Quantum PCP conjecture

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1 Introduction

The classical *probabilistically checkable proofs* (or **PCP**) has its importance in the case of NP complete problems and the verification of their solutions. Authors thought of some properties of classical physics failing in some aspects and tried to come up with a logical reasoning by introducing the idea of **q-PCP**.

2 Quantum Hamiltonian Complexity

The paper describes about analogy been drawn between classical computational complexity theory and very much unrelated branch of science, **condensed body physics**. In classical computational complexity theory, we have *constraint satisfactory problems* (or **CSP**) that are collection of clauses/ constraints which are build up of boolean variables and each clause has atmost k variables (**k-CSP**). NP complete problems are the ones for which though there is no efficient solution algorithm but there exist an algorithm by which a proposed solution can be verified in polynomial time. By **Cook-Levin theorem**, it is NP hard to decide whether a CSP instance has a satisfiable model.
A condensed matter consists of many interacting particles, which may have different configurations and so different energy levels. The energy of the system is determined by an operator called the *Hamiltonian*. We want to find

out the configuration with lowest energy out of all the configurations and also the minimal energy.
Hamiltonian is the sum of *local* terms, namely terms that determine the energy of a small number of "neighboring" particles and the total energy is the sum of energies from these local terms.

Each of these local terms act non-trivially only on a small group of particles, and trivially on the rest of the system. So we can think of it as a constraint acting on those particle and this constraint is not valid outside this small group. Some particles in the group will act in order to increase energy and some will try to decrease it. The particles acting in order to increase the energy of local terms seems as satisfying literal and the ones decreasing energy seems like non-satisfying literals of a clause. So the small group of particles is analogous to a clause consisting of small number of literals, and all these local terms combine to get a system which is analogous to the collection of clauses forming a boolean formula. Our aim is to find the minimal energy state and minimal energy, which is analogous to finding an assignment to boolean variables such that fewest clauses are violated. This is mathematically shown ahead.
Let H be k-local Hamiltonian on a system of n particles, out of which, it acts non-trivially on k particles.

$$H = \sum_{i=1}^{m} H_i$$

where H_i is Hermitian of norm not greater than 1. Let $|\Psi\rangle$ denote the configuration of the system, so the energy is denoted by,

$$\langle\Psi|H|\Psi\rangle = \sum_{i=1}^{m} \langle\Psi|H_i|\Psi\rangle$$

The *eigenvalues* of H are the energy levels of the system. Our main concern is to find the lowest energy or the *ground state* of the system.
We can frame this problem along with two real numbers a and b, which have the *promise gap*, b being the bigger one. The output is *whether the smallest eigenvalue is smaller than a or all eigenvalues are larger than b.* Considering the analogy we can say that the LH problem with a constant gap (γ¿0) is QMA-hard under polynomial time reductions, QMA being the quantum

analog of NP.

Quantum PCP Conjecture: *For any language in QMA there exists a polynomial time quantum verifier, which acts on the classical input string x and a witness $|\xi\rangle$, a quantum state of poly($|x|$) qubits, such that the verifier accesses only O(1) qubits from the witness and decides on acceptance or rejection with constant error probability.*

3 Motivation

The trivial motivation given for introducing the conjecture is that it is a natural analogue of the classical PCP theorem. Classical PCP theorem is proved by Dinur's Gap amplification proof. What if we quantized it?

3.1 Dinur's Gap Amplification for PCP Theorem

For any alphabet size d, there exist constants $0 < \gamma < 1$ and $W > 1$, together with an efficient algorithm that takes a constraint graph $G = (V, E)$ with alphabet size d, and transforms it to another constraint graph $G' = (V', E')$ with universal alphabet size d_0 such that the following holds:

- $|E'| \leq W|E|$ and $|V'| \leq W|V|$,
- (completeness) if UNSAT(G) = 0 then UNSAT(G') = 0,
- (soundness) if UNSAT(G) > 0 then UNSAT(G') $\leq$ min(2·UNSAT(G), γ)

3.2 Quantify Gap Amplification

For quantifying Dinur;'s Gap Amplification following chnages are made:

- The analog of constraint grapg $G = (V, E)$ is a a 2-local Hamiltonian H_G acting on d-dimensional particles placed on the vertices of G, where to every edge $e \in E$ we associate a 2-local projection H_e acting on the two adjacent particles.

$$H_G = \sum_{e \in E} H_e \, (quantum \, constarint \, graph)$$

- the quantum unsat-value of a state $|\psi\rangle$ with respect to H_G:

$$\mathrm{QUNSAT}_{\psi(H_G)} := \frac{\langle\psi|H|\psi\rangle}{m} = \frac{1}{m}\sum_{i=1}^{m}\langle\psi|H_i|\psi\rangle$$

For quantifying Dinur;'s Gap Amplification following problems occur: G is an expander graph in classical gap amplification, done through degree reduction. In quantum version there are two problems:

Entanglement: The particle we would like to "copy" is potentially entangled with additional vertices, and its state is unknown; it is unclear how to map the state of the particle and the rest of the system to a state in which there are more "identical copies" of that particle. In each step new variables are introduced, and consistency checks added; quantizing each such check presents an additional challenge.

In classical PCP proofs, one often uses some kind of error correcting code to check that we are inside the code, and a second to check that the original constraints.

4 Brand$\tilde{a}$o-Harrow's limitations

- ¡1-¿ The general approach is to identify conditions under which the approximation problem associated with the qPCP conjecture is inside NP, and therefore cannot be QMA-hard.
- ¡2-¿ They identify specific parameters of a 2-local Hamiltonian (as well as its groundstate) such that when the parameters lie in a certain range there is guaranteed to exist a product state whose average energy is within the relative promise gap Γ of the ground energy of the Hamiltonian.
- ¡3-¿ Such a product state can then serve as a classical witness, putting the approximation problem in NP

5 NLTS Conjucture

There exists a universal constant$c > 0$, an integer k and a family of k-local Hamiltonians $\{H(n)\}_n^\infty = 1\ n = 1$ such that for any n, $H^{(n)}$ acts on n particles, and all states of average energy less than c above the average ground energy with respect to $H^{(n)}$ are non-trivial.

- It was suggested by because of the difficulties in proving or disproving the qPCP conjecture, by choosing a characterization of multipartite entanglement through the notion of non-trivial states.

- A state $|\psi\rangle$ is said to be trivial if it is the output of a constant depth quantum circuit applied to the input state $|0^n\rangle$.

- We claim that the family of Hamiltonians produced by the qPCP reduction (or an infinite subfamily of it) satisfies the NLTS requirements for $c = \gamma$.

6 Is it really hard?

As the temperature of a physical system is cooled down, its state slowly evolves towards the ground state of the Hamiltonian that describes the system. Hence nature is by itself "computing" the ground state, or at least a mixture of low-energy states, of that Hamiltonian. But the qPCP asserts that approximating the energy of any such state is computationally hard, even for quantum computers — how can this computational hardness be reconciled with the fact that nature apparently solves the same problem on a day-to-day basis?

References

[1] Thomas Vidick Dorit Aharonov, Itai Arad. *The Quantum PCP Conjecture*. Cornell University, 2013.