Problem Set 2

Due: November 2, 2023 (11:59 PM EST)

Please turn in a PDF with solutions typed in LaTeX. You may collaborate on this problem set, but please attempt the problems yourself first. It goes without saying, please do not try to look up solutions online. Feel free to use Wikipedia, etc. for material pertaining to reasonable course prerequisites (e.g. basic probability, linear algebra, etc.) If you find a bug in a problem, please let me know.

Exercise 1 (Correlation Decay for Matchings). *Let $G = (V, E)$ be a graph, and $\lambda \geq 0$ be a fixed nonnegative real number. Recall the monomer-dimer Gibbs distribution is given by*

$$\mu(M) \propto \lambda^{|M|}, \qquad \forall \text{ matchings } M \subseteq E.$$

For every vertex $v \in V$, we write $v \leftarrow 1$ (resp. $v \leftarrow 0$) for the event that v is (resp. is not) is matched to some other vertex. Similarly, for $S \subseteq V$, we write $\sigma : V \to \{0,1\}$ to denote the event that $v \leftarrow \sigma(v)$ for all $v \in S$.

Assume G has maximum degree Δ. Prove that there exist constants $\delta = \delta(\lambda, \Delta) \in (0,1)$ and $C = C(\lambda, \Delta) \in \mathbb{R}_{\geq 0}$ such that the following decay of correlations property holds for μ: For every $v \in V$, $S \subseteq V \setminus \{v\}$ and $\tau, \sigma : S \to \{0,1\}$,

$$|\mu(v \leftarrow 1 \mid \tau) - \mu_G(v \leftarrow 1 \mid \sigma)| \leq C \cdot (1-\delta)^{\mathrm{dist}(v,S)}.$$

Note that this correlation decay result implies a FPTAS *for estimating the matching polynomial in bounded-degree graphs which is different from Barvinok's polynomial interpolation algorithm.*

Exercise 2 (Zero-Freeness for Matchings). *Consider the multivariate* edge matching polynomial

$$\mathcal{M}_G(\boldsymbol{z}) = \sum_{\substack{M \subseteq E \\ \text{matching}}} \prod_{e \in M} z_e.$$

Assume G has maximum degree Δ. Prove that there exists a constant $\eta = \eta(\Delta) > 0$ such that $\mathcal{M}_G$ is stable w.r.t. an η-strip around the nonnegative real axis $\{\boldsymbol{z} \in \mathbb{C}^E : \mathrm{dist}(z_e, \mathbb{R}_{\geq 0}) < \eta, \forall e \in E\}$.

Exercise 3 (Hardcore Model on the Random d-Regular Graph). *Let L, R be two sets with $|L| = |R| = n$. Fix a positive integer $d \in \mathbb{N}$, and form a random bipartite graph $G = (V = L \sqcup R, E)$ on $2n$ vertices by taking a union of d uniformly random perfect matchings.*[1] *In this exercise, we consider the hardcore model on the random bipartite graph G with fugacity $\lambda \in \mathbb{R}_{\geq 0}$.*

(a) *Fix arbitrary partitions $A_1 \sqcup \cdots \sqcup A_q$ and $B_1 \sqcup \cdots \sqcup B_q$ of L, R, respectively. Let $\boldsymbol{\gamma} \in \mathbb{R}_{\geq 0}^{q \times q}$ be a probability distribution on pairs $i, j \in [q]$ such that $\gamma_{ij} n$ is integral for all i, j, and for convenience, write $\alpha_i = \sum_{j=1}^q \gamma_{ij}$ (resp. $\beta_j = \sum_{i=1}^q \gamma_{ij}$) for the corresponding row (resp. column) sums of $\boldsymbol{\gamma}$. We say a perfect matching M between L and R* satisfies *$\boldsymbol{\gamma}$ if*

$$\#\{uv \in M : u \in A_i, v \in B_j\} = \gamma_{ij} \cdot n, \qquad \forall i, j \in [q].$$

Prove the following formula

$$\Pr_M[M \text{ satisfies } \boldsymbol{\gamma}] = \frac{\left[\prod_{i=1}^q \binom{\alpha_i n}{\boldsymbol{\gamma}(i,\cdot)n}\right]\left[\prod_{j=1}^q \binom{\beta_j n}{\boldsymbol{\gamma}(\cdot,j)n}\right]}{\binom{n}{\boldsymbol{\gamma}}},$$

where the probability is taken w.r.t. a uniformly random perfect matching M.

[1] For our purposes, it won't matter whether or not we remove duplicated edges. This is the *configuration* model, and is contiguous to the uniformly random d-regular graph.

(b) *For fixed* $(\alpha, \beta) \in \mathcal{T} \stackrel{\text{def}}{=} \{(x, y) \in [0,1]^2 : x + y \leq 1\}$ *(such that* $\alpha n, \beta n$ *are integral), define*

$$Z_G^{\alpha,\beta} \stackrel{\text{def}}{=} \sum_{\substack{I \subseteq V \text{ independent} \\ |I \cap L| = \alpha n, |I \cap R| = \beta n}} \lambda^{|I|}.$$

Establish the following first moment formula

$$\mathbb{E}_G\left[Z_G^{\alpha,\beta}\right] = \lambda^{(\alpha+\beta)n}\binom{n}{\alpha n}\binom{n}{\beta n} \cdot \left[\frac{\binom{n}{\alpha n, \beta n}}{\binom{n}{\alpha n}\binom{n}{\beta n}}\right]^d.$$

In particular, using the approximation $\binom{n}{\boldsymbol{p}n} \approx \exp(n \cdot H_e(\boldsymbol{p}))$ *for the multinomial where* $H_e(\boldsymbol{p}) = -\sum_{i=1}^{\ell} p_i \ln(p_i)$ *and* $\boldsymbol{p}$ *is any distribution* $[\ell]$*, we have* $\mathbb{E}_G\left[Z_G^{\alpha,\beta}\right] \approx \exp\left(n \cdot \Phi_{d,\lambda}(\alpha, \beta)\right)$ *where*

$$\Phi_{d,\lambda}(\alpha, \beta) \stackrel{\text{def}}{=} (\alpha+\beta)\ln(\lambda) + H_e(\alpha) + H_e(\beta) + d \cdot \left[H_e(\alpha, \beta) - H_e(\alpha) - H_e(\beta)\right]. \tag{1}$$

(c) *Show that the stationary points* (α^*, β^*) *of* $\Phi_{d,\lambda}$ *satisfy the univariate tree recursion for the hardcore model on the infinite* $(d-1)$*-ary tree in the following sense:*

$$\frac{1-\alpha^*-\beta^*}{1-\beta^*} = f_{d,\lambda}\left(\frac{1-\alpha^*-\beta^*}{1-\alpha^*}\right)$$

and vice versa (with the roles of α^*, β^* *swapped), where*

$$f_{d,\lambda}(p) \stackrel{\text{def}}{=} \frac{1}{1+\lambda \cdot p^{d-1}}.$$

Further argue that no global maximizers of $\Phi_{d,\lambda}$ *occur on the boundary of* $\mathcal{T}$*, and that any maximizer of* $\Phi_{d,\lambda}$ *must satisfy* $\frac{\alpha}{1-\alpha} \cdot \frac{\beta}{1-\beta} < \frac{1}{(d-1)^2}$*.*

(Aside: The fact that one can recover the tree recursion from first-order stationarity conditions of some variational problem is an interesting and generic phenomenon.)

(d) *Assume that the first moment prediction is correct, i.e. that with probability* $1 - o(1)$ *over the choice of* G*, we have* $Z_G^{\alpha,\beta} = \exp\left(n \cdot \Phi_{d,\lambda}(\alpha,\beta) + o(n)\right)$ *for all* $(\alpha, \beta) \in \mathcal{T}$*.*[2] *Under this assumption, prove that if* $\lambda > \lambda_c(d)$*, then with probability* $1 - o(1)$ *over* G*, Glauber dynamics requires* $\exp(\Omega(n))$*-steps to mix. Here, recall that* $\lambda_c(d) = \frac{(d-1)^{d-1}}{(d-2)^d}$ *is the uniqueness threshold for the hardcore model on the infinite* $(d-1)$*-ary tree.*

[2]This is highly nontrivial to establish because the expression for the second moment is much more complicated.