Scale Invariant Electrodynamics

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Abstract

In this note, we perform the algebraic steps necessary to obtain scale invariant form of Maxwell equations. This form is useful when using the popular FDTD package MEEP, and they are also useful when formulating the mode based solutions to Maxwell equations.

1 Theory

We begin by writing the standard form of Maxwell equations in the SI units as follows,

$$
\begin{aligned}
\epsilon_0 \nabla_x \cdot \varepsilon \boldsymbol{E}(\boldsymbol{x}, t) - \rho &= 0, &(1)\\
\mu_o \nabla_x \cdot \mu \boldsymbol{H} &= 0, &(2)\\
\nabla_x \times \boldsymbol{E} + \mu_0 \mu \frac{\partial \boldsymbol{H}}{\partial t} &= 0, &(3)\\
\nabla_x \times \boldsymbol{H} - \epsilon_0 \varepsilon \frac{\partial \boldsymbol{E}}{\partial t} &= \boldsymbol{J}. &(4)
\end{aligned}
$$

The consitutive relations are in the form $\boldsymbol{D} = \epsilon_0(1+\chi)\boldsymbol{E}$, and $\boldsymbol{B} = \mu_0 \mu \boldsymbol{H}$. Recall the free space impedance definition with units Voltage/Current, and the speed of light,

$$
\begin{aligned}
Z_0 &= \sqrt{\frac{\mu_0}{\epsilon_0}},\\
c &= \frac{1}{\sqrt{\epsilon_0 \mu_0}}.
\end{aligned}
$$

Pick a length scale a, and define spatial and temporal variables as

$$
\begin{aligned}
\boldsymbol{x} &= a\boldsymbol{\xi},\\
t &= \frac{a}{c}\tau,\\
\omega &= \frac{c}{a}\nu
\end{aligned}
$$

The derivatives with respect to these variables transform as,

$$
\begin{aligned}
\frac{\partial f(\boldsymbol{x})}{\partial \boldsymbol{x}} &= \frac{1}{a}\frac{\partial f(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}},\\
\frac{\partial f(t)}{\partial t} &= \frac{c}{a}\frac{\partial f(\tau)}{\partial \tau}.
\end{aligned}
$$

Now substitue these transformations into the Maxwell equations above,

$$
\begin{aligned}
\nabla \times \boldsymbol{E}(\boldsymbol{\xi}, \tau) + c\mu_0 \mu \frac{\partial}{\partial \tau}\boldsymbol{H}(\boldsymbol{\xi}, \tau) &= 0,\\
\nabla \times \boldsymbol{H}(\boldsymbol{\xi}, \tau) - c\epsilon_0 \varepsilon \frac{\partial \boldsymbol{E}}{\partial \tau} &= a\boldsymbol{J}(\boldsymbol{\xi}, \tau).
\end{aligned}
$$

Note that

$$c\mu_0 = Z_0, \quad c\epsilon_0 = \frac{1}{Z_0}.$$

Using this in the curl equations,

$$\begin{aligned}
\nabla \times \boldsymbol{E}(\boldsymbol{\xi}, \tau) + Z_0 \mu \frac{\partial}{\partial \tau} \boldsymbol{H}(\boldsymbol{\xi}, \tau) &= 0, \\
\nabla \times \boldsymbol{H}(\boldsymbol{\xi}, \tau) - \frac{\varepsilon}{Z_0} \frac{\partial \boldsymbol{E}}{\partial \tau} &= a \boldsymbol{J}(\boldsymbol{\xi}, \tau).
\end{aligned}$$

We multiply by the length scale a and write,

$$\begin{aligned}
\nabla \times \left(\frac{a\boldsymbol{E}}{\sqrt{Z_0}} \right) + \mu \frac{\partial}{\partial \tau} \left(a \sqrt{Z_0} \boldsymbol{H} \right) &= 0, \\
\nabla \times \left(a \sqrt{Z_0} \boldsymbol{H} \right) - \varepsilon \frac{\partial}{\partial \tau} \left(\frac{a\boldsymbol{E}}{\sqrt{Z_0}} \right) &= a^2 \sqrt{Z_0} \boldsymbol{J}.
\end{aligned}$$

We now define the scaled fields, with units $\sqrt{\text{Voltage} \times \text{Current}} = \sqrt{\text{Power}}$ for each field,

$$\begin{aligned}
\boldsymbol{\mathcal{E}}(\boldsymbol{\xi}, \tau) &= \frac{a}{\sqrt{Z_0}} \boldsymbol{E}, && (5) \\
\boldsymbol{\mathcal{H}}(\boldsymbol{\xi}, \tau) &= a \sqrt{Z_0} \boldsymbol{H}, && (6) \\
\boldsymbol{\mathcal{J}}(\boldsymbol{\xi}, \tau) &= a^2 \sqrt{Z_0} \boldsymbol{J}. && (7)
\end{aligned}$$

The Maxwell equations for these fields are

$$\begin{aligned}
\nabla \times \boldsymbol{\mathcal{E}} + \mu \frac{\partial \boldsymbol{\mathcal{H}}}{\partial \tau} &= 0, && (8) \\
\nabla \times \boldsymbol{\mathcal{H}} - \varepsilon \frac{\partial \boldsymbol{\mathcal{E}}}{\partial \tau} &= \boldsymbol{\mathcal{J}}. && (9)
\end{aligned}$$

This yields the wave equations,

$$\begin{aligned}
\frac{1}{\varepsilon} \nabla \times \frac{1}{\mu} \nabla \times \boldsymbol{\mathcal{E}} + \frac{\partial^2 \boldsymbol{\mathcal{E}}}{\partial \tau^2} &= \frac{\partial}{\partial \tau} \boldsymbol{\mathcal{J}}, && (10) \\
\frac{1}{\mu} \nabla \times \frac{1}{\varepsilon} \nabla \times \boldsymbol{\mathcal{H}} + \frac{\partial^2 \boldsymbol{\mathcal{H}}}{\partial \tau^2} &= \nabla \times \frac{1}{\varepsilon} \boldsymbol{\mathcal{J}}. && (11)
\end{aligned}$$

In frequency domain,

$$\begin{aligned}
\nabla \times \boldsymbol{\mathcal{E}} - i\nu\mu \boldsymbol{\mathcal{H}} &= 0, \\
\nabla \times \boldsymbol{\mathcal{H}} + i\bar{\omega}\varepsilon \boldsymbol{\mathcal{E}} &= \boldsymbol{\mathcal{J}}.
\end{aligned}$$

and the wave equations

$$\begin{aligned}
\varepsilon^{-1} \nabla \times \mu^{-1} \nabla \times \boldsymbol{\mathcal{E}} - \nu^2 \boldsymbol{\mathcal{E}} &= i\nu\varepsilon^{-1} \boldsymbol{J}, && (12) \\
\mu^{-1} \nabla \times \varepsilon^{-1} \nabla \times \boldsymbol{\mathcal{H}} - \nu^2 \boldsymbol{\mathcal{H}} &= \nabla \times \varepsilon^{-1} \boldsymbol{\mathcal{J}}.
\end{aligned}$$

Poynting Vector is

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}^* = a^{-2} \boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{H}}^* \equiv a^{-2} \boldsymbol{\mathcal{S}}.$$

Power dissipation is,

$$P = -\frac{1}{2}\Re\int \boldsymbol{E}^{*}\cdot\boldsymbol{J}d^{3}\boldsymbol{x} = -\frac{1}{2}\Re\int \boldsymbol{\mathcal{E}}^{*}\cdot\boldsymbol{\mathcal{J}}d^{3}\boldsymbol{\xi} \equiv \boldsymbol{\mathcal{P}}.$$

Thus the integrated quantities retain their original form.In other words, we solve Maxwell equations(8) and (9) with the relative permittivity ε and permeability μ and current density $\boldsymbol{\mathcal{J}}$. This is equivalent to solving these equations for $\boldsymbol{E}$ and $\boldsymbol{H}$ with the source current density $\boldsymbol{J} = a^{-2}Z_0^{-1/2}\boldsymbol{\mathcal{J}}$. In particular, the integrated power can be computed by either set of fields and current densities.

The solution to (12) is given by

$$\boldsymbol{\mathcal{E}} = i\bar{\omega}\left[\boldsymbol{\Theta}-\nu^{2}\right]^{-1}\varepsilon^{-1}\boldsymbol{J}$$

Let $\boldsymbol{\mathcal{E}}_n(\boldsymbol{\xi},\tau)$ be eigenfunctions of the operator $\boldsymbol{\Theta}$

$$\boldsymbol{\Theta}\boldsymbol{\mathcal{E}}_n \equiv \varepsilon^{-1}\nabla\times\mu^{-1}\nabla\times\boldsymbol{\mathcal{E}} = \nu_n^{2}\boldsymbol{\mathcal{E}}_n.$$

Then,

$$\begin{aligned}
\int_{\Omega}\boldsymbol{\mathcal{E}}_n^{*}\cdot\varepsilon\boldsymbol{\Theta}\boldsymbol{\mathcal{E}}_m &= \int_{\Omega}\boldsymbol{\mathcal{E}}_n^{*}\cdot\nabla\times\left(\mu^{-1}\nabla\times\boldsymbol{\mathcal{E}}_m\right)\\
&= \int_{\partial\Omega}\boldsymbol{\mathcal{E}}_n^{*}\times\mu^{-1}\nabla\times\boldsymbol{\mathcal{E}}_m + \int_{\Omega}\left(\mu^{-1}\nabla\times\boldsymbol{\mathcal{E}}_n^{*}\right)\cdot\nabla\times\boldsymbol{\mathcal{E}}_m\\
&= \int_{\partial\Omega}\boldsymbol{\mathcal{E}}_n^{*}\times\left(\mu^{-1}\nabla\times\boldsymbol{\mathcal{E}}_m\right) - \int_{\Omega}\boldsymbol{\mathcal{E}}_m\times\left(\mu^{-1}\nabla\times\boldsymbol{\mathcal{E}}_n^{*}\right) + \int_{\Omega}\boldsymbol{\mathcal{E}}_m\cdot\left(\nabla\times\mu^{-1}\nabla\times\boldsymbol{\mathcal{E}}_n^{*}\right).
\end{aligned}$$

Therefore, we get the relation

$$\int_{\Omega}\boldsymbol{\mathcal{E}}_n^{*}\cdot\varepsilon\boldsymbol{\Theta}\boldsymbol{\mathcal{E}}_m = \int_{\Omega}\boldsymbol{\mathcal{E}}_m\cdot\varepsilon\boldsymbol{\Theta}\boldsymbol{\mathcal{E}}_n^{*} + \int_{\partial\Omega}\left[\boldsymbol{\mathcal{E}}_n^{*}\times\left(\mu^{-1}\nabla\times\boldsymbol{\mathcal{E}}_m\right) - \boldsymbol{\mathcal{E}}_m\times\left(\mu^{-1}\nabla\times\boldsymbol{\mathcal{E}}_n^{*}\right)\right]$$

Substituting Faraday's law for the boundary term,

$$\int_{\Omega}\boldsymbol{\mathcal{E}}_n^{*}\cdot\varepsilon\boldsymbol{\Theta}\boldsymbol{\mathcal{E}}_m = \int_{\Omega}\boldsymbol{\mathcal{E}}_m\cdot\varepsilon\boldsymbol{\Theta}\boldsymbol{\mathcal{E}}_n^{*} + i\nu\int_{\partial\Omega}\left[\boldsymbol{\mathcal{E}}_n^{*}\times\boldsymbol{\mathcal{H}}_m + \boldsymbol{\mathcal{E}}_m\times\boldsymbol{\mathcal{H}}_n^{*}\right].$$