E}_{\sigma_{L(n)} \sim \mu_n} [\mathcal{M}_n^2 + \mathcal{M}_n] && (*) \\ &= \mathbb{E}_{\sigma_{L(n)} \sim \mu_n} [\mathcal{M}_n^2]. && (\text{Using } \mathbb{E}_{\sigma_{L(n)} \sim \mu_n} [\mathcal{M}_n] = 0 \text{ by symmetry}) \end{aligned}$$



For  $(*)$ , we used the fact that  $\Pr[\sigma_r = +1 \mid \sigma_{L(n)}] + \Pr[\sigma_r = -1 \mid \sigma_{L(n)}] = 1$  implies

$$2\Pr[\sigma_r = +1 \mid \sigma_{L(n)}] = 1 + (\Pr[\sigma_r = +1 \mid \sigma_{L(n)}] - \Pr[\sigma_r = -1 \mid \sigma_{L(n)}]).$$

This proves the first identity. For the second identity, we use the Law of Total Expectation to obtain

$$\begin{aligned}\mathbb{E}_{\sigma_{L(n)} \sim \mu_n^+} [\mathcal{M}_{n-1}^{(i)}] &= \theta \cdot \mathbb{E} [\mathcal{M}_{n-1}^{(i)} \mid u_i \text{ copies } r \text{ directly}] + (1 - \theta) \cdot \mathbb{E} [\mathcal{M}_{n-1}^{(i)} \mid \sigma_{u_i} \sim \text{Unif}\{\pm 1\}] \\ &= \theta \cdot \mathbb{E}_{\sigma_{L(n-1)} \sim \mu_{n-1}^+} [\mathcal{M}_{n-1}] + (1 - \theta) \cdot \mathbb{E}_{\sigma_{L(n-1)} \sim \mu_{n-1}} [\mathcal{M}_{n-1}] \\ &= \theta \cdot \mathbb{E}_{\sigma_{L(n-1)} \sim \mu_{n-1}} [\mathcal{M}_{n-1}^2].\end{aligned}\quad (\text{First identity})$$

For the final identity, we again use the Law of Total Expectation, similar to the above, to obtain

$$\mathbb{E}_{\sigma_{L(n)} \sim \mu_n^+} \left[ \left( \mathcal{M}_{n-1}^{(i)} \right)^2 \right] = \theta \cdot \mathbb{E}_{\sigma_{L(n-1)} \sim \mu_{n-1}^+} [\mathcal{M}_{n-1}^2] + (1 - \theta) \cdot \mathbb{E}_{\sigma_{L(n-1)} \sim \mu_{n-1}} [\mathcal{M}_{n-1}^2].$$

But  $\mathbb{E}_{\sigma_{L(n-1)} \sim \mu_{n-1}^+} [\mathcal{M}_{n-1}^2] = \mathbb{E}_{\sigma_{L(n-1)} \sim \mu_{n-1}^-} [\mathcal{M}_{n-1}^2] = \mathbb{E}_{\sigma_{L(n-1)} \sim \mu_{n-1}} [\mathcal{M}_{n-1}^2]$  by symmetry of the assignments  $\pm 1$ , and so the above is equal to  $\mathbb{E}_{\sigma_{L(n-1)} \sim \mu_{n-1}} [\mathcal{M}_{n-1}^2]$ .  $\square$

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## A Proofs for Second Moment Total Variation Bounds

*Proof of Lemma 3.2.* Without loss of generality, we may shift  $X, Y, W$  by the same constant to ensure they have mean zero, while preserving the total variation distance between  $\text{Law}(X), \text{Law}(Y)$ . Hence, we assume  $\mathbb{E}[X] = \mathbb{E}[Y] = \mathbb{E}[W] = 0$  in the remainder of the proof.

Observe that  $\text{supp}(W) = \text{supp}(X) \cup \text{supp}(Y)$ , and so

$$\begin{aligned}\mathcal{D}_{\text{TV}}(\text{Law}(X), \text{Law}(Y)) &= \frac{1}{2} \sum_{\omega \in \text{supp}(W)} |\Pr[X = \omega] - \Pr[Y = \omega]| \\ &= \sum_{\omega \in \text{supp}(W)} \frac{|\Pr[X = \omega] - \Pr[Y = \omega]|}{2 \cdot \Pr[W = \omega]} \cdot \Pr[W = \omega] \\ &= \mathbb{E}[|f(W)|],\end{aligned}$$

where  $f(\omega) \stackrel{\text{def}}{=} \frac{\Pr[X=\omega] - \Pr[Y=\omega]}{\Pr[X=\omega] + \Pr[Y=\omega]}$  takes values in the interval  $[-1, 1]$ . Hence, we may lower bound the above by

$$\begin{aligned} \mathbb{E}[f(W)^2] &\geq \frac{\mathbb{E}[W \cdot f(W)]^2}{\mathbb{E}[W^2]} && \text{(Cauchy-Schwarz)} \\ &= \frac{1}{\text{Var}(W)} \cdot \left( \sum_{\omega \in \text{supp}(W)} \omega \cdot \Pr[W = \omega] \cdot \frac{\Pr[X = \omega] - \Pr[Y = \omega]}{2 \cdot \Pr[W = \omega]} \right)^2 \\ &&& (\mathbb{E}[W] = 0 \text{ and definition of } f) \\ &= \frac{1}{4} \cdot \frac{(\mathbb{E}[X] - \mathbb{E}[Y])^2}{\text{Var}(W)}. \end{aligned}$$

For the second claim, observe that  $f : \Omega \rightarrow \mathbb{R}$  is any function, then letting  $X = f(x)$  for  $x \sim \mu$  and  $Y = f(y)$  for  $y \sim \nu$ , we have

$$\mathcal{D}_{\text{TV}}(\mu, \nu) \geq \mathcal{D}_{\text{TV}}(\text{Law}(X), \text{Law}(Y)) \geq \frac{1}{4} \cdot \frac{(\mathbb{E}_\mu[f] - \mathbb{E}_\nu[f])^2}{\text{Var}_{\frac{\mu+\nu}{2}}(f)}.$$

Since  $f$  was arbitrary, the proof is complete.  $\square$

*Proof of Lemma 3.3.* The key idea is to use an intermediate quantity, namely the  $\chi^2$ -squared divergence, which is defined as

$$\chi^2(\nu \parallel \mu) \stackrel{\text{def}}{=} \mathbb{E}_{\omega \sim \mu} \left[ \left( 1 - \frac{\nu(\omega)}{\mu(\omega)} \right)^2 \right].$$

Note that this is simply the variance of the function  $\frac{d\nu}{d\mu}(\omega) \stackrel{\text{def}}{=} \frac{\nu(\omega)}{\mu(\omega)}$ , often referred to as the *density* of  $\nu$  with respect to  $\mu$ . With this, we have by Cauchy-Schwarz that

$$\mathcal{D}_{\text{TV}}(\mu, \nu)^2 = \frac{1}{4} \cdot \mathbb{E}_{\omega \sim \mu} \left[ \left| 1 - \frac{\nu(\omega)}{\mu(\omega)} \right| \right]^2 \leq \frac{1}{4} \cdot \chi^2(\nu \parallel \mu).$$

On the other hand, we claim that the  $\chi^2$ -squared divergence admits the variational formula

$$\chi^2(\nu \parallel \mu) = \sup_{f: \Omega \rightarrow \mathbb{R}} \frac{(\mathbb{E}_\mu[f] - \mathbb{E}_\nu[f])^2}{\text{Var}_\mu(f)}.$$

To see this, observe that for any  $f$ , we have

$$\begin{aligned} \mathbb{E}_\mu[f] - \mathbb{E}_\nu[f] &= \mathbb{E}_\nu(f - \mathbb{E}_\mu[f]) \\ &= \mathbb{E}_\mu \left[ \frac{d\nu}{d\mu} \cdot (f - \mathbb{E}_\mu[f]) \right] && \text{(Change of Measure)} \\ &= \mathbb{E}_\mu \left[ \left( \frac{d\nu}{d\mu} - 1 \right) \cdot (f - \mathbb{E}_\mu[f]) \right] && \text{(Using } \mathbb{E}_\mu[f - \mathbb{E}_\mu[f]] = 0) \end{aligned}$$

It follows that

$$\begin{aligned} (\mathbb{E}_\mu[f] - \mathbb{E}_\nu[f])^2 &= \mathbb{E}_\mu \left[ \left( \frac{d\nu}{d\mu} - 1 \right) \cdot (f - \mathbb{E}_\mu[f]) \right]^2 \\ &\leq \mathbb{E}_\mu \left[ \left( \frac{d\nu}{d\mu} - 1 \right)^2 \right] \cdot \mathbb{E}_\mu [(f - \mathbb{E}_\mu[f])^2] && \text{(Cauchy-Schwarz)} \\ &= \text{Var}_\mu \left( \frac{d\nu}{d\mu} \right) \cdot \text{Var}_\mu(f) \\ &= \chi^2(\nu \parallel \mu) \cdot \text{Var}_\mu(f). && \text{(Definition of } \chi^2(\nu \parallel \mu)) \end{aligned}$$

Rearranging proves that  $\chi^2(\nu \parallel \mu)$  is lower bounded by the given variational formula. To prove equality, we need to exhibit a function  $f : \Omega \rightarrow \mathbb{R}$  which achieves equality. For this, we simply use the density  $\frac{d\nu}{d\mu}$ . Plugging in this function, we have

$$\mathbb{E}_\mu \left[ \frac{d\nu}{d\mu} \right] - \mathbb{E}_\nu \left[ \frac{d\nu}{d\mu} \right] = 1 - \mathbb{E}_\mu \left[ \left( \frac{d\nu}{d\mu} \right)^2 \right] = -\text{Var}_\mu \left( \frac{d\nu}{d\mu} \right) = -\chi^2(\nu \parallel \mu).$$

Plugging this back into the ratio in the variational expression completes the proof.  $\square$

## B Proof of Theorem 5.2

We express  $\Pr[\sigma_r = +1 \mid \sigma_{L(n)} = \tau]$  in terms of the quantities  $\Pr[\sigma_{u_i} = \pm 1 \mid \sigma_{L_i(n-1)} = \tau_i]$ . To do this, first observe that because the assignments  $\sigma_{L_1(n-1)}, \dots, \sigma_{L_d(n-1)}$  are independent conditioned on the value of  $\sigma_r$ , we have

$$\Pr[\sigma_{L(n)} = \tau \mid \sigma_r = +1] = \prod_{i=1}^d \Pr[\sigma_{L_i(n-1)} = \tau_i \mid \sigma_r = +1].$$

Now, using the Law of Total Probability,

$$\begin{aligned} & \Pr[\sigma_{L_i(n-1)} = \tau_i \mid \sigma_r = +1] \\ &= (1 - \epsilon) \cdot \Pr[\sigma_{L_i(n-1)} = \tau_i \mid \sigma_{u_i} = +1] + \epsilon \cdot \Pr[\sigma_{L_i(n-1)} = \tau_i \mid \sigma_{u_i} = -1] \\ &= 2 \cdot \Pr[\sigma_{L_i(n-1)} = \tau_i] \cdot \left( 1 + \theta \cdot \mathcal{M}_{n-1}^{(i)}(\tau_i) \right). \end{aligned}$$

These calculations all hold mutatis mutandis for the case  $\sigma_r = -1$ . Since

$$\Pr[\sigma_r = +1 \mid \sigma_{L(n)} = \tau] = \frac{\Pr[\sigma_{L(n)} = \tau \mid \sigma_r = +1]}{\Pr[\sigma_{L(n)} = \tau \mid \sigma_r = +1] + \Pr[\sigma_{L(n)} = \tau \mid \sigma_r = -1]},$$

which follows by Bayes' Rule and  $\Pr[\sigma_r = +1] = \Pr[\sigma_r = -1] = 1/2$ , plugging in the above calculations and canceling out the factors of  $2 \cdot \Pr[\sigma_{L_i(n-1)} = \tau_i]$  yield

$$\begin{aligned} \Pr[\sigma_r = +1 \mid \sigma_{L(n)} = \tau] &= \frac{\prod_{i=1}^d \left( 1 + \theta \cdot \mathcal{M}_{n-1}^{(i)}(\tau_i) \right)}{\prod_{i=1}^d \left( 1 + \theta \cdot \mathcal{M}_{n-1}^{(i)}(\tau_i) \right) + \prod_{i=1}^d \left( 1 - \theta \cdot \mathcal{M}_{n-1}^{(i)}(\tau_i) \right)} \\ \Pr[\sigma_r = -1 \mid \sigma_{L(n)} = \tau] &= \frac{\prod_{i=1}^d \left( 1 - \theta \cdot \mathcal{M}_{n-1}^{(i)}(\tau_i) \right)}{\prod_{i=1}^d \left( 1 + \theta \cdot \mathcal{M}_{n-1}^{(i)}(\tau_i) \right) + \prod_{i=1}^d \left( 1 - \theta \cdot \mathcal{M}_{n-1}^{(i)}(\tau_i) \right)}. \end{aligned}$$

This is one way of expressing the *belief propagation* equations, although they are more commonly written in terms of  $\epsilon$  and the conditional marginal probabilities  $\Pr[\sigma_{u_i} = \pm 1 \mid \sigma_{L_i(n-1)} = \tau_i]$  rather than  $\theta$  and  $\mathcal{M}_{n-1}^{(i)}(\tau_i)$ . Subtracting these two expressions and dividing both the numerator and denominator by common factors then yield the recursion for  $\mathcal{M}_n(\tau)$  stated in Theorem 5.2.