Quantum PCP Conjucture

Dorit Aharonov, Itai Arad, Thomas Vidick

Kuldeep & Preet

August 20, 2023

IIT KANPUR
Indian Institute of Technology Kanpur

Contents

What is qPCP?

qPCP stands for the **Quantum Probabilistically Checkable Proofs**.

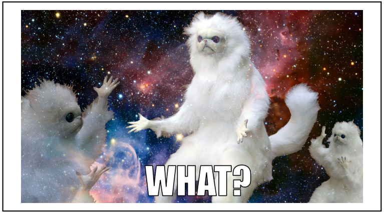
qPCP Conjucture

There exists constants $k, d \in \mathbb{N}$ and $0 < \delta < 1$ such that the following holds. Given integers n, m and k-local Hamiltonians H_i, it is QMA-hard to find an additive $\pm\delta$ approximation to the quantity

$$\varepsilon_0(H) = \min_{|\psi\rangle} \frac{1}{m} \sum_{i=1}^{m} \langle\psi|H_i|\psi\rangle,$$

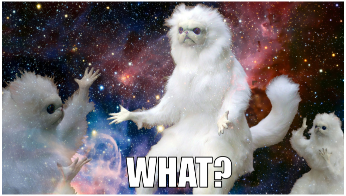
where the minimum is taken over all unit vectors $|\psi\rangle \in (\mathbb{C}^d)^{\otimes n}$.

What is qPCP?



- Is it hard to understand?

What is qPCP?



- Is it hard to understand?
- Well, it is **harder** to prove...

Problem it tackles

- **Condensed matter physics** is the study of properties of condensed phases of matter, such as solids or liquids.

Problem it tackles

- **Condensed matter physics** is the study of properties of condensed phases of matter, such as solids or liquids.
- **Quantum many-body physics** models the material as a lattice of interacting particles, each governed by the laws of quantum mechanics.

Problem it tackles

- **Condensed matter physics** is the study of properties of condensed phases of matter, such as solids or liquids.
- **Quantum many-body physics** models the material as a lattice of interacting particles, each governed by the laws of quantum mechanics.
- Interactions between the particles are typically nearest-neighbor; in addition there might be some global constraints, such as a magnetic field, that act independently and identically on each of the particles.

Problem it tackles

- **Condensed matter physics** is the study of properties of condensed phases of matter, such as solids or liquids.
- **Quantum many-body physics** models the material as a lattice of interacting particles, each governed by the laws of quantum mechanics.
- Interactions between the particles are typically nearest-neighbor; in addition there might be some global constraints, such as a magnetic field, that act independently and identically on each of the particles.
- Can one extract global properties of the material from such a fine-grained modelization in a computationally tractable way?

Modelling the problem

- The **Hamiltonian** H of a system is an operator corresponding to the total energy of that system, including both kinetic energy and potential energy.

Modelling the problem

- The **Hamiltonian** H of a system is an operator corresponding to the total energy of that system, including both kinetic energy and potential energy.
- $H = \sum_i H_i$, where H_i is the Hermitian operator, which acts on some neighbouring particles of the system.

Modelling the problem

- The **Hamiltonian** H of a system is an operator corresponding to the total energy of that system, including both kinetic energy and potential energy.
- $H = \sum_i H_i$, where H_i is the Hermitian operator, which acts on some neighbouring particles of the system.
- Given a state $|\psi\rangle$, its energy with respect to the Hamiltonian H is defined to be $\langle\psi|H|\psi\rangle = \sum_{i=1}^{m} \langle\psi|H_i|\psi\rangle$

Modelling the problem

- The **Hamiltonian** H of a system is an operator corresponding to the total energy of that system, including both kinetic energy and potential energy.
- $H = \sum_i H_i$, where H_i is the Hermitian operator, which acts on some neighbouring particles of the system.
- Given a state $|\psi\rangle$, its energy with respect to the Hamiltonian H is defined to be $\langle\psi|H|\psi\rangle = \sum_{i=1}^{m} \langle\psi|H_i|\psi\rangle$
- Eigenvalues of H are the different values that the energy of the system can take, and its eigenvectors the corresponding states.

Modelling the problem

- The **Hamiltonian** H of a system is an operator corresponding to the total energy of that system, including both kinetic energy and potential energy.
- $H = \sum_i H_i$, where H_i is the Hermitian operator, which acts on some neighbouring particles of the system.
- Given a state $|\psi\rangle$, its energy with respect to the Hamiltonian H is defined to be $\langle\psi|H|\psi\rangle = \sum_{i=1}^{m} \langle\psi|H_i|\psi\rangle$
- Eigenvalues of H are the different values that the energy of the system can take, and its eigenvectors the corresponding states.
- The ground state energy is given by the least eigenvalue of the Hamiltonian H. The ground state is given by its corresponding eigenvector.

The k-local Hamiltonian (LH) problem

Input

$H_1, \cdots, H_m$, a set of m Hermitian matrices each acting on k qudits out of an n-qudit system and satisfying $||H_i|| \leq 1$. Each matrix entry is specified by poly(n)-many bits. Apart from the H_i we are also given two real numbers, a and b (again, with polynomially many bits of precision) such that $\Gamma := b$ - $a > 1/poly(n)$. Γ is referred to as the absolute promise gap.

Output

Is the smallest eigenvalue of $H = H_1 + H_2 \cdots + H_m$ smaller than a or are all its eigenvalues larger than b?

The k-local Hamiltonian (LH) problem

If z is an assignment to the n variables, which satisfies a clause C_i,

- $H_i|z\rangle = 0$: $|z\rangle$ has 0 energy with respect to that clause
- $H_i|z\rangle = |z\rangle$: $|z\rangle$ has 1 energy with respect to that clause

The k-local Hamiltonian (LH) problem

- Have we seen a similar thing before?

The k-local Hamiltonian (LH) problem

- Have we seen a similar thing before?
- yes, **Constraint Satisfactory Problem**

The k-local Hamiltonian (LH) problem

- Have we seen a similar thing before?
- yes, **Constraint Satisfaction Problem**

k-CSP

Definition

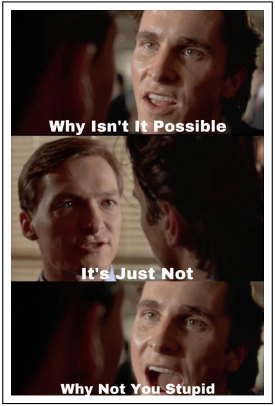
A k-CSP is a formula on n Boolean (or over a larger alphabet) variables, composed of m constraints, or clauses, each acting on at most k variables, where k should be thought of as a small constant (say, 2 or 3). By a constraint, we mean some restriction on assignments to the k variables which excludes one or more of the 2^k possibilities.

The **Cook–Levin theorem** states that the Boolean satisfiability problem is **NP-complete**.

But what if we smartly approximate the solution, in such a way it satisfy most of the constarints (approx 99%)?

A small issue

But, that's not possible :(



PCP Theorem

- **The PCP theorem says that any mathematical proof can be written in a special "PCP" format such that it can be verified, with arbitrarily high probability, by sampling only a few symbols in the proof.**

PCP Theorem

- **The PCP theorem says that any mathematical proof can be written in a special "PCP" format such that it can be verified, with arbitrarily high probability, by sampling only a few symbols in the proof.**
- Every NP-statement has a probabilistically checkable proof, i.e. a proof which can be "spot-checked" by reading only a constant number of bits from the proof (about $log(n)$ coin flips).

PCP Theorem

- **The PCP theorem says that any mathematical proof can be written in a special "PCP" format such that it can be verified, with arbitrarily high probability, by sampling only a few symbols in the proof.**
- Every NP-statement has a probabilistically checkable proof, i.e. a proof which can be "spot-checked" by reading only a constant number of bits from the proof (about $log(n)$ coin flips).
- Even approximation of NP-complete problems is NP-hard.

k-Local Hamiltonian Problem

QMA Complexity Class

A language $L \subseteq \{0,1\}^*$ is in QMA if there exists a quantum polynomial time algorithm V (called the verifier) and a polynomial $p()$ such that:

- $\forall x \in L$ there exists a state $|\xi_i\rangle$ on $p(|x|)$ qubits such that V accepts the pair of inputs $(x, |\xi_i\rangle)$ with probability at least $2/3$.
- $\forall x \notin L$ and for all states $|\xi_i\rangle$ on $p(|x|)$ qubits, V accepts $(x, |\xi_i\rangle)$ with probability at most $1/3$.

- The k-local Hamiltonian problem is QMA complete (quantum analog of NP Complete)
- Can we aprroximate the problem?

TAKE A GUESS

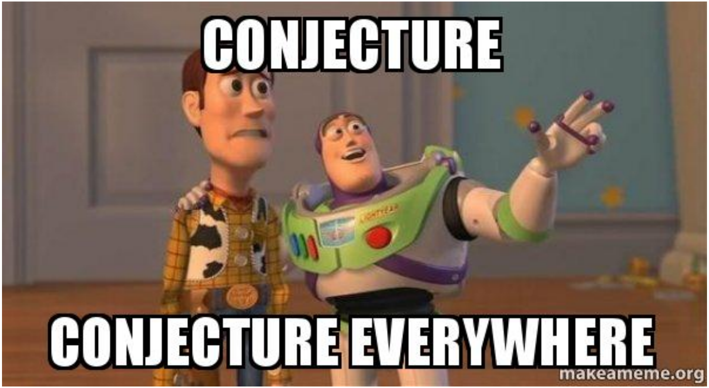
Quantum PCP Conjucture

qPCP Conjucture

There exists constants $k, d \in \mathbb{N}$ and $0 < \delta < 1$ such that the following holds. Given integers n, m and k-local Hamiltonians H_i, it is **QMA-hard** to find an additive $\pm\delta$ approximation to the quantity

$$\varepsilon_0(H) = \min_{|\psi\rangle} \frac{1}{m} \sum_{i=1}^{m} \langle\psi|H_i|\psi\rangle,$$

where the minimum is taken over all unit vectors $|\psi\rangle \in (\mathbb{C}^d)^{\otimes n}$.


CONJECTURE
CONJECTURE EVERYWHERE
makeameme.org

It is only a conjucture

- The trivial motivation given for introducing the conjecture is that it is a natural analogue of the classical PCP theorem.

It is only a conjucture

- The trivial motivation given for introducing the conjecture is that it is a natural analogue of the classical PCP theorem.
- Classical PCP theorem is proved by Dinur's Gap amplification proof.

It is only a conjucture

- The trivial motivation given for introducing the conjecture is that it is a natural analogue of the classical PCP theorem.
- Classical PCP theorem is proved by Dinur's Gap amplification proof.
- What if we quantized it?

Dinur's Gap Amplification

Dinur's Gap Amplification

For any alphabet size d, there exist constants $0 < \gamma < 1$ and $W > 1$, together with an efficient algorithm that takes a constraint graph $G = (V, E)$ with alphabet size d, and transforms it to another constraint graph $G' = (V', E')$ with universal alphabet size d_0 such that the following holds:

- $|E'| \leq W|E|$ and $|V'| \leq W|V|$,
- (completeness) if UNSAT(G) $=$ 0 then UNSAT(G') $=$ 0,
- (soundness) if UNSAT(G) $>$ 0 then UNSAT(G') $\leq$ min(2·UNSAT(G), γ)

Quantify Gap Amplification

For quantifying Dinur;'s Gap Amplification following chnages are made:

- The analog of constraint grapg $G = (V, E)$ is a a 2-local Hamiltonian H_G acting on d-dimensional particles placed on the vertices of G, where to every edge $e \in E$ we associate a 2-local projection H_e acting on the two adjacent particles.

$$H_G = \sum_{e \in E} H_e \,(quantum\, constarint\, graph)$$

- the quantum unsat-value of a state $|\psi\rangle$ with respect to H_G:

$$\text{QUNSAT}_{\psi(H_G)} := \frac{\langle\psi|H|\psi\rangle}{m} = \frac{1}{m}\sum_{i=1}^{m}\langle\psi|H_i|\psi\rangle$$

Quantify Gap Amplification

For quantifying Dinur;'s Gap Amplification following problems occur: G is an expander graph in classical gap amplification, done through degree reduction. In quantum version there are two problems:
Entanglement: The particle we would like to "copy" is potentially entangled with additional vertices, and its state is unknown; it is unclear how to map the state of the particle and the rest of the system to a state in which there are more "identical copies" of that particle. In each step new variables are introduced, and consistency checks added; quantizing each such check presents an additional challenge.
In classical PCP proofs, one often uses some kind of error correcting code to check that we are inside the code, and a second to check that the original constraints are satisfied. Quantumly, however, this seems impossible.

Brandão-Harrow's limitations on qPCP

- The general approach is to identify conditions under which the approximation problem associated with the qPCP conjecture is inside NP, and therefore cannot be QMA-hard.

Brandão-Harrow's limitations on qPCP

- The general approach is to identify conditions under which the approximation problem associated with the qPCP conjecture is inside NP, and therefore cannot be QMA-hard.
- They identify specific parameters of a 2-local Hamiltonian (as well as its groundstate) such that when the parameters lie in a certain range there is guaranteed to exist a product state whose average energy is within the relative promise gap Γ of the ground energy of the Hamiltonian.

Brandão-Harrow's limitations on qPCP

- The general approach is to identify conditions under which the approximation problem associated with the qPCP conjecture is inside NP, and therefore cannot be QMA-hard.
- They identify specific parameters of a 2-local Hamiltonian (as well as its groundstate) such that when the parameters lie in a certain range there is guaranteed to exist a product state whose average energy is within the relative promise gap Γ of the ground energy of the Hamiltonian.
- Such a product state can then serve as a classical witness, putting the approximation problem in NP

NLTS Conjucture

There exists a universal constant$c > 0$, an integer k and a family of k-local Hamiltonians $\{H(n)\}_n^\infty = 1$ $n = 1$ such that for any n, $H^{(n)}$ acts on n particles, and all states of average energy less than c above the average ground energy with respect to $H^{(n)}$ are non-trivial.

- It was suggested by because of the difficulties in proving or disproving the qPCP conjecture, by choosing a characterization of multipartite entanglement through the notion of non-trivial states.
- A state $|\psi\rangle$ is said to be trivial if it is the output of a constant depth quantum circuit applied to the input state $|0^n\rangle$.

NLTS Conjucture

There exists a universal constant, $c > 0$, an integer k and a family of k-local Hamiltonians $\{H(n)\}_{n=1}^{\infty}$ such that for any n, $H^{(n)}$ acts on n particles, and all states of average energy less than c above the average ground energy with respect to $H^{(n)}$ are non-trivial.

- We claim that the family of Hamiltonians produced by the qPCP reduction (or an infinite subfamily of it) satisfies the NLTS requirements for $c = \gamma$.

Is it really hard?

As the temperature of a physical system is cooled down, its state slowly evolves towards the ground state of the Hamiltonian that describes the system. Hence nature is by itself "computing" the ground state, or at least a mixture of low-energy states, of that Hamiltonian. But the qPCP asserts that approximating the energy of any such state is computationally hard, even for quantum computers — how can this computational hardness be reconciled with the fact that nature apparently solves the same problem on a day-to-day basis?

References

- A quantum PCP theorem?
- The Quantum PCP Conjecture

Thank You!
Any Questions?