ISTD 01.001 - Introduction to Probability and Statistics: Jan-Apr 2017

Review problems: Week 10

April 14, 2018

Problem 1

We know for the exponential distribution $f(x) = \lambda \exp(-\lambda x)$ that the mean is given as $\frac{1}{\lambda}$, so if the mean is 4, then $\lambda = 1/4$

The serving distribution for one day is given as $f(x) = \frac{1}{4}\exp(-\frac{1}{4}x)$. The probability for **one day** to be served in less than 3 minutes is given as

$$P(X < 3) = \int_0^3 \lambda \exp(-\lambda x) dx = -\exp(-\frac{1}{4}x)\Big|_0^3 = 1 - \exp(-3/4)$$

The event of being served on k days out of n days in less than 3 minutes is given as a binomial distribution with $p = 1 - \exp(-3/4)$.

The event of being served on at least 4 days out of 6 days in less than 3 minutes, is the event

$$\begin{aligned} P(Y \geq 4) &= P(Y = 4) + P(Y = 5) + P(Y = 6) = \sum_{k=4}^{6} \binom{n}{k} p^k (1-p)^{6-k} \\ &= \frac{6 \cdot 5}{2} p^4 (1-p)^2 + 6 \cdot p^5 (1-p)^1 + 1 \cdot p^6 \approx 0.3969 \end{aligned}$$

Problem 2

$$f(x_1, x_2) = \frac{c}{1 + 2\sqrt{x_1^2 + x_2^2}}$$

1. determine c. Plug in polar coordinates $t(\alpha, r) = (r\cos(\alpha), r\sin(\alpha))$. This is a 1-to-1 mapping onto the target space for $\alpha \in [0, 2\pi), r \in [0, 3)$. $|det(Dt)|(r, \alpha) = r$, so

$$
\begin{aligned}
&\int_{\sqrt{x_1^2+x_2^2}<3} \frac{c}{1+2\sqrt{x_1^2+x_2^2}}\ dx_1\ dx_2 = 1 \\
=&\int_{\alpha=0}^{2\pi}\int_{r=0}^{3} \frac{c}{1+2\sqrt{r^2}} r\ dr\ d\alpha \\
=&2\pi c\int_{r=0}^{3} \frac{r}{1+2r}\ dr \\
=&\pi c\int_{r=0}^{3} \frac{2r}{1+2r}\ dr \\
=&\pi c\int_{r=0}^{3} \frac{1+2r}{1+2r} - \frac{1}{1+2r}\ dr \\
=&\pi c\int_{r=0}^{3} 1 - \frac{1}{1+2r}\ dr \\
=&\pi c\ (r - \frac{1}{2}\ln(1+2r))\Big|_0^3 = \pi c(3 - \frac{1}{2}ln(7)) \\
=&\pi c(3 - \frac{1}{2}ln(7)) = 1 \\
\Leftrightarrow& c = \frac{1}{\pi(3-\frac{1}{2}ln(7))} \\
\Rightarrow& f(x_1, x_2) = \frac{1}{\pi(3-\frac{1}{2}ln(7))}\frac{1}{1+\sqrt{x_1^2+x_2^2}}
\end{aligned}
$$

Windspeeds in $[0, 3]$**:**

The expectation for this will be:

$$
\begin{aligned}
&\int_{0<\sqrt{x_1^2+x_2^2}<3} \sqrt{x_1^2+x_2^2}\frac{c}{1+2\sqrt{x_1^2+x_2^2}}\ dx_1\ dx_2 \\
=&\pi c\int_{r=0}^{3} r\cdot\frac{2r}{1+2r}\ dr \\
=&\pi c\int_{r=0}^{3} r\cdot(\frac{1+2r}{1+2r} - \frac{1}{1+2r})\ dr \\
=&\pi c\int_{r=0}^{3} r\cdot(1 - \frac{1}{1+2r})\ dr \\
=&\pi c\int_{r=0}^{3} r - \frac{1}{2}\frac{2r}{1+2r}\ dr \\
=&\pi c\int_{r=0}^{3} r - \frac{1}{2}(\frac{1+2r}{1+2r} - \frac{1}{1+2r})\ dr \\
=&\pi c\int_{r=0}^{3} r - \frac{1}{2}(1 - \frac{1}{1+2r})\ dr
\end{aligned}
$$

$$
\begin{aligned}
&= \pi c \int_{r=0}^{3} r - \frac{1}{2} + \frac{1}{2}\frac{1}{1+2r}\, dr \\
&= \pi c\, (r^2/2 - \frac{1}{2}r + \frac{1}{4}ln(1+2r))\Big|_0^3 \\
&= \pi c(\frac{9}{2} - \frac{3}{2} + \frac{1}{4}ln(7)) \\
&= \frac{\pi(3 + \frac{1}{4}ln(7))}{\pi(3 - \frac{1}{2}ln(7)} \\
&= \frac{3 + \frac{1}{4}ln(7)}{3 - \frac{1}{2}ln(7)}
\end{aligned}
$$

Windspeeds in $[2,3]$**:**

Here one can interpret the wording in two different ways - providing one writes it explicitly what the meaning is. If you dont write what you meant, please do not complain about deductions from the side of TAs - when they have to make the assumption for you! **Be aware of your assumptions and make them explicit.**

Interpretation 1: I am interested in windspeeds in $[2,3]$, thus my density needs to be rescaled so that it integrates up to one for the set of windspeeds such that $2 <$ speed < 3.

Then you need to determine c_2 such that

$$
\pi c_2\, (r - \frac{1}{2}\ln(1+2r))\Big|_2^3 = \pi c(3 - \frac{1}{2}\ln(7) - 2 + \frac{1}{2}\ln(5)) = 1 \Leftrightarrow c_2 = \frac{1}{\pi}\frac{1}{1 + \frac{1}{2}\ln(5/7)}
$$

and compute the expectation with this new constant c_2, same as above, but over the interval $[2,3]$:

$$
\begin{aligned}
E[s] &= \pi c_2\, (r^2/2 - \frac{1}{2}r + \frac{1}{4}ln(1+2r))\Big|_2^3 \\
&= \frac{1}{1 + \frac{1}{2}\ln(5/7)}(\frac{9-4}{2} - \frac{1}{2} + \frac{1}{4}\ln(7) - \frac{1}{4}\ln(5)) \\
&= \frac{2 + \frac{1}{4}\ln(7/5)}{1 + \frac{1}{2}\ln(5/7)}
\end{aligned}
$$

Interpretation 2:

You could claim that the wording means the expectation of a function $r(x_1,x_2) = \begin{cases} \sqrt{x_1^2 + x_2^2} & \text{if } \sqrt{x_1^2 + x_2^2} > 2 \\ 0 & \text{else} \end{cases}$

In that case, you compute the expectation with the old constant c instead. Pls be aware of such ambiguities!

In that case

$$E[s] = \frac{2 + \frac{1}{4}\ln(7/5)}{3 - \frac{1}{2}ln(7)}$$