# Reflection at Planar Interface in Uniaxial Media

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## 1 Dispersion in Uniaxial Media

Consider a monochromatic wave with angular frequency $\omega$, and let $k_0 = \omega/c$ be the corresponding wavenumber in vacuum. We will also use the free space impedance $Z_0 = \sqrt{\mu_0/\epsilon_0}$.

We consider a planar interface and define $\hat{\zeta}$ to be the surface normal defininig the orientation. We will denote as "1" the region *into* which the normal $\hat{\zeta}$ points. With this orientation, we define the electric and magnetic fields in region $\alpha$ as

$$\begin{aligned}
\boldsymbol{E}_\alpha(\boldsymbol{r}) &= \boldsymbol{E}_\alpha^- e^{i\boldsymbol{k}_\alpha^-\cdot\boldsymbol{r}} + \boldsymbol{E}_\alpha^+ e^{i\boldsymbol{k}_\alpha^+\cdot\boldsymbol{r}}, \quad (1)\\
\boldsymbol{H}_\alpha(\boldsymbol{r}) &= \boldsymbol{H}_\alpha^- e^{i\boldsymbol{k}_\alpha^-\cdot\boldsymbol{r}} + \boldsymbol{H}_\alpha^+ e^{i\boldsymbol{k}_\alpha^+\cdot\boldsymbol{r}}, \quad (2)
\end{aligned}$$

where the $\pm$ superscripts are defined such that

$$\begin{aligned}
\boldsymbol{k}_\alpha^\sigma &= \boldsymbol{\kappa} + \sigma w_\alpha \hat{\zeta}, \quad (3)\\
\boldsymbol{\kappa}\cdot\hat{\zeta} &= 0.
\end{aligned}$$

Note also that we have 2 orthogonal axes $\hat{\kappa}$ and $\hat{\zeta}$ already, and to make a right-handed coordinate system, we define the third vector,

$$\hat{s} = \hat{\kappa}\times\hat{\zeta}.$$

With this notation, we define TE mode to be the one in which $\boldsymbol{\kappa}\cdot\boldsymbol{E}_\alpha^\sigma = 0$ and TM mode to be the one where $\boldsymbol{\kappa}\cdot\boldsymbol{H}_\alpha^\sigma = 0$. We let the $\epsilon_0\boldsymbol{\varepsilon}_\alpha$ and $\mu_0\boldsymbol{\mu}_\alpha$ be permittivity and permeability tensors respectively. The source-free Maxwell equations give the relationships

$$\begin{aligned}
\boldsymbol{k}_\alpha^\sigma\cdot\boldsymbol{E}_\alpha^\sigma &= 0, \quad (4)\\
\boldsymbol{k}_\alpha^\sigma\cdot\boldsymbol{\mu}\boldsymbol{H}_\alpha^\sigma &= 0, \quad (5)\\
\omega\mu_0\boldsymbol{\mu}\boldsymbol{H}_\alpha^\sigma &= \boldsymbol{k}_\alpha^\sigma\times\boldsymbol{E}_\alpha^\sigma, \quad (6)\\
-\omega\epsilon_0\boldsymbol{\varepsilon}\boldsymbol{E}_\alpha^\sigma &= \boldsymbol{k}_\alpha^\sigma\times\boldsymbol{H}_\alpha^\sigma. \quad (7)
\end{aligned}$$

Combining equations (6) and (7), we get the *dispersion relations,*

$$\begin{aligned}
\boldsymbol{k}_\alpha^\sigma\times\boldsymbol{\varepsilon}_\alpha^{-1}\boldsymbol{k}_\alpha^\sigma\times\boldsymbol{H}_\alpha^\sigma + \frac{\omega^2}{c^2}\boldsymbol{\mu}_\alpha\boldsymbol{H}_\alpha^\sigma &= 0, \quad (8)\\
\boldsymbol{k}_\alpha^\sigma\times\boldsymbol{\mu}_\alpha^{-1}\boldsymbol{k}_\alpha^\sigma\times\boldsymbol{E}_\alpha^\sigma + \frac{\omega^2}{c^2}\boldsymbol{\varepsilon}_\alpha\boldsymbol{E}_\alpha^\sigma &= 0. \quad (9)
\end{aligned}$$

At this point we make the assumption that $\boldsymbol{\mu}$ and $\boldsymbol{\varepsilon}$ have the form

$$\begin{aligned}\boldsymbol{\varepsilon} &= \varepsilon_{\|}\hat{\zeta}\hat{\zeta} + \varepsilon_{\perp} I_{\perp}, \\ \boldsymbol{\mu} &= \mu_{\|}\hat{\zeta}\hat{\zeta} + \mu_{\perp} I_{\perp}, \\ \text{where } I_{\perp} &= \hat{s}\hat{s} + \hat{\kappa}\hat{\kappa}.\end{aligned}$$

This also gives the result

$$\begin{aligned}\boldsymbol{\varepsilon}^{-1} &= \varepsilon_{\|}^{-1}\hat{\zeta}\hat{\zeta} + \varepsilon_{\perp}^{-1} I_{\perp}, \\ \boldsymbol{\mu}^{-1} &= \mu_{\|}^{-1}\hat{\zeta}\hat{\zeta} + \mu_{\perp}^{-1} I_{\perp}.\end{aligned}$$

We define refractive indices

$$n_{\perp} = \sqrt{\mu_{\perp}\varepsilon_{\perp}}, \text{ and, } n_{\|} = \sqrt{\mu_{\|}\varepsilon_{\|}}.$$

It is convenient to also define an anisotropy measure,

$$\gamma_E = \sqrt{\frac{\varepsilon_{\perp}}{\varepsilon_{\|}}}, \tag{10}$$

$$\gamma_H = \sqrt{\frac{\mu_{\perp}}{\mu_{\|}}}. \tag{11}$$

We will also make use of the geometric means,

$$\begin{aligned}\bar{\varepsilon} &= \sqrt{\varepsilon_{\|}\varepsilon_{\perp}}, \\ \bar{\mu} &= \sqrt{\mu_{\|}\mu_{\perp}}.\end{aligned}$$

In terms of these quantities,

$$\begin{aligned}\boldsymbol{\varepsilon} &= \bar{\varepsilon}\left[\frac{1}{\gamma_E} I_{\perp} + \gamma_E \hat{\zeta}\hat{\zeta}\right], \\ \boldsymbol{\mu} &= \bar{\mu}\left[\frac{1}{\gamma_H} I_{\perp} + \gamma_H \hat{\zeta}\hat{\zeta}\right], \\ \boldsymbol{\varepsilon}^{-1} &= \frac{1}{\bar{\varepsilon}}\left[\gamma_E I_{\perp} + \frac{1}{\gamma_E}\hat{\zeta}\hat{\zeta}\right], \\ \boldsymbol{\mu}^{-1} &= \frac{1}{\bar{\mu}}\left[\gamma_H I_{\perp} + \frac{1}{\gamma_H}\hat{\zeta}\hat{\zeta}\right].\end{aligned}$$

We also note that $\hat{p}$-polarization can be defined based on (7) as follows. Let $\boldsymbol{H}_{\alpha}^{\sigma} = H_{\alpha}^{\sigma}\hat{s}$ , and use (7) to determine the electric field,

$$\boldsymbol{E}_{\alpha}^{\sigma} = -\left(\frac{Z_0 H_{\alpha}^{\sigma}}{k_0}\right)\boldsymbol{\varepsilon}^{-1}\boldsymbol{k}_{\alpha}^{\sigma} \times \hat{s} = \left(\frac{Z_0 H_{\alpha}^{\sigma}}{k_0}\right)\boldsymbol{\varepsilon}^{-1}\hat{s} \times \boldsymbol{k}_{\alpha}^{\sigma} \tag{12}$$

Let us now consider the inverse dielectric tensor acting on the cross product,

$$\boldsymbol{\varepsilon}^{-1}\hat{s} \times \boldsymbol{k}_{\alpha}^{\sigma} = \boldsymbol{\varepsilon}^{-1}\left(\hat{s} \times \boldsymbol{\kappa} + \sigma w_{\alpha}\hat{s} \times \hat{\zeta}\right)$$

$$= \left[\frac{\kappa}{\varepsilon_\perp}\hat{\zeta} - \frac{\sigma w_\alpha}{\varepsilon_\parallel}\hat{\kappa}\right]$$
$$= \frac{1}{\bar{\varepsilon}_\alpha}\frac{\left[\frac{\kappa}{\gamma_{\alpha,E}}\hat{\zeta} - \sigma\gamma_{\alpha,E}w_\alpha\hat{\kappa}\right]}{\sqrt{\kappa^2\gamma_{\alpha,E}^{-2} + w_\alpha^2\gamma_{\alpha,E}^2}}.$$

We define the unit vector as $\hat{p}^\sigma$,

$$\hat{p}^\sigma_\alpha = \frac{\left[\kappa\hat{\zeta} - \sigma\gamma_{\alpha,E}^2 w_\alpha\hat{\kappa}\right]}{\sqrt{\kappa^2 + \gamma_{\alpha,E}^4 w_\alpha^2}}. \tag{13}$$

Note that in isotrpic materials, $\gamma_{\alpha,E} = 1$ and we recover the definition, $\hat{p}^\sigma = (\kappa\hat{\zeta} - \sigma w\hat{\kappa})/|\boldsymbol{k}|$. In the general case, the expression for the electric field for a TM wave is,

$$\boldsymbol{E}^\sigma_\alpha = \frac{Z_0}{k_0}\frac{\sqrt{\kappa^2 + w_\alpha^2\gamma_{\alpha,E}^4}}{\bar{\varepsilon}_\alpha}H^\sigma_\alpha\hat{p}^\sigma_\alpha. \tag{14}$$

In isotropic materials, we obtain,

$$\boldsymbol{E}^\sigma_\alpha = \frac{Z_0}{n_\alpha}H^\sigma_\alpha\hat{p}^\sigma_\alpha.$$

Next we write the cross product of the wave-vector with a vector in the representation given by the basis above,

$$\begin{aligned}\left(\boldsymbol{\kappa} + w\hat{\zeta}\right) \times \left(v_s\hat{s} + v_\kappa\hat{\kappa} + v_\zeta\hat{\zeta}\right) &= \left(\kappa v_\zeta - w v_\kappa\right)\hat{s} + w v_s\hat{\kappa} - \kappa v_s\hat{\zeta} \\ &= \begin{bmatrix} 0 & -w & \kappa \\ w & 0 & 0 \\ -\kappa & 0 & 0 \end{bmatrix}\begin{bmatrix} v_s \\ v_\kappa \\ v_\zeta \end{bmatrix}.\end{aligned}$$

When we substitute this result into (8). In a given region $\alpha$

$$\begin{bmatrix} 0 & -w & \kappa \\ w & 0 & 0 \\ -\kappa & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & -\varepsilon_\perp^{-1}w & \varepsilon_\perp^{-1}\kappa \\ \varepsilon_\perp^{-1}w & 0 & 0 \\ -\varepsilon_\parallel^{-1}\kappa & 0 & 0 \end{bmatrix}\boldsymbol{H}^\sigma + \frac{\omega^2}{c^2}\begin{bmatrix} \mu_\perp & & \\ & \mu_\perp & \\ & & \mu_\parallel \end{bmatrix}\boldsymbol{H}^\sigma = 0.$$

We removed the subscript $\alpha$ with the underestanding that the equation applies only inside one homogeneous region. We now simplify this into a single matrix,

$$\begin{bmatrix} -\left(\varepsilon_\perp^{-1}w^2 + \varepsilon_\parallel^{-1}\kappa^2\right) + \frac{\omega^2}{c^2}\mu_\perp & 0 & 0 \\ 0 & -\varepsilon_\perp^{-1}w^2 + \frac{\omega^2}{c^2}\mu_\perp & \varepsilon_\perp^{-1}w\kappa \\ 0 & \varepsilon_\perp^{-1}w\kappa & -\varepsilon_\perp^{-1}\kappa^2 + \frac{\omega^2}{c^2}\mu_\parallel \end{bmatrix}\boldsymbol{H}^\sigma_\alpha = 0.$$

The first row gives the dispersion relation for $s$-polarized magnetic field wave,

$$\frac{\omega^2}{c^2}\mu_\perp - \left(\varepsilon_\perp^{-1}w^2 + \varepsilon_\parallel^{-1}\kappa^2\right) = 0.$$

For the last two, we must set the determinant of the 2x2 system to zero,

$$\left(-\varepsilon_\perp^{-1} w^2 + \frac{\omega^2}{c^2}\mu_\perp\right)\left(-\varepsilon_\perp^{-1}\kappa^2 + \frac{\omega^2}{c^2}\mu_\parallel\right) - \left(\varepsilon_\perp^{-1} w\kappa\right)^2 \quad = \quad 0.$$

Expanding the first term, we get

$$\frac{\omega^2}{c^2}\mu_\parallel\mu_\perp - \varepsilon_\perp^{-1}(\mu_\parallel w^2 + \mu_\perp\kappa^2) \quad = \quad 0.$$

The two dispersion relations, one for TM and one for TE wave, respectively, are

$$\varepsilon_\perp\left[\frac{\omega^2}{c^2}\mu_\perp - \left(\frac{w^2}{\varepsilon_\perp} + \frac{1}{\varepsilon_\parallel}\kappa^2\right)\right] \quad = \quad 0, \tag{15}$$

$$\mu_\perp\left[\frac{\omega^2}{c^2}\varepsilon_\perp - \left(\frac{w^2}{\mu_\perp} + \frac{1}{\mu_\parallel}\kappa^2\right)\right] \quad = \quad 0. \tag{16}$$

It is clear that if we had started with (9), we will simply switch the roles of $H, E$ and $\mu, \varepsilon$, and the above two equations would have followed for TM and TE waves again.

In terms of the anisotropy and refractive index,

$$\frac{\omega^2}{c^2}n_\perp^2 - \left(w^2 + \gamma_E^2\kappa^2\right) \quad = \quad 0, \tag{17}$$

$$\frac{\omega^2}{c^2}n_\perp^2 - \left(w^2 + \gamma_H^2\kappa^2\right) \quad = \quad 0. \tag{18}$$

## 2 Transmission and Reflection at an Interface

### TM

We first consider TM waves for which we write, using (2)

$$\boldsymbol{H}_\alpha \quad = \quad \hat{s}\left[H_\alpha^+ e^{iw_\alpha\zeta} + H_\alpha^- e^{-iw_\alpha\zeta}\right] e^{i\boldsymbol{\kappa}\cdot\boldsymbol{r}_\perp}. \tag{19}$$

The second term in the expression for $\boldsymbol{E}_\alpha$ is parallel to $\hat{\zeta}$. We impose continuity of parallel components of both fields (recall that in the absence of surface currents, $\boldsymbol{H}$ is continuous in parallel components). We assume that the source is located in region 1 away from the interface and therefore waves exist in both directions with respect to $\hat{\zeta}$ in region 1, but only in the direction opposite to $\hat{\zeta}$ in region 2.

We obtain, setting $\zeta = 0$ in the local coordinate system at the interface,

$$H_1^+ + H_1^- \quad = \quad H_2^-, \tag{20}$$

$$-H_1^+ + H_1^- \quad = \quad \frac{w_2}{w_1}\frac{\varepsilon_{\perp 1}}{\varepsilon_{\perp 2}}H_2^-. \tag{21}$$

This yields transmission and reflection coefficients for TM waves,

$$t_H = \frac{H_2^-}{H_1^-} = \frac{2w_1\varepsilon_{\perp 2}}{w_1\varepsilon_{2\perp}+w_2\varepsilon_{1\perp}}, \tag{22}$$

$$r_H = \frac{H_1^+}{H_1^-} = \frac{w_1\varepsilon_{2\perp}-w_2\varepsilon_{1\perp}}{w_1\varepsilon_{2\perp}+w_2\varepsilon_{1\perp}}, \tag{23}$$

$$w_\alpha = \sqrt{\frac{\omega^2}{c^2}\mu_{\alpha\perp}\varepsilon_{\alpha\perp}-\frac{\varepsilon_{\alpha\perp}}{\varepsilon_{\alpha\|}}\kappa^2}. \tag{24}$$

To get the reflection and transmission coefficients for the electric field, we make use of (12), (13), and (14),

$$\boldsymbol{E}_\alpha = \frac{Z_0}{k_0}\frac{\sqrt{\kappa^2+w_\alpha^2\gamma_{\alpha,E}^4}}{\bar{\varepsilon}_\alpha}\left[\hat{p}_\alpha^+ H_\alpha^+ e^{iw_\alpha\zeta}+\hat{p}_\alpha^- H_\alpha^- e^{-iw_\alpha\zeta}\right].$$

The reflection and transmission amplitudes are

$$\begin{aligned} t_E &= \frac{\sqrt{\kappa^2+w_2^2\gamma_{2,E}^4}}{\sqrt{\kappa^2+w_1^2\gamma_{1,E}^4}}\frac{\bar{\varepsilon}_1}{\bar{\varepsilon}_2}t_H = \left[\frac{\bar{\varepsilon}_1 n_{2_\perp}}{\bar{\varepsilon}_2 n_{1_\perp}}\frac{\sqrt{\kappa^2+w_2^2\gamma_{2,E}^4}}{\sqrt{\kappa^2+w_1^2\gamma_{1,E}^4}}\right]\left(\frac{2w_1 n_{1\perp}n_{2\perp}}{w_1\varepsilon_{2\perp}+w_2\varepsilon_{1\perp}}\right), \\ r_E &= r_H. \end{aligned}$$

We observe that in the expression for $t_E$, the term square brackets reduces to unity for isotropic materials and the second term becomes identical to Fresnel transsmission formula. Thus the of reflection and transmission coefficients for the electric field in this case are,

$$\begin{aligned} t_E &= \frac{n_1}{n_2}t_H = \frac{2w_1 n_1 n_2}{w_1\varepsilon_2+w_2\varepsilon_1}, \\ r_E &= r_H. \end{aligned}$$

## TE

For the TE case, the E-field is parallel to the interface, and we project $\boldsymbol{H}$ field into the interface plane since the normal component is continuous,

$$\begin{aligned} \boldsymbol{E}_\alpha &= \hat{s}\left[E_\alpha^+ e^{iw_\alpha\zeta}+E_\alpha^- e^{-iw_\alpha\zeta}\right]e^{i\boldsymbol{\kappa}\cdot\boldsymbol{r}_\perp}. \\ I_\perp\boldsymbol{H}_\alpha &= \frac{w_{te,\alpha}\mu_\perp^{-1}}{\omega\epsilon_0}\left(\hat{\zeta}\times\hat{s}\right)\left[E_\alpha^+ e^{iw_\alpha\zeta}-E_\alpha^- e^{-iw_\alpha\zeta}\right]e^{i\boldsymbol{\kappa}\cdot\boldsymbol{r}_\perp}. \end{aligned}$$

Again, imposing the continuity of these amplitudes across the interface we obtain the dual relations,

$$E_1^+ + E_1^- = E_2^-, \tag{25}$$

$$-E_1^+ + E_1^- = \frac{w_2\mu_{1\perp}}{w_1\mu_{2\perp}}E_2^-. \tag{26}$$

| | $w$ | | $t$ | $r$ |
|---|---|---|---|---|
| TE | $w_\alpha = \sqrt{\frac{\omega^2}{c^2}\mu_{\alpha\perp}\varepsilon_{\alpha\perp} - \frac{\varepsilon_{\alpha\perp}}{\varepsilon_{\alpha\parallel}}\kappa^2}$ | $\boldsymbol{E}$ | $\frac{2w_1\mu_{2\perp}}{w_1\mu_{2\perp}+w_2\mu_{1\perp}}$ | $\frac{w_1\mu_{2\perp}-w_2\mu_{1\perp}}{w_1\mu_{2\perp}+w_2\mu_{1\perp}}$ |
| | | $\boldsymbol{H}$ | | $r_E$ |
| TM | $w_\alpha = \sqrt{\frac{\omega^2}{c^2}\mu_{\alpha\perp}\varepsilon_{\alpha\perp} - \frac{\mu_{\alpha\perp}}{\mu_{\alpha\parallel}}\kappa^2}$ | $\boldsymbol{E}$ | $\left[\frac{\bar{\varepsilon}_1 n_{2_\perp}}{\bar{\varepsilon}_2 n_{1_\perp}}\frac{\sqrt{\kappa^2+w_2^2\gamma_{2,E}^4}}{\sqrt{\kappa^2+w_1^2\gamma_{1,E}^4}}\right]\frac{2w_1 n_{1_\perp} n_{2\perp}}{w_1\varepsilon_{2\perp}+w_2\varepsilon_{1\perp}}$ | $r_H$ |
| | | $\boldsymbol{H}$ | $\frac{2w_1\varepsilon_{2\perp}}{w_1\varepsilon_{2\perp}+w_2\varepsilon_{1\perp}}$ | $\frac{w_1\varepsilon_{2\perp}-w_2\varepsilon_{1\perp}}{w_1\varepsilon_{2\perp}+w_2\varepsilon_{1\perp}}$ |

Tab. 1: Fresnel Coefficients for anisotrpic material

These equations give the solution,

$$
\begin{aligned}
\frac{E_2^-}{E_1^-} &= \frac{2w_1\mu_{2\perp}}{w_1\mu_{2\perp} + w_2\mu_{1\perp}}, \\
\frac{E_1^+}{E_1^-} &= \frac{w_1\mu_{2\perp} - w_2\mu_{1\perp}}{w_1\mu_{2\perp} + w_2\mu_{1\perp}}, \\
w_\alpha &= \sqrt{\frac{\omega^2}{c^2}\mu_{\alpha\perp}\varepsilon_{\alpha\perp} - \frac{\mu_{\alpha\perp}}{\mu_{\alpha\parallel}}\kappa^2}.
\end{aligned}
$$

We summarize these in table 1.

# 3 Surface Plasmon Modes

## 3.1 Electric

Equations (20) and (21) yield a solution for incident field $H_1^-$. There is a solution to these equations when $H_1^- = 0$, and that is a mode of the system. Substituting $H_1^- = 0$ into these equations we get,

$$
\begin{aligned}
H_1^+ &= H_2^-, \\
-H_1^+ &= \frac{w_2}{w_1}\frac{\varepsilon_{\perp 1}}{\varepsilon_{\perp 2}} H_2^-.
\end{aligned}
$$

These equations imply that the necessary condition is

$$
\frac{w_1}{\varepsilon_{\perp 1}} = -\frac{w_2}{\varepsilon_{\perp 2}}. \tag{27}
$$

On a planar interface $\kappa$ is conserved. Equation (24) and the above condition then imply,

$$
\frac{\omega^2}{c^2}\frac{\mu_{1\perp}}{\varepsilon_{1\perp}} - \frac{\mu_{1\perp}}{\mu_{1\parallel}}\frac{1}{\varepsilon_{1\perp}^2}\kappa^2 = \frac{\omega^2}{c^2}\frac{\mu_{2\perp}}{\varepsilon_{2\perp}} - \frac{\mu_{2\perp}}{\mu_{2\parallel}}\frac{1}{\varepsilon_{2\perp}^2}\kappa^2.
$$

Solving for $\kappa$, we get

$$
\kappa^2(\omega) = \frac{\omega^2}{c^2}\frac{\left(\mu_{2\perp}\varepsilon_{1\perp} - \mu_{1\perp}\varepsilon_{2\perp}\right)\varepsilon_{2\perp}\varepsilon_{1\perp}}{\gamma_{2H}^2\varepsilon_{1\perp}^2 - \gamma_{1H}^2\varepsilon_{2\perp}^2}.
$$

Let us introduce the geometric mean (10) and the anisotropy measure (11) into this expression, and obtain

$$\begin{aligned}\kappa^2(\omega) &= \frac{\omega^2}{c^2}\frac{\left(\mu_{2\|}\gamma_{2H}^2\varepsilon_{1\perp}-\mu_{1\|}\gamma_{1H}^2\varepsilon_{2\perp}\right)\varepsilon_{2\perp}\varepsilon_{1\perp}}{\left(\gamma_{2H}\varepsilon_{1\perp}-\gamma_{1H}\varepsilon_{2\perp}\right)\left(\gamma_{2H}\varepsilon_{1\perp}+\gamma_{1H}\varepsilon_{2\perp}\right)}\\ &= \frac{\omega^2}{c^2}\frac{\varepsilon_{2\perp}\varepsilon_{1\perp}}{\left(\gamma_{2H}\varepsilon_{1\perp}+\gamma_{1H}\varepsilon_{2\perp}\right)}\left[\left\langle\mu_{\|}\right\rangle+\Delta\mu_{\|}\frac{\gamma_{2H}\varepsilon_{1\perp}+\gamma_{1H}\varepsilon_{2\perp}}{\gamma_{2H}\varepsilon_{1\perp}-\gamma_{1H}\varepsilon_{2\perp}}\right].\end{aligned}$$

where we defined the average of the geometric means of the permeabilities in each region,

$$\langle\mu\rangle = \frac{1}{2}\left[\mu_{2\|}\gamma_{2H}+\mu_{1\|}\gamma_{1H}\right] = \frac{1}{2}\left[\sqrt{\mu_{2\|}\mu_{2\perp}}+\sqrt{\mu_{1\|}\mu_{1\perp}}\right],$$

and the difference beteween the geometric means,

$$\Delta\mu = \mu_{2\|}\gamma_{2H}-\mu_{1\|}\gamma_{1H} = \sqrt{\mu_{2\|}\mu_{2\perp}}-\sqrt{\mu_{1\|}\mu_{1\perp}}.$$

By cancelling out the common factors, we get

$$\kappa(\omega) = \frac{\omega}{c}\sqrt{\frac{\langle\mu\rangle\,\varepsilon_{2\perp}\varepsilon_{1\perp}}{\gamma_{2H}\varepsilon_{1\perp}+\gamma_{1H}\varepsilon_{2\perp}}}\sqrt{1+\frac{\Delta\mu}{\langle\mu\rangle}\frac{\gamma_{2H}\varepsilon_{1\perp}+\gamma_{1H}\varepsilon_{2\perp}}{\gamma_{2H}\varepsilon_{1\perp}-\gamma_{1H}\varepsilon_{2\perp}}}. \tag{28}$$

In the case of non-magnetic media, $\langle\mu\rangle=1$, and $\Delta\mu=0$, so that we obtain,

$$\kappa(\omega) = \frac{\omega}{c}\sqrt{\frac{\varepsilon_{2\perp}\varepsilon_{1\perp}}{\gamma_{2H}\varepsilon_{1\perp}+\gamma_{1H}\varepsilon_{2\perp}}}. \tag{29}$$

This is formally identical to the expression obtained for the conventional surface-plasmon dispersion at a planar interface with the additional anisotropy factors $\gamma_{jH}$. We recover the conventional formula for isotropic media by setting $\gamma_{jH}=1$. For an interface of dielectric with Drude metal, we set $\varepsilon_1$ as constant, and $\varepsilon_2=1-\omega_p^2/(\omega^2+i\eta\omega)$. The dispersion in this case is

$$\kappa(\omega) = \frac{\omega\sqrt{\varepsilon_1}}{c}\sqrt{\frac{\omega^2-\omega_p^2+i\eta\omega}{\left(1+\varepsilon_1\right)\omega^2-\omega_p^2+i\eta\left(1+\varepsilon_1\right)\omega}}. \tag{30}$$

We consider two limits. At frequencies much smaller than the plasma frequency, $\kappa$ corresponds to the total wavenumber of a wave travelling along the surface in the

$$\begin{aligned}\lim_{\omega/\omega_p\to 0}\kappa(\omega) &= \frac{\omega\sqrt{\varepsilon_1}}{c}\lim_{\omega/\omega_p\to 0}\sqrt{\frac{1-\omega^2/\omega_p^2-i\eta\omega/\omega_p^2}{1-\left(1+\varepsilon_1\right)\omega^2/\omega_p^2-i\eta\left(1+\varepsilon_1\right)\omega/\omega_p^2}}\\ &= \frac{\omega\sqrt{\varepsilon_1}}{c}\left[1+\frac{\varepsilon_1}{2}\left(\frac{\omega^2}{\omega_p^2}+i\frac{\eta\omega}{\omega_p^2}\right)\right].\end{aligned}$$

In the limit of large frequency,

$$\begin{aligned}\lim_{\omega_p/\omega\to 0}\kappa(\omega) &= \frac{\omega}{c}\sqrt{\frac{\varepsilon_1}{1+\varepsilon_1}}\lim_{\omega/\omega_p\to 0}\sqrt{\frac{1-\omega_p^2/\omega^2-i\eta/\omega}{1-\omega_p^2/\omega^2/\left(1+\varepsilon_1\right)-i\eta/\omega}}\\ &= \frac{\omega}{c}\sqrt{\frac{\varepsilon_1}{1+\varepsilon_1}}\left[1-\frac{1}{2}\left(\frac{\varepsilon_1}{1+\varepsilon_1}\right)\left(\frac{\omega_p^2}{\omega^2}+\frac{i\eta}{\omega}\right)\right].\end{aligned}$$

The surface normal components in this case are,

$$
\begin{aligned}
w_1 &= \frac{\omega}{c}\sqrt{\frac{\varepsilon_1^2}{1+\varepsilon_1}} \\
w_2 &= -\frac{\omega}{c}\sqrt{\frac{\left(1-\omega_p^2/\omega^2\right)\left(1+\varepsilon_1\right)-\varepsilon_1}{1+\varepsilon_1}} \\
&\approx -\frac{\omega}{c}\frac{1}{\sqrt{1+\varepsilon_1}}\left[1-\frac{\omega_p^2}{2\omega^2}\left(1+\varepsilon_1\right)\right]
\end{aligned}
$$

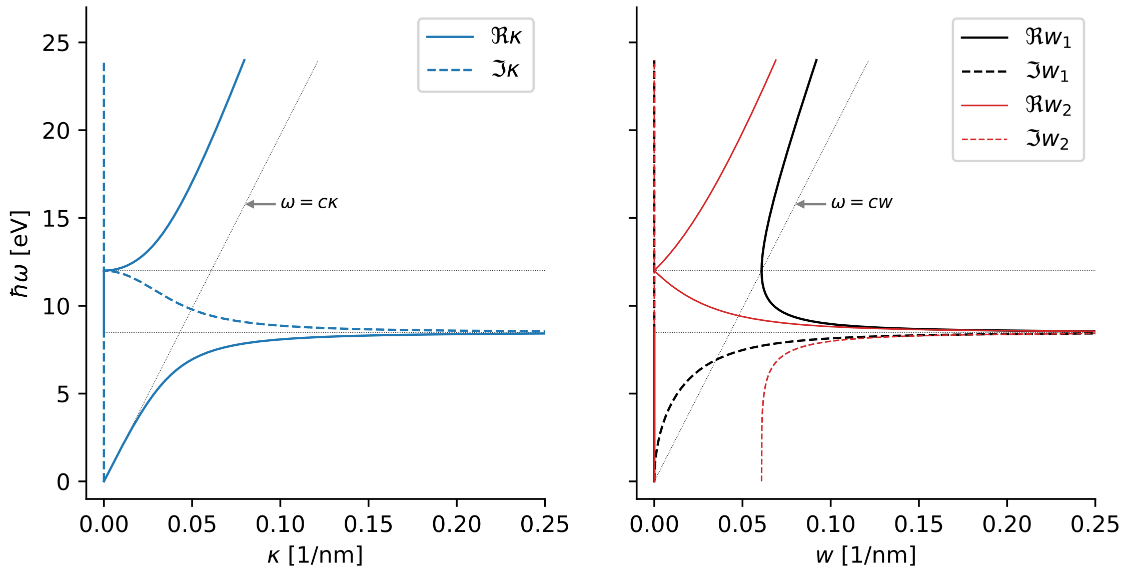


Fig. 1: Plasmon dispersion with a Drude metal and air interface. The plasma frequency for the Drude model parameters are $\omega_p = 12$ eV and $\eta = 0$ and the dielectric half space has, $\varepsilon_1 = 1$. Left panel: frequency dispersion with respect to the in-plane wavevector. Right panel: frequency dispersion with respect to $w_\alpha$ . Due to $\eta = 0$, the wavevectors are either purely real or purely imaginary. Note that in the lower branch, $\omega < \omega_p/\sqrt{1+\varepsilon_1}$, the wave is entirely localized to the interface since $w_\alpha$ are both purely imaginary and large. In the upper branch, $\omega > \omega_p$, the wave is fully delocalized as both $\kappa$ and $w_\alpha$ are purely real.

## 3.2 Magnetic

In this case we seek the solution to (25) and (26) for $E_1^- = 0$, which gives the condition,

$$
\frac{w_1}{\mu_{1\perp}} = -\frac{w_2}{\mu_{2\perp}}.
$$

From the dispersion of $w_\alpha$ in the TE case,

$$\left(\frac{\gamma_{2E}^2}{\mu_{2\perp}^2}-\frac{\gamma_{1E}^2}{\mu_{1\perp}^2}\right)\kappa^2 = \frac{\omega^2}{c^2}\left(\frac{\varepsilon_{2\perp}}{\mu_{2\perp}}-\frac{\varepsilon_{1\perp}}{\mu_{1\perp}}\right).$$

Solving for $\kappa^2$

$$\kappa^2(\omega) = \frac{\omega^2}{c^2}\frac{\left(\varepsilon_{2\|}\gamma_{2E}^2\mu_{1\perp}-\varepsilon_{1\|}\gamma_{1E}^2\mu_{2\perp}\right)\mu_{2\perp}\mu_{1\perp}}{\left(\gamma_{2E}\mu_{1\perp}-\gamma_{1E}\mu_{2\perp}\right)\left(\gamma_{2E}\mu_{1\perp}+\gamma_{1E}\mu_{2\perp}\right)}.$$

Introducing the mean and difference of the geometric means of the anisotropic dielectric values, we obtain

$$\kappa(\omega) = \frac{\omega}{c}\sqrt{\frac{\langle\varepsilon\rangle\,\mu_{2\perp}\mu_{1\perp}}{\gamma_{2E}\mu_{1\perp}+\gamma_{1E}\mu_{2\perp}}}\sqrt{1+\frac{\Delta\varepsilon}{\langle\varepsilon\rangle}\frac{\gamma_{2E}\mu_{1\perp}+\gamma_{1E}\mu_{2\perp}}{\gamma_{2E}\mu_{1\perp}-\gamma_{1E}\mu_{2\perp}}}. \tag{31}$$

## 4 Reflectionless Interface

For an interface to have zero reflection independent of the polarization and direction of incidence, the Fresnel equations imply that the following two equations must be satistfied simultaneously for all $\kappa$,

$$\begin{aligned}\frac{1}{\mu_{\perp 1}}\sqrt{\frac{\omega^2}{c^2}\mu_{1\perp}\varepsilon_{1\perp}-\frac{\mu_{1\perp}}{\mu_{1\|}}\kappa^2} &= \frac{1}{\mu_{\perp 2}}\sqrt{\frac{\omega^2}{c^2}\mu_{2\perp}\varepsilon_{2\perp}-\frac{\mu_{2\perp}}{\mu_{2\|}}\kappa^2},\\ \frac{1}{\varepsilon_{\perp 1}}\sqrt{\frac{\omega^2}{c^2}\mu_{1\perp}\varepsilon_{1\perp}-\frac{\varepsilon_{1\perp}}{\varepsilon_{1\|}}\kappa^2} &= \frac{1}{\varepsilon_{\perp 2}}\sqrt{\frac{\omega^2}{c^2}\mu_{2\perp}\varepsilon_{2\perp}-\frac{\varepsilon_{2\perp}}{\varepsilon_{2\|}}\kappa^2}.\end{aligned}$$

Let us assume that the properties of region 1 are fixed, and we can choose arbitrary properties in region 2. One solution to these is the perfectly matched layer, PML, in which we define a *single* parameter $\gamma$ such that,

$$\varepsilon_{2\perp} = \gamma\varepsilon_{1\perp},\;\; \mu_{2\perp}=\gamma\mu_{1\perp},\;\; \varepsilon_{2\|}=\frac{1}{\gamma}\varepsilon_{1\|},\;\; \mu_{2\|}=\frac{1}{\gamma}\mu_{1\|}. \tag{32}$$

Substuting these values in the above equations, we get

$$\begin{aligned}\frac{1}{\mu_{\perp 1}}\sqrt{\frac{\omega^2}{c^2}\mu_{1\perp}\varepsilon_{1\perp}-\frac{\mu_{1\perp}}{\mu_{1\|}}\kappa^2} &= \frac{1}{\mu_{\perp 1}\gamma}\sqrt{\frac{\omega^2}{c^2}\mu_{1\perp}\varepsilon_{1\perp}\gamma^2-\gamma^2\kappa^2},\\ \frac{1}{\varepsilon_{\perp 1}}\sqrt{\frac{\omega^2}{c^2}\mu_{1\perp}\varepsilon_{1\perp}-\frac{\varepsilon_{1\perp}}{\varepsilon_{1\|}}\kappa^2} &= \frac{1}{\varepsilon_{\perp 2}\gamma}\sqrt{\frac{\omega^2}{c^2}\mu_{1\perp}\varepsilon_{1\perp}\gamma^2-\gamma^2\kappa^2}.\end{aligned}$$

Taking the same branch of the square root on both sides, we see that $\gamma$ cancels out on the right hand side, and the equations are satisfied. This is the anisotropic material formulation of *universal perfectly matched layer* (UPML). Any arbitrary $\gamma$ works, and the optimum value is dictated by the particular discretization of the domain. We also note that if we let $\gamma = iq$,

$$\begin{aligned}w_2 &= iqw_1,\\ \Rightarrow \boldsymbol{E}_2 &= \boldsymbol{E}_2(\zeta=0)e^{-q\zeta},\end{aligned}$$

so that the fields decay away into region 2 without any reflection at the interface.

Furthermore, since all interfaces with the property equation (32) are reflectionless, we can imagine a series of interfaces such that we increase $\gamma$ with each slice. This can be useful in slowly increasing the absorption in the PML if large values at the first interface cause convergence problems.