50.039 – Theory and Practice of Deep Learning

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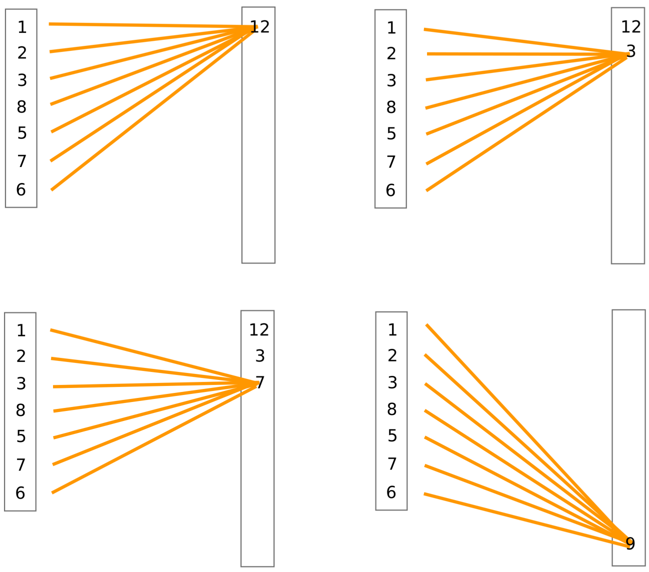
Week 03: Convolutional Neural networks

[The following notes are compiled from various sources such as textbooks, lecture materials, Web resources and are shared for academic purposes only, intended for use by students registered for a specific course. In the interest of brevity, every source is not cited. The compiler of these notes gratefully acknowledges all such sources.]

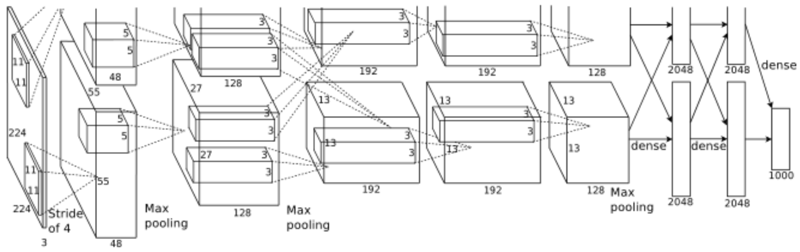
1 Convolutional Neural networks

See also for example: `http://neuralnetworksanddeeplearning.com/chap6.html`.

1.1 a fully connected layer, 1 input channel



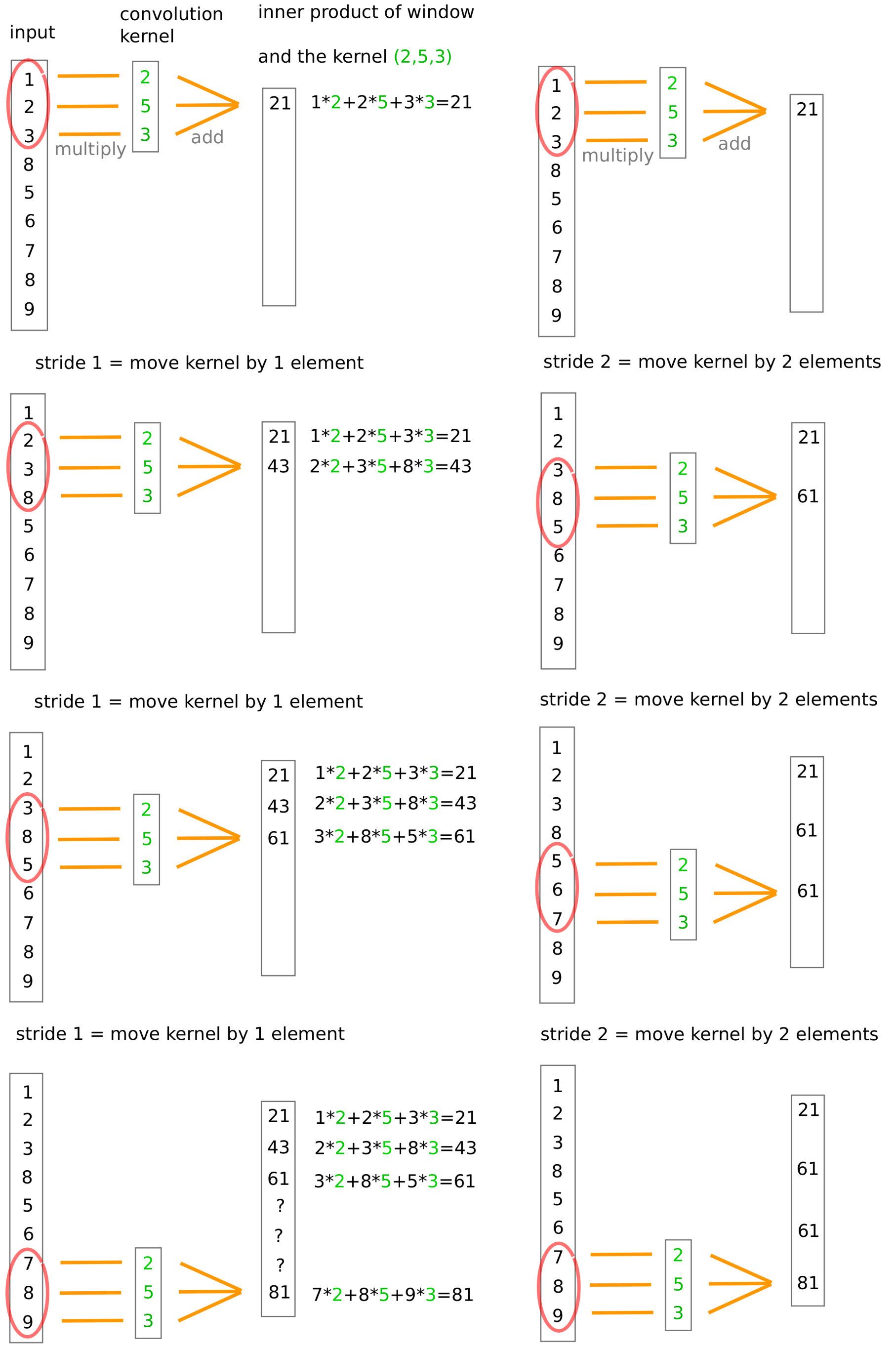
number of weights grows with the number of elements in the input and the output



taken from: Alex Krizhevsky et al., NIPS2012 `http://www.cs.toronto.edu/~fritz/absps/imagenet.pdf`

- how to connect neurons sparsely?
- **key idea**: in images neighbor pixels tend to be related! So we connect only neighboring neurons in the input.

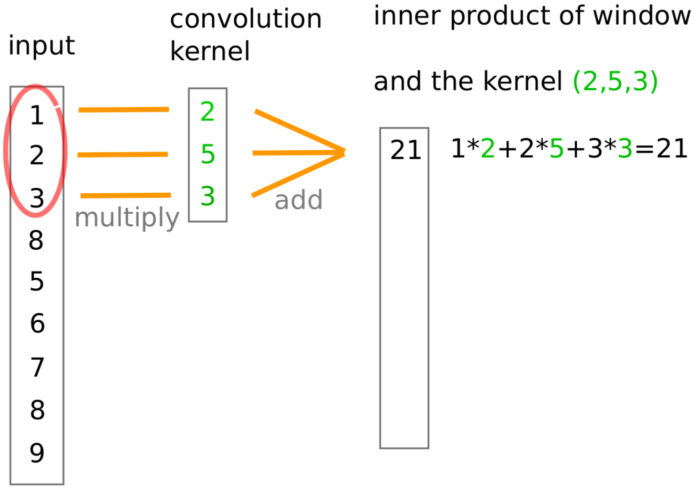
1.2 1-d convolutions, 1 input channel



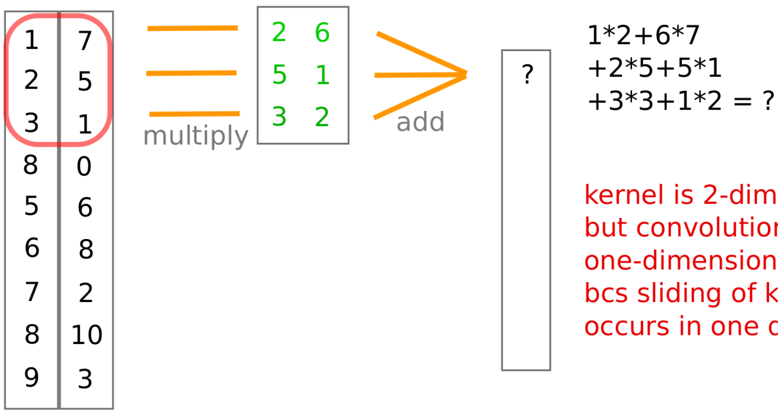
no padding of input:
outputsize = inputsize -2 = inputsize -(kernelsize-1)

no padding of input:
outputsize = ceil ((inputsize -2)/2)
= ceil((inputsize -(kernelsize-1))/stride)

1.3 1-d convolutions, 2+ input channels

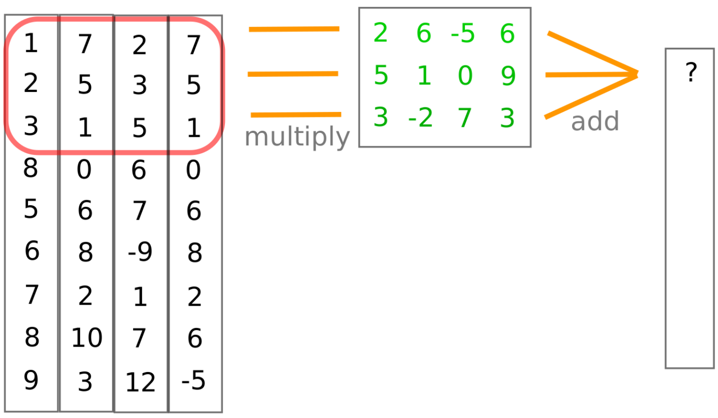


2 input channels, kernel is of size (nchannels, kernel size) = (2,3)

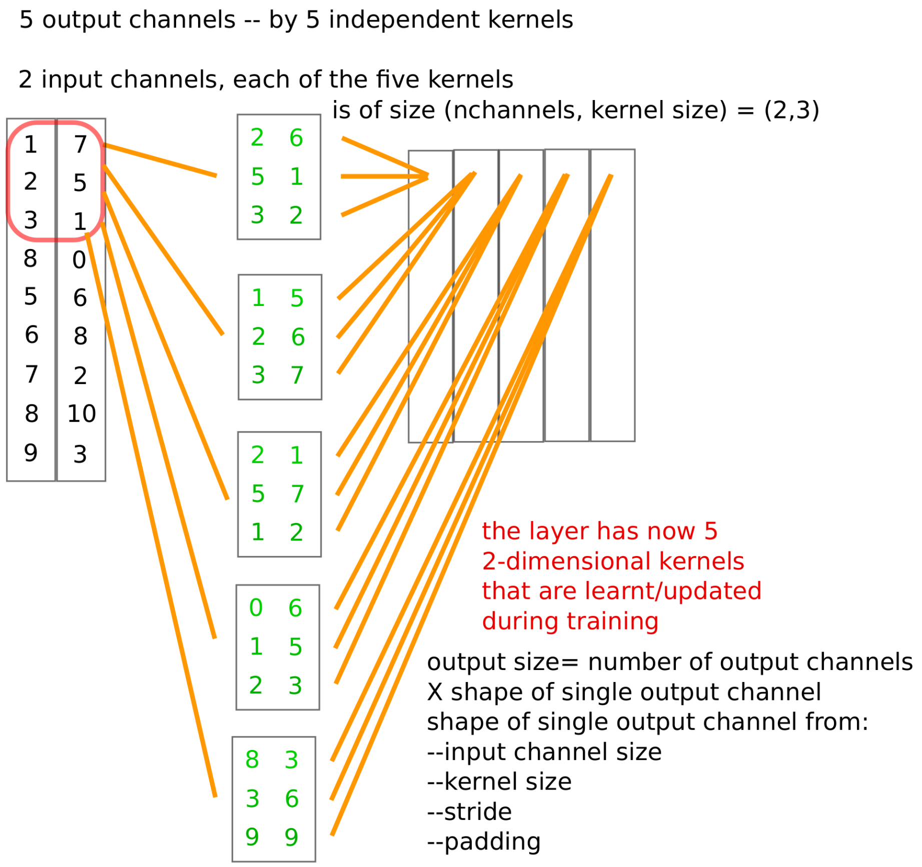


kernel is 2-dim object,
but convolution is
one-dimensional
bcs sliding of kernel
occurs in one dimension

4 input channels, kernel is of size (nchannels, kernel size) = (4,3)



1.4 1-d convolutions, one whole convolution layer (multiple output channels)



1.5 why convolutions I?

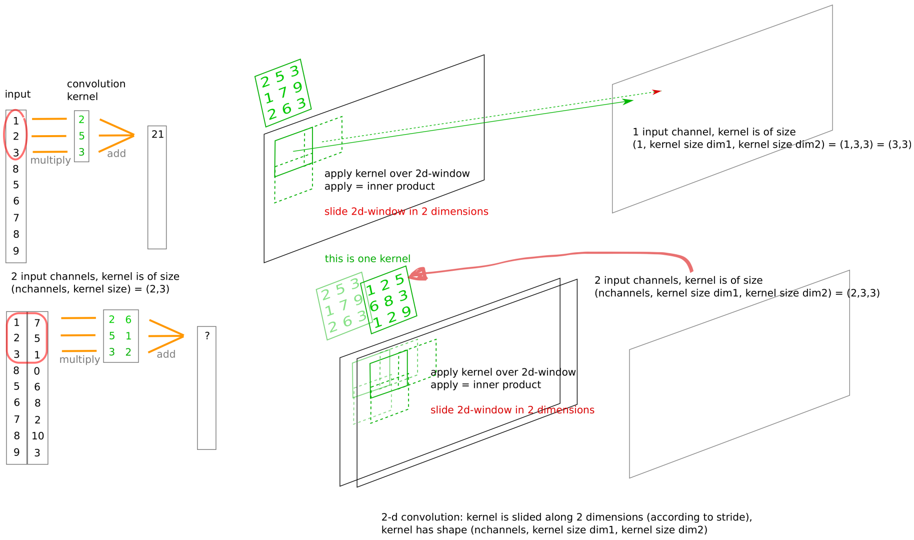
The neural net example code for mnist has linear (fully connected) layers. In it: each neuron of layer l is connected to each neuron of layer $l+1$.

- Fully connected: Assume we have N_l neurons in layer l and N_{l+1} neurons in layer $l+1$: parameters have dimensionality =?
- Locally connected: Assume we have N_l neurons in layer l and N_{l+1} neurons in layer $l+1$, each output neuron takes input from a patch of 5 neighbors: parameters have dimensionality ?
- 1-d convolution with kernel size 5: parameters have dimensionality 5, no matter how many inputs or outputs in the sliding dimension
- convolutions have a small parameter dimensionality, independent of input and output size in the sliding dimension (number of output channels matters though)

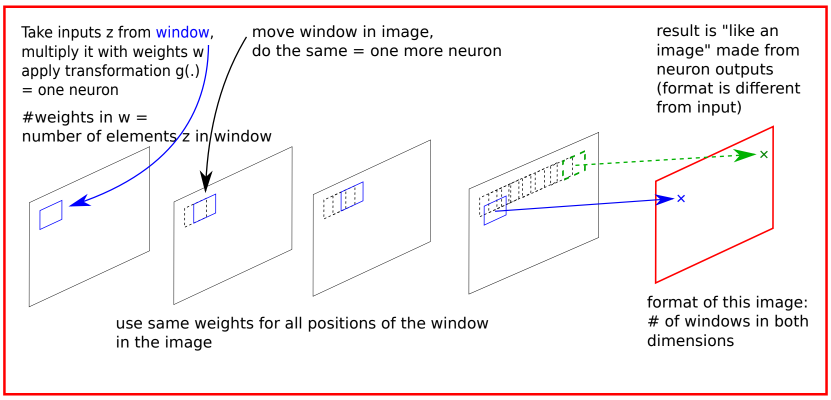
The philosophy:

- machine learning in general: keep number of parameters limited relative to training set size
- deep learning: better stack simple functions in deeply in many layers than learning in one layer something very complex.

1.6 2-d convolutions, 2+ input channels



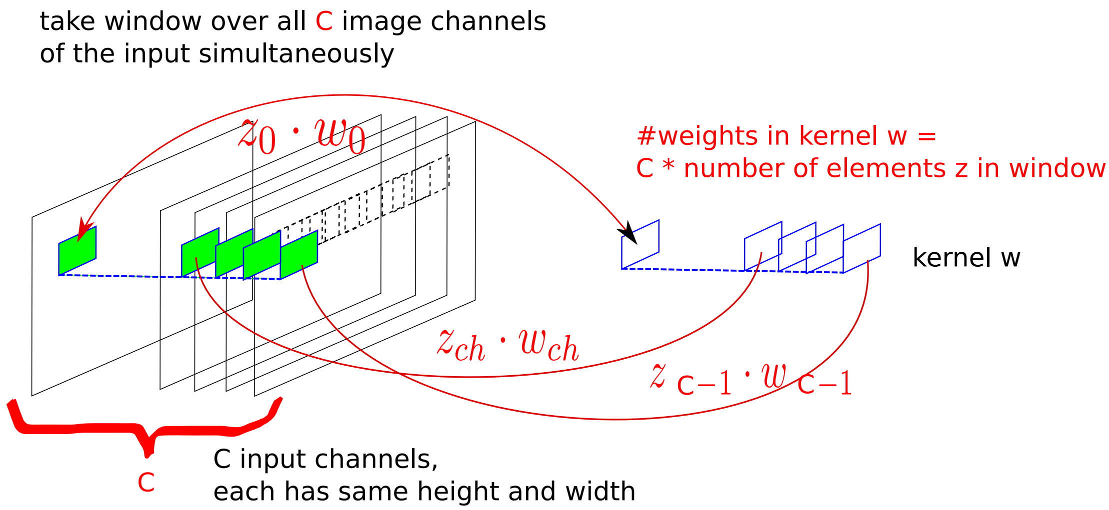
A simplified convolution (one input channel only) can be drawn like this:



- a convolution at one fixed region (region means here: rectangular window) x of an image is an inner product $w \cdot x$ between kernel weights w and this region x - this is a single real number.
- we have applied the matrix in one point, resulting it in one real number as output. now can slide the kernel w along both axes (height and widths) - results in a matrix of outputs

- This means: we slide the convolutional kernel w over the image and apply the inner product over many windows. **In convolution we apply the same w across all locations** for computing inner products.
- the parameters to be learned during training are the values of the kernel w!

can apply to images with multiple input channels - conv kernel has one depth dimension more than sliding dimensions, result still one channel image. A less simplified convolution (multiple input channels, one output channel) can be drawn like this:



$$output(h, w) = z_0 w_0 + \ldots + z_{ch} w_{ch} + \ldots + z_{C-1} w_{C-1}$$

sum of C inner products -- one for each input channel

Above shows convolution for a convolution kernel with 1 output channels. Finally, one convolution layer usually uses O independent convolutional kernels, resulting in O channels as output.

1.7 why convolutions II?

Convolution implements a battery of localized pattern detectors.

- kernel matrix is some pattern (Krizhevsky paper, Zeiler paper)
- apply convolution, get $y = kernel \cdot inputwindow + b$

  inner product is a similarity measure, high positive for inputs parallel to kernel, zero for inputs orthogonal, high negative for inputs antiparallel to kernel
- apply convolution with activation function $g(\cdot)$ in the next layer – result for one window has formula $g(kernel \cdot inputwindow + b)$, detector for patterns in input channels similar to the kernel

- detection is performed all over across the sliding space (1-d,2-d,3-d,n-d)

1.8 why convolutions III?

- Think of the image in the above graphic not as an image, but as a grid of signals of detectors. Why does that makes sense?

  An RGB-input image itself is a signal of detectors (camera sensors)

  $w \cdot inputwindow$ is a similarity between these two. A neuron output $g(w \cdot inputwindow + b)$ can be seen as a detector for some kind of structure encoded by w. It gives a high signal for some inputs, and a low signal for other inputs.

  Then every input pixel in above graphic is a signal of some detector for a kind of part/structure, e.g. a car wheel, a car window.

  A convolution can be seen as weighted sum of inputs $conv(pos1, pos2) = \sum_{i,j} w_{i,j} z_{pos1+i,pos2+j}$. So it is a weighted sum of signals of detectors over different positions.

  A localized $k \times k$ combination of neurons allows to learn a combination of parts that are *neighboring*. Why it is ok to learn a neighboring combination of parts?

- Semantically meaningfully Parts in an image (eye, fur, leg, whole dog) form usually a connected, neighboring region in an image [1].So **a convolution at one fixed output point learns to combine neighboring parts.**

- another thought: by stacking convolution layers one can look at regions that get larger with every layer:

- If one stacks two such layers by $3x3$ kernels, then the first layer looks at 3×3, but the next layer looks at 5×5 regions (and the third 7×7). So each layer looks at regions in the input image with a larger size.

  Neurons at high layers look at very large regions of the input image

- convolutions can be applied as 1d-convolution to any sequence data, like time series, language sentences, DNA sequences.

[1] What would be a counterexample?

1.9 Kernel parameters and their influence on the output shape

What is the size of the output matrix ? Suppose we use 2d convolutions and inputs are $M \times N$. First observation: analysis can be done separately for every sliding dimension.

> Takeaway: there are three parameters influencing the output size: padding, kernel size, and stride.

It is clear what kernel size is – the shape of w.

shape without padding:

Problem: if we apply a kernel to a window, it must fit into it. If we use a 3×3 convolution kernel w, then at the borders could start only at element 1 **(index starts at 0)**, and must end at $M - 2$ $(N - 2)$.

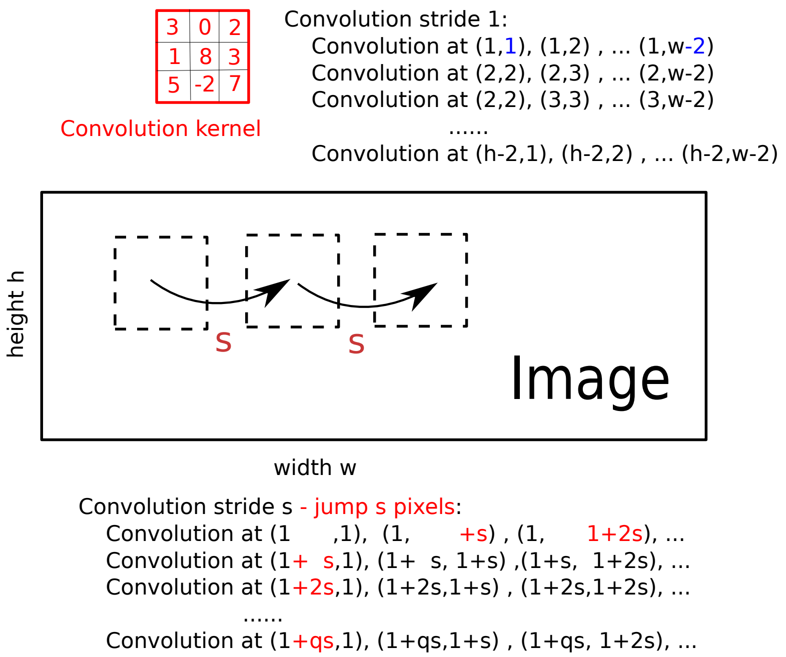
Result is: if we move w always by one pixel (=stride 1), then the output is $(M - 2) \times (N - 2)$. **That can be drawn**

For a 5×5 kernel: then at the borders could start only at element #2, and must end at $M - 3$ $(N - 3)$, so output shape will be $(M - 4) \times (N - 4)$.

> In general without padding, if we apply a kernel of size $k \times k$, and we move w always by a stride of 1, then the output shape will be $(M - k + 1) \times (N - k + 1)$

What is if we use a stride s larger than one ?

> stride is the number of input elements/pixels which w is moved in every step



For a $k \times k$ kernel we start with the first convolution window placed at $(k-1)/2$, move always $(k-1)/2, (k-1)/2+s, (k-1)/2+2s, (k-1)/2+3s$ and end at the largest index c such that $c + (k-1)/2 \leq M-1$ holds.

How many sliding windows do we have?

$$
\begin{aligned}
M &= k, \ldots, k+s-1 && \mapsto 1,\ M = k \text{ image as large as kernel} \\
M &= k+s, \ldots, k+2s-1 && \mapsto 2 \\
M &= k+2s, \ldots, k+3s-1 && \mapsto 3 \\
M &= k+3s, \ldots, k+4s-1 && \mapsto 4 \\
M &= k+4s, \ldots, k+5s-1 && \mapsto 5
\end{aligned}
$$

lets look at an example

$$
\begin{aligned}
M = k+2s, \ldots, k+3s-1 &\mapsto 3 \\
M-k+1 = 2s+1, \ldots, 3s &\mapsto 3 \\
(M-k+1)/s \in [2.x, 3] &\mapsto 3
\end{aligned}
$$

Therefore: this function can be given as:

$$ceil((M - ksize + 1)/s)$$

> If we have an input of length M in one dimension, then the output size in that dimension for a kernel of size $ksize$ and stride s without padding is given as: $ceil((M - ksize + 1)/s)$

Example: $5 \times 7, ksize = (3 \times 1), s = 2$, then: output is $ceil(5-3+1)/2 \times ceil(7-1+1)/2 = 2 \times 4$

shape with padding:
Padding means to add for every dimension at both ends of an input a layer of zeros. Example: 2d-input, pad 2 means:

0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0						0	0
0	0						0	0
0	0			5x5			0	0
0	0						0	0
0	0						0	0
0	0						0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0

Padding of 2

- add two columns at the beginning
- add two columns at the end
- add two rows at the beginning
- add two rows at the end
- $M \times N \to (M+4) \times (N+4)$

In general: padding by r changes the input shape $M \times N \to (M+2r) \times (N+2r)$

If we pad by r, then the image dimension increases from M to $M+2r$, therefore:

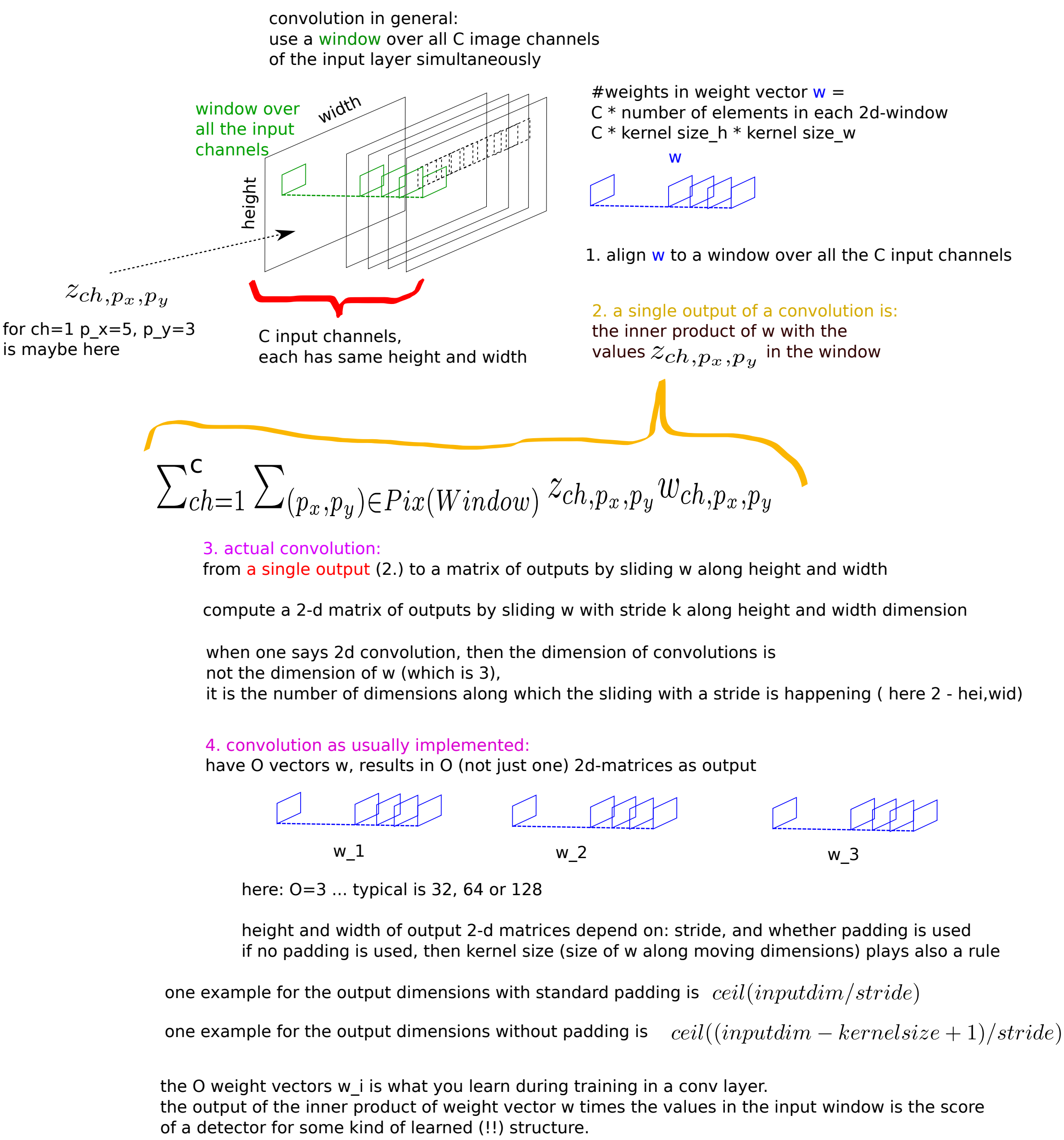
the output kernel size for a kernel of size $ksize$ and stride s with padding of r is given as: $ceil((M - ksize + 1 + 2r)/s)$

Standard padding: Standard padding is used if we pad for a kernel of size $ksize$ by a pad value r such that $r = (ksize - 1)/2$.
In such a case the output shape is $ceil(M/s)$.

Observations:

- stride s shrinks an output shape much more than kernel size $ksize$ does ... see $ceil((M - ksize + 1)/s)$
- if we use standard padding (which is adaptive), then kernel size has no influence on the output shape

- Convolution with stride s takes an image with height (h, w) and creates a downsampled image with dimensions being approximately $(h/s, w/s)$



https://github.com/vdumoulin/conv_arithmetic

2d-Convolution as recapitulation:
A 2d-convolutional layer applies a convolutional kernel w over a multi-channel

image (that is an 3-dimensional array having format $C \times width \times height$). Application means here: the kernel is slided along 2-dimensions (height,width) of the multi-channel image according to a stride. Everytime it stops over a rectangular window, an inner product between that kernel and the rectangular window is computed. This inner product is a real number. By sliding the kernel, one obtains as output one matrix per kernel. Using multiple kernels results in a multi-channel image with format $\#kernels \times newwidth \times newheight$. The weights w for all the kernels are learnt during neural network training.

In convolution we apply one weight vector w over many positions in one set of input channels – why sharing a w across the image makes sense ?
the inner product between a window of the input layer and the weight w can be seen as using w as a "detector".

- One wants to learn the detector over all regions in the image.
- when one has found a good detector, one wants to apply the same detector over all regions in the image
- for these reasons w is shared across the image.
- convolutional neural nets: combine **neighboring neurons** into a neuron in the next layer
- original paper: *Neocognitron: A self-organizing neural network model for a mechanism of pattern recognition unaffected by shift in position.* K. Fukushima, Biological Cybernetics, 1980
- one further idea: related to summing up $\sum_i w_{ij} z_i$, see next slides

How does a learned w look like?

One can consider the filters in the paper "Visualizing and Understanding Convolutional Networks",Matthew D. Zeiler and Rob Fergus, ECCV 2014

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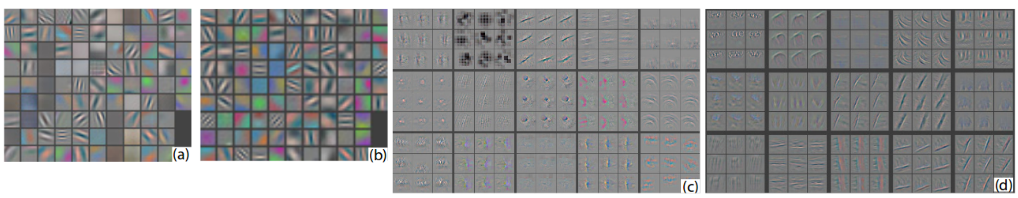


Fig. 5. (a): 1st layer features without feature scale clipping. Note that one feature dominates. (b): 1st layer features from Krizhevsky *et al.* [18]. (c): Our 1st layer features. The smaller stride (2 vs 4) and filter size (7x7 vs 11x11) results in more distinctive features and fewer "dead" features. (d): Visualizations of 2nd layer features from Krizhevsky *et al.* [18]. (e): Visualizations of our 2nd layer features. These are cleaner, with no aliasing artifacts that are visible in (d).

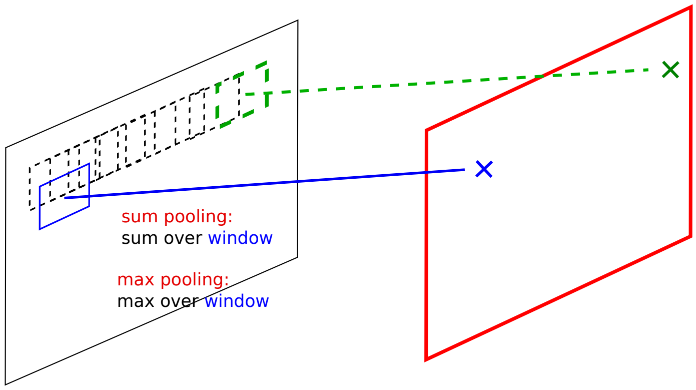
See also works by Anh Mai Nguyen, Jeff Clune, U Wyoming.

- each filter kernel w works like a detector for some structure by computing an inner product.
- the detector is applied over a window. the window gets slided. everytime the detector is applied to an array of $2-d$ images (at the input: RGB image are 3 2-d images, in higher layers one can have many more channels than 3).
- learn the values of w by backpropagation

1.10 Pooling

Its related to convolution, but has no parameters to be learned. in case of multi channels: each channel separately treated, usually not combined.

- **Sum pooling:** Sums up all the elements over a window and returns a real number. Same sliding window approach with kernel size (= window size) and stride – yields then a matrix as output.



- Equivalent to a Filter kernel $w = const$
- same as convolution: works with M input channels – but usually each channel is pooled separately!
- Idea: Often after convolution+activation ... average of detector outputs
- **Max pooling:** Computes a max up all the elements over a window and returns a real number. Same sliding window approach with kernel size (= window size) and stride – yields then a matrix as output.
- Equivalent to a Filter kernel $w = const$ and replace aggregation: sum gets replaced by a max (see that you could plug in other aggregation operations).
- Idea: Often after convolution+activation ... highest detector output over a field of view

1.11 Old-style (pre-batchnorm and residual connections) Neural Network structure for Computer vision tasks

- **Convolution-Relu-Pooling** Repeated. Last $1-2$ layers are a **fully connected** layer.
- ReLU: Rectified linear $g(x) = max(x, 0)$. Nowadays more alternatives: leaky ReLU, eLU, SeLU. Sigmoids are used for bounded regression outputs, not common for vision.