Extreme Mass Ratio Binaries

An extreme mass ratio binary consists of a stellar-mass compact object (the secondary) that orbits around a supermassive black hole (the primary). The ratio of the mass of the secondary to that of the primary is in the range 10^{-6} to 10^{-4}.

Since the mass of the secondary is negligible compared to the mass of the primary, its motion in the gravitational field of the primary at the zeroth order can be approximated by geodesic motion [1].

Geodesic Motion

In the theory of General Relativity, the world lines of free test particles in a gravitational field are geodesics of the spacetime [2]. They are the straightest possible curves of extremal length that connect two distinct spacetime points. The *Lagrangian* of a free particle with rest mass μ can be written as

$$\mathcal{L} = \frac{1}{2}\mu g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta, \tag{1}$$

where the overhead dot represents differentiation with respect to proper time τ. The equations of motion can be obtained from the Euler-Lagrange equation [3],

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\lambda}\right) - \frac{\partial \mathcal{L}}{\partial x^\lambda} = 0. \tag{2}$$

The metric tensor field g on the spacetime manifold $\mathcal{M}$ is given by

$$g = g_{\alpha\beta} dx^\alpha \otimes dx^\beta, \tag{3}$$

where $g_{\alpha\beta} \coloneqq g\left(\partial_\alpha, \partial_\beta\right)$. The invariant line element ds^2 can be obtained from the metric by considering an infinitesimal displacement vector $dx^\alpha \dfrac{\partial}{\partial x^\alpha}$ [4],

$$ds^2 = g\left(dx^\alpha \frac{\partial}{\partial x^\alpha}, dx^\beta \frac{\partial}{\partial x^\beta}\right) = dx^\alpha dx^\beta g\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right) = g_{\alpha\beta} dx^\alpha dx^\beta. \tag{4}$$

The line element is related to the proper time through

$$d\tau^2 = -ds^2. \tag{5}$$

From (4) and (5), we arrive at the following result:

$$g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = -1. \tag{6}$$

Inspecting (1) and (6), we conclude that the Lagrangian of a free test particle is

$$\mathcal{L} = -\frac{1}{2}\mu. \tag{7}$$

The purely kinetic Lagrangian remains invariant during the particle's motion due to the fact that its rest mass is a conserved quantity [5].

Stationary & Axisymmetric Spacetimes

Equilibrium configurations of rotating bodies possessing axial symmetry can be described by *stationary* and *axisymmetric* spacetimes [2]. If a spacetime is stationary (and axially symmetric), it is possible to find coordinates such that all the components of the metric are independent of the timelike coordinate (and the angle around the axis of symmetry) [6]. The line element associated with a general stationary, axisymmetric spacetime can be written in Boyer–Lindquist coordinates (t, r, θ, ϕ) as

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2, \tag{8}$$

where all the metric components are functions of r and θ only. The Lagrangian of a free test particle moving in this spacetime is obtained from (1),

$$\mathcal{L} = \frac{1}{2}\mu\left(g_{tt}\dot{t}^2 + 2g_{t\phi}\dot{t}\dot{\phi} + g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2 + g_{\phi\phi}\dot{\phi}^2\right). \tag{9}$$

The Lagrangian is independent of t and ϕ since the spacetime is stationary and axially symmetric. Therefore,

$$\left(\frac{\partial\mathcal{L}}{\partial t}\right) = \left(\frac{\partial\mathcal{L}}{\partial\phi}\right) = 0. \tag{10}$$

Let us now calculate the equations of motion of the particle using (2). For $x^0 = t$,

$$\frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{t}}\right) - \left(\frac{\partial\mathcal{L}}{\partial t}\right) = 0.$$

Using (10), the equation shown above reduces to

$$\frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{t}}\right) = 0$$
$$\implies \frac{d}{d\tau}\left(g_{tt}\dot{t} + g_{t\phi}\dot{\phi}\right) = 0.$$

Thus, we obtain a constant of motion

$$E \coloneqq g_{tt}\dot{t} + g_{t\phi}\dot{\phi}, \tag{11}$$

which corresponds to the energy per unit mass (specific energy) of the particle. When we calculate the equations of motion for $x^3 = \phi$, we obtain another constant of motion,

$$L_z \coloneqq g_{t\phi}\dot{t} + g_{\phi\phi}\dot{\phi}. \tag{12}$$

It corresponds to the component of the orbital angular momentum per unit mass along the direction of the symmetry axis. For the sake of brevity, E and L_z from now on will be referred to as specific energy and specific angular momentum of the particle. Solving (11) and (12) for $\dot{t}$ and $\dot{\phi}$, we get

$$\dot{t} = \left(\frac{g_{\phi\phi}E + g_{t\phi}L_z}{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}\right), \tag{13}$$

$$\dot{\phi} = -\left(\frac{g_{t\phi}E + g_{tt}L_z}{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}\right). \tag{14}$$

To obtain the change in the ϕ coordinate of the particle with respect to the coordinate time t, we make use of the chain rule for derivatives and then substitute the expressions for $\dot{t}$ and $\dot{\phi}$ from (13) and (14). Thus,

$$v_\phi := \frac{d\phi}{dt} = \frac{d\phi/d\tau}{dt/d\tau} = \frac{\dot{\phi}}{\dot{t}} = -\left(\frac{g_{t\phi}E + g_{tt}L_z}{g_{\phi\phi}E + g_{t\phi}L_z}\right). \tag{15}$$

The Euler-Lagrange equation for the r coordinate is

$$\frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{r}}\right) - \frac{\partial\mathcal{L}}{\partial r} = 0. \tag{16}$$

Let us now calculate the terms in the above equation one by one,

$$\frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{r}}\right) = \frac{d}{d\tau}\left(\frac{1}{2}\mu\left(2g_{rr}\dot{r}\right)\right) = \frac{1}{2}\mu\left(2\dot{g}_{rr}\dot{r} + 2g_{rr}\ddot{r}\right). \tag{17}$$

Since g_{rr} is a function of only r and θ,

$$\dot{g}_{rr} := \frac{d}{d\tau}\left(\dot{g}_{rr}\right) = \dot{r}\partial_r g_{rr} + \dot{\theta}\partial_\theta g_{rr}.$$

Using the above expression for $\dot{g}_{rr}$ in (17), we get

$$\frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{r}}\right) = \frac{1}{2}\mu\left(2g_{rr}\ddot{r} + 2\dot{r}^2\partial_r g_{rr} + 2\dot{r}\dot{\theta}\partial_\theta g_{rr}\right). \tag{18}$$

Also,

$$\frac{\partial\mathcal{L}}{\partial r} = \frac{1}{2}\mu\left(\dot{t}^2\partial_r g_{tt} + 2\dot{t}\dot{\phi}\partial_r g_{t\phi} + \dot{r}^2\partial_r g_{rr} + \dot{\theta}^2\partial_r g_{\theta\theta} + \dot{\phi}^2\partial_r g_{\phi\phi}\right). \tag{19}$$

From (16), (18) and (19), we arrive at the following equation of motion for the r coordinate:

$$2g_{rr}\ddot{r} + 2\dot{r}\dot{\theta}\partial_\theta g_{rr} + \dot{r}^2\partial_r g_{rr} - \dot{t}^2\partial_r g_{tt} - 2\dot{t}\dot{\phi}\partial_r g_{t\phi} - \dot{\theta}^2\partial_r g_{\theta\theta} - \dot{\phi}^2\partial_r g_{\phi\phi} = 0. \tag{20}$$

Similarly, the equation of motion corresponding to the θ coordinate is obtained as

$$2g_{\theta\theta}\ddot{\theta} + 2\dot{r}\dot{\theta}\partial_r g_{\theta\theta} - \dot{r}^2\partial_\theta g_{rr} - \dot{t}^2\partial_\theta g_{tt} - 2\dot{t}\dot{\phi}\partial_\theta g_{t\phi} + \dot{\theta}^2\partial_\theta g_{\theta\theta} - \dot{\phi}^2\partial_\theta g_{\phi\phi} = 0. \tag{21}$$

The ordinary differential equations in (20) and (21) are with respect to the proper time. Let us express them with respect to the coordinate time. The derivatives of the r and θ coordinates with respect to t are

$$\begin{aligned} v_r &:= \frac{dr}{dt} = \frac{dr/d\tau}{dt/d\tau} = \frac{\dot{r}}{\dot{t}}, \\ v_\theta &:= \frac{d\theta}{dt} = \frac{d\theta/d\tau}{dt/d\tau} = \frac{\dot{\theta}}{\dot{t}}, \end{aligned} \tag{22}$$

and the rate of change of $\dot{t}$ with respect to proper time τ is

$$\ddot{t} := \frac{d\dot{t}}{d\tau} = \frac{\partial\dot{t}}{\partial r}\dot{r} + \frac{\partial\dot{t}}{\partial\theta}\dot{\theta} = \frac{\partial\dot{t}}{\partial r}v_r\dot{t} + \frac{\partial\dot{t}}{\partial\theta}v_\theta\dot{t}. \tag{23}$$

The rate of change of $\dot{r}$ with respect to τ is

$$\ddot{r} := \frac{d\dot{r}}{d\tau} = \frac{d}{d\tau}\left(v_r\dot{t}\right) = \frac{dv_r}{d\tau}\dot{t} + v_r\ddot{t}$$

$$\implies \ddot{r} = \dot{t}^2 \frac{dv_r}{dt} + v_r \ddot{t}. \tag{24}$$

From (20), (22) and (24), we get

$$\begin{aligned} 2g_{rr}\left(\dot{t}^2\frac{dv_r}{dt} + v_r\ddot{t}\right) =& -2v_r v_\theta \dot{t}^2 \partial_\theta g_{rr} - v_r^2 \dot{t}^2 \partial_r g_{rr} + \dot{t}^2 \partial_r g_{tt} \\ &+ 2\dot{t}\dot{\phi}\partial_r g_{t\phi} + v_\theta^2 \dot{t}^2 \partial_r g_{\theta\theta} + \dot{\phi}^2 \partial_r g_{\phi\phi}. \end{aligned}$$

Thus, the change in the r coordinate of the particle with respect to the coordinate time is given by the equations

$$\begin{aligned} \frac{dr}{dt} &= v_r, \\ \frac{dv_r}{dt} &= -\frac{v_r}{\dot{t}^2}\ddot{t} - \frac{v_r v_\theta \partial_\theta g_{rr}}{g_{rr}} - \frac{v_r^2 \partial_r g_{rr}}{2g_{rr}} + \frac{\partial_r g_{tt}}{2g_{rr}} + \frac{v_\phi \partial_r g_{t\phi}}{g_{rr}} + \frac{v_\theta^2 \partial_r g_{\theta\theta}}{2g_{rr}} + \frac{v_\phi^2 \partial_r g_{\phi\phi}}{2g_{rr}}. \end{aligned} \tag{25}$$

Similarly, the equations that govern the change in the θ coordinate of the particle with respect to the coordinate time can be obtained as

$$\begin{aligned} \frac{d\theta}{dt} &= v_\theta, \\ \frac{dv_\theta}{dt} &= -\frac{v_\theta}{\dot{t}^2}\ddot{t} - \frac{v_r v_\theta \partial_r g_{\theta\theta}}{g_{\theta\theta}} + \frac{v_r^2 \partial_\theta g_{rr}}{2g_{\theta\theta}} + \frac{\partial_\theta g_{tt}}{2g_{\theta\theta}} + \frac{v_\phi \partial_\theta g_{t\phi}}{g_{\theta\theta}} - \frac{v_\theta^2 \partial_\theta g_{\theta\theta}}{2g_{\theta\theta}} + \frac{v_\phi^2 \partial_\theta g_{\phi\phi}}{2g_{\theta\theta}}. \end{aligned} \tag{26}$$

In a general stationary and axisymmetric spacetime, (6) can be written as

$$g_{tt}\dot{t}^2 + 2g_{t\phi}\dot{t}\dot{\phi} + g_{rr}\dot{r} + g_{\theta\theta}\dot{\theta}^2 + g_{\phi\phi}\dot{\phi}^2 = -1. \tag{27}$$

Substituting the expressions for $\dot{t}$ and $\dot{\phi}$ from (13) and (14), we obtain

$$\begin{aligned} &\frac{g_{tt}\left(g_{\phi\phi}^2 E^2 + g_{t\phi}^2 L_z^2 + 2g_{\phi\phi}g_{t\phi}EL_z\right)}{\left(g_{t\phi}^2 - g_{tt}g_{\phi\phi}\right)^2} - \frac{2g_{t\phi}\left(g_{\phi\phi}E + g_{t\phi}L_z\right)\left(g_{t\phi}E + g_{tt}L_z\right)}{\left(g_{t\phi}^2 - g_{tt}g_{\phi\phi}\right)^2} \\ &\qquad + g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2 + \frac{g_{\phi\phi}\left(g_{t\phi}^2 E^2 + g_{tt}^2 L_z^2 + 2g_{t\phi}g_{tt}EL_z\right)}{\left(g_{t\phi}^2 - g_{tt}g_{\phi\phi}\right)^2} = -1. \end{aligned}$$

The equation shown above can be simplified into the following form:

$$g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2 + \frac{g_{tt}g_{\phi\phi}\left(g_{\phi\phi}E^2 + g_{tt}L_z^2 + 2g_{t\phi}EL_z\right) - g_{t\phi}^2\left(g_{\phi\phi}E^2 + g_{tt}L_z^2 + 2g_{t\phi}EL_z\right)}{\left(g_{t\phi}^2 - g_{tt}g_{\phi\phi}\right)^2} = -1$$

$$\implies g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2 + 1 + \frac{g_{\phi\phi}E^2 + g_{tt}L_z^2 + 2g_{t\phi}EL_z}{g_{tt}g_{\phi\phi} - g_{t\phi}^2} = 0.$$

Dividing the equation by g_{rr} yields

$$\dot{r}^2 + \frac{g_{\theta\theta}}{g_{rr}}\dot{\theta}^2 + \frac{1}{g_{rr}}\left(1 + \frac{g_{\phi\phi}E^2 + g_{tt}L_z^2 + 2g_{t\phi}EL_z}{g_{tt}g_{\phi\phi} - g_{t\phi}^2}\right) = 0$$

$$\implies \dot{r}^2 + \frac{g_{\theta\theta}}{g_{rr}}\dot{\theta}^2 + V_{\mathit{eff}} = 0, \tag{28}$$

where V_{eff} is the *effective potential* and it is defined as

$$V_{eff} \coloneqq \frac{1}{g_{rr}} \left(1 + \frac{g_{\phi\phi}E^2 + g_{tt}L_z^2 + 2g_{t\phi}EL_z}{g_{tt}g_{\phi\phi} - g_{t\phi}^2} \right). \tag{29}$$

Thus, we obtain a constraint equation for the motion of the free test particle. It is evident from (28) that $V_{eff} = 0$ whenever $\dot{r}$ and $\dot{\theta}$ become zero simultaneously. The curve in the (r, θ)-plane defined by vanishing of the effective potential is called the *Curve of Zero Velocity* (CZV). Using (22), the constraint equation can be expressed with respect to the coordinate time t,

$$\begin{aligned} v_r^2 \dot{t}^2 + \frac{g_{\theta\theta}}{g_{rr}} v_\theta^2 \dot{t}^2 + V_{eff} &= 0 \\ \implies v_r^2 + \frac{g_{\theta\theta}}{g_{rr}} v_\theta^2 + V_{eff}\, \dot{t}^{-2} &= 0. \end{aligned} \tag{30}$$

Example Spacetimes

The spacetimes described by the Schwarzschild metric and the Kerr metric are stationary and axisymmetric. They are vacuum solutions to the Einstein field equations [7].

Schwarzschild metric

It describes the spacetime around an uncharged, non-rotating, spherically symmetric black hole. The non-vanishing metric components in the Schwarzschild coordinates (t, r, θ, ϕ) are:

$$\begin{aligned} g_{tt} &= -\left(1 - \frac{2M}{r}\right) \\ g_{rr} &= \left(1 - \frac{2M}{r}\right)^{-1} \\ g_{\theta\theta} &= r^2 \\ g_{\phi\phi} &= r^2 \sin^2(\theta) \end{aligned} \tag{31}$$

Kerr metric

It describes the spacetime around an uncharged, rotating black hole. The non-vanishing metric components in the Boyer–Lindquist coordinates (t, r, θ, ϕ) are:

$$\begin{aligned} g_{tt} &= -\left(1 - \frac{2Mr}{\rho^2}\right) \\ g_{t\phi} &= -\frac{2Mar\sin^2(\theta)}{\rho^2} \\ g_{rr} &= \frac{\rho^2}{\Delta} \\ g_{\theta\theta} &= \rho^2 \\ g_{\phi\phi} &= \frac{\sin^2(\theta)}{\rho^2}\left[\left(r^2 + a^2\right)^2 - a^2\Delta\sin^2(\theta)\right] \end{aligned} \tag{32}$$

where

$$\begin{aligned}\Delta(r) &= r^2 - 2Mr + a^2 \\ \rho^2\left(r,\theta\right) &= r^2 + a^2\cos^2\left(\theta\right)\end{aligned} \tag{33}$$

When the spin parameter a vanishes, the Schwarzschild metric is recovered.

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