LRR Reynolds Stress Model

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1 Reynolds Stress Model

I have used the Finite Volume Method (FVM) approach. For the given problem, half channel height (i.e. upto $y_{max} = \delta$) is considered. Wall functions are implemented in the RSM. Hence, the first inner node is placed at the location $y_p = 0.1\delta$ which corresponds to the region of $30 \leq y^+ \leq 40$. After the first node, a uniform grid has been deployed.

Initialising RSM: $k - \varepsilon$ model with wall functions is ran for the first 200 iterations where the following initial guess values are used:

$k = 5$ x $10^{-3},\ \varepsilon = 10^{-5},\ U =$ randomly initialised between [0, 1] values

The obtained velocity field U and other turbulent scalar quantities $\left(\overline{u_1^2},\ \overline{u_2^2},\ \overline{u_3^2},\ \overline{u_1u_2},\ k,\ \varepsilon\right)$ are used as the initial condition for RSM.

Boundary conditions applied are:

At $y = 0$ (wall): $\tau_w = \mu\dfrac{\partial U}{\partial y} = \rho u_\tau^2$

At $y_p = 0.1\delta$ (near-wall region): $\varepsilon = \dfrac{u_\tau^3}{\kappa y_p},\ \overline{u_1^2} = 3.67u_\tau^2,\ \overline{u_2^2} = 0.83u_\tau^2,\ \overline{u_3^2} = 2.17u_\tau^2,\ \overline{u_1u_2} = -u_\tau^2$

where $u_\tau = \dfrac{\kappa U_p}{ln\left(Eu_\tau y_p/\nu\right)},\ \kappa = 0.41,\ E = 9,\ \nu = \dfrac{1}{Re_\tau}$

At $y = \delta$ (symmetry): $\dfrac{\partial U}{\partial y} = 0,\ \dfrac{\partial \varepsilon}{\partial y} = 0,\ \dfrac{\partial\left(\overline{u_1u_1}\right)}{\partial y} = 0,\ \dfrac{\partial\left(\overline{u_2u_2}\right)}{\partial y} = 0,\ \dfrac{\partial\left(\overline{u_3u_3}\right)}{\partial y} = 0,\ \overline{u_1u_2} = 0$

Model constants:

$c_\mu = 0.09,\ c_1 = 1.8,\ c_2 = 0.6,\ c_1' = 0.5,\ c_2' = 0.3,\ c_{1\varepsilon} = 1.44,\ c_{2\varepsilon} = 1.92,\ \sigma_k = 1,\ \sigma_\varepsilon = 1.3$

Considering a 1D fully developed turbulent channel flow, the governing equations and its discretization is given below.

1.1 U equation

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\partial}{\partial y}\left[\nu\frac{\partial U}{\partial y} - \overline{u_1u_2}\right]$$

Discretized equation is:

$$a_pU_p = a_NU_N + a_SU_S + b_U$$

$$\text{where } a_N = \frac{\nu}{\delta y_n},\ a_S = \frac{\nu}{\delta y_s},$$

$$a_p = a_N + a_S,\ b_U = \left\{-\frac{1}{\rho}\frac{\partial P}{\partial x} - \frac{\partial\left(\overline{u_1u_2}\right)}{\partial y}\right\}\Delta y_p$$

1.2 ε equation

$$0=\frac{\partial}{\partial y}\left[\left(\nu+\frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial\varepsilon}{\partial y}\right]+\frac{c_{1\varepsilon}P_k\varepsilon}{k}-\frac{c_{2\varepsilon}\varepsilon^2}{k}$$

$$\text{where } \nu_t=\frac{c_\mu k^2}{\varepsilon},\ P_k=\frac{1}{2}P_{11},\ P_{11}=-2\overline{u_1u_2}\frac{\partial U}{\partial y},\ k=\frac{1}{2}\left(\overline{u_1u_1}+\overline{u_2u_2}+\overline{u_3u_3}\right)$$

Discretized equation is:

$$a_p\varepsilon_p=a_N\varepsilon_N+a_S\varepsilon_S+b_\varepsilon$$

$$\text{where } a_N=\frac{\left(\nu+\frac{\nu_t}{\sigma_\varepsilon}\right)_n}{\delta y_n},\ a_S=\frac{\left(\nu+\frac{\nu_t}{\sigma_\varepsilon}\right)_s}{\delta y_s},$$

$$a_p=a_N+a_S-S_p\Delta y_p,\ S_p=-\frac{c_2\varepsilon^*}{k},\ b_\varepsilon=\frac{c_1P_k\varepsilon^*}{k}\Delta y_p$$

1.3 $\overline{u_1u_1}$ equation

$$0=\frac{\partial}{\partial y}\left[\left(\nu+\frac{\nu_t}{\sigma_k}\right)\frac{\partial\left(\overline{u_1u_1}\right)}{\partial y}\right]+P_{11}+\phi_{11}-\varepsilon_{11}$$

$$\text{where } P_{11}=-2\overline{u_1u_2}\frac{\partial U}{\partial y},\ \phi_{11}=\phi_{11,1}+\phi_{11,2}+\phi_{11,w_1}+\phi_{11,w_2},\ \varepsilon_{11}=\frac{2}{3}\varepsilon$$

$$\phi_{11,1}=-c_1\frac{\varepsilon}{k}\left(\overline{u_1u_1}-\frac{2}{3}k\right),\ \phi_{11,2}=-c_2\left(-2\overline{u_1u_2}\frac{\partial U}{\partial y}-\frac{2}{3}P_k\right)$$

$$\phi_{11,w_1}=c_1\frac{\varepsilon}{k}\overline{u_2u_2}f,\ \phi_{11,w_2}=\frac{2}{3}c_2c_2'P_kf,\ f=\frac{k^{1.5}}{2.55y_p\varepsilon}$$

Discretized equation is:

$$a_p\left(\overline{u_1u_1}\right)_p=a_N\left(\overline{u_1u_1}\right)_N+a_S\left(\overline{u_1u_1}\right)_S+b$$

$$\text{where } a_N=\frac{\left(\nu+\frac{\nu_t}{\sigma_k}\right)_n}{\delta y_n},\ a_S=\frac{\left(\nu+\frac{\nu_t}{\sigma_k}\right)_s}{\delta y_s},$$

$$a_p=a_N+a_S-S_p\Delta y_p,\ b=S_c\Delta y_p$$

$$S_p=-\frac{c_1\varepsilon}{k},\ S_c=2\overline{u_1u_2}\frac{\partial U}{\partial y}\left(c_2-1\right)+\frac{2}{3}\varepsilon\left(c_1-1\right)+\frac{2}{3}c_2P_k\left(c_2'f+1\right)+\frac{c_1'\varepsilon}{k}\overline{u_2u_2}f$$

1.4 $\overline{u_2u_2}$ expression

$$0=\frac{\partial}{\partial y}\left[\left(\nu+\frac{\nu_t}{\sigma_k}\right)\frac{\partial\left(\overline{u_2u_2}\right)}{\partial y}\right]+P_{22}+\phi_{22}-\varepsilon_{22}$$

$$\text{where } P_{22}=0,\ \phi_{22}=\phi_{22,1}+\phi_{22,2}+\phi_{22,w_1}+\phi_{22,w_2},\ \varepsilon_{22}=\frac{2}{3}\varepsilon$$

$$\phi_{22,1}=-c_1\frac{\varepsilon}{k}\left(\overline{u_2u_2}-\frac{2}{3}k\right),\ \phi_{22,2}=\frac{2}{3}c_2P_k$$

$$\phi_{22,w_1}=-2c_1'\frac{\varepsilon}{k}\overline{u_2u_2}f,\ \phi_{22,w_2}=\frac{2}{3}c_2P_k\left(-2c_2'f\right)$$

Discretized equation is:

$$a_p\left(\overline{u_2u_2}\right)_p=a_N\left(\overline{u_2u_2}\right)_N+a_S\left(\overline{u_2u_2}\right)_S+b$$

$$\text{where } a_N = \frac{\left(\nu + \frac{\nu_t}{\sigma_k}\right)_n}{\delta y_n},\ a_S = \frac{\left(\nu + \frac{\nu_t}{\sigma_k}\right)_s}{\delta y_s},$$

$$a_p = a_N + a_S - S_p \Delta y_p,\ b = S_c \Delta y_p$$

$$S_p = -\frac{\varepsilon}{k}\left(c_1 + 2c_1' f\right),\ S_c = \frac{2}{3}\varepsilon\left(c_1 - 1\right) + \frac{2}{3}c_2 P_k \left(1 - 2c_2' f\right)$$

1.5 $\overline{u_3u_3}$ expression

$$0 = \frac{\partial}{\partial y}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial\left(\overline{u_3u_3}\right)}{\partial y}\right] + P_{33} + \phi_{33} - \varepsilon_{33}$$

$$\text{where } P_{33} = 0,\ \phi_{33} = \phi_{33,1} + \phi_{33,2} + \phi_{33,w_1} + \phi_{33,w_2},\ \varepsilon_{33} = \frac{2}{3}\varepsilon$$

$$\phi_{33,1} = -c_1\frac{\varepsilon}{k}\left(\overline{u_3u_3} - \frac{2}{3}k\right),\ \phi_{33,2} = \frac{2}{3}c_2 P_k$$

$$\phi_{33,w_1} = c_1'\frac{\varepsilon}{k}\overline{u_2u_2}f,\ \phi_{33,w_2} = \frac{2}{3}c_2 P_k\left(c_2' f\right)$$

Discretized equation is:

$$a_p\left(\overline{u_3u_3}\right)_p = a_N\left(\overline{u_3u_3}\right)_N + a_S\left(\overline{u_3u_3}\right)_S + b$$

$$\text{where } a_N = \frac{\left(\nu + \frac{\nu_t}{\sigma_k}\right)_n}{\delta y_n},\ a_S = \frac{\left(\nu + \frac{\nu_t}{\sigma_k}\right)_s}{\delta y_s},$$

$$a_p = a_N + a_S - S_p \Delta y_p,\ b = S_c \Delta y_p$$

$$S_p = -\frac{c_1\varepsilon}{k},\ S_c = \frac{2}{3}\varepsilon\left(c_1 - 1\right) + \frac{2}{3}c_2 P_k\left(c_2' f + 1\right) + \frac{c_1'\varepsilon}{k}\overline{u_2u_2}f$$

1.6 $\overline{u_1u_2}$ expression

$$0 = \frac{\partial}{\partial y}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial\left(\overline{u_1u_2}\right)}{\partial y}\right] + P_{12} + \phi_{12} - \varepsilon_{12}$$

$$\text{where } P_{12} = -\overline{u_2u_2}\frac{\partial U}{\partial y},\ \phi_{12} = \phi_{12,1} + \phi_{12,2} + \phi_{12,w_1} + \phi_{12,w_2},\ \varepsilon_{12} = 0$$

$$\phi_{12,1} = -c_1\frac{\varepsilon}{k}\overline{u_1u_2},\ \phi_{12,2} = c_2\overline{u_2u_2}\frac{\partial U}{\partial y}$$

$$\phi_{12,w_1} = -\frac{3}{2}\frac{c_1'\varepsilon}{k}\overline{u_1u_2}f,\ \phi_{12,w_2} = c_2\overline{u_2u_2}\frac{\partial U}{\partial y}\left(-1.5c_2' f\right)$$

Discretized equation is:

$$a_p\left(\overline{u_1u_2}\right)_p = a_N\left(\overline{u_1u_2}\right)_N + a_S\left(\overline{u_1u_2}\right)_S + b$$

$$\text{where } a_N = \frac{\left(\nu + \frac{\nu_t}{\sigma_k}\right)_n}{\delta y_n},\ a_S = \frac{\left(\nu + \frac{\nu_t}{\sigma_k}\right)_s}{\delta y_s},$$

$$a_p = a_N + a_S,\ b = S_c \Delta y_p$$

$$S_c = \overline{u_2u_2}\frac{\partial U}{\partial y}\left(c_2 - 1 - 1.5c_2c_2' f\right) - \overline{u_1u_2}\frac{\varepsilon}{k}\left(c_1 + 1.5c_1' f\right)$$

2 Results

2.1 Plots

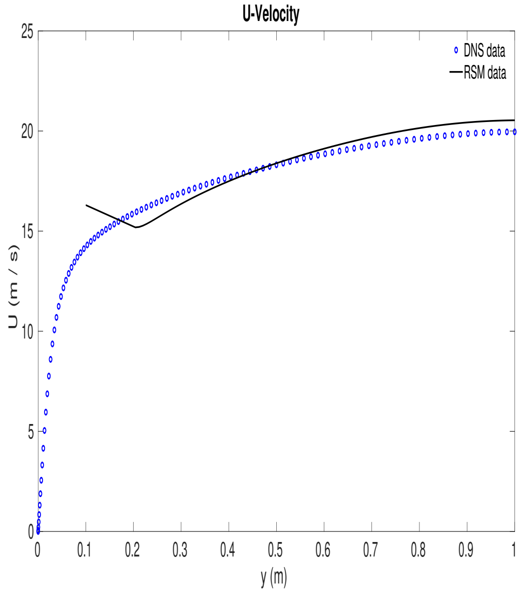

Figure 1: U Velocity

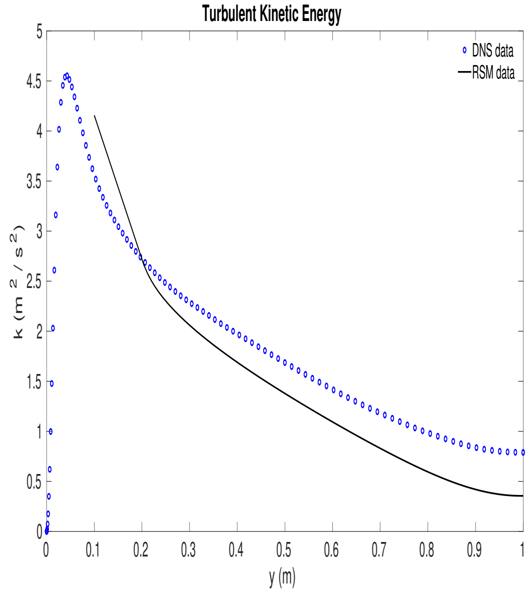

Figure 2: Turbulent Kinetic Energy

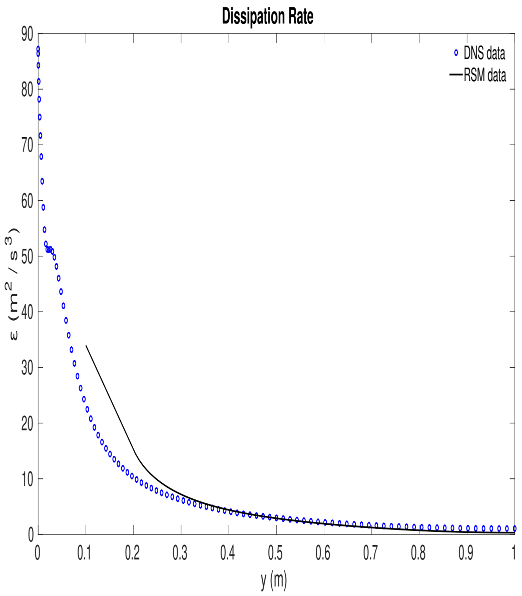

Figure 3: Dissipation Rate

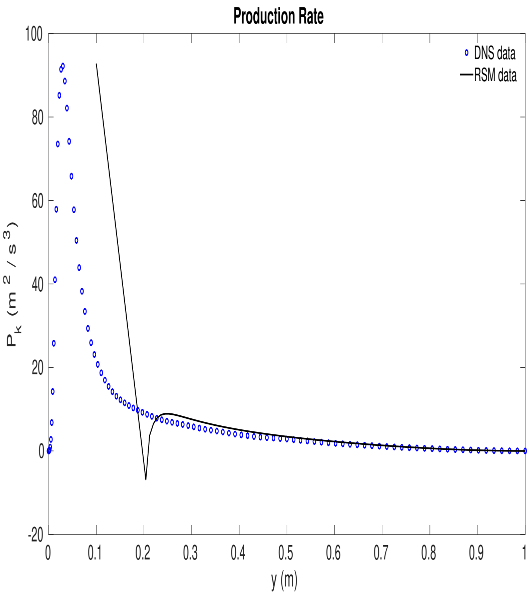

Figure 4: Production Rate

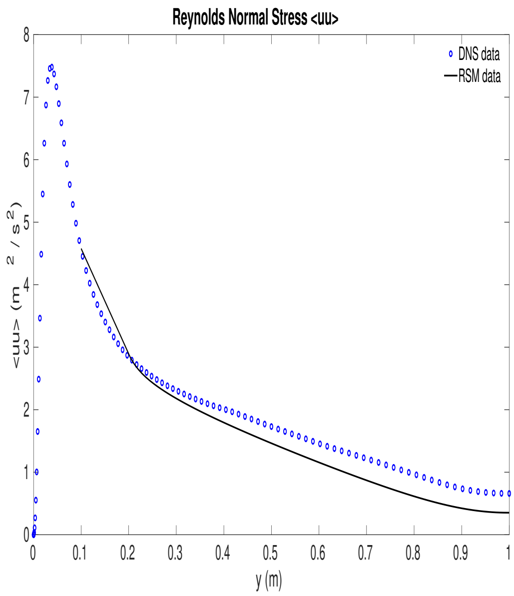

Figure 5: Reynolds Normal Stress $\overline{u_1u_1}$

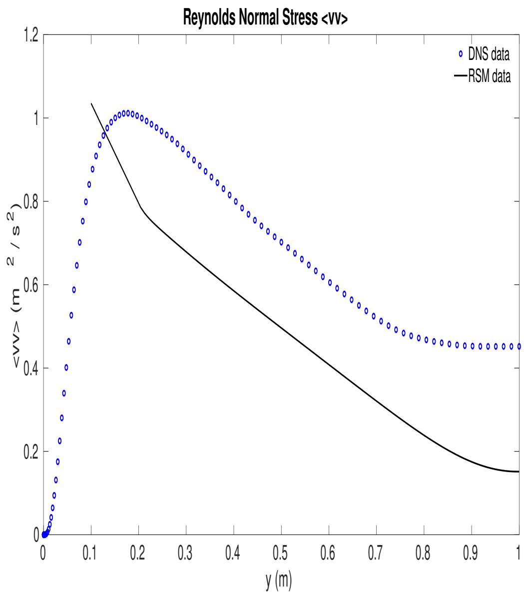

Figure 6: Reynolds Normal Stress $\overline{u_2u_2}$

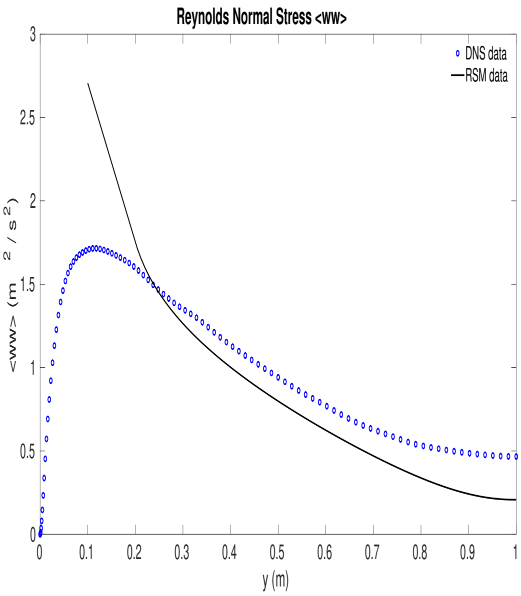

Figure 7: Reynolds Normal Stress $\overline{u_3u_3}$

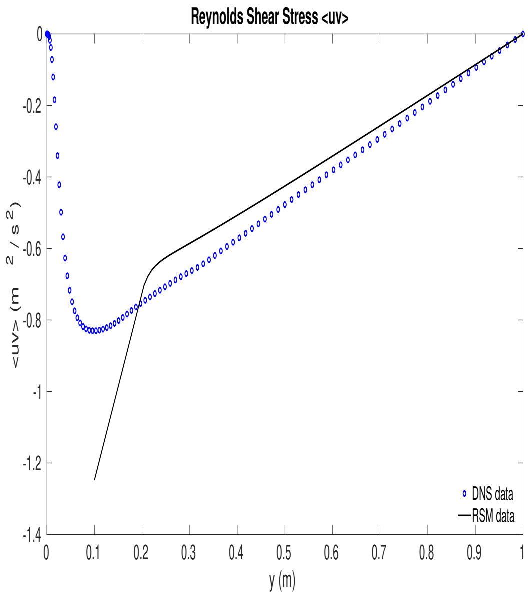

Figure 8: Reynolds Normal Stress $\overline{u_1u_2}$

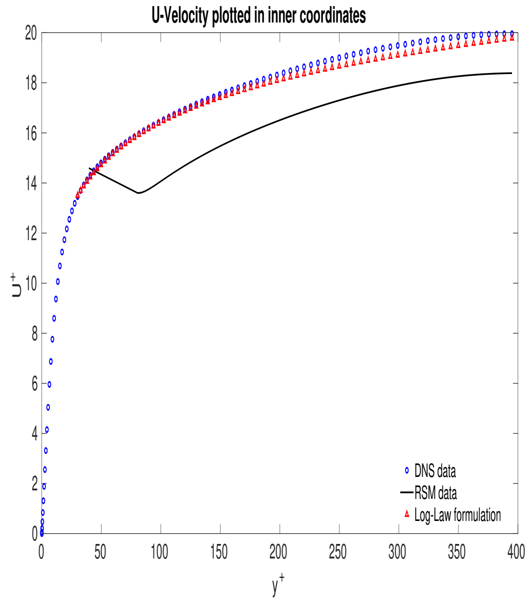

Figure 9: U Velocity inner scaling plot

2.2 Wall Shear Stress

Analytical: $\tau_w = 1.0,\ u_\tau = 1.0$

In RSM: $\tau_w = 1.24733,\ u_\tau = 1.11684$

2.3 Results Discussion

The implemented Reynolds Stress Model has over-predicted the u_τ value in comparison with the DNS data. Hence, it led to mismatch of the data near the wall; i.e. over-predicted the scalar quantities. However, in the regions away from the wall, the model has predicted fairly good results by capturing the profile shapes of the turbulent quantities and the trend they follow.

The whole idea of implementing wall functions is that we do not want to introduce any grid points in the linear sub-layer and obtain reasonable results. This will reduce the computational effort by placing the first grid point in the inertial sub-layer. It is not recommended to use the wall functions if one needs the data (like coefficient of drag/lift, wall shear stress, skin friction coefficient, etc) on the wall. As a consequence, this led to incorrect prediction for wall shear stress by the RSM model.