Lecture 17: Spectral Independence from Zero-Freeness

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In this lecture, we return to (multivariate) zero-freeness of the partition function. We show that it implies spectral independence and hence, fast mixing of Glauber dynamics.

1 The Generating Polynomial and its Zero-Freeness

Let μ be a probability distribution over $2^{[n]}$ (which of course can be identified with $\{\pm 1\}^n$). Define its *generating polynomial* as

$$g_\mu(\boldsymbol{z}) \stackrel{\mathsf{def}}{=} \sum_{S \subseteq [n]} \mu(S) \cdot \boldsymbol{z}^S, \tag{1}$$

where we use the shorthand $\boldsymbol{z}^S \stackrel{\mathsf{def}}{=} \prod_{i \in S} z_i$. This is a *multiaffine* polynomial of degree (at most) n. Its logarithm is essentially the cumulant generating function $\mathcal{L}_\mu$ of μ we previously saw, except with a change of variables to obtain a polynomial. One of the central ideas in the study of the *geometry of polynomials* is the following.

Theme 1.1. *It is fruitful to relate the analytic/algebraic properties of g_μ to the probabilistic/combinatorial properties of μ itself.*

For instance, a consequence of Barvinok's algorithm is that whenever g_μ admits a large zero-free region, then we have efficient algorithms for estimating g_μ.[1] In this lecture, we will be interested in deducing correlation bounds given zero-freeness of g_μ. The following beautiful theorem, which we will not prove, is an example of such a result.

Theorem 1.2 ([BBL09]). *Suppose g_μ is* real stable, *i.e. that $g_\mu \neq 0$ whenever $\operatorname{Im} z_i > 0$ for all $i \in [n]$. Then μ is* negatively correlated *in the sense that*

$$\Pr_{S \sim \mu}[j \in S \mid i \in S] \leq \Pr_{S \sim \mu}[j \in S \mid i \notin S], \qquad \forall i \neq j.$$

Real stability of the generating polynomial of μ is sometimes referred to as the *strongly Rayleigh property*. It turns out to be an extremely robust notion of negative correlation, one which has found many applications; see e.g. [Pem12] and references therein. We also previously saw that negatively correlated distributions on (homogeneous) set systems are 1-spectrally independent. The main result of this lecture is to establish a more direct connection between zero-freeness and spectral independence, one which does not require zero-freeness w.r.t. an entire half-plane.

Recall the following notion of multivariate zero-freeness we used previously.

Definition 1 (Stability). *Let $\Gamma_1, \ldots, \Gamma_n \subseteq \mathbb{C}$ be subsets of the complex plane. We say a multivariate polynomial $p(z_1, \ldots, z_n)$ is $\Gamma_1 \times \cdots \times \Gamma_n$-stable if $p(\boldsymbol{z}) \neq 0$ whenever $z_i \in \Gamma_i$ for all $i = 1, \ldots, n$. If $\Gamma_1 = \cdots = \Gamma_n = \Gamma$ for some $\Gamma \subseteq \mathbb{C}$, then we simply say p is Γ-stable.*

Theorem 1.3 ([CLV21]; building on [Ali+21]). *Suppose there exists a constant $\delta > 0$ such that g_μ is stable w.r.t. the open radius-δ disk $\mathbb{D}(1, \delta)$ around 1. If in addition the marginals of μ are bounded in the sense that there is a constant $0 < \mathscr{B} \leq 1/2$ such that $\Pr_{S \sim \mu}[i \in S], \Pr_{S \sim \mu}[i \notin S] \geq \mathscr{B}$ for all $i \in [n]$, then*

$$\sum_{j=1}^{n} |\Psi_\mu(i \to j)| \leq \frac{4}{\mathscr{B}(1 - \mathscr{B})\delta^2}, \qquad \forall i \in [n].$$

In particular, μ is $O(1/\mathscr{B}\delta^2)$-spectrally independent.

[1] This is not entirely true, since Barvinok's algorithm also requires that that zero-free region contains a point at which computing g_μ is easy. For instance, we do not have FPTAS for estimating arbitrary real stable polynomials.

Remark 1. The point $\mathbf{1}$ for zero-freeness isn't special. One could look at stability w.r.t. an open radius-δ disk around any other point $\boldsymbol{\lambda} \in \mathbb{R}^n_{\geq 0}$, in which case we'd get spectral independence for the tilted distribution $\mu_{\boldsymbol{\lambda}}(S) \propto \mu(S) \cdot \boldsymbol{\lambda}^S$.

Theorem 1.4 ([Ali+21])**.** *Suppose there exists a constant $\alpha > 0$ such that g_μ is stable w.r.t. the sector*

$$S_\alpha \overset{\mathsf{def}}{=} \left\{ re^{i\theta} : |\theta| < \alpha\pi/2, r > 0 \right\}. \tag{2}$$

around the nonnegative real axis with aperture $\alpha\pi$. Then

$$\sum_{j=1}^{n} \left| \Pr_{S\sim\mu}[j \in S \mid i \in S] - \Pr_{S\sim\mu}[j \in S \mid i \notin S] \right| \leq \frac{2}{\alpha}, \qquad \forall i \in [n],$$

and μ is $\left(\frac{2}{\alpha} - 1\right)$-spectrally independent. Furthermore, the same inequality holds for all exponential tilts of μ, i.e. distributions of the form $\mu_{\boldsymbol{\lambda}}(S) \propto \mu(S) \cdot \boldsymbol{\lambda}^S$ for some $\boldsymbol{\lambda} \in \mathbb{R}^n_{\geq 0}$.

Remark 2. In the special case $\alpha = 1$, S_α becomes the open right half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. Polynomials which are stable w.r.t. S_1 are often called *Hurwitz stable*.

The rough intuition behind these statements is the following: Since the correlations of μ are given by second-order derivatives of $\log g_\mu$ at $\mathbf{1}$, these correlations are small if $\log g_\mu(\mathbf{1})$ is "smooth" in some sense. This is only the case if $\mathbf{1}$ is far away from the zeros of g_μ. For more results accommodating more general zero-free regions, see [Ali+21; CLV21].

1.1 Applications

Before we prove Theorems 1.3 and 1.4, let us mention a few applications.

Example 1 (Hardcore Model in Tree Uniqueness). We previously mentioned that Peters–Regts [PR19] established stability of the multivariate independence polynomial $Z_G(\boldsymbol{\lambda})$ of a graph of maximum degree Δ in a neighborhood of the interval $[0, \lambda_c(\Delta))$, where $\lambda_c(\Delta)$ is again the uniqueness threshold w.r.t. the infinite Δ-regular tree. In this regime, Barvinok's algorithm furnishes an FPTAS. Combining this zero-freeness result with Theorem 1.3 yields an alternative proof of $O(1)$-spectral independence of the hardcore Gibbs measure in the uniqueness regime, albeit with worse quantitative bounds.

Example 2 (Monomer-Dimer Model). Recall we previously showed that the univariate matching polynomial $\mathcal{M}_G(z) = \sum_{M \subseteq E \text{ matching}} z^{|M|}$ real-rooted. This is the Heilmann–Lieb Theorem [HL72], and more in depth analysis reveals that for every nonnegative vector of edge weights $\boldsymbol{\lambda} \in \mathbb{R}^E_{\geq 0}$, the multivariate (vertex) matching polynomial

$$\mathcal{M}_G(\boldsymbol{z}) \overset{\mathsf{def}}{=} \sum_{\substack{M \subseteq E \\ \text{matching}}} \prod_{e \in M} \lambda_e \prod_{v \text{ unmatched}} z_v,$$

is *Hurwitz stable*; see e.g. [BB09]. This in particular implies that for any graph $G = (V, E)$, the monomer-dimer Gibbs distribution satisfies the correlation bounds

$$\sum_{v \in V} \left| \Pr_M[v \text{ matched} \mid u \text{ matched}] - \Pr_M[v \text{ matched} \mid u \text{ unmatched}] \right| \leq 2.$$

Note that no assumptions on the degree were made.

There are also additional applications to determinantal point processes, even subgraphs, edge covers, antiferromagnetic spin systems on line graphs, etc. [Ali+21; CLV21].

2 A Little Complex Analysis

To formalize the above intuition, we will leverage the following standard fact from complex analysis, which captures the "rigidity" of smooth complex functions.

Lemma 2.1 (Schwarz–Pick)**.** *Let $f : \mathbb{D}(0,1) \to \mathbb{D}(0,1)$ is a univariate holomorphic function. Then* $|f'(0)| \leq 1 - |f(0)|^2 \leq 1$.

In light of the Schwarz–Pick lemma, our strategy will be to construct such a univariate holomorphic function f such that f maps $\mathbb{D}(0,1)$ into itself, and $|f'(0)| \approx \sum_{j=1}^{n} |\Psi_\mu(i \to j)|$, perhaps up to constants depending on $\mathscr{B}, \delta$.

Proof of Theorem 1.3. Fix an arbitrary $i \in [n]$, and define

$$F_i(\boldsymbol{z}) \stackrel{\text{def}}{=} \frac{\partial_{z_i} \log g_\mu(\boldsymbol{z})}{\Pr_{S\sim\mu}[i \in S] \cdot \Pr_{S\sim\mu}[i \notin S]}.$$

Note that $\partial_{z_j} F_i(\mathbf{1}) = \Psi_\mu(i \to j)$ for all $j \in [n]$. Our goal is to construct appropriate maps $\psi : \mathbb{C} \to \mathbb{C}$ and $\boldsymbol{\varphi} : \mathbb{C} \to \mathbb{C}^n$ such that the composition $f(z) = \psi(F_i(\varphi_1(z), \ldots, \varphi_n(z)))$ is holomorphic and satisfies $f(\mathbb{D}(0,1)) \subseteq \mathbb{D}(0,1)$. If we have such $\psi, \boldsymbol{\varphi}$, then by applying the Schwarz–Pick Lemma,

$$\begin{aligned}
1 &\geq |f'(0)| && \text{(Lemma 2.1)}\\
&= |\psi'(F_i(\boldsymbol{\varphi}(0)))| \cdot \langle \nabla F_i(\boldsymbol{\varphi}(0)), \boldsymbol{\varphi}'(0)\rangle && \text{(Chain Rule)}\\
&= |\psi'(F_i(\boldsymbol{\varphi}(0)))| \cdot \sum_{j=1}^{n} \Psi_\mu(i \to j) \cdot \varphi_j'(0).
\end{aligned}$$

A natural and simple choice is to take $\psi, \boldsymbol{\varphi}$ to be *affine* functions.

- To ensure that f is holomorphic, we use $\boldsymbol{\varphi}$ to map $\mathbb{D}(0,1)$ into the region of stability of g_μ, which we assumed is $\mathbb{D}(1,\delta)$. For convenience, let us allow an epsilon of room. Let

  $$\varphi_j(z) \stackrel{\text{def}}{=} 1 + \frac{\delta}{2} s_j z,$$

  where $s_j = \operatorname{sign}(\Psi_\mu(i \to j))$. This ensures that $\varphi_j(\mathbb{D}(0,1)) \subseteq \mathbb{D}(1,\delta/2)$ for all $j \in [n]$,

  $$\sum_{j=1}^{n} \Psi_\mu(i \to j) \cdot \varphi_j'(0) = \frac{\delta}{2} \cdot \sum_{j=1}^{n} |\Psi_\mu(i \to j)|.$$

  In particular, this choice for $\boldsymbol{\varphi}$ alone already implies

  $$\sum_{j=1}^{n} |\Psi_\mu(i \to j)| \leq \frac{2}{\delta \cdot |\psi'(F_i(\mathbf{1}))|}. \tag{3}$$

- Now, we design ψ. In order to apply the Schwarz–Pick Lemma, we need ψ to map the image of $\mathbb{D}(0,1)$ under $F_i \circ \boldsymbol{\varphi}$ back into $\mathbb{D}(0,1)$. This is the tricky part, since we must understand the image of F_i. This is also where it is convenient to have an epsilon of room from the definition of $\boldsymbol{\varphi}$, since we only need to consider $F_i(\mathbb{D}(1,\delta/2))$ instead of $F_i(\mathbb{D}(1,\delta))$.

  Claim 2.2. *The image of $\mathbb{D}(1,\delta/2)$ under F_i is contained in $\mathbb{D}\left(0, \frac{2}{\mathscr{B}(1-\mathscr{B})\delta}\right)$.*

  Once we have this, an obvious choice for ψ is to scale everything down by a factor of $\frac{\mathscr{B}(1-\mathscr{B})\delta}{2}$. In particular, we let $\psi(z) = \frac{\mathscr{B}(1-\mathscr{B})\delta}{2} z$. Plugging this into Eq. (3) immediately implies the theorem. All that remains is to justify Claim 2.2.

$\square$

Proof of Claim 2.2. Since we assumed the marginal bound $\Pr_{S\sim\mu}[i \in S] \geq \mathscr{B}$, the claim is equivalent to showing that the image of $\mathbb{D}(1,\delta/2)$ under $\boldsymbol{z} \mapsto \partial_i \log g_\mu(\boldsymbol{z})$ is contained in $\mathbb{D}(0, 2/\delta)$.

We go by contradiction, making crucial use linearity of g_μ in each of its variables. Fix $z_1, \ldots, z_n \in \mathbb{D}(1,\delta/2)$, and write $y = \partial_i \log g_\mu(\boldsymbol{z})$. We wish to show that $|y| < 2/\delta$. Since

$$y = \partial_i \log g_\mu(\boldsymbol{z}) = \frac{\partial_i g_\mu(\boldsymbol{z})}{g_\mu(\boldsymbol{z})},$$

rearranging yields

$$g_\mu(\boldsymbol{z}) - \frac{1}{y} \cdot \partial_i g_\mu(\boldsymbol{z}) = 0. \tag{4}$$

Suppose for contradiction that $|y| \geq \frac{2}{\delta}$. Then $\left|-\frac{1}{y}\right| \leq \frac{\delta}{2}$. We use this and Eq. (4) to construct a new vector $\boldsymbol{z}' \in \mathbb{D}(1,\delta)^n$ such that $g_\mu(\boldsymbol{z}') = 0$, contradicting $\mathbb{D}(1,\delta)$-stability of g_μ.

Define $\boldsymbol{z}'$ by $z_j' = z_j$ for all $j \neq i$, and $z_i' = z_i - \frac{1}{y}$. Since $|z_i - 1| < \frac{\delta}{2}$ and $\left|-\frac{1}{y}\right| \leq \frac{\delta}{2}$, we have $\boldsymbol{z}' \in \mathbb{D}(1,\delta)^n$. Furthermore, since g_μ is linear in each of its variables,

$$\begin{aligned} g_\mu(\boldsymbol{z}') &= \underbrace{(g_\mu(\boldsymbol{z}) - z_i \cdot \partial_i g_\mu(\boldsymbol{z}))}_{\text{Monomials without } i} + \underbrace{\left(z_i - \frac{1}{y}\right) \cdot \partial_i g_\mu(\boldsymbol{z})}_{\text{Monomials with } i} \\ &= g_\mu(\boldsymbol{z}) - \frac{1}{y} \cdot \partial_i g_\mu(\boldsymbol{z}) \\ &= 0 && \text{(By Eq. (4))} \end{aligned}$$

□

3 Better Maps for Sectors

If we impose more structure on our zero-free regions, then we can construct much better $\psi, \boldsymbol{\varphi}$ and prove better bounds. We just need to understand where how the map $y \mapsto -\frac{1}{y}$ changes our zero-free region, and how to map between these regions and the unit disk $\mathbb{D}(0,1)$.

Proof of Theorem 1.4. Since μ and $\mu_{\boldsymbol{\lambda}}$ have the same generating polynomials up to rescaling the variables by nonnegative coefficients, sector stability also holds for $g_{\mu_{\boldsymbol{\lambda}}}$. Hence, without loss of generality, we just prove spectral independence for μ itself.

Fix an arbitrary $i \in [n]$, and define[2]

$$F_i(\boldsymbol{z}) = \log\left(\frac{\partial_i g_\mu(\boldsymbol{z})}{(1 - z_i \partial_i) g_\mu(\boldsymbol{z})}\right).$$

A direct calculation reveals that $\partial_{z_j} F_i(\mathbf{1}) = \Pr_{S\sim\mu}[j \in S \mid i \in S] - \Pr_{S\sim\mu}[j \in S \mid i \notin S]$ for all $j \in [n]$. We construct appropriate maps $\psi : \mathbb{C} \to \mathbb{C}$ and $\boldsymbol{\varphi} : \mathbb{C} \to \mathbb{C}^n$ such that the composition $f(z) = \psi(F_i(\varphi_1(z), \ldots, \varphi_n(z)))$ satisfies the assumptions of the Schwarz–Pick Lemma, since then we'd have

$$1 \geq |f'(0)| = |\psi'(F_i(\boldsymbol{\varphi}(0)))| \cdot \sum_{j \neq i} \Psi_\mu(i \to j) \cdot \varphi_j'(0). \tag{5}$$

- Since g_μ is stable w.r.t. the sector S_α, we use Möbius transformations and exponential maps instead of affine functions. More specifically, take

$$\begin{aligned} \varphi_j(z) &\stackrel{\mathsf{def}}{=} g(s_j z)^\alpha = \left(\frac{1 + s_j z}{1 - s_j z}\right)^\alpha \\ \text{where} \qquad g(x) &= \frac{1+x}{1-x} \qquad \text{and} \qquad s_j = \operatorname{sign}(\Psi_\mu(i \to j)), \qquad \forall j \in [n]. \end{aligned} \tag{6}$$

  The point is that $\varphi_j(\mathbb{D}(0,1)) \subseteq S_\alpha$ since the inner Möbius function g maps $\mathbb{D}(0,1)$ to the right half-plane S_1, and then taking the αth power scales down the angle. A quick calculation reveals that $\varphi_j'(z) = 2 s_j \alpha \cdot \left(\frac{1+s_j z}{1-s_j z}\right)^{\alpha-1} \cdot \frac{1}{(1-s_j z)^2}$ and so plugging this into Eq. (5) and using $\boldsymbol{\varphi}(0) = \mathbf{1}$ gives

$$\sum_{j \neq i} |\Psi_\mu(i \to j)| \leq \frac{1}{2\alpha} \cdot \frac{1 - \psi'(F_i(\mathbf{1}))^2}{|\psi'(F_i(\mathbf{1}))|}. \tag{7}$$

- Now let us argue about the image of F_i, which will then tell us how to construct ψ.

[2] If the numerator inside the logarithm were multiplied by z_i, and if $\boldsymbol{z} \in \mathbb{R}^n_{\geq 0}$, then we'd exactly have the marginal ratio of i under the tilted measure $\mu_{\boldsymbol{z}}$.

Claim 3.1. *For every $z_1, \dots, z_n \in S_\alpha$, we have that*

$$\frac{\partial_i g_\mu(\boldsymbol{z})}{(1 - z_i \partial_i) g_\mu(\boldsymbol{z})} \notin -S_\alpha.$$

In particular, the image of S_α under F_i is contained within the strip

$$\left\{ z \in \mathbb{C} : |\mathrm{Im}\, z| < \left(1 - \frac{\alpha}{2}\right) \pi \right\}. \tag{8}$$

Before we prove this claim, let us finish the proof by constructing ψ. Let

$$\psi(z) = g^{-1}\left(\exp\left(\frac{1/2}{1 - \alpha/2} \cdot z \right) \right),$$

where $g^{-1}(z) = \frac{z-1}{z+1}$ is the inverse of the Möbius transformation g we used in the definition of $\boldsymbol{\varphi}$ above. The point is that the inner exponential maps the strip in Claim 3.1 to the right half-plane S_1, and then g^{-1} maps this right half-plane to $\mathbb{D}(0, 1)$. Another quick calculation reveals that

$$\psi'(z) = \frac{2 \exp\left(\frac{1/2}{1-\alpha/2} \cdot z \right)}{\left(1 + \exp\left(\frac{1/2}{1-\alpha/2} \cdot z \right) \right)^2} \cdot \frac{1/2}{1 - \alpha/2} = \frac{1/2}{1 - \alpha/2} \cdot \frac{1}{2} \left(1 - \psi(z)^2 \right).$$

Combined with Eq. (7) yields

$$\sum_{j \neq i} |\Psi_\mu(i \to j)| \leq \frac{2}{\alpha} - 1.$$

Adding back $\Psi_\mu(i \to i) = 1$ to both sides concludes the proof.

□

Proof of Claim 3.1. Since the image of $-S_\alpha$ under the exponential map $z \mapsto \exp(z)$ is the strip in Eq. (8), the first claim indeed implies the second. Let $y = \frac{\partial_i g_\mu(\boldsymbol{z})}{(1 - z_i \partial_i) g_\mu(\boldsymbol{z})}$ and suppose for contradiction that $y \in S_{-\alpha}$. Then $-\frac{1}{y} \in S_\alpha$, whence

$$\begin{aligned} 0 &= (1 - z_i \partial_i) g_\mu(\boldsymbol{z}) - \frac{1}{y} \cdot \partial_i g_\mu(\boldsymbol{z}) && \text{(Rearranging)} \\ &= g_\mu\left(-\frac{1}{y}, \boldsymbol{z}_{-i} \right). && \text{(g_μ is multiaffine)} \end{aligned}$$

This contradicts S_α-stability of g_μ. □

References

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