

Robotic Navigation and Exploration

Unit 3: Kinematic Model & Path Tracking Control

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直播連結 https://www.youtube.com/@NTHURNE-I9v

Outline

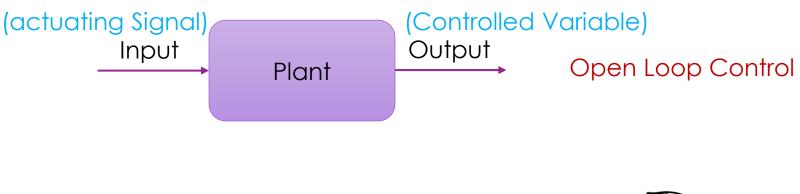
- Basics of Control System for Automobile
- PID (Proportional-Integral-Derivative) Control
- Kinematic Model
- Differential Drive
- Pure-Pursuit Control
- Bicycle Model
 - Pure Pursuit Control
 - Stanley Control (Path Coordinate and Control Stabilization)
 - Linear Quadratic Regulator (LQR)

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Control Theory: Open Loop Control

- Control System: the mechanism that affects the future state of a system
- Control Theory: a strategy to change input to desired output

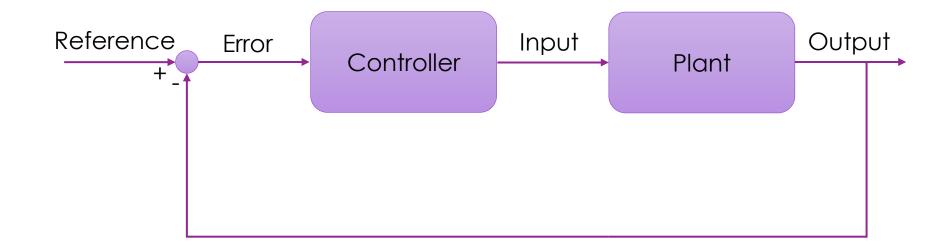






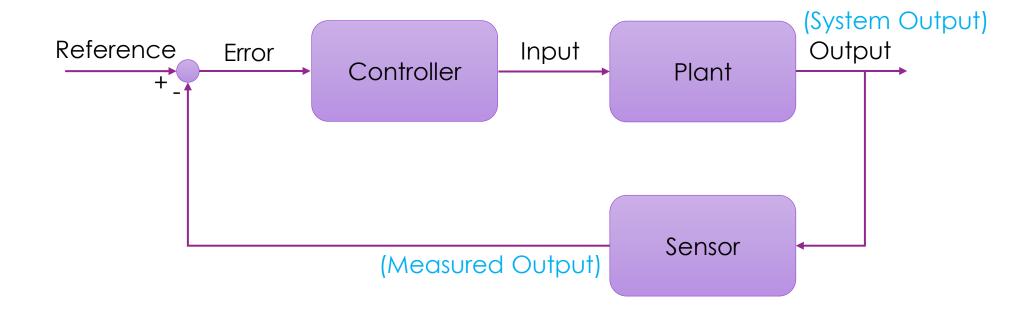


Control Theory: Close Loop Control



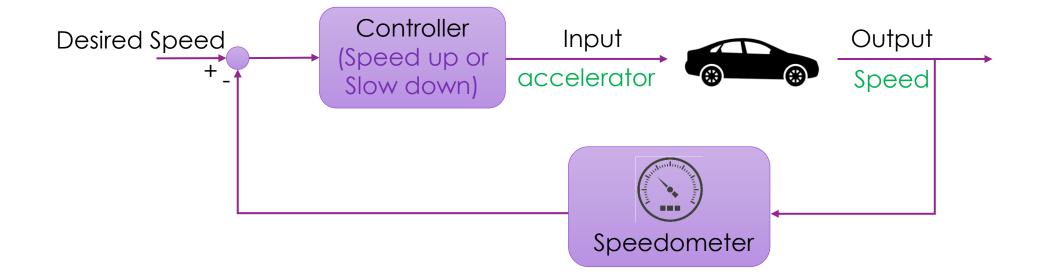
Close Loop Control (Feedback Control)

Control Theory: Close Loop Control



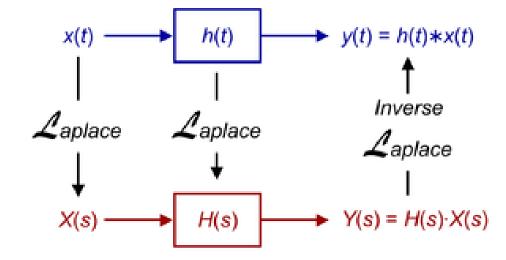
Close Loop Control (Feedback Control)

Control Theory: Car Example



Linear Time Invariant System

Time domain



Frequency domain

Laplace transform

$$\mathcal{L}\left\{f(t)\right\} = \int_{0^{-}}^{\infty} e^{-st} f(t) dt$$

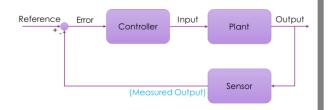
$$= \left[\frac{f(t)e^{-st}}{-s}\right]_{0^{-}}^{\infty} - \int_{0^{-}}^{\infty} \frac{e^{-st}}{-s} f'(t) dt \quad \text{(by parts)}$$

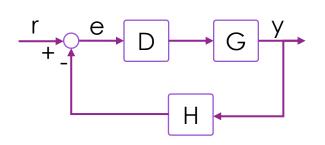
$$= \left[-\frac{f(0^{+})}{-s}\right] + \frac{1}{s} \mathcal{L}\left\{f'(t)\right\},$$

Basic Laplace Transform Pairs

Signal or Function	f(t)	F(s)
Impulse	$\delta(t)$	1
Step	$u(t)=1, t\geq 0$	1 s
Ramp	$r(t)=t, t\geq 0$	$\frac{1}{s^2}$
Exponential	$e^{-\alpha t}$ $e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$
Damped Ramp	te ^{-at}	$\frac{1}{(s+\alpha)^2}$
Sine	$\sin(\beta t)$	$\frac{\beta}{s^2+\beta^2}$
Cosine	$\cos(\beta t)$	$\frac{s}{s^2 + \beta^2}$
Damped Sine	$e^{-\alpha t}\sin(\beta t)$	$\frac{\beta}{(s+\alpha)^2+\beta^2}$
Damped Cosine	$e^{-\alpha t}\cos(\beta t)$	$\frac{s+\alpha}{(s+\alpha)^2+\beta^2}$
Simple Complex Pole	see next pg	see next pg

Linear Time Invariant System





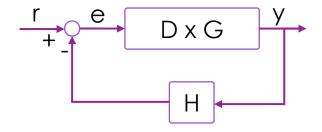
$$e = r-yH$$

 $y = e \cdot D \cdot G$ $e = \frac{y}{DG}$

$$r-yH = \frac{y}{DG}$$

$$(DG)(r-yH)=y$$

DGr=y(1+DGH) or
$$y = \frac{DGr}{1+DGH}$$





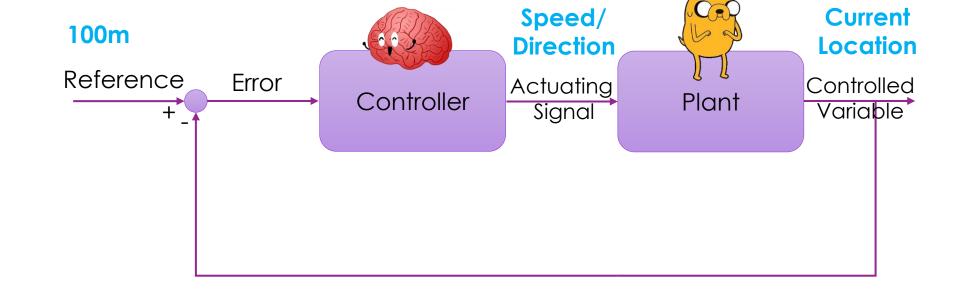


Plant

Open Loop

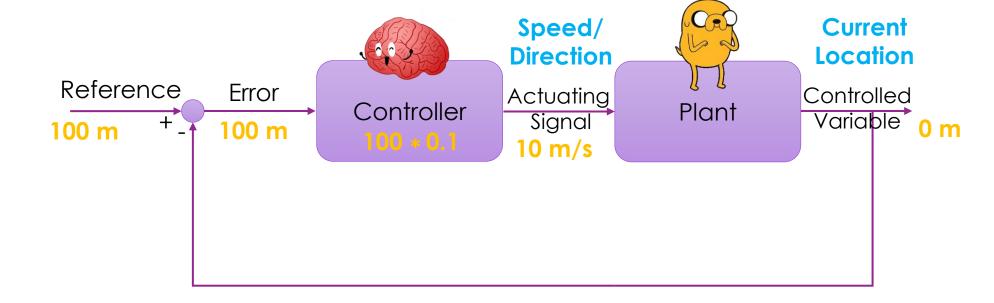
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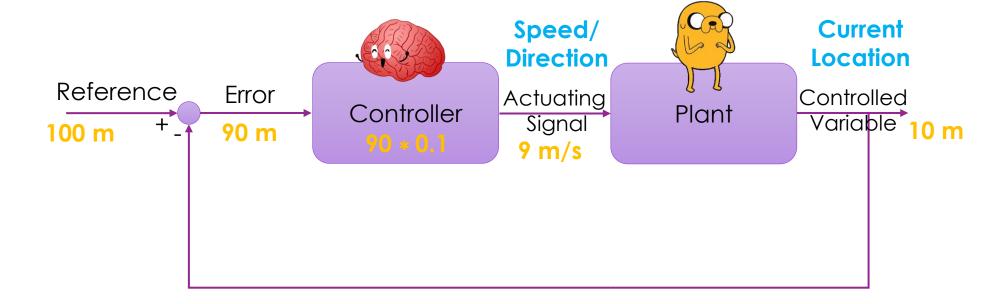


100 m (desired location)



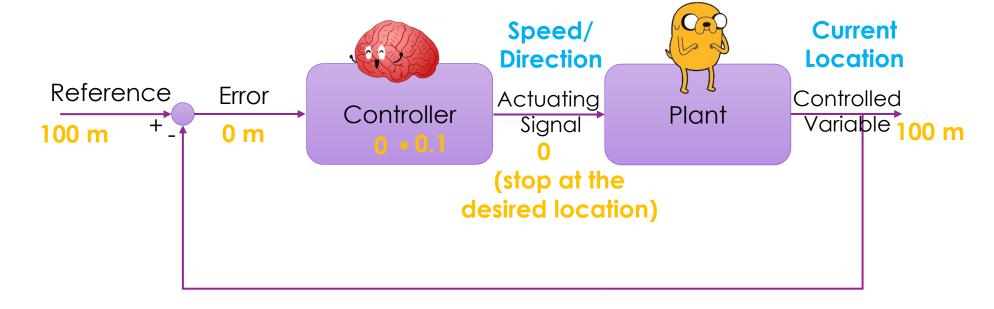


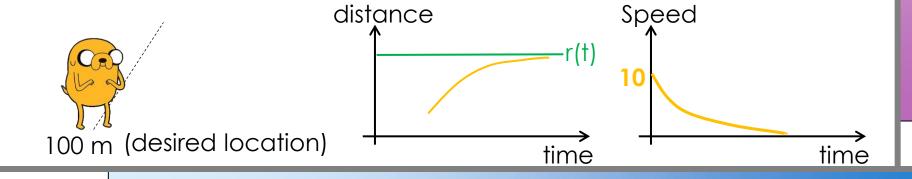
100 m (desired location)





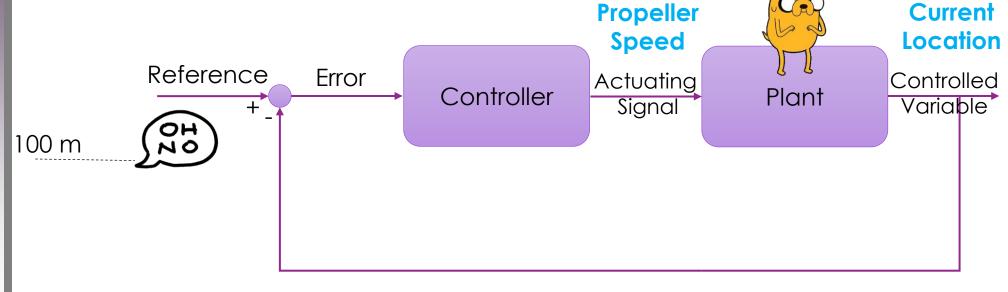
100 m (desired location)





0 m

PID Control: Problem of Proportional Gain





state

error

= 200 rpm100 * 2 Steady 40 * 5 = 200 rpm

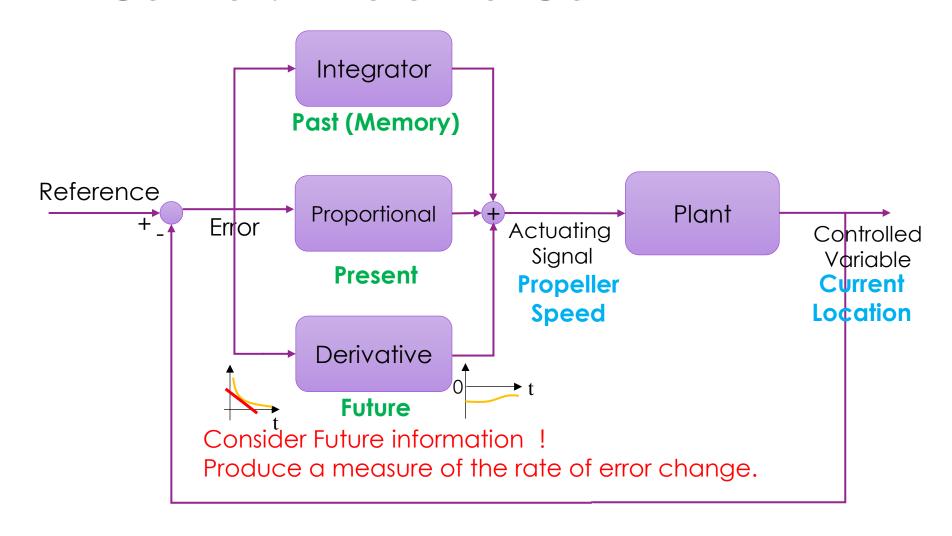
> 20 * 10 = 200 rpm

> * 100 = 200 rpm

Idea: Consider past information!

PID Control: Integral Gain Overshooting! Consider past information! Sum up non-zero steady state error over time distance Negative **Error** Integrator → What if >200 rpm? Past (Memory) time Reference 200 Proportional Plant **Error** Actuating Controlled Signal Variable **Present Propeller** Current **Speed** Location

PID Control: Differential Gain



PID Control

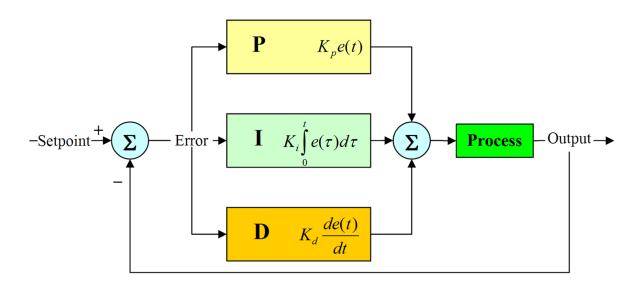
Proportional / Integral / Derivative Control

Continuous Form:

$$Output = K_p e(t) + K_i \int_0^t e_t dt + K_d \frac{de(t)}{dt}$$

Discrete Form:

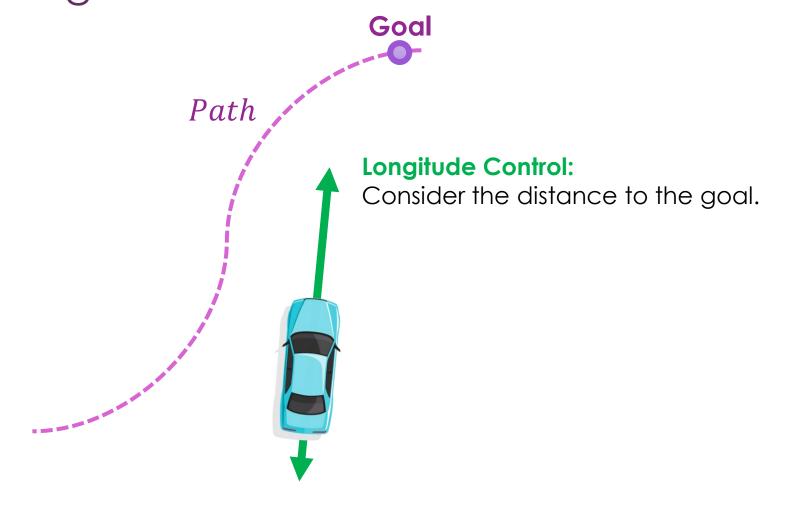
Output =
$$K_p e(t) + K_i \sum_{t=0}^{t} e_t + K_d(e(t) - e(t-1))$$



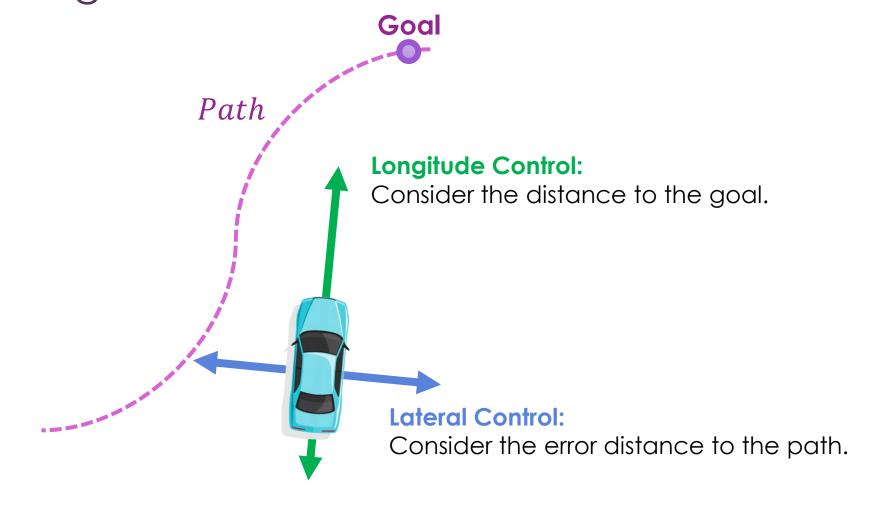
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Path Tracking Problem



Path Tracking Problem



Basic Kinematic Model

State:

Rotation Matrix:

$$\xi_I = \begin{bmatrix} x^- \\ y \\ \theta \end{bmatrix}$$

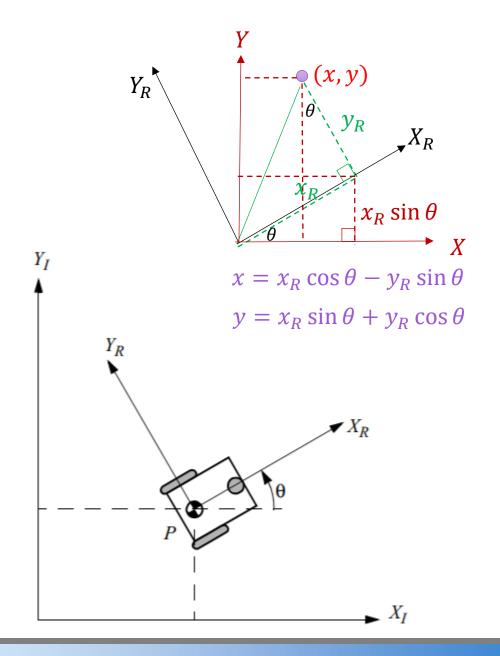
$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1 \end{bmatrix}$$

Kinematic Model:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = R(\theta) \begin{bmatrix} \dot{x_R} \\ \dot{y_R} \\ \dot{\theta} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ 0 \\ \omega \end{bmatrix}$$

$$= \begin{bmatrix} v\cos(\theta) \\ v\sin(\theta) \\ v\sin(\theta) \end{bmatrix}$$



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Left Motor speed

Right Motor speed



y_R ψ_1 v

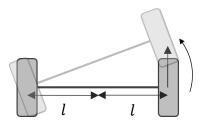
Differential Drive Vehicle (cont.)

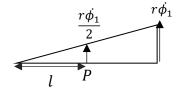
Right Wheel:

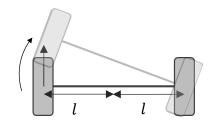
$$x_{R1} = \frac{r\dot{\phi_1}}{2}$$
$$\omega_1 = \frac{r\dot{\phi_1}}{2l}$$

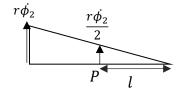
Left Wheel:

$$\begin{aligned}
x_{R2} &= \frac{r\dot{\phi_2}}{2} \\
\omega_2 &= \frac{-r\dot{\phi_2}}{2l}
\end{aligned}$$









Kinematic model for differential drive:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = R(\theta)^{-1} \begin{bmatrix} \dot{x_R} \\ \dot{y_R} \\ \dot{\theta} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x_R} \\ \dot{y_R} \\ \dot{\theta} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r\dot{\phi_1} \\ \frac{1}{2} + r\dot{\phi_2} \\ 0 \\ \frac{r\dot{\phi_1}}{2l} - \frac{r\dot{\phi_2}}{2l} \end{bmatrix}$$

Differential Drive Vehicle

• Given target velocity v and angular velocity ω

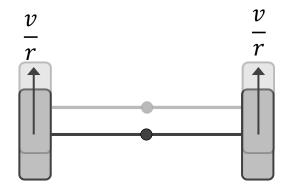
$$\begin{cases} v = \frac{r\dot{\phi}_1}{2} + \frac{r\dot{\phi}_2}{2} \\ \omega = \frac{r\dot{\phi}_1}{2l} - \frac{r\dot{\phi}_2}{2l} \end{cases}$$

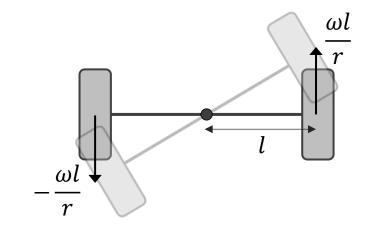
$$\dot{\phi_2} = \left(v - \frac{r\dot{\phi_1}}{2}\right)\frac{2}{r} = \frac{2v}{r} - \dot{\phi_1}$$

$$\omega = \frac{r\dot{\phi_1}}{2l} - \frac{r\left(\frac{2v}{r} - \dot{\phi_1}\right)}{2l} = \frac{r\dot{\phi_1} - v}{l}$$

$$\dot{\phi_1} = \frac{v}{r} + \frac{\omega l}{r}$$

$$\dot{\phi_2} = \frac{v}{r} - \frac{\omega l}{r}$$





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Pure Pursuit Control

- Concept:
 - Modify the angular velocity to let the center achieve a point on path

$$\alpha = \arctan\left(\frac{y - y_g}{x - x_g}\right) - \theta$$

$$\frac{L_d}{\sin(2\alpha)} = \frac{R}{\sin\left(\frac{\pi}{2} - \alpha\right)}$$

$$R = \frac{L_d \sin\left(\frac{\pi}{2} - \alpha\right)}{\sin(2\alpha)} = \frac{L_d \cos(\alpha)}{2\sin(\alpha)\cos(\alpha)} = \frac{L_d}{2\sin(\alpha)}$$

$$\omega = \frac{v}{R} = \frac{2v\sin(\alpha)}{L_d}$$

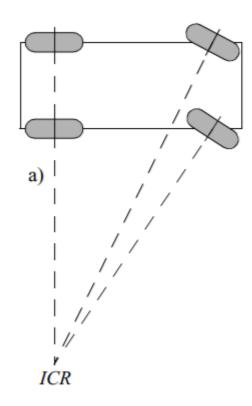
 2α

 L_d usually set to $(kv + L_{fc})$, where k, L_{fc} are parameters.

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Speed and Steering Control



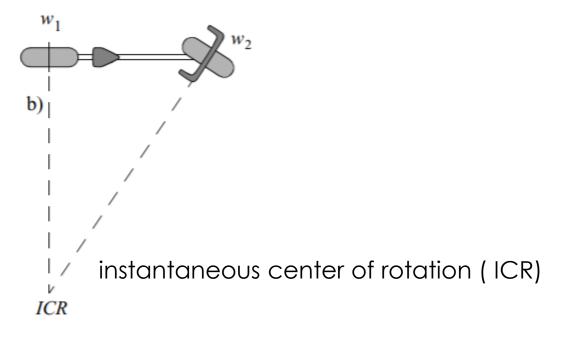


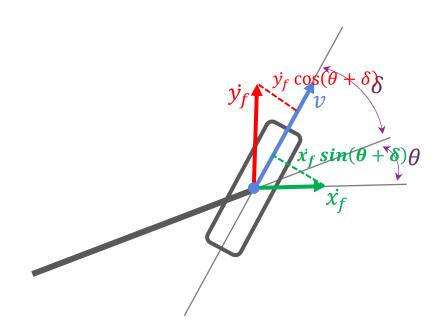
Figure 3.12
(a) Four-wheel with car-like Ackerman steering. (b) bicycle.

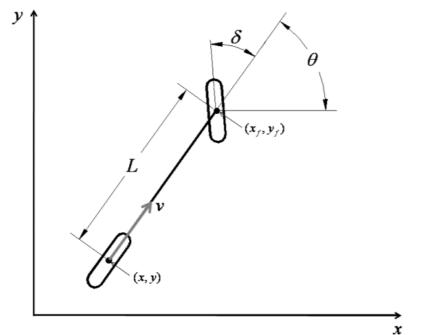
nonholonomic constraint equations

$$\dot{x_f}\sin(\theta + \delta) - \dot{y_f}\cos(\theta + \delta) = 0 \qquad (1) \text{ Front Wheel}$$

 $\dot{x}\sin(\theta) - \dot{y}\cos(\theta) = 0$

(2) Rear Wheel





nonholonomic constraint equations

$$\dot{x}_f \sin(\theta + \delta) - \dot{y}_f \cos(\theta + \delta) = 0$$
 (1) Front Wheel $\dot{x} \sin(\theta) - \dot{y} \cos(\theta) = 0$ (2) Rear Wheel

(2) Rear Wheel



$$x_f = x + L\cos(\theta)$$

 $y_f = y + L\sin(\theta)$



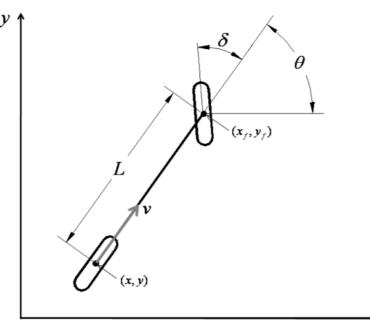
Eliminating front wheel position from (1)

$$0 = (\dot{x} - \dot{\theta}L\sin(\theta))\sin(\theta + \delta) - (\dot{y} + \dot{\theta}L\cos(\theta))\cos(\theta + \delta)$$

$$= \dot{x}\sin(\theta + \delta) - \dot{y}\cos(\theta + \delta) - \dot{\theta}L\sin(\theta)(\sin(\theta)\cos(\delta) + \cos(\theta)\sin(\delta))$$

$$-\dot{\theta}L\cos(\theta)(\cos(\theta)\cos(\delta) - \sin(\theta)\sin(\delta))$$

$$= \dot{x}\sin(\theta + \delta) - \dot{y}\cos(\theta + \delta) - \dot{\theta}L\cos(\delta) \quad (3)$$



nonholonomic constraint equations

$$\dot{x}\sin(\theta + \delta) - \dot{y}\cos(\theta + \delta) - \dot{\theta}L\cos(\delta) = 0$$
 (3)
$$\dot{x}\sin(\theta) - \dot{y}\cos(\theta) = 0$$
 (2)

Rear wheel satisfied the constrain (2) when

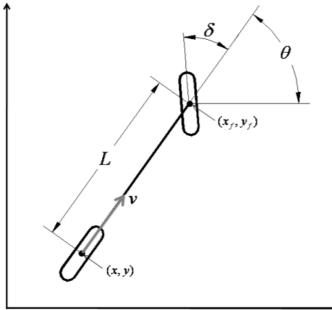
$$\dot{x} = v\cos(\theta) \qquad (4)
\dot{y} = v\sin(\theta) \qquad (5)$$

Applying (4)(5) to (3)

$$\dot{\theta} = \frac{\dot{x}\sin(\theta + \delta) - \dot{y}\cos(\theta + \delta)}{L\cos(\delta)}$$

$$= \frac{v\cos(\theta)(\sin(\theta)\cos(\delta) + \cos(\theta)\sin(\delta)) - v\sin(\theta)(\cos(\theta)\cos(\delta) - \sin(\theta)\sin(\delta))}{L\cos(\delta)}$$

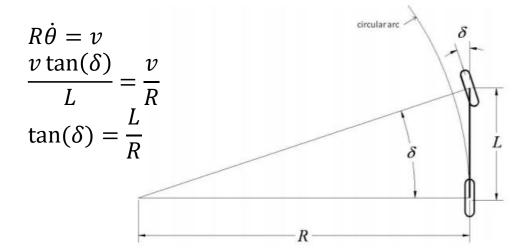
$$= \frac{v(\cos^{2}(\theta) + \sin^{2}(\theta))\sin(\delta)}{L\cos(\delta)} = \frac{v\tan(\delta)}{L}$$

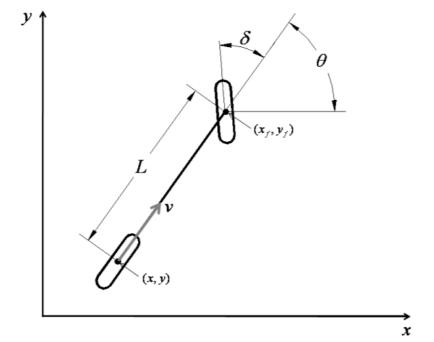


Kinematic Model

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ \tan(\delta) \\ L \end{bmatrix} v$$

Some Property





Pure Pursuit Control for Bicycle Model

- Concept:
 - Control the steer to let the rear wheel achieve a point on the path.

$$\alpha = \arctan\left(\frac{y - y_g}{x - x_g}\right) - \theta$$

$$R = \frac{L_d \sin\left(\frac{\pi}{2} - \alpha\right)}{\sin(2\alpha)} = \frac{L_d \cos(\alpha)}{2\sin(\alpha)\cos(\alpha)} = \frac{L_d}{2\sin(\alpha)}$$

$$\tan(\delta) = \frac{L}{R}$$

$$\delta = \arctan\left(\frac{L}{R}\right) = \arctan\left(\frac{2L\sin(\alpha)}{L_d}\right)$$

 (g_x, g_y) circular arc 2α

 L_d usually set to $(kv + L_{fc})$, where k, L_{fc} are parameters.

Stanley Control

- Concept:
 - Exponential stability for front wheel feedback
- Differential of error distance

$$\dot{e} = v_f sin(\delta - \theta_e)$$

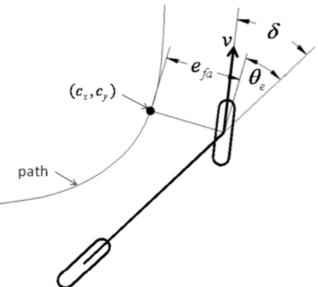
To achieve exponential stability to path, we can set

$$\dot{e} = -ke$$
, where k>0

$$-ke = v_f sin(\delta - \theta_e)$$

$$\delta = \arcsin\left(-\frac{ke}{v_f}\right) + \theta_e$$





• It is not defined when |-ke/vf| > 1. We can modify the control law to $\delta = \arctan\left(-\frac{ke}{v_{\ell}}\right) + \theta_{e}$, which satisfy the local exponential stability (LES).

LQR Control

- If we use the motion model with more complex form (e.g. dynamic model), it is hard to directly analyze the error function.
- Linear Quadratic Regulator (LQR) introduce the concept of cost function, and try to solve the optimization problem when the motion model is linear form and the cost function is quadratic form.
- The formulation of LQR problem:
 - Define state **x** and control **u**, the motion model is $\dot{x} = Ax + Bu$.
 - The cost function is setting to the quadratic form $c = \underline{x^T Q x} + \underline{u^T R u}$

State Error Minimum Control

, in which ${\bf Q}$ is the state weighting matrix and ${\bf R}$ is the control weighting matrix.

– The total objective function of an episode $J = \int_0^{Ter} [x(t)^T Q x(t) + u(t)^T R u(t)] dt + x^T (Ter) S x(Ter)$

• The goal is to find the optimal control \mathbf{u}^* which minimize the total object function: $\min_{u} J = \min_{u} \int_{0}^{T} x(t)^{T} Qx(t) + u(t)^{T} Ru(t) dt + x(T)^{T} Sx(T)$

• To solve this problem, we first introduce the concept of optimal principle. If we have a optimal control sequence $[u_t^*, u_{t+1}^*, u_{t+2}^*, ..., u_T^*]$, then the subsequence $[u_{t+1}^*, u_{t+2}^*, ..., u_T^*]$ is also an optimal control sequence.

 Follow the concept, we can apply dynamic programming to recursively solve the optimal control from terminal state to current time.

- However, we do not know the terminal time or even the terminal time is infinite in most time. In this case, we can solve the LQR using the recursive relation of value function.
- Introduce the value function V(x), which is the summing of the future cost.
 We can write down the recursive form of the discrete time value function:

$$V(x_t) = \min_{\mathbf{u}} \{ x_t^T Q x_t + u_t R u_t + V(x_{t+1}) \}$$

• We can guess the value function to be quadratic form $V(x_t) = x_t^T P_t x_t$ (which P is symmetric positive-definite), and apply the linear motion model $Ax_t + Bu_t$ to value function:

$$\begin{split} V(x_t) &= \min_{\mathbf{u}} \{x_t^T Q x_t + u_t R u_t + x_{t+1}^T P_{t+1} x_{t+1} \} \\ &= \min_{\mathbf{u}} \{x_t^T Q x_t + u_t R u_t + (A x_t + B u_t)^T P_{t+1} (A x_t + B u_t) \} \\ &= \min_{\mathbf{u}} \{x_t^T (Q + A^T P_{t+1} A) x_t + 2 x^T A^T P B u + u_t^T (R + B^T P_{t+1} B) u_t \} \end{split}$$

Solve the minimum equation

$$V(x_{t}) = x_{t}^{T} P_{t} x_{t} = \min_{\mathbf{u}} \{ x_{t}^{T} (Q + A^{T} P_{t+1} A) x_{t} + 2 x^{T} A^{T} P B u + u_{t}^{T} (R + B^{T} P_{t+1} B) u_{t} \}$$

$$\frac{\partial}{\partial u} \left[x_{t}^{T} (Q + A^{T} P_{t+1} A) x_{t} + 2 x^{T} A^{T} P B u_{t}^{*} + u_{t}^{*T} (R + B^{T} P_{t+1} B) u_{t}^{*} \right] = 0$$

$$2 (x^{T} A^{T} P_{t+1} B)^{T} + 2 (R + B^{T} P_{t+1} B) u_{t}^{*} = 0$$

$$u_{t}^{*} = -(R + B^{T} P_{t+1} B)^{-1} B^{T} P_{t+1} A x_{t}$$

Apply u* to the value function, and get the equation of P

$$\begin{aligned} x_t^T P_t x_t &= x_t^T (Q + A^T P_{t+1} A - A^T P_{t+1} B (R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A) x_t \\ P_t &= Q + A^T P_{t+1} A - A^T P_{t+1} B (R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A \\ & \text{Discrete Algebra Riccati Equation (DARE)} \end{aligned}$$

Remark: In continuous case, $\dot{P} = -PA - A^TP + PBR^{-1}P - Q$ is the Continuous Algebra Riccati Equation (CARE)

Given discrete Riccati algebra equation

$$P_{t} = Q + A^{T} P_{t+1} A - A^{T} P_{t+1} B (R + B^{T} P_{t+1} B)^{-1} B^{T} P_{t+1} A$$

Suppose the value function is time-invariant, then

$$P = Q + A^T P A - A^T P B (R + B^T P B)^{-1} B^T P A$$

• In practice, we can first initialize $P^{(0)}=Q$, then iteratively apply the Riccati equation on until converge :

```
INITIALIZE: P \leftarrow Q

REPEAT
P_{next} \leftarrow Q + A^T PA - A^T PB(R + B^T PB)^{-1}B^T PA
\epsilon \leftarrow ||P_{next} - P||
IF \epsilon < threshold THEN
return P_{next}
ENDIF
P \leftarrow P_{next}
END
```

LQR Control for Kinematic Model

- Take an example to solve the LQR optimal control of the kinematic model.
- Define State: $x = [e, \dot{e}, \theta, \dot{\theta}]$, and set the matrix Q and R
- The linear approximate of kinematic motion model:

$$\frac{d}{dt} \begin{bmatrix} e \\ \dot{e} \\ \dot{\theta} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 1 & dt & 0 & 0 \\ 0 & 0 & v & 0 \\ 0 & 0 & 1 & dt \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e \\ \dot{e} \\ \dot{\theta} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ v \tan(\delta) \\ L \end{bmatrix} \approx \begin{bmatrix} 1 & dt & 0 & 0 \\ 0 & 0 & v & 0 \\ 0 & 0 & 1 & dt \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e \\ \dot{e} \\ \dot{\theta} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ v \\ L \end{bmatrix} \delta = Ax + Bu$$

Solve the DARE to get the P matrix

$$P = Q + A^T P A - A^T P B (R + B^T P B)^{-1} B^T P A$$

Finally, we can get the optimal control

$$u_t^* = -(R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A x_t$$

LQR Control

- Following the steps:
 - Construct the matrix A, B, X of the linear approximation model.

$$Ax + Bu = \begin{bmatrix} 1 & dt & 0 & 0 \\ 0 & 0 & v & 0 \\ 0 & 0 & 1 & dt \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e \\ \dot{e} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{v}{L} \end{bmatrix} \delta \text{ (Bicycle Model)}$$

$$A \qquad X \qquad B \qquad U$$

– Solve DARE and get the matrix P of the value function:

$$P = Q + A^T P A - A^T P B (R + B^T P B)^{-1} B^T P A$$

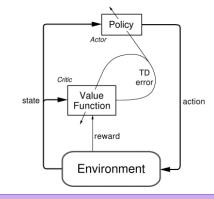
– Compute the optimal control:

$$u_t^* = -(R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A x_t$$

Review of Control Algorithms

$$\delta = K_p e(t) + K_i \sum_{i=0}^{t} e_t + K_d(e(t) - e(t-1))$$

$$\delta = \arctan\left(-\frac{ke}{v_f}\right) + \theta_e$$



PID Control

Stanley Control

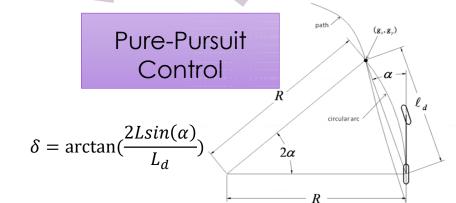
Model-free Reinforcement Learning

Apply the kinematic property.

Consider the progressive stability.

More complex motion model.

Don't need model. Non-linear case.



LQR Control

DARE:

$$P_t = Q + A^T P_{t+1} A - A^T P_{t+1} B (R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A$$

Q&A