On the Coupling of Decimation Operator with Subdivision Schemes for Multi-scale Analysis

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Abstract. Subdivision schemes [5,11] are powerful tools for the fast generation of refined sequences ultimately representing curves or surfaces. Coupled with decimation operators, they generate multi-scale transforms largely used in signal/image processing [1,3] that generalize the multi-resolution analysis/wavelet framework [8]. The flexibility of subdivision schemes (a subdivision scheme can be non-stationary, non-homogeneous, position-dependent, interpolating, approximating, non-linear...) (e.g. [3]) is balanced, as a counterpart, by the fact that the construction of suitable consistent decimation operators is not direct and easy.

In this paper, we first propose a generic approach for the construction of decimation operators consistent with a given linear subdivision. A study of the so-called prediction error within the multi-scale framework is then performed and a condition on the subdivision mask to ensure a fast decay of this error is established. Finally, the cases of homogeneous Lagrange interpolatory subdivision, spline subdivision, subdivision related to Daubechies scaling functions (and wavelets) and some recently developed non stationary non interpolating schemes are revisited.

Keywords: Multi-scale analysis · Decimation · Subdivision

1 Introduction

Since the eighties [5,10,11] and even earlier [9], subdivision schemes have been developed, analyzed and used with very popular applications such as curve generation, image processing or animation movies. One of their advantages stands in the flexibility of the construction of their masks. In many ways, subdivision schemes are connected to the refinement process associated to the wavelet framework [6,8,15] and have to be coupled with a so called decimation operator [12] to fully generalize the classical multi-resolution approach. In practice, this coupling has been successfully used for image compression [1,3] since it offers an efficient

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compromise between sparsity of the decomposition and quality of the reconstruction of an image. However, if the decimation operator is trivially defined as a subsampling when working with interpolatory subdivision schemes, it is not the case in other situations. As an example, one can mention the non-stationary penalized Lagrange subdivision scheme [14] for which there is, up to now, no available decimation operator in the literature due to the mixing between interpolating and non interpolating subdivisions. Many approximating subdivision schemes can not be used for compression for the same reason. More generally, even if a decimation operator is known, it is important in practice to be able to derive a large choice of decimation masks in order to exhibit the most relevant one according to specific objectives. It is the case for example in image compression where a criterion based on the stability of the multi-resolution transforms could be used to select decimation operators.

Therefore, the goal of our work is to propose a generic approach for the construction of decimation operators related to a given subdivision scheme. In this paper, we focus on linear and homogeneous subdivision schemes as a starting step. We first provide an overview on the multi-scale and subdivision frameworks (Sect. 2). Then we revisit in Sect. 3 the fundamental aspects of consistency and provide a generic construction of decimation operators of minimal length mask that we call elementary decimation operators. We then describe how to generate all the possible decimation operators from them. Section 4 is then devoted to a theoretical analysis of the so-called prediction error. We show that the decay of this quantity is fully controlled by the subdivision operator. Finally, several examples and applications are considered in Sect. 5.

2 Combining Multi-scale Transforms and Subdivision Schemes

2.1 Multi-scale Analysis and Multi-scale Transform of Harten

A multi-scale analysis is characterized by the introduction of a family of separable spaces $(V^j)_{j\in\mathbb{Z}}$ (j is a scale parameter) and two families of operators $(D^j_{j+1})_{j\in\mathbb{Z}}$ and $(P^{j+1}_j)_{j\in\mathbb{Z}}$ connecting two successive spaces V^j and V^{j+1}. For each value of j, the decimation operator, D^j_{j+1}, maps $f^{j+1}=\left(f^{j+1}_k\right)_{k\in\mathbb{Z}}\in V^{j+1}$ to an element $f^j=\left(f^j_k\right)_{k\in\mathbb{Z}}\in V^j$; the prediction operator, P^{j+1}_j, maps $f^j\in V^j$ to an element of V^{j+1}. If f^j is obtained after decimation of f^{j+1}, $P^{j+1}_j f^j$ does not usually coincide with f^{j+1}. However the following consistency condition is required:

$$D^j_{j+1}P^{j+1}_j = I_{V^j} \tag{1}$$

where I_{V^j} stands for the identity operator in V^j. In order to recover f^{j+1} after a decimation and a prediction, a sequence of prediction errors $e^{j+1} = \left(e_k^{j+1}\right)_{k\in\mathbb{Z}}$ is introduced and defined as:

$$e_k^{j+1} = f_k^{j+1} - \left(P_j^{j+1} D_{j+1}^j f^{j+1}\right)_k = \left(\left(I_{V^{j+1}} - P_j^{j+1} D_{j+1}^j\right) f^{j+1}\right)_k \quad . \tag{2}$$

The multi-scale transform of Harten [12] is then constructed as follows. Focussing on a level j, we note $\tilde{h} = D_{j+1}^j$ and $h = P_j^{j+1}$. Thanks to (1) and (2) that can be reformulated as $\tilde{h}h = I_{V^j}$ and $e^{j+1} = \left(I_{V^{j+1}} - h\tilde{h}\right) f^{j+1}$, it comes out that $\tilde{h}e^{j+1} = 0$. When $\tilde{h}$ is linear, we get $e^{j+1} \in Ker(\tilde{h}) = W^j$. We call $\tilde{g}$ the operator that associates e^{j+1} to its decomposition on a basis of W^j and g the canonical injection from W^j to V^{j+1}. We note $d^j = \tilde{g}e^{j+1}$ and $e^{j+1} = gd^j$. Then $\left(h, \tilde{h}, g, \tilde{g}\right)$ satisfy:

$$\begin{cases} \tilde{g}g = I_{W^j}, \\ h\tilde{h} + g\tilde{g} = I_{V^{j+1}}, \\ \tilde{g}h = 0, \\ \tilde{h}g = 0. \end{cases}$$

One-scale decomposition and reconstruction transforms are then classically sketched as follows [15]:

$$\begin{tikzcd} f^{j+1} \arrow[d, shift right, "\tilde{h}"'] \arrow[dr, shift left, "\tilde{g}"'] & \\ f^j = \tilde{h} f^{j+1} \arrow[u, shift right, "h"'] & d^j = \tilde{g} e^{j+1} \text{ with } e^{j+1} = (I - h\tilde{h}) f^{j+1} \arrow[ul, shift left, "g"'] \end{tikzcd}$$

Iterating this process and denoting $J_0 < j$, two multi-scale decomposition and reconstruction transforms can be finally constructed as:

$$\text{decomposition:} \quad f^{j+1} \mapsto \{f^{J_0}, d^{J_0}, \ldots, d^j\}, \tag{3}$$

$$\text{reconstruction:} \quad \{f^{J_0}, d^{J_0}, \ldots, d^j\} \mapsto f^{j+1} \quad . \tag{4}$$

These two transforms are of prime importance in data analysis and compression. In this context, their stability with respect to small perturbations is a key property that is recalled in the following definition.

Definition 1
The decomposition transform is said to be stable with regards to the norm $||\cdot||$ *if there exists a constant* C *such that for all* j *and for all* (f^j, f^j_ϵ), *if* $f^j \mapsto \left\{f^{J_0}, d^{J_0}, ..., d^{j-1}\right\}$ *and* $f^j_\epsilon \mapsto \left\{f^{J_0}_\epsilon, d^{J_0}_\epsilon, ..., d^{j-1}_\epsilon\right\}$, *then,*

$$sup\left\{||f^{J_0}_\epsilon - f^{J_0}||,\ \{||d^m_\epsilon - d^m||,\ m < j\}\right\} \leq C||f^j_\epsilon - f^j|| \ . \tag{5}$$

The reconstruction transform is said to be stable with regards to the norm $||\cdot||$ *if there exists a constant* C *such that for all* $j > J_0$ *and for all* $\{f^{J_0}, d^{J_0}, ..., d^{j-1}\}$ *and* $\left\{f^{J_0}_\epsilon, d^{J_0}_\epsilon, ..., d^{j-1}_\epsilon\right\}$, *if* $\{f^{J_0}, d^{J_0}, ..., d^{j-1}\} \mapsto f^j$ *and* $\left\{f^{J_0}_\epsilon, d^{J_0}_\epsilon, ..., d^{j-1}_\epsilon\right\} \mapsto f^j_\epsilon$, *then,*

$$||f^j_\epsilon - f^j|| \leq C\ sup\left\{||f^{J_0}_\epsilon - f^{J_0}||,\ \{||d^m_\epsilon - d^m||,\ m < j\}\right\} \ . \tag{6}$$

From the previous definition, it turns out that the stability of the multi-scale transforms fully depends on the choice of the four operators $(h, \tilde{h}, g, \tilde{g})$. Indeed the following results hold as a direct consequence of the definition of $(h, \tilde{h}, g, \tilde{g})$.

Proposition 1
Assuming that the one-scale decomposition and reconstruction transforms are constructed from the linear operators $(h, \tilde{h}, g, \tilde{g})$, *if* $f^j \mapsto \left\{f^{j-1}, d^{j-1}\right\}$ *and* $f^j_\epsilon \mapsto \left\{f^{j-1}_\epsilon, d^{j-1}_\epsilon\right\}$, *then,*

$$||f^{j-1} - f^{j-1}_\epsilon|| \leq ||\tilde{h}||\ ||f^j - f^j_\epsilon||, \tag{7}$$

$$||d^{j-1} - d^{j-1}_\epsilon|| \leq ||\tilde{g}||\ ||I_{V^j} - h\tilde{h}||\ ||f^j - f^j_\epsilon||, \tag{8}$$

moreover, if $\{f^{j-1}, d^{j-1}\} \mapsto f^j$ *and* $\left\{f^{j-1}_\epsilon, d^{j-1}_\epsilon\right\} \mapsto f^j_\epsilon$, *then*

$$||f^j - f^j_\epsilon|| \leq (||h|| + ||g||)\ sup\left\{||f^{j-1} - f^{j-1}_\epsilon||, ||d^{j-1} - d^{j-1}_\epsilon||\right\}, \tag{9}$$

where for simplicity, $||\cdot||$ *denotes the vector or operator norm.*

The family of prediction operators $(P^{j+1}_j)_{j\in\mathbb{Z}}$ plays a key role in the efficiency of the multi-scale process (3)–(4). In this paper, we focus on linear local operators. There exists several approaches to construct them. The first one exploits the classical multi-scale analysis and wavelet framework [8]. In this case, the coefficients involved in the linear combination are directly deduced from the scaling relation connecting scaling functions at different levels. Starting from functional (continuous) spaces then moving to separable (discrete) ones is then required. On the contrary, a second approach consists in defining explicitly the connection between V^j and V^{j+1} without specifying scaling functions and wavelets. This can be performed using subdivision schemes that are briefly described in the next section.

2.2 Subdivision and Decimation Schemes

The definition of a binary subdivision scheme [11] is first recalled.

Definition 2
A (univariate) subdivision scheme S is defined as a linear operator $S : l^\infty(\mathbb{Z}) \to l^\infty(\mathbb{Z})$ constructed from a real-valued sequence $(h_k)_{k\in\mathbb{Z}}$ having a finite number of non zero values such that

$$(f_k)_{k\in\mathbb{Z}} \in l^\infty(\mathbb{Z}) \mapsto ((Sf)_k)_{k\in\mathbb{Z}} \in l^\infty(\mathbb{Z}) \quad \textit{with} \quad (Sf)_k = \sum_{l\in\mathbb{Z}} h_{k-2l} f_l \ .$$

The set of non zero values of $(h_k)_{k\in\mathbb{Z}}$ is called the mask of S and is denoted M_h.

From the definition of a subdivision scheme, it is easy to verify that a subdivision scheme reproduces constant[1] if and only if

$$\sum_{l\in\mathbb{Z}} h_{2l} = \sum_{l\in\mathbb{Z}} h_{2l+1} = 1 \ . \tag{10}$$

This property is also called shift invariance for constant and is assumed to be verified for all the subdivision schemes considered in this paper.

Subdivision can be iterated from an initial sequence $(f_k^{J_0})_{k\in\mathbb{Z}}$ to generate $(f_k^j)_{k\in\mathbb{Z}}$ for $j \geq J_0$ as

$$f^{j+1} = Sf^j, j \geq J_0 \ . \tag{11}$$

The mask plays a key role in the subdivision process and there exists many ways to construct it ([2,4,7,11,14] for example). In this work, we focus on homogeneous (the mask does not depend on k) and stationary (the mask does not depend on j) schemes.

Expression (11) can be interpreted as a prediction relation. There is therefore a one-to-one correspondence between subdivision and local prediction operators that will be exploited in this work.

The connection with the multi-scale framework is then fully achieved by introducing in the following definition the notion of binary decimation scheme.

Definition 3
A (univariate) decimation scheme D is defined as a linear operator $D : l^\infty(\mathbb{Z}) \to l^\infty(\mathbb{Z})$ constructed from a real-valued sequence $(\tilde{h}_k)_{k\in\mathbb{Z}}$ having a finite number of non zero values such that

$$(f_k)_{k\in\mathbb{Z}} \in l^\infty(\mathbb{Z}) \mapsto ((Df)_k)_{k\in\mathbb{Z}} \in l^\infty(\mathbb{Z}) \quad \textit{with} \quad (Df)_k = \sum_{l\in\mathbb{Z}} \tilde{h}_{l-2k} f_l \ .$$

The set of non zero values of $(\tilde{h}_k)_{k\in\mathbb{Z}}$ is called the mask of D and denoted $M_{\tilde{h}}$.

Moreover, similarly to subdivision schemes, a decimation scheme reproduces constant if and only if

$$\sum_{k\in\mathbb{Z}} \tilde{h}_k = 1 \ . \tag{12}$$

[1] a scheme U is said to reproduce constant if $\forall k, f_k = C \implies \forall k, (Uf)_k = C$.

The subdivision framework leads to a large choice for the family $(P_j^{j+1})_{j\in\mathbb{Z}}$ thanks to the flexibility in the construction of the mask. This is not the case when considering wavelet multi-resolution analysis, since the prediction and the decimation are fixed once scaling functions and wavelets are specified. However, for a given subdivision scheme, the construction of a decimation mask leading to a family of consistent decimation operators satisfying (1) is more involved. This topic is addressed in the next section where a new method to generate decimation masks associated to a fixed subdivision is proposed.

3 Construction of Decimation Operators

The first part of this section (Sect. 3.1) is devoted to the derivation of a condition on the subdivision and decimation masks to ensure the consistency property for the associated operators. Then, we propose in Sect. 3.2 a generic approach to construct decimations consistent with a fixed subdivision.

3.1 Consistency Condition

Proposition 2
Let h be a prediction operator constructed from the mask of a subdivision scheme i.e. $\forall j$, $\forall \left(f_l^j\right)_{l\in\mathbb{Z}}$,

$$\left(hf^j\right)_k = \sum_{l\in\mathbb{Z}} h_{k-2l} f_l^j \quad . \tag{13}$$

Let $\tilde{h}$ be a decimation operator constructed from the mask of a decimation scheme i.e. $\forall j$, $\forall \left(f_k^{j+1}\right)_{k\in\mathbb{Z}}$,

$$\left(\tilde{h}f^{j+1}\right)_l = \sum_{k\in\mathbb{Z}} \tilde{h}_{k-2l} f_k^{j+1}. \tag{14}$$

Then h and $\tilde{h}$ satisfy the consistency relation (1) if and only if

$$\forall j \in \mathbb{Z},\ \sum_{i\in\mathbb{Z}} h_i \tilde{h}_{i+2j} = \delta_{j,0} \quad . \tag{15}$$

where $\left(\delta_{j,0}\right)_{j\in\mathbb{Z}}$ is the Dirac sequence.

Proof
According to (13) and (14), the consistency condition (1) implies that $\forall \left(f_m^j\right)_{m\in\mathbb{Z}}$,

$$\forall m \in \mathbb{Z},\quad f_m^j = \sum_{k\in\mathbb{Z}} \tilde{h}_{k-2m} \sum_{l\in\mathbb{Z}} h_{k-2l} f_l^j = \sum_{l\in\mathbb{Z}} (\sum_{k\in\mathbb{Z}} \tilde{h}_{k-2m} h_{k-2l}) f_l^j$$

which is equivalent to

$$\forall m \in \mathbb{Z}, \quad \sum_{k\in\mathbb{Z}} \tilde{h}_{k-2m} h_{k-2l} = \delta_{m,l}$$

that leads to (15). □

Remark 1
According to Eqs. (10), (12) and (15), if a subdivision scheme and a decimation scheme are consistent and if the subdivision scheme reproduces constants, then the decimation scheme also reproduces constants.

We end up this section by providing results related to convex combination and translation of consistent decimation operators.

Definition 4
If $\tilde{h}$ is a decimation operator constructed from the sequence $(\tilde{h}_k)_{k\in\mathbb{Z}}$, for all $t \in \mathbb{Z}$ we call $T_t(\tilde{h})$ the decimation operator related to the sequence $(\tilde{h}_{k-t})_{k\in\mathbb{Z}}$.

Then, the following proposition holds:

Proposition 3
Let h be a prediction operator constructed from the sequence $(h_k)_{k\in\mathbb{Z}}$,

1. *if $\tilde{h}^0, \tilde{h}^1$ are two decimation operators of sequences $(\tilde{h}^0_k)_{k\in\mathbb{Z}}$ and $(\tilde{h}^1_k)_{k\in\mathbb{Z}}$ consistent with h, then $\forall\lambda \in \mathbb{R}$, $\lambda\tilde{h}^0 + (1-\lambda)\tilde{h}^1$ is consistent with h;*
2. *if $\tilde{h}^0, \tilde{h}^1, \tilde{h}^2$ are three decimation operators of sequences $(\tilde{h}^0_k)_{k\in\mathbb{Z}}$, $(\tilde{h}^1_k)_{k\in\mathbb{Z}}$ and $(\tilde{h}^2_k)_{k\in\mathbb{Z}}$ consistent with h, then $\forall\lambda \in \mathbb{R}$ and $\forall t \in \mathbb{Z}$, $\tilde{h}^0 + \lambda T_{2t}(\tilde{h}^1) - \lambda T_{2t}(\tilde{h}^2)$ is consistent with h.*

Proof

1. We have

$$\sum_k h_{k+2j}(\lambda\tilde{h}^0_k + (1-\lambda)\tilde{h}^1_k) = \lambda\delta_{j,0} + (1-\lambda)\delta_{j,0} = \delta_{j,0} \ .$$

2. Similarly,

$$\sum_k h_{k+2j}(\tilde{h}^0_k + \lambda\tilde{h}^1_{k-2t} - \lambda\tilde{h}^2_{k-2t}) = \delta_{j,0} + \lambda\delta_{j+t,0} - \lambda\delta_{j+t,0} = \delta_{j,0} \ .$$

□

Corollary 1
Subdivision operator h being fixed, let $\{\tilde{h}^i\}_{i\in\mathcal{I}}$ be a set of consistent decimation operators, a general consistent decimation operator can be constructed as

$$\sum_{t\in\mathcal{T}} \sum_{i\in\mathcal{I}} c_{i,t} T_{2t}(\tilde{h}^i) \tag{16}$$

with

$$\forall t \in \mathcal{T}, \sum_{i\in\mathcal{I}} c_{i,t} = \delta_{t,0},\ 0 \in \mathcal{T} \subset \mathbb{Z} \ .$$

3.2 Generic Approach

In this section, we focus on the construction of all the decimation operators consistent with a given subdivision operator. We first construct elementary operators with masks of minimal number of non zero values. Then, we show how all consistent decimation operators can be recovered using linear combinations of translated versions of elementary operators.

The two following propositions provide the construction of the elementary operators. Proposition 4 gives a generic approach to get consistent decimation operators for a subdivision mask of even or odd length. Then, Proposition 5 deals with further consideration for subdivision masks of odd length.

Proposition 4
Let h be a prediction operator constructed from the mask

$$M_h = \{h_{n-2\alpha}, h_{n-2\alpha+1}, \ldots, h_n, h_{n+1}\}$$

of length $2(\alpha+1)$ with $h_{n-2\alpha}h_{n+1} \neq 0$ or of length $2\alpha+1$ with $h_{n-2\alpha} = 0$ and $h_{n-2\alpha+1}h_{n+1} \neq 0$.
We note H_{M_h} the following matrix,

$$H_{M_h} = \begin{bmatrix} h_n & h_{n-2} & \cdots & h_{n-2\alpha} & 0 & \cdots & 0 \\ h_{n+1} & h_{n-1} & \cdots & h_{n-2\alpha+1} & 0 & \cdots & 0 \\ 0 & h_n & h_{n-2} & \cdots & h_{n-2\alpha} & \cdots & 0 \\ 0 & h_{n+1} & h_{n-1} & \cdots & h_{n-2\alpha+1} & \cdots & 0 \\ \vdots & & & \vdots & & & \\ 0 & 0 & \cdots & h_n & h_{n-2} & \cdots & h_{n-2\alpha} \\ 0 & 0 & \cdots & h_{n+1} & h_{n-1} & \cdots & h_{n-2\alpha+1} \end{bmatrix}.$$

If $det(H_{M_h}) \neq 0$, there exists 2α consistent elementary decimation operators which masks are of length not larger than 2α. These masks are given by each row of $H_{M_h}^{-1}$.

Proof
First, let us assume that M_h is of even length ($h_{n-2\alpha}h_{n+1} \neq 0$) and denote formally for any integer $m \in \mathbb{Z}$,

$$M_{\tilde{h}} = \{\tilde{h}_{n-m}, \tilde{h}_{n-m+1}, \ldots, \tilde{h}_{n-m+2\alpha-2}, \tilde{h}_{n-m+2\alpha-1}\},$$

the mask of a consistent decimation operator of length not larger than 2α. Here the parameter n controls the centering of the mask M_h. The parameter m is related to the shift between the masks M_h and $M_{\tilde{h}}$.

If $\tilde{h}$ is consistent with h then the consistency condition (15) is verified. It can be written as

$$[\tilde{h}_{n-m}, \tilde{h}_{n-m+1}, \ldots, \tilde{h}_{n-m+2\alpha-2}, \tilde{h}_{n-m+2\alpha-1}] \begin{bmatrix} h_{n-m-2j} \\ h_{n-m+1-2j} \\ \vdots \\ h_{n-m+2\alpha-2-2j} \\ h_{n-m+2\alpha-1-2j} \end{bmatrix} = \delta_{j,0} \quad . \tag{17}$$

To ensure that (17) makes sense with a given M_h, we should have

$$\{h_{n-m-2j}, h_{n-m+1-2j}, \dots, h_{n-m+2\alpha-1-2j}\} \bigcap \{h_{n-2\alpha}, h_{n-2\alpha+1}, \dots, h_{n+1}\} \neq \varnothing,$$

which means

$$\begin{cases} n-m+2\alpha-1-2j \geq n-2\alpha \\ n-m-2j \leq n+1 \end{cases} .$$

and leads to

$$-\frac{m+1}{2} \leq j \leq -\frac{m+1}{2} + 2\alpha \ .$$

When m is odd, (17) corresponds to $2\alpha+1$ linear equations for $j \in \{-\frac{m+1}{2}, \dots, -\frac{m+1}{2}+2\alpha\}$ including

$$\tilde{h}_{n-m} h_{n+1} = \delta_{m,-1} \quad \text{for} \quad j = \frac{m+1}{2}$$

and

$$\tilde{h}_{n-m+2\alpha-1} h_{n-2\alpha} = \delta_{m,4\alpha-1} \quad \text{for} \quad j = \frac{m+1}{2} + 2\alpha.$$

Since $h_{n+1}h_{n-2\alpha} \neq 0$, it necessarily leads to $\tilde{h}_{n-m}\tilde{h}_{n-m+2\alpha-1} = 0$. If $\tilde{h}_{n-m} = 0$, then $M_{\tilde{h}}$ is equivalent to $\{\tilde{h}_{n-m'}, \tilde{h}_{n-m'+1}, \dots, \tilde{h}_{n-m'+2\alpha-2}, \tilde{h}_{n-m'+2\alpha-1}\}$ where m' is even by considering

$$\{\tilde{h}_{n-m+1}, \dots, \tilde{h}_{n-m+2\alpha-2}, \tilde{h}_{n-m+2\alpha-1}, 0\},$$

and replacing $m-1$ by m'. The same kind of argument holds when $\tilde{h}_{n-m+2\alpha-1} = 0$ by considering

$$\{0, \tilde{h}_{n-m}, \dots, \tilde{h}_{n-m+2\alpha-2}, \tilde{h}_{n-m+2\alpha-2}\} \ .$$

Therefore, m can always be considered as even without losing generality. Since m is even, (17) leads to 2α linear equations for $j \in \{-\frac{m}{2}, -\frac{m}{2}+1, \dots, -\frac{m}{2}+2\alpha-1\}$ that can be written as

$$[\tilde{h}_{n-m}, \tilde{h}_{n-m+1}, \dots, \tilde{h}_{n-m+2\alpha-2}, \tilde{h}_{n-m+2\alpha-1}] H_{M_h} = [\delta_{m,0}, \delta_{m-2,0}, \dots, \delta_{m-4\alpha+2,0}]$$

with

$$H_{M_h} = \begin{bmatrix} h_n & h_{n-2} & \cdots & h_{n-2\alpha} & 0 & \cdots & 0 \\ h_{n+1} & h_{n-1} & \cdots & h_{n-2\alpha+1} & 0 & \cdots & 0 \\ 0 & h_n & h_{n-2} & \cdots & h_{n-2\alpha} & \cdots & 0 \\ 0 & h_{n+1} & h_{n-1} & \cdots & h_{n-2\alpha+1} & \cdots & 0 \\ \vdots & & & \vdots & & & \\ 0 & 0 & \cdots & h_n & h_{n-2} & \cdots & h_{n-2\alpha} \\ 0 & 0 & \cdots & h_{n+1} & h_{n-1} & \cdots & h_{n-2\alpha+1} \end{bmatrix},$$

where the column index corresponds to parameter j.

For $m \in \{0, 2, \ldots, 4\alpha - 4, 4\alpha - 2\}$, Eq. (17) can be written as

$$\tilde{H}_{M_h} H_{M_h} = I_{2\alpha}$$

with

$$\tilde{H}_{M_h} = \begin{bmatrix} M_{\tilde{h}^0} \\ M_{\tilde{h}^2} \\ \vdots \\ M_{\tilde{h}^{4\alpha-4}} \\ M_{\tilde{h}^{4\alpha-2}} \end{bmatrix} = \begin{bmatrix} \tilde{h}^0_n & \tilde{h}^0_{n+1} & \cdots & \tilde{h}^0_{n+2\alpha-1} \\ \tilde{h}^2_{n-2} & \tilde{h}^2_{n-1} & \cdots & \tilde{h}^2_{n+2\alpha-3} \\ \vdots & & \vdots & \\ \tilde{h}^{4\alpha-4}_{n-4\alpha+4} & \tilde{h}^{4\alpha-4}_{n-4\alpha+5} & \cdots & \tilde{h}^{4\alpha-4}_{n-2\alpha+3} \\ \tilde{h}^{4\alpha-2}_{n-4\alpha+2} & \tilde{h}^{4\alpha-2}_{n-4\alpha+3} & \cdots & \tilde{h}^{4\alpha-2}_{n-2\alpha+1} \end{bmatrix}.$$

Each row of $\tilde{H}_{M_{\tilde{h}}}$ corresponds to a value of m and to a consistent decimation operator. Note that, specifically for the elementary decimation operators defined above, the superscript k for $\tilde{h}^k$ controls the shift between M_h and $M_{\tilde{h}^k}$. Since $det(H_{M_h}) \neq 0$, $\tilde{H}_{M_h} = H^{-1}_{M_h}$, that concludes the proof when M_h is of even length.

In the case of subdivision mask of odd length, the same proof can be conducted assuming $h_{n-2\alpha} = 0$ and the same matrix $\tilde{H}_{M_h}$ can be deduced if $det(H_{M_h}) \neq 0$. □

When the subdivision masks are of even length, the previous proposition provides 2α consistent elementary decimation operators. When the subdivision masks are of odd length, it turns out that the last row of $\tilde{H}_{M_h}$ can be obtained by a linear combination of translated versions of the decimation masks associated with the other rows. This leads to only $2\alpha - 1$ elementary decimation operators. It is stated by the next proposition.

Proposition 5
Let h be a prediction operator constructed from the mask

$$M'_h = \{h_{n-2\alpha+1}, h_{n-2\alpha+2}, \ldots, h_n, h_{n+1}\}$$

of length $2\alpha + 1, \alpha \geq 2$ with $h_{n-2\alpha+1} h_{n+1} \neq 0$.
We note $H'_{M'_h}$ the following matrix

$$H'_{M'_h} = \begin{bmatrix} h_n & h_{n-2} & \cdots & 0 & 0 & \cdots & 0 \\ h_{n+1} & h_{n-1} & \cdots & h_{n-2\alpha+1} & 0 & \cdots & 0 \\ 0 & h_n & h_{n-2} & \cdots & 0 & \cdots & 0 \\ 0 & h_{n+1} & h_{n-1} & \cdots & h_{n-2\alpha+1} & \cdots & 0 \\ \vdots & & & \vdots & & & \\ 0 & 0 & \cdots & h_n & h_{n-2} & \cdots & h_{n-2\alpha+2} \end{bmatrix}.$$

If $det(H'_{M'_h}) \neq 0$, there exists $2\alpha - 1$ consistent elementary decimation operators which masks are of length not larger than $2\alpha - 1$. These masks are given by each row of $H'^{-1}_{M'_h}$.

Proof
Following Proposition 4, we construct a similar matrix with $h_{n-2\alpha} = 0$

$$H_{M'_h} = \begin{bmatrix} h_n & h_{n-2} & \cdots & 0 & 0 & \cdots & 0 \\ h_{n+1} & h_{n-1} & \cdots & h_{n-2\alpha+1} & 0 & \cdots & 0 \\ 0 & h_n & h_{n-2} & \cdots & 0 & \cdots & 0 \\ 0 & h_{n+1} & h_{n-1} & \cdots & h_{n-2\alpha+1} & \cdots & 0 \\ \vdots & & & \vdots & & & \\ 0 & 0 & \cdots & h_n & h_{n-2} & \cdots & 0 \\ 0 & 0 & \cdots & h_{n+1} & h_{n-1} & \cdots & h_{n-2\alpha+1} \end{bmatrix} .$$

Since $h_{n-2\alpha+1} \neq 0$ and $det(H'_{M'_h}) \neq 0$, we have $det(H_{M'_h}) \neq 0$ and we can introduce

$$\tilde{H}_{M'_h} = \begin{bmatrix} M_{\tilde{h}^0} \\ M_{\tilde{h}^2} \\ \vdots \\ M_{\tilde{h}^{4\alpha-4}} \\ M_{\tilde{h}^{4\alpha-2}} \end{bmatrix} = \begin{bmatrix} \tilde{h}^0_n & \tilde{h}^0_{n+1} & \cdots & \tilde{h}^0_{n+2\alpha-2} & 0 \\ \tilde{h}^2_{n-2} & \tilde{h}^2_{n-1} & \cdots & \tilde{h}^2_{n+2\alpha-4} & 0 \\ \vdots & & \vdots & & \vdots \\ \tilde{h}^{4\alpha-4}_{n-4\alpha+4} & \tilde{h}^{4\alpha-4}_{n-4\alpha+5} & \cdots & \tilde{h}^{4\alpha-4}_{n-2\alpha+2} & 0 \\ \tilde{h}^{4\alpha-2}_{n-4\alpha+2} & \tilde{h}^{4\alpha-2}_{n-4\alpha+3} & \cdots & \tilde{h}^{4\alpha-2}_{n-2\alpha} & \tilde{h}^{4\alpha-2}_{n-2\alpha+1} \end{bmatrix}$$

with $\tilde{H}_{M'_h} = H^{-1}_{M'_h}$.

Note that the last row of $\tilde{H}_{M'_h}$ denoted $M_{\tilde{h}^{4\alpha-2}}$ is the only mask with a non-zero last term. Therefore $\tilde{h}^{4\alpha-2}_{n-2\alpha+1} \neq 0$ according to the consistency condition.

In the sequel, we show that the last row of $\tilde{H}_{M'_h}$ can be obtained by linear combinations of the translated versions of the above ones.

First, note that the set

$$\{\tilde{h}^0_{n+2\alpha-2}, \tilde{h}^2_{n+2\alpha-4}, \ldots, \tilde{h}^{4\alpha-6}_{n-2\alpha+4}\}$$

has at least one non-zero term, otherwise, according to the consistency condition, all terms in

$$\{\tilde{h}^0_{n+2\alpha-3}, \tilde{h}^2_{n+2\alpha-5}, \ldots, \tilde{h}^{4\alpha-6}_{n-2\alpha+3}\}$$

would be also zero which implies $det(\tilde{H}_{M'_h}) = 0$.

So there exists $\tilde{h}^{4\alpha-4-2t}_{n-2\alpha+2+2t} \neq 0$ for $t \in \{1, 2, \ldots, 2\alpha-2\}$. Introducing $\lambda = \tilde{h}^{4\alpha-4}_{n-2\alpha+2} / \tilde{h}^{4\alpha-4-2t}_{n-2\alpha+2+2t}$, we note

$$\tilde{h}^* = \tilde{h}^{4\alpha-4} + \lambda T_{-2t}(\tilde{h}^{4\alpha-2-2t}) - \lambda T_{-2t}(\tilde{h}^{4\alpha-4-2t})$$

which can have non-zero value from index $n-4\alpha+2$ to $n-2\alpha+2$. Calculating the last term gives

$$\tilde{h}^*_{n-2\alpha+2} = \tilde{h}^{4\alpha-4}_{n-2\alpha+2} + \lambda \tilde{h}^{4\alpha-2-2t}_{n-2\alpha+2+2t} - \lambda \tilde{h}^{4\alpha-4-2t}_{n-2\alpha+2+2t} = 0 \ .$$

It means that $\tilde{h}^*$ can have non-zero value from index $n-4\alpha+2$ to $n-2\alpha+1$.

Since $det(H_{M'_h}) \neq 0$, $\tilde{h}^{4\alpha-2}$ is the unique consistent operator with a mask of length not larger than 2α and admitting non-zero values from index $n-4\alpha+2$ to $n-2\alpha+1$. It therefore implies that $\tilde{h}^* = \tilde{h}^{4\alpha-2}$.

Thus, eliminating the last row and column of $H_{M'_h}$, we construct the matrix

$$H'_{M'_h} = \begin{bmatrix} h_n & h_{n-2} & \cdots & 0 & 0 & \cdots & 0 \\ h_{n+1} & h_{n-1} & \cdots & h_{n-2\alpha+1} & 0 & \cdots & 0 \\ 0 & h_n & h_{n-2} & \cdots & 0 & \cdots & 0 \\ 0 & h_{n+1} & h_{n-1} & \cdots & h_{n-2\alpha+1} & \cdots & 0 \\ \vdots & & & \vdots & & & \\ 0 & 0 & \cdots & h_n & h_{n-2} & \cdots & h_{n-2\alpha+2} \end{bmatrix}.$$

Since $det(H'_{M'_h}) \neq 0$, we then get elementary consistent decimation operator masks by considering the rows of $\tilde{H}'_{M'_h} = H'^{-1}_{M'_h}$. □

By construction, $\tilde{H}'_{M'_h}$ in Proposition 5 is a $(2\alpha-1)\times(2\alpha-1)$ sub matrix of $\tilde{H}_{M_h}$ introduced in Proposition 4. Therefore, for any subdivision scheme, $\tilde{H}_{M_h}$ provides all consistent elementary decimation masks. However, in practice, in the case of odd length (more than 3), the elementary operators will be obtained by restricting to $\tilde{H}'_{M'_h}$ that is denoted for simplicity $\tilde{H}'_{M_h}$ since a subdivision mask of odd length can be considered as a mask of even length by adding zero in front or behind.

The previous elementary decimation operators can then be used to construct any consistent decimation operator. Indeed we have the following proposition.

Proposition 6
Given a subdivision scheme h satisfying the hypotheses of Propositions 4 or 5, combining elementary decimation operators with formula (16) generates all the consistent decimation operators.

Proof
Let's first consider $M_h = \{h_{n-2\alpha}, h_{n-2\alpha+1}, \ldots, h_n, h_{n+1}\}$ the mask of a given operator h of length $2(\alpha+1)$ with $h_{n-2\alpha}h_{n+1} \neq 0$. Then, Proposition 4 provides 2α consistent elementary decimation operators of length 2α which can be denoted as

$$M_{\tilde{h}^{2i}} = \{\tilde{h}^{2i}_{n-2i}, \tilde{h}^{2i}_{n-2i+1}, \ldots, \tilde{h}^{2i}_{n-2i+2\alpha-2}, \tilde{h}^{2i}_{n-2i+2\alpha-1}\}$$

with $i = 0, 1, 2, \ldots, 2\alpha-1$.

Let $M_{\tilde{h}} = \{\tilde{h}_{m-2\beta}, \tilde{h}_{m-2\beta+1}, \ldots, \tilde{h}_m, \tilde{h}_{m+1}\}$ be the mask of an arbitrary decimation operator $\tilde{h}$ consistent with h. The length of $M_{\tilde{h}}$, $2\beta+2$, is supposed to be larger than 2α that is to say $\beta > \alpha - 1$, otherwise $\tilde{h}$ is an elementary operator itself and the proof is completed. Moreover, m in $M_{\tilde{h}}$ is always chosen to ensure $n-m$ even by assuming that $\tilde{h}_{m-2\beta}$ and $\tilde{h}_{m+1}$ can be zero. However, $\{\tilde{h}_{m-2\beta}, \tilde{h}_{m-2\beta+1}\} \neq \{0,0\}$ and $\{\tilde{h}_m, \tilde{h}_{m+1}\} \neq \{0,0\}$ are always guaranteed.

The consistency of h and $\tilde{h}$ implies directly $n-2\alpha-1 < m < n+2\beta+1$.

The aim is to prove that $\tilde{h}$ can be represented as a linear combination of translated version of $(\tilde{h}^{2i})_{0\leq i\leq 2\alpha-1}$.

This will be proved in two steps. The first step consists in writing $\tilde{h}$ as the sum of a term involving some $\tilde{h}^{2i}$ or its translated versions and of another one denoted $\tilde{h}^*$ that is a consistent decimation operator with a shorter mask than $M_{\tilde{h}}$. The second step is an iteration of this process until $\tilde{h}^*$ is an elementary decimation operator. We restrict the proof to the first step since the second one is straightforward.

The starting point is the consistency condition (15). Considering $j = \frac{m-n}{2} - \beta$ and $j = \frac{m-n}{2} + \alpha$, it leads to

$$h_n\tilde{h}_{m-2\beta} + h_{n+1}\tilde{h}_{m-2\beta+1} = \delta_{\frac{m-n}{2}-\beta,0} \ , \tag{18}$$

$$h_{n-2\alpha}\tilde{h}_m + h_{n-2\alpha+1}\tilde{h}_{m+1} = \delta_{\frac{m-n}{2}+\alpha,0} \ . \tag{19}$$

According to α and β, at least one of the two above RHS term is equal to zero. Let us suppose that the RHS of (18) is zero, i.e.

$$h_n\tilde{h}_{m-2\beta} + h_{n+1}\tilde{h}_{m-2\beta+1} = 0. \tag{20}$$

Since $h_{n+1} \neq 0$, we cannot have $\tilde{h}^{2i}_{n-2i} = 0$ for all $i \in \{1, 2, \ldots, 2\alpha - 1\}$ according to the consistency condition.

Let us introduce the two following operators with a mask of length 2α with $\tilde{h}'_{m-2\beta} \neq 0$,

$$M_{\tilde{h}'} = \{\tilde{h}'_{m-2\beta}, \tilde{h}'_{m-2\beta+1}, \ldots, \tilde{h}'_{2\alpha-2\beta+m-2}, \tilde{h}'_{2\alpha-2\beta+m-1}\} = T_{m-2\beta-n+2i}(\tilde{h}^{2i}),$$

$$M_{\tilde{h}''} = \{\tilde{h}''_{m-2\beta+2}, \tilde{h}''_{m-2\beta+3}, \ldots, \tilde{h}''_{2\alpha-2\beta+m}, \tilde{h}''_{2\alpha-2\beta+m+1}\} = T_{m-2\beta-n+2i}(\tilde{h}^{2i-2})$$

which are elementary operators with the same translation. The consistency condition implies

$$h_n\tilde{h}'_{m-2\beta} + h_{n+1}\tilde{h}'_{m-2\beta+1} = 0. \tag{21}$$

Considering (20) and (21), $\tilde{h}_{m-2\beta} = 0$ leads to $\tilde{h}_{m-2\beta+1} = 0$ which is not allowed. Moreover, $\tilde{h}_{m-2\beta+1} = 0$ implies $h_n = 0$ and then $\tilde{h}'_{m-2\beta+1} = 0$. Therefore there exists $\lambda \in \mathbb{R}/\{0\}$ such that

$$\lambda[\tilde{h}'_{m-2\beta}, \tilde{h}'_{m-2\beta+1}] = [\tilde{h}_{m-2\beta}, \tilde{h}_{m-2\beta+1}] \ . \tag{22}$$

According to Proposition 3, $\tilde{h}^* = \tilde{h} - \lambda\tilde{h}' + \lambda\tilde{h}''$ is consistent with h. Since $M_{\tilde{h}}$ has length $2\beta + 2$, M_{h^*} has length 2β from index $m - 2\beta + 2$ to $m + 1$.

If $\beta = \alpha$, $M_{\tilde{h}^*}$ and $M_{\tilde{h}''}$ have the same length and indices. According to Proposition 4, $\tilde{h}^* = \tilde{h}''$ and

$$\tilde{h} = \lambda\tilde{h}' + (1 - \lambda)\tilde{h}'' \ ,$$

which leads to the expected result with a zero second term.

If $\beta > \alpha$, $M_{\tilde{h}^*}$ has a shorter length than $M_{\tilde{h}}$ and

$$\tilde{h} = \lambda\tilde{h}' - \lambda\tilde{h}'' + \tilde{h}^* \ ,$$

that allows us to iterate by replacing $\tilde{h}$ with $\tilde{h}^*$ and then to conclude.

The above process actually eliminate the first two terms $\tilde{h}_{m-2\beta}, \tilde{h}_{m-2\beta+1}$ of $M_{\tilde{h}}$ using elementary decimation operators. If we suppose that the RHS of (19) is zero, a symmetrical similar process can be performed and the last two terms $\tilde{h}_m, \tilde{h}_{m+1}$ of $M_{\tilde{h}}$ will be eliminated.

To complete the proof in the case of subdivision mask of odd length, we suppose $h_{n+1} = 0$ in (20) and (21). It is straightforward that $\tilde{h}_{m-2\beta} = 0$, $\tilde{h}_{m-2\beta+1} \neq 0$ and then $\tilde{h}'_{m-2\beta} = 0$. Moreover, introducing $M_{h'}$ and $M_{h''}$ with $\tilde{h}'_{m-2\beta} = 0$ and $\tilde{h}'_{m-2\beta+1} \neq 0$, there exists $\lambda \neq 0$ verifying (22). □

Coming back to the multi-scale framework, the next section is devoted to a theoretical study of the prediction error involving consistent subdivision and decimation operators.

4 Analysis of the Prediction Error

The following result holds:

Proposition 7
Let $\{(V^j, h, \tilde{h})\}_{j\in\mathbb{Z}}$ define a multi-scale analysis with h a prediction operator constructed from the real sequence $(h_k)_{k\in\mathbb{Z}}$ and $\tilde{h}$ a decimation operator constructed from the real sequence $(\tilde{h}_k)_{k\in\mathbb{Z}}$. We assume that the associated multi-scale transform is applied from a fine scale J_{max} to a coarse one J_0.
If there exists $L \in \mathbb{N}$ such that $\forall n \in \{0, 1, \ldots, L\}$,

$$\forall k \in \mathbb{Z},\ \sum_{l\in\mathbb{Z}} h_{k-2l} \sum_{m\in\mathbb{Z}} \tilde{h}_{m-2l}(m-k)^n = \delta_{n,0} \tag{23}$$

then for sufficiently large $j \in [J_0, J_{max} - 1]$,

$$||e^j|| \leq C 2^{-(L+1)j}, \tag{24}$$

where C does not depend on j.

Proof
Condition (23) is equivalent to

$$\forall 1 \leq n \leq L, \quad k^n = \sum_l h_{k-2l} \sum_m \tilde{h}_{m-2l} m^n,$$

and implies that

$$\forall j \in \mathbb{Z}, \forall 1 \leq n \leq L, \quad (k2^{-j})^n - \sum_l h_{k-2l} \sum_m \tilde{h}_{m-2l}(m2^{-j})^n = 0 \ . \tag{25}$$

Moreover, for any j, one can introduce $f_j \in C^{L_0}(\mathbb{R})$ with $L_0 >> L$ such that $f_k^j = f_j(k2^{-j})$. We postpone to the end of the proof the construction of a particular f_j to get the expected result of the proposition.

Using Taylor expansion, it then comes out,

$$f_k^j = f_j(k2^{-j}) = \sum_{n=0}^{L+1} \frac{1}{n!} f_j^{(n)}(0)(k2^{-j})^n + o((2^{-j})^{L+1})$$

and the prediction error (2) can be rewritten

$$e_k^j = \sum_{n=1}^{L+1} \frac{1}{n!} f_j^{(n)}(0)\big((k2^{-j})^n - \sum_l h_{k-2l} \sum_m \tilde{h}_{m-2l}(m2^{-j})^n\big) + o((2^{-j})^{L+1}).$$

According to (25),

$$e_k^j = \frac{1}{(L+1)!} f_j^{(L+1)}(0)\big((k2^{-j})^{L+1} - \sum_l h_{k-2l} \sum_m \tilde{h}_{m-2l}(m2^{-j})^{L+1}\big) + o(2^{-j(L+1)}). \quad (26)$$

To finish the proof we introduce a particular f_j such that $\forall i \leq L_0$, $||f_j^{(i)}||_\infty$ is controlled independently of j that leads to a constant C independent of j such that $||e^j|| \leq C2^{-j(L+1)}$.

For any $j \in [J_0, J_{max} - 1]$, f_j is constructed from $f_{J_{max}} \in C^{L_0}(\mathbb{R})$.

More precisely, starting from $(f_k^{J_{max}})_{k\in\mathbb{Z}}$ with $f_k^{J_{max}} = f_{J_{max}}(k2^{-J_{max}})$, $(f_k^{J_{max}-1})_{k\in\mathbb{Z}}$ is written

$$f_k^{J_{max}-1} = (D_{J_{max}}^{J_{max}-1} f^{J_{max}})_k = \sum_{l\in\mathbb{Z}} \tilde{h}_{l-2k} f_l^{J_{max}} = \sum_{l\in\mathbb{Z}} \tilde{h}_l f_{l+2k}^{J_{max}} = \sum_{l\in\mathbb{Z}} \tilde{h}_l f_{J_{max}}((l+2k)2^{-J_{max}}),$$

$f_{J_{max}-1}$ can therefore be defined as $\forall x \in \mathbb{R}$

$$f_{J_{max}-1}(x) = \sum_l \tilde{h}_l f_{J_{max}}(l2^{-J_{max}} + x),$$

and it is straightforward that $\forall i \leq L_0$,

$$||f_{J_{max}-1}^{(i)}||_\infty \leq (\sum_l |\tilde{h}_l|)||f_{J_{max}}^{(i)}||_\infty \quad . \quad (27)$$

Iterating this process, $\forall j \in [J_0, J_{max} - 1]$,

$$||f_j^{(i)}||_\infty \leq (\sum_l |\tilde{h}_l|)^{J_{max}-j}||f_{J_{max}}^{(i)}||_\infty \quad . \quad (28)$$

Since $\sum_l |\tilde{h}_l| \geq 1$,

$$||f_j^{(i)}||_\infty \leq (\sum_l |\tilde{h}_l|)^{J_{max}-J_0}||f_{J_{max}}^{(i)}||_\infty,$$

and (26) leads to

$$||e^j|| \leq C2^{-(L+1)j}$$

with C independent of j. □

Remark 2
The bound in Inequality (27) depends on the L^∞ norm of the decimation operator that is also a key quantity controlling the stability of the multi-scale decomposition (Expression (7) of Proposition 1). Therefore, an heuristic criterion to select a decimation scheme after applying our generic approach can be based on the L^∞ norm of its mask. An optimal choice corresponds to a minimal norm equal to 1 that leads to a bound in Inequality (28) independent of J_{max} and J_0. We show in Sect. 5 some numerical examples related to the stability of the decimation operators associated to classical subdivision schemes.

Condition (23) involves both subdivision and decimation masks. Under the consistency Condition (15), it can be reformulated in a simpler way.

Proposition 8
Let h and $\tilde{h}$ be two consistent operators. Condition (23) is satisfied if and only if $\forall n \in \{0, 1, 2, \ldots, L\}$

$$\sum_{l\in\mathbb{Z}} h_{2l}(2l)^n = \sum_{l\in\mathbb{Z}} h_{2l+1}(2l+1)^n \quad . \tag{29}$$

Proof
Let us first introduce the following notations,

$E^n_{e,e} = \sum_l h_{2l} \sum_k \tilde{h}_{2k}(2k-2l)^n,$
$E^n_{o,o} = \sum_l h_{2l+1} \sum_k \tilde{h}_{2k+1}(2k-2l)^n,$
$E^n_{e,o} = \sum_l h_{2l} \sum_k \tilde{h}_{2k+1}(2k+1-2l)^n,$
$E^n_{o,e} = \sum_l h_{2l+1} \sum_k \tilde{h}_{2k}(2k-2l-1)^n.$

Condition (23) becomes $\forall 0 \le n \le L$,

$$E^n_{e,e} + E^n_{e,o} = \delta_{n,0},\; E^n_{o,e} + E^n_{o,o} = \delta_{n,0} \quad . \tag{30}$$

First we will prove that, the consistency constraint implies

$$\forall n \in \mathbb{N},\; E^n_{e,e} + E^n_{o,o} = \delta_{n,0} \quad . \tag{31}$$

It is easy to verify that for $n = 0$, $E^0_{e,e} + E^0_{o,o} = 1$. Moreover, for any $n \in \mathbb{N}^*$, the consistency condition leads to

$$\sum_j (\sum_i h_{i-2j}\tilde{h}_i)(2j)^n = \sum_j \delta_{j,0}(2j)^n,$$

$$\sum_i (\sum_j h_{i-2j}(2j)^n)\tilde{h}_i = 0 \quad .$$

Splitting the previous sum with respect to even and odd indices, we get

$$\sum_i (\sum_j h_{2i-2j}(2j)^n)\tilde{h}_{2i} + \sum_i (\sum_j h_{2i+1-2j}(2j)^n)\tilde{h}_{2i+1} = 0,$$

$$\sum_l h_{2l} \sum_k \tilde{h}_{2k}(2k-2l)^n + \sum_l h_{2l+1} \sum_k \tilde{h}_{2k+1}(2k-2l)^n = 0,$$

which is precisely,

$$E^n_{e,e} + E^n_{o,o} = 0 \ .$$

Considering (30) and (31), Condition (23) is then equivalent to $\forall n \in \{0, 1, 2, \dots, L\}$

$$\sum_l h_{2l} \sum_k \tilde{h}_{2k}(2k-2l)^n = \sum_l h_{2l+1} \sum_k \tilde{h}_{2k}(2k-2l-1)^n,$$

$$\sum_l h_{2l} \sum_k \tilde{h}_{2k+1}(2k+1-2l)^n = \sum_l h_{2l+1} \sum_k \tilde{h}_{2k+1}(2k-2l)^n,$$

which can be written as

$$\sum_l h_{2l} \sum_k \tilde{h}_{2k} \sum_{i=0}^{n} \binom{n}{i} (-1)^i (2k)^{n-i} (2l)^i = \sum_l h_{2l+1} \sum_k \tilde{h}_{2k} \sum_{i=0}^{n} \binom{n}{i} (-1)^i (2k)^{n-i} (2l+1)^i,$$

$$\sum_l h_{2l} \sum_k \tilde{h}_{2k+1} \sum_{i=0}^{n} \binom{n}{i} (-1)^i (2k+1)^{n-i} (2l)^i = \sum_l h_{2l+1} \sum_k \tilde{h}_{2k+1} \sum_{i=0}^{n} \binom{n}{i} (-1)^i (2k+1)^{n-i} (2l+1)^i,$$

leading to

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i (\sum_l h_{2l}(2l)^i - \sum_l h_{2l+1}(2l+1)^i) \sum_k \tilde{h}_{2k}(2k)^{n-i} = 0,$$

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i (\sum_l h_{2l}(2l)^i - \sum_l h_{2l+1}(2l+1)^i) \sum_k \tilde{h}_{2k+1}(2k+1)^{n-i} = 0,$$

which is equivalent to (29). □

In fact, (29) in Proposition 8 and (23) in Proposition 7 are related to polynomial quasi-reproduction and to polynomial reproduction. Full discussion on these topics is beyond the scope of this paper.

5 Examples and Applications

Several applications of the previous results are provided in this section. They are focussed on different goals. The first one is to revisit, with our generic approach, classical decimation operators associated with well-known subdivision schemes (Sects. 5.1, 5.2 and 5.3). A special attention is devoted to the stability of the generated decimations. We illustrate how our approach can be used to improve stability constants. The second one is to apply our method to the practical case of a newly developed subdivision scheme for which there is no available decimation operator in the literature (Sect. 5.4).

5.1 Lagrange Subdivision

Lagrange interpolation provides interpolatory subdivision as follows. Given $(l, r) \in \mathbb{N}^{*2}$ and introducing for each value of k, $\sum_{m=-l+1}^{r} L_m(x) f^j_{k+m}$, the polynomial of degree $(l+r-1)$ that interpolates the points $((m, f^j_{k+m}), -l+1 \leq m \leq r)$, the subdivision is given by

$$\begin{cases} f^{j+1}_{2k} &= f^j_k, \\ f^{j+1}_{2k+1} &= \sum_{m=-l+1}^{r} L_m(\frac{1}{2}) f^j_{k+m}. \end{cases}$$

Here, $\{L_m(x)\}_{m=-l+1,\ldots,r}$ stands for the Lagrange functions associated to the stencil $\{-l+1, \ldots, r\}$. The coefficients of the subdivision mask are then given by $h_0 = 1$ and $h_{2i+1} = L_{-i}(1/2), -r \leq i \leq l-1$.

We then have

Proposition 9
Given $(l, r) \in \mathbb{N}^{*2}$, the Lagrange subdivision mask satisfies

$$\sum_{i=-r}^{l-1} h_{2i}(2i)^n = \sum_{i=-r}^{l-1} h_{2i+1}(2i+1)^n = \delta_{n,0}, \quad n = 0, 1, \ldots, l+r-1, \tag{32}$$

which can be regarded as an enhanced condition (29) with $L = l+r-1$. Moreover, it is the only family $\{h_k\}_{k=-2r,\ldots,2l-1}$ satisfying (32).

Proof
Since $h_{2i} = \delta_{i,0}$ the left hand side term of (32) is $\delta_{n,0}$. Moreover, using the interpolatory property of the Lagrange subdivision for polynomials of degree less than n gives directly that $\sum_{i=-r}^{l-1} h_{2i+1}(2i+1)^n = \delta_{n,0}\ \forall n \in \{0, 1, 2, \ldots, l+r-1\}$. To prove uniqueness, we rewrite (32) as

$$A \begin{bmatrix} h_{-2r} \\ h_{-2r+2} \\ \vdots \\ h_{2l-4} \\ h_{2l-2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad \textit{and} \quad B \begin{bmatrix} h_{-2r+1} \\ h_{-2r+3} \\ \vdots \\ h_{2l-3} \\ h_{2l-1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

with

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ -2r & -2r+2 & \cdots & 2l-4 & 2l-2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (-2r)^{l+r-2} & (-2r+2)^{l+r-2} & \cdots & (2l-4)^{l+r-2} & (2l-2)^{l+r-2} \\ (-2r)^{l+r-1} & (-2r+2)^{l+r-1} & \cdots & (2l-4)^{l+r-1} & (2l-2)^{l+r-1} \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ -2r+1 & -2r+3 & \cdots & 2l-3 & 2l-1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (-2r+1)^{l+r-2} & (-2r+3)^{l+r-2} & \cdots & (2l-3)^{l+r-2} & (2l-1)^{l+r-2} \\ (-2r+1)^{l+r-1} & (-2r+3)^{l+r-1} & \cdots & (2l-3)^{l+r-1} & (2l-1)^{l+r-1} \end{bmatrix}.$$

Since A and B are invertible Vandermonde matrices the uniqueness is proved. □

Example

The 4-point centred Lagrange interpolating subdivision scheme is given by

$$\begin{cases} f_{2k}^{j+1} &= f_k^j \\ f_{2k+1}^{j+1} &= -\frac{1}{16}f_{k-1}^j + \frac{9}{16}f_k^j + \frac{9}{16}f_{k+1}^j - \frac{1}{16}f_{k+2}^j \end{cases}$$

corresponding to

$$M_h = \{h_3, h_2, h_1, h_0, h_{-1}, h_{-2}, h_{-3}, h_{-4}\} = \{-\frac{1}{16}, 0, \frac{9}{16}, 1, \frac{9}{16}, 0, -\frac{1}{16}, 0\} \ .$$

Applying Proposition 5 gives

$$\tilde{H}'_{M_h} = \begin{bmatrix} \tilde{h}_2^0 & \tilde{h}_3^0 & \tilde{h}_4^0 & \tilde{h}_5^0 & \tilde{h}_6^0 \\ \tilde{h}_0^2 & \tilde{h}_1^2 & \tilde{h}_2^2 & \tilde{h}_3^2 & \tilde{h}_4^2 \\ \tilde{h}_{-2}^4 & \tilde{h}_{-1}^4 & \tilde{h}_0^4 & \tilde{h}_1^4 & \tilde{h}_2^4 \\ \tilde{h}_{-4}^6 & \tilde{h}_{-3}^6 & \tilde{h}_{-2}^6 & \tilde{h}_{-1}^6 & \tilde{h}_0^6 \\ \tilde{h}_{-6}^8 & \tilde{h}_{-5}^8 & \tilde{h}_{-4}^8 & \tilde{h}_{-3}^8 & \tilde{h}_{-2}^8 \end{bmatrix} = \begin{bmatrix} 9 & -16 & 9 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 9 & -16 & 9 \end{bmatrix} .$$

Remark 3

The five elementary decimation operators defined above correspond to sub-sampling (lines 2, 3 *and* 4 *of* $\tilde{H}'_{M_h}$*) and polynomial extrapolations of degree* 3 *(from the positions* $(0, .5, 1, 2)$ *to* $x = -1$ *for line* 1 *and from the positions* $(-2, -1, -.5, 0)$ *to* $x = 1$ *for line 5). Note that as for all interpolatory subdivision, sub-sampling provides an optimally stable decimation with* $\sum_l |\tilde{h}_l| = 1$.

5.2 Compactly Supported Wavelet Subdivision

Wavelets and, more precisely, scaling functions for multi-resolutions [8], are known to provide subdivision operators.

Orthogonality and zero moment conditions translate on the scaling coefficient $M_{h'} = \{h'_0, h'_1, \ldots, h'_{2N-1}\}$ as [8]

$$\begin{cases} \sum_i h'_i h'_{i+2j} = 2\delta_{j,0} \\ \sum_i (-1)^i h'_i i^p = 0 \end{cases} \tag{33}$$

for $j \in \mathbb{Z}, p = 0, 1, \ldots, N-1$.

According to orthogonal compactly supported wavelet theory, the rescaled operators $h = \sqrt{2}h'$ and $\tilde{h} = \frac{1}{\sqrt{2}}h'$ are consistent subdivision/decimation operators. More precisely, compact support wavelets of length $2N$ constructed in [8] lead to the unique subdivision/decimation operators with the same mask (up to $\sqrt{2}$ rescaling) with exponential decay of the error (Proposition 7) corresponding to $L = N - 1$.

Example
For $N = 2$ we get from [8]

$$[h'_0 \quad h'_1 \quad h'_2 \quad h'_3] = [\frac{1+\sqrt{3}}{4\sqrt{2}} \quad \frac{3+\sqrt{3}}{4\sqrt{2}} \quad \frac{3-\sqrt{3}}{4\sqrt{2}} \quad \frac{1-\sqrt{3}}{4\sqrt{2}}] \ .$$

Applying Proposition 4 for $h = \sqrt{2}h'$ we get

$$\tilde{H}_{M_h} = \begin{bmatrix} \tilde{h}_2^0 & \tilde{h}_3^0 \\ \tilde{h}_0^2 & \tilde{h}_1^2 \end{bmatrix} = H_{M_h}^{-1} = \begin{bmatrix} \frac{3-\sqrt{3}}{4} & \frac{1+\sqrt{3}}{4} \\ \frac{1-\sqrt{3}}{4} & \frac{3+\sqrt{3}}{4} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{3+\sqrt{3}}{4} & -\frac{1+\sqrt{3}}{4} \\ \frac{-1+\sqrt{3}}{4} & \frac{3-\sqrt{3}}{4} \end{bmatrix}$$

and therefore two elementary decimation operators $\tilde{h}^0$ and $\tilde{h}^2$.

For $\lambda = \frac{1}{2}\frac{\sqrt{3}-1}{\sqrt{3}+1}$, the linear combination

$$\begin{aligned}[\tilde{h}_0 \quad \tilde{h}_1 \quad \tilde{h}_2 \quad \tilde{h}_3] &= \lambda[0 \quad 0 \quad \tilde{h}_2^0 \quad \tilde{h}_3^0] + (1-\lambda)[\tilde{h}_0^2 \quad \tilde{h}_1^2 \quad 0 \quad 0] \\ &= [\frac{1+\sqrt{3}}{8} \quad \frac{3+\sqrt{3}}{8} \quad \frac{3-\sqrt{3}}{8} \quad \frac{1-\sqrt{3}}{8}]\end{aligned}$$

provides $\tilde{h} = \frac{1}{\sqrt{2}}h'$. We also get that (29) is verified for $n \le N-1 = 1$.

5.3 B-Spline Subdivision

It is well known that the scaling relation satisfied by the B-spline basis functions of order m, $B_m(t) = \sum_k h_k^m B_m(2t-k)$ implies that any spline function $C(t) = \sum_k f_k^j B_m(2^j t - k)$ can also be written as $C(t) = \sum_k f_k^{j+1} B_m(2^{j+1}t - k)$ with

$$f_k^{j+1} = \sum_l h_{k-2l}^m f_k^j \quad k \in \mathbb{Z} \ .$$

From the definition $B_{m+1}(t) = \int_{t-1}^t B_m(\tau)d\tau$ and $B_0(t) = \chi_{[0,1]}$ with χ_ω the characteristic function of the domain ω, it follows that the mask of the B-spline subdivision of degree m is given by

$$h_k^m = \frac{1}{2^m}\binom{m+1}{k}, \quad k = 0, 1, \ldots, m+1 \ . \tag{34}$$

We then have the following

Proposition 10
Given $m \in \mathbb{N}$, the mask of the B-spline subdivision scheme satisfies condition (29) with $L = m$, i.e.

$$\sum_{l=0}^{\lceil m/2 \rceil} h_{2l}(2l)^n = \sum_{l=0}^{\lfloor m/2 \rfloor} h_{2l+1}(2l+1)^n, \quad n = 0, 1, \ldots, m \ . \tag{35}$$

Moreover, it is the only mask $\{h_k\}_{0 \le k \le m+1}$ satisfying (29) and (10) with $L = m$.

Proof
It is easy to verify by induction that

$$\sum_{k=0}^{m} \binom{m}{k} k^n (-1)^k = 0, \quad \forall n \leq m-1,$$

and splitting the previous sum with respect to even and odd indices leads to (35) where the mask is given by (34).
To prove uniqueness, we rewrite (35) incorporating (10) as

$$A \begin{bmatrix} h_0 \\ -h_1 \\ h_2 \\ -h_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad \textit{with} \quad A = \begin{bmatrix} 1 & -1 & 1 & -1 & \cdots \\ 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 2 & 3 & \cdots \\ 0 & 1^2 & 2^2 & 3^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix} .$$

After a short calculus, the determinant of A can be written as the sum of strictly positive determinants of Vandermonde matrices. Therefore, $det(A) \neq 0$, that concludes the proof. □

Example 1
For $m = 3$ we get

$$\begin{cases} f_{2k}^{j+1} &= \frac{3}{4} f_k^j + \frac{1}{4} f_{k+1}^j \\ f_{2k+1}^{j+1} &= \frac{1}{4} f_k^j + \frac{3}{4} f_{k+1}^j \end{cases}$$

and $M_h = \{h_1, h_0, h_{-1}, h_{-2}\} = \{\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{4}\}$. We obtain two elementary decimation masks from

$$\tilde{H}_{M_h} = \begin{bmatrix} \tilde{h}_0^0 & \tilde{h}_1^0 \\ \tilde{h}_{-2}^2 & \tilde{h}_{-1}^2 \end{bmatrix} = H_{M_h}^{-1} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

and other solutions for any $\lambda \in \mathbb{R}$ as

$$\begin{aligned} [\tilde{h}_{-2} \quad \tilde{h}_{-1} \quad \tilde{h}_0 \quad \tilde{h}_1] &= \lambda [0 \quad 0 \quad \tilde{h}_0^0 \quad \tilde{h}_1^0] + (1-\lambda)[\tilde{h}_{-2}^2 \quad \tilde{h}_{-1}^2 \quad 0 \quad 0] \\ &= [-\tfrac{1}{2}(1-\lambda) \quad \tfrac{3}{2}(1-\lambda) \quad \tfrac{3}{2}\lambda \quad -\tfrac{1}{2}\lambda] \end{aligned} .$$

Remark 4
For splines of order 3*, the subdivision given above coincides with the values at* $1/4$ *and* $3/4$ *of the polynomial of degree* 1 *interpolating* $(0, f_k^j), (1, f_{k+1}^j)$*. The elementary decimations correspond to the extrapolations from the right (first line of* $H_{M_h}^{-1}$*) or from the left (second line of* $H_{M_h}^{-1}$*).*

Example 2
For splines of order 5 we get

$$\begin{cases} f_{2k}^{j+1} &= \frac{5}{16} f_{k-1}^j + \frac{5}{8} f_k^j + \frac{1}{16} f_{k+1}^j \\ f_{2k+1}^{j+1} &= \frac{1}{16} f_{k-1}^j + \frac{5}{8} f_k^j + \frac{5}{16} f_{k+1}^j \end{cases}$$

and $M_h = \{h_3, h_2, h_1, h_0, h_{-1}, h_{-2}\} = \{\frac{1}{16}, \frac{5}{16}, \frac{5}{8}, \frac{5}{8}, \frac{5}{16}, \frac{1}{16}\}$. We obtain four elementary decimation masks from

$$\tilde{H}_{M_h} = \begin{bmatrix} \tilde{h}_0^0 & \tilde{h}_1^0 & \tilde{h}_2^0 & \tilde{h}_3^0 \\ \tilde{h}_{-2}^2 & \tilde{h}_{-1}^2 & \tilde{h}_0^2 & \tilde{h}_1^2 \\ \tilde{h}_{-4}^4 & \tilde{h}_{-3}^4 & \tilde{h}_{-2}^4 & \tilde{h}_{-1}^4 \\ \tilde{h}_{-6}^6 & \tilde{h}_{-5}^6 & \tilde{h}_{-4}^6 & \tilde{h}_{-3}^6 \end{bmatrix} = H_{M_h}^{-1} = \begin{bmatrix} \frac{5}{16} & \frac{5}{8} & \frac{1}{16} & 0 \\ \frac{1}{16} & \frac{5}{8} & \frac{5}{16} & 0 \\ 0 & \frac{5}{16} & \frac{5}{8} & \frac{1}{16} \\ 0 & \frac{1}{16} & \frac{5}{8} & \frac{5}{16} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{35}{8} & -\frac{47}{8} & \frac{25}{8} & -\frac{5}{8} \\ -\frac{5}{8} & \frac{25}{8} & -\frac{15}{8} & \frac{3}{8} \\ \frac{3}{8} & -\frac{15}{8} & \frac{25}{8} & -\frac{5}{8} \\ -\frac{5}{8} & \frac{25}{8} & -\frac{47}{8} & -\frac{35}{8} \end{bmatrix} .$$

Other solutions can be constructed following (16) using for instance $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ as

$$\begin{aligned} &[\tilde{h}_{-6} \quad \tilde{h}_{-5} \quad \tilde{h}_{-4} \quad \tilde{h}_{-3} \quad \tilde{h}_{-2} \quad \tilde{h}_{-1} \quad \tilde{h}_0 \quad \tilde{h}_1 \quad \tilde{h}_2 \quad \tilde{h}_3] \\ &= \lambda_1[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \tilde{h}_0^0 \quad \tilde{h}_1^0 \quad \tilde{h}_2^0 \quad \tilde{h}_3^0] \\ &+ \lambda_2[0 \quad 0 \quad 0 \quad 0 \quad \tilde{h}_{-2}^2 \quad \tilde{h}_{-1}^2 \quad \tilde{h}_0^2 \quad \tilde{h}_1^2 \quad 0 \quad 0] \\ &+ \lambda_3[0 \quad 0 \quad \tilde{h}_{-4}^4 \quad \tilde{h}_{-3}^4 \quad \tilde{h}_{-2}^4 \quad \tilde{h}_{-1}^4 \quad 0 \quad 0 \quad 0 \quad 0] \\ &+ \lambda_4[\tilde{h}_{-6}^6 \quad \tilde{h}_{-5}^6 \quad \tilde{h}_{-4}^6 \quad \tilde{h}_{-3}^6 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0] . \end{aligned}$$

Remark 5
The values $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \{\frac{1}{100}, \frac{47}{300}, \frac{47}{60}, \frac{1}{20}\}$ *minimize* $\sum_{i=-6}^{3} |\tilde{h}_i|$ *to the value* $\frac{163}{40}$. *Following Remark 2, the corresponding decimation operator gets a smaller stability constant than any of the elementary decimation for which the stability constants are* $(14, 6, 6, 14)$.

5.4 Penalized Lagrange Subdivision

We finally consider a non-stationary (i.e. depending on the scale j) subdivision scheme recently introduced in [14] and focus in the sequel on the associated consistent decimation masks generated by our approach.

Using the notations of [14], the scheme is here constructed from a polynomial $P_j(x) = 100(2^{-2j})x^2 - 2^{-4j}x^4$ and a vector of penalization $C = (0, 2, 0, 0)$.

Denoting $M_{h^{(j)}} = \{h_3^{(j)}, h_2^{(j)}, h_1^{(j)}, h_0^{(j)}, h_{-1}^{(j)}, h_{-2}^{(j)}, h_{-3}^{(j)}, h_{-4}^{(j)}\}$, it first comes out that

$$\lim_{j \to -\infty} M_{h^{(j)}} = \{-\frac{1}{16}, 0, \frac{9}{16}, 1, \frac{9}{16}, 0, -\frac{1}{16}, 0\}, \tag{36}$$

and

$$\lim_{j \to +\infty} M_{h^{(j)}} = \{\frac{1}{3}, \frac{1}{8}, 0, 0, 1, \frac{9}{8}, -\frac{1}{3}, -\frac{1}{4}\} . \tag{37}$$

Therefore, according to the scale j, the subdivision evolves from the classical interpolatory Lagrange subdivision (36) to a non-interpolatory one of Lagrange-type. Indeed, the coefficients in (37) are the point values at $x = 0$ or $x = \frac{1}{2}$ of the Lagrange functions associated with the stencil $\{-1, 1, 2\}$.

According to Proposition 4, it is then possible to generate for each $j \in \mathbb{Z}$ the matrix of associated consistent elementary decimation masks. As an example, Fig. 1 displays the evolution of the third row of $\tilde{H}_{M_h}$ for $j \in [-10, 10]$.

It appears that the decimation mask quickly converges towards its asymptotical limit which is a sub-sampling ($\{0, 0, 0, 0, 1, 0\}$) when $j \to -\infty$ and $\{-\frac{54}{107}, \frac{144}{107}, -\frac{9}{107}, \frac{24}{107}, -\frac{6}{107}, \frac{8}{107}\}$ when $j \to +\infty$. It is also interesting to notice that, as expected, these decimations are consistent with the asymptotical subdivision schemes associated with the masks (36) and (37) respectively.

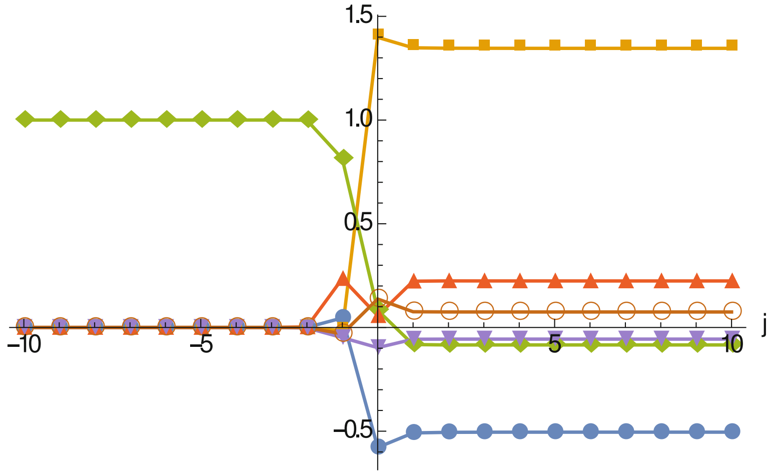

Fig. 1. Six coefficients of the mask of a decimation operator consistent with the penalized Lagrange subdivision scheme for scale $-10 \le j \le 10$.

6 Conclusion

We have shown, in this paper, that a construction of all the decimation operators consistent with a given linear and homogeneous subdivision operator can be performed. The proposed generic approach first leads to the construction of elementary decimation operators by exploiting the consistency condition. Then, all the possible consistent decimation operators can be obtained by linear combinations of translated versions of elementary ones. A theoretical analysis has been performed to provide an error bound for the so-called prediction error. It turned out that the decay rate is controlled by the subdivision scheme only. Moreover, the bound depends on the stability constant of the decimation operator that leads to the proposition of an heuristic strategy for the selection of the decimation mask among the family of consistent ones. Several applications have been performed in order to show the interest of our approach. It first appeared that our method is coherent with previous constructions of decimation operators associated with well-known subdivision schemes. Moreover, it allows to go beyond such constructions by offering a large choice of decimation masks that can be used to improve the stability of multi-scale transforms. Finally, it provides an efficient tool to design decimation operators for non classical subdivision schemes. Some generalizations to position dependent schemes as well as to some non linear subdivision operators are in progress [13].

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