Subdivision Schemes and Multiresolution Analyses: Focus on the Shifted Lagrange and Shifted PPH Schemes

Zhiqing Kui, Jean Baccou, and Jacques Liandrat

Abstract Subdivision schemes have been extensively developed since the eighties with very powerful applications for surface generation. To be implemented for compression, subdivision schemes have to be coupled with decimation operators sharing some consistency relation and with detail operators. The flexibility of subdivision schemes (they can be non-stationary, position or zone dependent, non-linear,…) makes that the construction of consistent decimation operators is a difficult task. In this paper, following the first results introduced in Kui et al. (On the coupling of decimation operator with subdivision schemes for multi-scale analysis. In: Lecture notes in computer science, vol. 10521. Springer, Berlin, pp. 162–185, 2016), we present the construction of multiresolution analyses connected to general subdivision schemes with detailed application to a non-interpolatory linear scheme called shifted Lagrange (Dyn et al., A C^2 four-point subdivision scheme with fourth order accuracy and its extensions. In: Mathematical methods for curves and surfaces: Tromsø 2004. Citeseer, 2005) and its non-linear version called shifted PPH (Amat et al., Math. Comput. 80:959–959, 2011).

Keywords Subdivision schemes · Multiresolutions · Decimation · Non-linear

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1 Introduction

Since 50 years, subdivision schemes have been developed, analyzed and used with very powerful applications such as curve generation, image processing or animation movies. One of their advantages stands in the flexibility of the construction of their masks. One can easily derive various types of subdivision schemes adapted to specific goals; one can cite for instance the development of position dependent schemes devoted to the treatment of local singularities in the data [3] or the development of non-linearly perturbed subdivision schemes for removing undesirable behaviours [1]. When used for compression, subdivision schemes are plugged into a multiresolution framework that involves a so-called decimation operator [9]. However, although the decimation operator is trivially defined as a sub-sampling when handling interpolatory subdivision schemes, it is a difficult task to derive it in other situations.

Following the first results presented in [10], the contribution of this paper is a generic approach to derive decimation operators that will be called consistent with a given general subdivision scheme. We will focus on two examples related to the shifted Lagrange subdivision scheme (that is linear and non-interpolatory) introduced in [8] and to the shifted PPH subdivision scheme (that is non-linear and non-interpolatory) introduced in [2]. After an overview on subdivision schemes and multiresolution analyses (Sect. 2), we focus on the construction of decimation operators (Sect. 3) and on the so-called prediction errors and details (Sect. 4). Section 5 is devoted to numerical tests obtained using the shifted Lagrange and the shifted PPH multiresolutions.

2 Subdivision Schemes and Associated Multiresolution Transform

2.1 Subdivision Schemes

2.1.1 General Construction

We consider binary subdivision schemes [7] defined as operators $S : l^\infty(\mathbb{Z}) \to l^\infty(\mathbb{Z})$ constructed from real-valued sequences $(h_k)_{k\in\mathbb{Z}}$ having a finite number of non-zero values such that $(f_k)_{k\in\mathbb{Z}} \in l^\infty(\mathbb{Z}) \mapsto ((Sf)_k)_{k\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$ with

$$(Sf)_k = \sum_{l\in\mathbb{Z}} h_{k-2l} f_l. \tag{1}$$

The set of non-zero values of $(h_k)_{k\in\mathbb{Z}}$ is called the mask of S.

Subdivision is generally iterated starting from an initial sequence $(f_k^{J_0})_{k\in\mathbb{Z}}$ to generate $(f_k^j)_{k\in\mathbb{Z}}$ for $j \geq J_0$ as

$$f^{j+1} = Sf^j,\, j \geq J_0 \ . \tag{2}$$

One of the advantages of subdivision schemes stands in the flexibility associated with the choice of the mask when changing position (k), scale (j) and data (f^j). The simplest strategy is to consider the same mask for every position, scale and data [7] and to set $h_{2k} = \delta_{k,0}$. It leads to linear (i.e. the mask is independent of f^j), uniform (i.e. the mask is independent of k), stationary (i.e. the mask is independent of j) and interpolatory (i.e. $f_{2k}^{j+1} = f_k^j$) schemes. The main limitation of this type of construction is the generation of Gibbs oscillations for discontinuous data. A better strategy is then to work with non-uniform or non-interpolatory schemes. The shifted Lagrange scheme [8] is non-interpolatory. Moreover, for each position and scale, the construction of the subdivision can be adapted according to the regularity of the data to specifically handle discontinuities. Among various schemes following this strategy, one can mention the ENO [4] or PPH[1] [1] approach: for each j and k, the subdivision depends on the values of the sequence $(f_l^j)_{l\in\mathbb{Z}}$ for l in the vicinity of k. It leads to non-linear schemes.

In this paper, we focus on these types of schemes and more specifically on the shifted Lagrange and shifted PPH schemes. The construction of PPH has been first performed in an interpolatory framework [1] by modifying the classical linear Lagrange 4-point interpolating scheme [6]. Then, a non-interpolatory version (called shifted PPH) has been introduced in [2]. The corresponding definitions are recalled in the remaining of this section.

2.1.2 Shifted Lagrange and Shifted PPH Schemes

The situation we consider here is a specific case where a non-linear subdivision scheme is constructed as a perturbation of a linear one.

We start from the 4-point Lagrange interpolation and consider its shifted version. More precisely, for any value of $k \in \mathbb{Z}$ we build

$$P_k(x) = L_{-1}(x) f_{k-1} + L_0(x) f_k + L_1(x) f_{k+1} + L_2(x) f_{k+2},$$

where $\{L_n(x)\}_{-1\leq n\leq 2}$ denotes the degree 3 Lagrange interpolatory polynomial associated with the stencil $\{-1, 0, 1, 2\}$.

[1] As it will become clear in the next section, this type of scheme is derived from a perturbation of a linear one. Therefore, a non-linear mask cannot be constructed but it is straightforward to extend the definition of a subdivision scheme provided by Expression (1).

The 4-point shifted Lagrange linear scheme is given by:

$$\begin{cases} (S_{LA}f)_{2k} &= P_k(\frac{1}{4}) \\ (S_{LA}f)_{2k+1} &= P_k(\frac{3}{4}) \end{cases} . \tag{3}$$

The shifted PPH scheme is constructed introducing $N_k(x)$, a degree 3 polynomial depending non-linearly on $(f_{k-1}, f_k, f_{k+1}, f_{k+2})$ and substituting in (3) P_k by $P_k + N_k$.

More precisely, if

$$A(x, y) = \frac{x+y}{2}, \quad H(x, y) = \frac{xy}{x+y}\,(sgn(xy) + 1)$$

where sgn denotes the sign function and $D_k = (f_{k+1} - f_k) - (f_k - f_{k-1})$ then, if $|D_k| \le |D_{k+1}|$,

$$N_k(x) = 2L_2(x)\,(H(D_k, D_{k+1}) - A(D_k, D_{k+1}))\,,$$

if $|D_k| > |D_{k+1}|$,

$$N_k(x) = 2L_{-1}(x)\,(H(D_k, D_{k+1}) - A(D_k, D_{k+1}))\,.$$

Introducing the non-linear perturbation S_N defined by

$$\begin{cases} (S_N f)_{2k} &= N_k(\frac{1}{4}) \\ (S_N f)_{2k+1} &= N_k(\frac{3}{4}) \end{cases} ,$$

the shifted PPH subdivision scheme can be written as

$$S_{PPHA}f = S_{LA}f + S_N f. \tag{4}$$

Expression (2) can be interpreted as a two-scale relation. It is therefore possible to establish a connection between subdivision schemes and local prediction operators in a multiresolution framework. This framework as well as the associated precisions required to construct subdivision-based multiresolution transforms are recalled in the next section.

2.2 Multiresolution Transform

A Harten multiresolution analysis [9] is characterized by the family of triplets $\left((V^j, D^j_{j+1}, P^{j+1}_j)\right)_{j\in\in\mathbb{Z}}$ where V^j is a separable space (j is a scale parameter) and D^j_{j+1} (resp. P^{j+1}_j) is a decimation (resp. prediction) operator connecting

V^{j+1} (resp. V^j) to V^j (resp. V^{j+1}). If $f^j \in V^j$ is obtained after decimation of $f^{j+1} \in V^{j+1}$, $P_j^{j+1} f^j$ does not usually coincide with f^{j+1}. However the following consistency condition is required:

$$D_{j+1}^j P_j^{j+1} = I_{V^j} \tag{5}$$

where I_{V^j} stands for the identity operator in V^j.

In order to recover f^{j+1} after a decimation and a prediction, a sequence of prediction errors $e^{j+1} = \left(e_k^{j+1}\right)_{k \in \mathbb{Z}}$ is introduced and defined as:

$$e_k^{j+1} = f_k^{j+1} - \left(P_j^{j+1} D_{j+1}^j f^{j+1}\right)_k = \left(\left(I_{V^{j+1}} - P_j^{j+1} D_{j+1}^j\right) f^{j+1}\right)_k \tag{6}$$

The mapping $f^{j+1} \mapsto \{f^j, e^{j+1}\}$ is a key ingredient for the multiresolution transform.

Focussing on a given j and denoting $\tilde{h} = D_{j+1}^j$ and $h = P_j^{j+1}$, the consistency relation (5), $\tilde{h}h = I_{V^j}$, leads in the linear case to the fact that $\tilde{h}e^{j+1} = 0$. Introducing d^j and o^j such that $\forall k \in \mathbb{Z}, d_k^j = e_{2k+1}^{j+1}$ and $\forall k \in \mathbb{Z}, o_k^j = e_{2k}^{j+1}$, we get a linear relation of type $L^j o^j = R^j d^j$ where L^j and R^j are two linear operators. As soon as L^j is invertible, this relation allows to recover o^j from d^j, and therefore e^{j+1} from d^j. One can then substitute the previous mapping by the following bijective one $f^{j+1} \mapsto \{f^j, d^j\}$.

In the non-linear case, one can not, in general, deduce such a bijection from the consistency relation. We postpone to Sect. 4 for the construction of this bijection in the case of the shifted PPH scheme.

Finally, for a fixed level $J_0 \leq j$, the multiresolution decomposition of a sequence f^{j+1} is the element $\{f^{J_0}, d^{J_0}, d^{J_0+1}, \ldots, d^j\}$. The advantage of this representation stands in the fact that generally the norm of the details d^l decays exponentially with the level l. Similarly a reconstruction transform can be introduced to recover f^{j+1} from $\{f^{J_0}, d^{J_0}, d^{J_0+1}, \ldots, d^j\}$.

Exploiting (2), the prediction from V^j to V^{j+1} can be performed using a subdivision scheme. The main advantage of the corresponding operator is that it inherits the interesting properties of the scheme such as the flexibility in the construction of the mask. One can therefore introduce interpolatory or non-interpolatory, linear or non-linear predictions according to the data, that generally improve the classical wavelet-based multi-analyses framework [5]. However, the full specification of the multiresolution transforms is more involved. The construction of a decimation still remains difficult to tackle for non-interpolatory schemes since, generally, a subsampling operator does not satisfy the consistency property (5). Moreover, as mentioned in the previous section, the storage of the prediction error is well described for linear operators but cannot be extended to the non-linear framework. Therefore, we propose in the next sections new contributions to handle these two open questions with a specific application to the shifted Lagrange and shifted PPH schemes recalled in Sect. 2.1.

3 Decimation Operators

3.1 The Linear Case

A linear and uniform decimation operator $\tilde{h}_L$ is defined through a sequence $(\tilde{h}_k)_{k\in\mathbb{Z}}$ having a finite number of non-zero values as $(f_k)_{k\in\mathbb{Z}} \in l^\infty(\mathbb{Z}) \mapsto ((Df)_k)_{k\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$ with

$$(Df)_k = \sum_{l\in\mathbb{Z}} \tilde{h}_{l-2k} f_l \quad .$$

The set of non-zero values of $(\tilde{h}_k)_{k\in\mathbb{Z}}$ is called the mask of D and denoted $M_{\tilde{h}}$.

A generic method to construct the mask of a decimation scheme consistent with a given linear uniform subdivision scheme has been proposed in [10]. The main results are recalled in the two following propositions. The first one is devoted to the construction of elementary operators (i.e. with masks of minimal number of non-zero values) while the second one describes how all consistent decimation operators can be recovered using linear combinations of translated versions of elementary operators.[2]

Proposition 1 *Let h be a prediction operator whose mask is constructed from the sequence*

$$\{h_{n-2\alpha}, h_{n-2\alpha+1}, \ldots, h_n, h_{n+1}\}$$

with $h_{n-2\alpha}h_{n+1} \neq 0$ (i.e. mask of even length) or with $h_{n-2\alpha} = 0$ and $h_{n-2\alpha+1}h_{n+1} \neq 0$ (i.e. mask of odd length).

Introducing $H_{M_h} =$

$$\begin{bmatrix} h_n & h_{n-2} & \cdots & h_{n-2\alpha} & 0 & \cdots & 0 \\ h_{n+1} & h_{n-1} & \cdots & h_{n-2\alpha+1} & 0 & \cdots & 0 \\ 0 & h_n & h_{n-2} & \cdots & h_{n-2\alpha} & \cdots & 0 \\ 0 & h_{n+1} & h_{n-1} & \cdots & h_{n-2\alpha+1} & \cdots & 0 \\ \vdots & & & \vdots & & & \\ 0 & 0 & \cdots & h_n & h_{n-2} & \cdots & h_{n-2\alpha} \\ 0 & 0 & \cdots & h_{n+1} & h_{n-1} & \cdots & h_{n-2\alpha+1} \end{bmatrix},$$

if $det(H_{M_h}) \neq 0$, there exists at most 2α consistent elementary decimation operators whose masks are of length not larger than 2α. These masks are given by each row of $H_{M_h}^{-1}$.

[2]For any decimation operator $\tilde{h}$ constructed from $(\tilde{h}_k)_{k\in\mathbb{Z}}$ and any integer t, we define $T_t(\tilde{h})$ the translated decimation operator related to the sequence $(\tilde{h}_{k-t})_{k\in\mathbb{Z}}$.

Proposition 2 *A subdivision operator h being fixed and satisfying the hypotheses of Proposition 1, we denote* $\{\tilde{h}^i\}_{1\leq i\leq 2\alpha}$ *the set of the elementary consistent decimation operators.*

Then, all the consistent decimation operators can be constructed as

$$\sum_{t\in\mathfrak{T}}\sum_{i\in\mathfrak{I}} c_{i,t}\,T_{2t}(\tilde{h}^i) \tag{7}$$

with

$$\forall t\in\mathfrak{T}\subset\mathbb{Z},\ \sum_{i\in\mathfrak{I}} c_{i,t}=\delta_{t,0},\ \ \textit{and}\ 0\in\mathfrak{T}\ .$$

The large choice of decimation schemes offered by this approach is of prime importance in practice since it allows to optimize some specific characteristics of the scheme according to given objectives. In image compression for example, it is important to select decimation operators with minimal L^∞-norm because this norm is involved in the stability of the multiresolution transforms.

We finally conclude this section by applying our approach to the shifted Lagrange linear subdivision scheme for which, up to now, there was no consistent decimation operator available in the literature.

Decimation operators consistent with the shifted Lagrange scheme

From (3), the shifted Lagrange scheme is given by:

$$\begin{cases}(S_{LA}f)_{2k}=-\frac{7}{128}f_{k-1}+\frac{105}{128}f_k+\frac{35}{128}f_{k+1}-\frac{5}{128}f_{k+2},\\ (S_{LA}f)_{2k+1}=-\frac{5}{128}f_{k-1}+\frac{35}{128}f_k+\frac{105}{128}f_{k+1}-\frac{7}{128}f_{k+2}.\end{cases}$$

and its mask is therefore

$$M_h=\{h_{-4},h_{-3},h_{-2},h_{-1},h_0,h_1,h_2,h_3\}$$

$$=\{-\frac{5}{128},-\frac{7}{128},\frac{35}{128},\frac{105}{128},\frac{105}{128},\frac{35}{128},-\frac{7}{128},-\frac{5}{128}\}.$$

A straightforward application of Proposition 1 leads to the set of consistent elementary decimation operators whose masks correspond to each row of the following matrix:

$$\tilde{H}_{LA}=\begin{bmatrix}\frac{24367}{1152} & -\frac{63605}{1152} & \frac{31115}{576} & -\frac{10325}{576} & -\frac{4165}{1152} & \frac{2975}{1152}\\ \frac{2975}{1152} & -\frac{4165}{1152} & \frac{1771}{576} & -\frac{565}{576} & -\frac{245}{1152} & \frac{175}{1152}\\ \frac{175}{1152} & -\frac{245}{1152} & \frac{875}{576} & -\frac{245}{576} & -\frac{133}{1152} & \frac{95}{1152}\\ \frac{95}{1152} & -\frac{133}{1152} & -\frac{245}{576} & \frac{875}{576} & -\frac{245}{1152} & \frac{175}{1152}\\ \frac{175}{1152} & -\frac{245}{1152} & -\frac{565}{576} & \frac{1771}{576} & -\frac{4165}{1152} & \frac{2975}{1152}\\ \frac{2975}{1152} & -\frac{4165}{1152} & -\frac{10325}{576} & \frac{31115}{576} & -\frac{63605}{1152} & \frac{24367}{1152}\end{bmatrix}.$$

Following Proposition 2, one can also increase the length of the decimation mask in order to reduce the L^∞-norm of the operator. If for example the length is fixed to 8, the mask of minimal L^∞-norm is given by:

$$\begin{aligned} M_{\tilde{h}} &= \{\tilde{h}_{-4}, \tilde{h}_{-3}, \tilde{h}_{-2}, \tilde{h}_{-1}, \tilde{h}_0, \tilde{h}_1, \tilde{h}_2, \tilde{h}_3\} \\ &= \{\frac{95}{2304}, -\frac{133}{2304}, -\frac{35}{256}, \frac{1505}{2304}, \frac{1505}{2304}, -\frac{35}{256}, -\frac{133}{2304}, \frac{95}{2304}\}. \end{aligned} \tag{8}$$

This decimation operator is often considered since it has the same length as the initial subdivision scheme and it is symmetrical.

3.2 Extension to the Non-linear Perturbation Case

Generalizing Expression (4) and using the notations of the subdivision-based multiresolution framework, the non-linear subdivision scheme considered in this paper satisfies:

$$hf^j = h_L f^j + h_N f^j, \tag{9}$$

where h_L is the prediction associated to a linear subdivision scheme and h_N stands for a non-linear perturbation operator.

Starting from a sequence $f^{j+1} \in V^{j+1}$, we can construct a consistent non-linear decimation operator by solving the following fixed-point equation:

$$f^j = \tilde{h}_L f^{j+1} - \tilde{h}_L h_N f^j, \tag{10}$$

where $\tilde{h}_L$ is a linear decimation operator consistent with h_L and constructed by the method described in the previous section.

The contractivity of the operator $\tilde{h}_L h_N$ which means that there exists $c \in \mathbb{R}$, $|c| < 1$, such that $\forall (u, v) \in (l^\infty(\mathbb{Z}))^2$,

$$||\tilde{h}_L h_N u - \tilde{h}_L h_N v|| \le c||u - v||,$$

is required to ensure the existence and uniqueness of f^j according to the Banach fixed-point theorem.

Moreover, $f^j = \lim_{n\to+\infty} (f^j)_n$ where $((f^j)_n)_{n\in\mathbb{Z}}$ is constructed by induction:

$$\begin{cases} (f^j)_0 &= \tilde{h}_L f^{j+1} \\ (f^j)_{n+1} &= \tilde{h}_L f^{j+1} - \tilde{h}_L h_N (f^j)_n, \end{cases} . \tag{11}$$

Concerning the consistency of the decimation operator defined above, if $f^{j+1} = hg^j$, the unique solution of the fixed-point equation denoted $\hat{f}^j$ satisfies:

$$\hat{f}^j = \tilde{h}_L hg^j - \tilde{h}_L h_N \hat{f}^j = g^j + \tilde{h}_L h_N g^j - \tilde{h}_L h_N \hat{f}^j.$$

Since g^j is solution of the previous equation, it is the unique solution. Therefore $g^j = \tilde{h} f^{j+1} = \tilde{h} hg^j$ that leads to the consistency of $\tilde{h}$.

When h is the shifted PPH non-linear scheme, the following proposition provides that, for a suitable choice of $\tilde{h}_L$, the fixed-point is unique and can be reached using fixed-point iterations:

Proposition 3 *If h_L is the prediction operator associated with the shifted Lagrange linear scheme and $\tilde{h}_L$ is the consistent decimation whose mask is given by*

$$\begin{aligned}&\{\tilde{h}_{-6}, \tilde{h}_{-5}, \tilde{h}_{-4}, \tilde{h}_{-3}, \tilde{h}_{-2}, \tilde{h}_{-1}, \tilde{h}_0, \tilde{h}_1, \tilde{h}_2, \tilde{h}_3, \tilde{h}_4, \tilde{h}_5\}\\ =&\{\frac{19}{16128}, -\frac{19}{11520}, \frac{19}{576}, -\frac{19}{576}, -\frac{2623}{16128}, \frac{7639}{11520},\\ &\frac{7639}{11520}, -\frac{2623}{16128}, -\frac{19}{576}, \frac{19}{576}, -\frac{19}{11520}, \frac{19}{16128}\}\end{aligned}$$

then the operator $\tilde{h}_L h_N$ is contractive with a Lipschitz constant bounded by $\frac{19657}{20160}$.

4 Prediction Errors and Details

4.1 Construction of the Details

As mentioned in Sect. 2.2, the construction of the details is straightforward in the linear case and is based on a splitting of the prediction error that belongs to the kernel of the decimation operator. One can not apply such a construction in the non-linear case. However, for the shifted PPH scheme, the same idea can be exploited to address the problem of prediction error storage. It is stated by the following proposition.

Proposition 4 *Let $h = h_L + h_N$ be a non-linear subdivision operator defined by (4) and $\tilde{h}$ be a consistent decimation operator constructed from Eq. (10).*

For all $f^{j+1} \in l^\infty(\mathbb{Z})$, the associated prediction error e^{j+1} satisfies

$$\tilde{h}_L e^{j+1} = 0, \tag{12}$$

and

$$\tilde{h} e^{j+1} = 0. \tag{13}$$

Proof Under the contractivity condition, if $\hat{f}^j = \tilde{h} f^{j+1}$ denotes the unique solution to Eq. (10), the prediction error can be written as

$$\begin{aligned} e^{j+1} &= f^{j+1} - h\tilde{h} f^{j+1} \\ &= f^{j+1} - h_L \hat{f}^j - h_N \hat{f}^j \\ &= (I - h_L\tilde{h}_L) f^{j+1} - (I - h_L\tilde{h}_L)h_N \hat{f}^j. \end{aligned}$$

Then Eq. (12) is straightforward using the consistency condition linking h_L and $\tilde{h}_L$.

Assuming $w^j = \tilde{h} e^{j+1}$,

$$w^j = \tilde{h}_L e^{j+1} - \tilde{h}_L h_N w^j = -\tilde{h}_L h_N w^j.$$

According to the Banach fixed-point theorem, $w^j = 0$ is the unique solution which leads to (13). □

Therefore, following what has been done in Sect. 2.2 and introducing the corresponding operators L_L^j and R_L^j, a bijective mapping $f^{j+1} \mapsto (f^j, d^j)$ can be derived leading to the details d^j.

4.2 Prediction Error Decay

The following proposition, borrowed from [10], explains that for the linear case and a couple of consistent operators $(h, \tilde{h})$, the decay of the prediction error with the scale is fully controlled by the polynomial approximation property of the subdivision operator. Indeed, we have:

Proposition 5 *Let $(V^j, h, \tilde{h})_{j\in\mathbb{Z}}$ define a linear multiresolution analysis with h a subdivision operator associated to the real sequence $(h_k)_{k\in\mathbb{Z}}$ and $\tilde{h}$ a decimation operator constructed from the real sequence $(\tilde{h}_k)_{k\in\mathbb{Z}}$. We assume that the associated multi-scale transform is applied from a fine scale J_{max} to a coarse one J_0.*

If there exists $L \in \mathbb{N}$ such that $\forall n \in \{0, 1, \ldots, L\}$,

$$\sum_{l\in\mathbb{Z}} h_{2l}(2l)^n = \sum_{l\in\mathbb{Z}} h_{2l+1}(2l+1)^n \quad , \tag{14}$$

then for sufficiently large $j \in [J_0, J_{max} - 1]$,

$$||e^j|| \leq C2^{-(L+1)j}, \tag{15}$$

where C does not depend on j.

In the case of the 4-point shifted Lagrange scheme, it is easy to verify that $\forall n \in \{0, 1, 2, \ldots, 4\}$,

$$\sum_{i=-1}^{2} L_i(\frac{1}{4})(-2i)^n = \sum_{i=-1}^{2} L_i(\frac{3}{4})(-2i+1)^n,$$

and it follows from the previous proposition that the decay rate of the associated prediction error is 5.

When the subdivision is non-linear the consistency and the polynomial approximation property can not be combined to provide the decay of the prediction error, and therefore of the details. A direct analysis must be performed. In this paper, a numerical study is provided in the next section.

5 Numerical Results

This section provides a series of tests in order to evaluate the linear (shifted Lagrange) and non-linear (shifted PPH) multiresolutions. The results obtained using the Lagrange interpolatory scheme with mask

$$M_h = \{-1/16, 0, 9/16, 1, 9/16, 0, -1/16, 0\}$$

and consistent decimation $M_{\tilde{h}} = \{1\}$ are also provided as a comparison.

Initial data are sampled from related functions. Figure 1 deals with a piecewise regular function with a discontinuity at $x_0 = 0.5$ (Left). The multiresolution transform of this function is performed using the linear consistent decimation operator given by Proposition 3 while the non-linear one satisfies the fixed-point equation (10). Figure 1 (right) displays the convergence rate of the fixed-point in algorithm (11). The slope of the curve exhibits the convergence rate. It appears that very few iterations (less than 17) are required for convergence.

In Fig. 2, the prediction error associated to regular data (sampled from the sin function) is numerically estimated and plotted versus the scale. For each approach, a multi-scale decomposition transform is applied from a fine level $J_{max} = 12$ to a coarse one $J_0 = 7$. It appears that the decay rate is larger for the linear approach (slope of 5.0379, to be compared with the theoretical value of 5) than for the non-linear one (slope of 4.21979).This can be explained by the presence of the non-linear perturbation term that reduces the degree of polynomial approximation. As a comparison, the interpolatory approach has a slope of 4.00717, to be compared with the theoretical value of 4.

For the functions of Fig. 3, we consider in Fig. 4 the prediction errors around the point $x_0 = 0, 5$. It appears that the stronger the discontinuity the better is the related performance of the non-linear scheme. This good behaviour of the non-linear scheme is confirmed by the plots in Fig. 5.

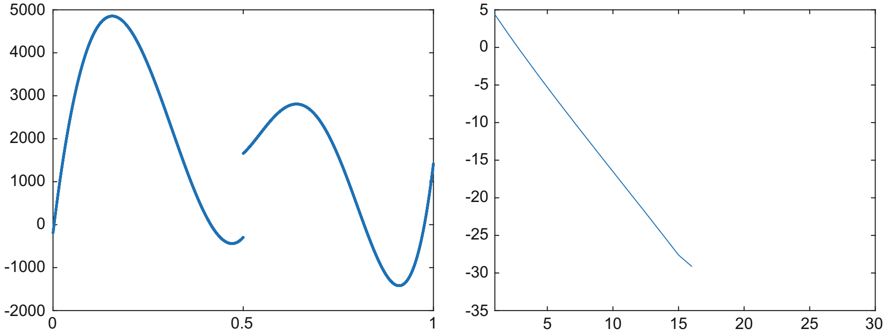

Fig. 1 Left: Test function, Right: $\log ||f_n^j - f_{n-1}^j||_\infty$ versus n for the fixed-point algorithm (11)

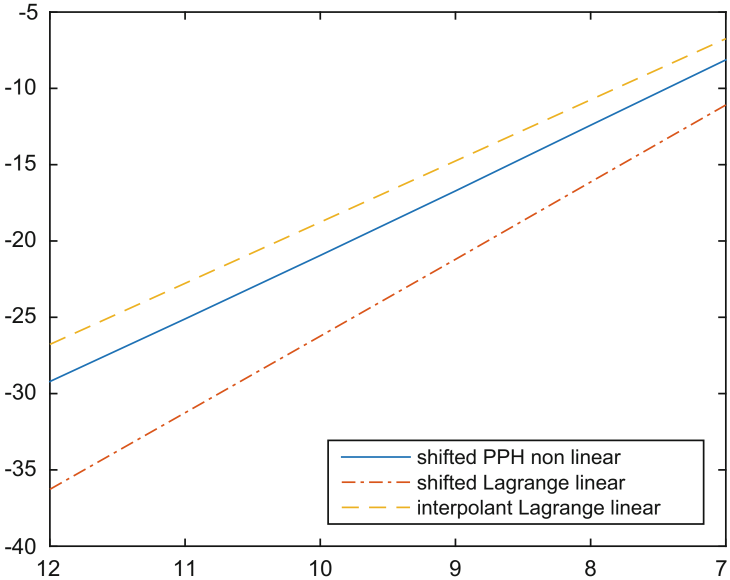

Fig. 2 Log of the prediction error versus scale

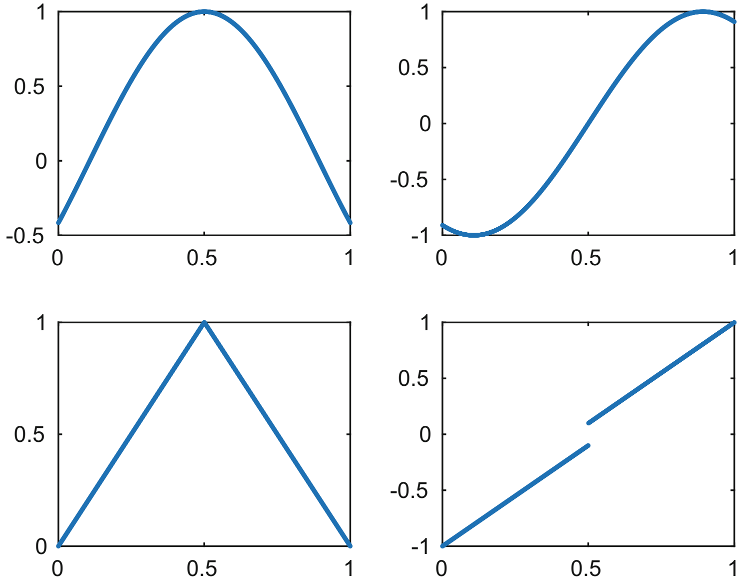

Fig. 3 Test functions

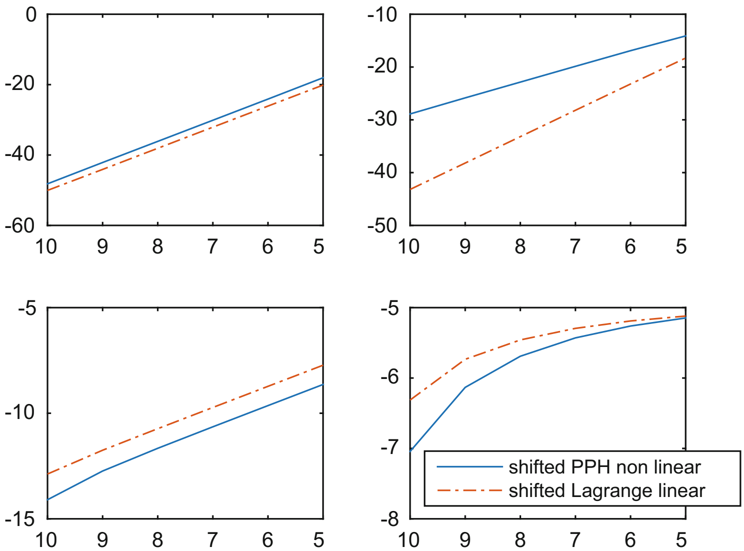

Fig. 4 Log of the prediction error versus scale

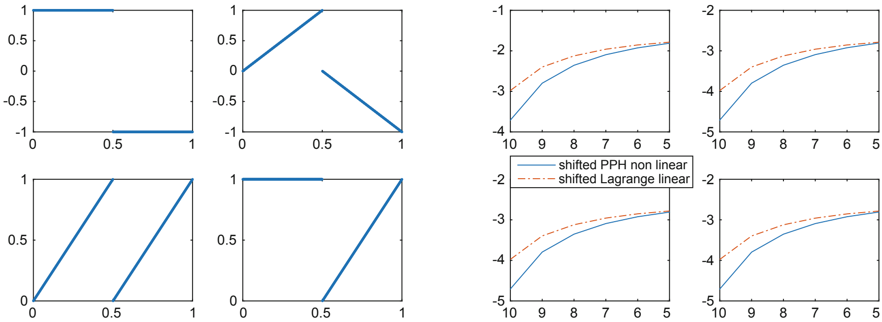

Fig. 5 Left: Test functions, Right: Log of the prediction error versus scale

6 Conclusion

In this paper, we have defined the different elements required to construct a multiresolution analysis associated to linear or non-linear subdivision schemes. The key points are the generation of consistent decimation operators and the definition of the details starting from the prediction error. The construction of decimation is performed inverting a matrix in the linear case and next solving a fixed-point problem in the non-linear case. The detail operators are constructed exploiting a specific relation implying a linear decimation. We have described the construction of two schemes (shifted Lagrange and shifted PPH) for which no multiresolution was available before. The different numerical results show the interest of such frameworks for the compression of data.

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