ISTD 01.001 - Introduction to Probability and Statistics: Jan-Apr 2017

Review problems: Week 12

April 7, 2018

Problem 1:
Think here clearly. Probability is defined over sets. The space of outcomes is

$\{-1, 1\} \times [0, 2]$.

So we need to define $P((X, Y) \in A), A \subset \{-1, 1\} \times [0, 2]$. All such sets A can be written as $A = \{-1\} \times A_1 \cup \{1\} \times A_2$ such that we have A_1, A_2 is either $\emptyset$ or $A_i \subset [0, 2]$.

Therefore

$$P((X, Y) \in A) = P((X, Y) \in (\{-1\} \times A_1) \cup (\{+1\} \times A_2))$$

But the union $\cup$ is a disjoint one. so

$$\begin{aligned} P((X,Y) \in (\{-1\} \times A_1) \cup (\{1\} \times A_2)) &= P((X,Y) \in \{-1\} \times A_1) + P((X,Y) \in \{1\} \times A_2) \\ &= P(X = -1, Y \in A_1) + P(X = 1, Y \in A_2) \\ &= c \int_{A_1} 1[x == -1](x+7)(1+y^3)dy + c \int_{A_2} 1[x == +1] \frac{1}{2} x 6 y^2 dy \\ \text{just plug in the x-values here } &= c \int_{A_1} 6(1+y^3)dy + c \int_{A_2} 3y^2 dy \end{aligned}$$

The whole space would be $\{-1\} \times [0, 2] \cup \{1\} \times [0, 2]$, so for getting c we need to compute

$$\begin{aligned} 1 &= c \int_0^2 6(1+y^3)dy + c \int_0^2 3y^2 dy \\ &= c(6y + \frac{6}{4} y^4 + y^3 \Big|_0^2) = c(12 + 24 + 8) \\ c &= \frac{1}{44} \end{aligned}$$

This is about 1.

2. the probability is defined over sets $A \subset \{-1, 1\} \times [0, 2]$ consisting of pairs (x, y). So clearly the marginal over X takes only values $-1, +1$.

3. For the same reason So clearly the marginal over Y takes values $[0,2]$, and likely will have a density function.

4. The set $X = -1$ can be represented as $\{-1\} \times [0,2]$, so

$$\begin{aligned} P(X=-1) &= c\int_0^2 6(1+y^3)dy \\ &= c(6y + \frac{6}{4}y^4\Big|_0^2) = \frac{36}{44}, \Rightarrow P(X=+1) = 1 - \frac{36}{44} = \frac{8}{44} \end{aligned}$$

5. This can be read off from

$$P((X,Y) \in (\{-1\} \times A_1) \cup (\{1\} \times A_2)) =$$

$$= c\int_{A_1} 1[x == -1](x+7)y^2(1+y^3)dy + c\int_{A_2} 1[x == +1](x+2)6y^2dy.$$

Note here

$$\begin{aligned} P(Y \in A^*) &= P((X,Y) \in (\{-1\} \times A^*) \cup (\{1\} \times A^*)) \\ &= P((X,Y) \in \{-1\} \times A^*) + P((X,Y) \in \{1\} \times A^*) \\ &= c\int_{A^*} 6(1+y^3)dy + c\int_{A^*} 3y^2dy \\ &= c\int_{A^*} 6 + 6y^3 + 3y^2dy \end{aligned}$$

so all we need to do is: sum here over all possible values of X

$$f_Y(y) = \frac{1}{44}(6 + 6y^3 + 3y^2)$$

6. and 7. – This is just putting components together, but note
the conditional distribution of X given Y must be defined over $\{-1,+1\}$. In fact $P(X|Y=y) = f(x,y)/f_Y(y)$, so

$$\begin{aligned} P(X=+1|Y=y) = f(x=+1,y)/f_Y(y) &= \frac{3y^2}{6+6y^3+3y^2} \\ P(X=-1|Y=y) = f(x=-1,y)/f_Y(y) &= \frac{6(1+y^3)}{6+6y^3+3y^2} \end{aligned}$$

pretty unpleasantly looking, and obviously non-independent.

The other conditional *density* is nicer. Note that Y takes values in an interval, so $f_{Y|X}$ must be defined for $y \in [0,2]$ for every value of X (which is just $-1,+1$)

$$\begin{aligned} f_{Y|X}(x=-1,y) = f(x=-1,y)/P_X(x=-1) &= \frac{\frac{1}{44}6(1+y^3)}{\frac{36}{44}} = \frac{1}{36}6(1+y^3) \\ f_{Y|X}(x=+1,y) = f(x=+1,y)/P_X(x=+1) &= \frac{\frac{1}{44}3y^2}{\frac{8}{44}} = \frac{1}{8}3y^2 \end{aligned}$$

Problem 2:

$$f(x,y,z) = c(12x + 18z)y$$

Clearly $X, Z \perp Y$, $X \perp Y$, $Z \perp Y$.

$$\begin{aligned}
&c\int_0^4 (12x + 18z)ydxdydz \\
&= c\int_0^4 (6x + 9z)16dxdz \\
&= 16c\int_0^4 (3x^2 + 9zx)\big|_0^4 dz \\
&= 16c\int_0^4 48 + 36zdz \\
&= 16c(\,48z + 18z^2\big|_0^4) = c16(48 * 4 + 18 * 16) \\
c &= \frac{1}{7680}
\end{aligned}$$

Lets get the marginals f_X, f_Y We have:

$$\begin{aligned}
f_{X,Y}(x,y) &= c\int_0^4 f(x,y,z)dz = \int_0^4 12xy + 18yzdz \\
&= c(12xyz + 9yz^2)\ |_{z=0}^4 = c(48xy + 144y) \\
f_X(x) &= \int_{y=0}^4 c(48xy + 144y)dy = c(24xy^2 + 72y^2) \\
&= c(24 * 16x + 72 * 16) \\
f_Y(y) &= \int_{x=0}^4 c(48xy + 144y)dx \\
&= c(24x^2y + 144xy)\ |_{x=0}^4 = c(24 * 16 + 144 * 4)y
\end{aligned}$$

So:

$$\begin{aligned}
E[X] &= c\int_0^4 24 * 16x^2 + 72 * 16x = c(8 * 16x^3 + 72 * 8x^2|_0^4) \\
&= c(8 * 16 * 64 + 72 * 8 * 16) = \frac{6.8}{3} \\
E[X^2] &= c\int_0^4 24 * 16x^3 + 72 * 16x^2 = c(24 * 4x^4 + 24 * 16x^3|_0^4) \\
&= \frac{24 * 4 * 256 + 24 * 16 * 64}{7680} = 6.4 \\
Var[X] &= E[X^2] - E[X]^2 = \frac{11.36}{9}
\end{aligned}$$

$$E[Y] = 960c \int_0^4 y^2 = 320c * 64 = \frac{8}{3}$$
$$E[Y^2] = 960c \int_0^4 y^3 = 240c * 256 = 8$$
$$Var[Y] = E[Y^2] - E[Y]^2 = \frac{8}{9}$$

For $Cov(X, Z)$:

$$\begin{aligned}
f_{X,Z}(x,z) &= c16(6x + 9z) \\
f_Z(z) &= c16(48 + 36z) \\
E[Z] &= 16c \int_0^4 48z + 36z^2 \\
&= 16c(24z^2 + 12z^3) = 16c(24 * 16 + 12 * 64) = 2.4 \\
E[Z^2] &= 16c \int_0^4 48z^2 + 36z^3 \\
&= 16c(16z^3 + 9z^4) = 16c(16 * 64 + 9 * 256) = \frac{20.8}{3} \\
E[XZ] &= \int \int c16(6x^2 z + 9z^2 x) \\
&= \int_x 16c(3x^2 z^2 + 3z^3 x)_{z=0}^4 = \int_x 16c(48x^2 + 3 * 64x) \\
&= 16c(16 * 64 + 3 * 32 * 16) = \frac{16}{3} \\
Cov(X, Z) &= \frac{16}{3} - \frac{6.8}{3} 2.4 = \frac{-0.32}{3} \\
\rho_{X,Z} &= \frac{-0.32}{3\sqrt{\frac{11.36}{9}(\frac{20.8}{3} - 2.4^2)}} \approx -0.08765
\end{aligned}$$

Problem3:

$$f(x,y) = c\frac{x}{1 + x^2}\frac{1}{1 + y^2}$$

$$\begin{aligned}
1 &= c\int_{x=1}^2 \frac{x}{1 + x^2} \int_{y=3}^4 \frac{1}{1 + y^2} \\
&= c\frac{1}{2}\ln(1 + x^2)|_1^2 \cdot arctan(y)|_3^4 \\
c &= \frac{2}{\ln(5/2)(arctan(4) - arctan(3))}
\end{aligned}$$

Yep, they are independent, thus $f_X = f_{X|Y}$ and

$$f_X(x) = \frac{2}{\ln(5/2)}\frac{x}{1 + x^2}$$

$$
\begin{aligned}
E[Y^2] &= \frac{1}{arctan(4) - arctan(3)} \int_{y=3}^{4} \frac{y^2}{1+y^2} \\
&= \frac{1}{arctan(4) - arctan(3)} \int_{y=3}^{4} (\frac{1+y^2}{1+y^2} - \frac{1}{1+y^2}) \\
&= \frac{1}{arctan(4) - arctan(3)} (1 - (arctan(4) - arctan(3))) \\
&= \frac{1}{arctan(4) - arctan(3)} - 1
\end{aligned}
$$