Master Equations

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Abstract

These notes summarize technicalities in the derivation of master equations, and a "standard" template that makes it easier to write equations of motion easy.

1 Step-By-Step Standard Born-Markov form

I start with a Hamiltonian

$$H = \sum_n \hbar\omega_n |n\rangle \langle n| + \sum_\alpha S_\alpha B_\alpha + H_b, \tag{1.1}$$

where S_α act on system, and B_α act on the Hilbert space of the bath. H_b is the Hamiltonian for the bath. It is convenient to define another system operator, which lumps together matrix elements of S_α between states that have equal energy differences,

$$\begin{aligned} A_\alpha(\omega) &= \sum_{\varepsilon'-\varepsilon=\hbar\omega} \Pi(\varepsilon)\hat{S}_\alpha \Pi(\varepsilon') \\ A^\dagger_\alpha(\omega) &= \sum_{\varepsilon-\varepsilon'=\hbar\omega} \Pi(\varepsilon')\hat{S}^\dagger_\alpha \Pi(\varepsilon). \end{aligned}$$

Herer $\Pi(\varepsilon)$ is the projection operator for states with energy equal to ε. Now switch to the interaction picture. It is easy to see that the operators $A_\alpha(\omega)$ and $A^\dagger_\alpha(\omega)$ transform to

$$A_\alpha(\omega, t) = e^{-i\omega t} A_\alpha(\omega), \quad A^\dagger_\alpha(\omega, t) = e^{i\omega t} A^\dagger_\alpha(\omega).$$

The interaction Hamiltonian in this picture becomes

$$\begin{aligned} H_I(t) &= \sum_\alpha A_\alpha(\omega) e^{i\omega t} B_\alpha(t) = \sum_\alpha A^\dagger_\alpha(\omega) e^{-i\omega t} B^\dagger_\alpha(t), \\ \text{where } B_\alpha(t) &= e^{-iH_b t} B_\alpha e^{iH_b t}. \end{aligned}$$

Both forms of $H_I(t)$ are used below. The equation of motion for the *full* density matrix in the interaction picture is,

$$i\hbar \frac{d}{dt} \hat{\varrho}_I(t) = \left[\hat{H}_I(t), \hat{\varrho}_I(t)\right]. \tag{1.2}$$

Substituting the solution formally on the right hand side yields,

$$\frac{d}{dt}\hat{\varrho}_I(t) = -\frac{i}{\hbar}\left[\hat{H}_I(t''),\hat{\varrho}_I(0)\right] - \frac{1}{\hbar^2}\int_0^t dt''\left[\hat{H}_I(t),\left[\hat{H}_I(t''),\hat{\varrho}_I(t'')\right]\right].$$

Now coarse grain this over a time t_c. The left hand side of (1.2) gives

$$\int_{t-t_c/2}^{t+t_c/2}\frac{d}{dt'}\hat{\varrho}_I(t')dt' = \frac{\hat{\varrho}_I(t+t_c/2)-\hat{\varrho}_I(t-t_c/2)}{t_c} \approx \frac{d}{dt}\hat{\varrho}_I(t).$$

Applying *uncorrelated system-bath approximation*,

$$\hat{\varrho}_I(t) = \hat{\rho}(t)\otimes\hat{R},$$

where $\hat{R}$ is the bath statistical operator in its initial state.

Take the trace over the bath on both sides, denoted by $\langle . \rangle$. It is shown in the appendix that the second term on the right hand side becomes,

$$-\sum_\omega\sum_{\alpha\beta} A^\dagger_\alpha(\omega)A_\beta(\omega)\hat{\rho}(t)\int_0^\infty d\tau K_{\alpha\beta}(\tau)e^{i\omega\tau} + \hat{\rho}(t)A^\dagger_\alpha(\omega)A_\beta(\omega)\int_{-\infty}^0 K_{\alpha\beta}(\tau)e^{i\omega\tau}$$
$$- A_\beta(\omega)\hat{\rho}(t)A^\dagger_\alpha(\omega)\left[\int_{-\infty}^0 K_{\alpha\beta}(\tau)e^{i\omega\tau} + \int_0^\infty K_{\alpha\beta}(\tau)e^{i\omega\tau}\right].$$

$$K_{\alpha\beta}(\tau) \equiv \frac{1}{\hbar^2}\left\langle B^\dagger_\alpha(t)B_\beta(t-\tau)\hat{R}\right\rangle.$$

independent of t by stationarity.

Now define the Fourier transform

$$\gamma_{\alpha\beta}(\omega) = \frac{1}{\hbar^2}\int_{-\infty}^{+\infty} d\tau K(\tau)e^{i\omega\tau} = \frac{1}{\hbar^2}\int_{-\infty}^{+\infty} d\tau e^{i\omega\tau}\left\langle B^\dagger_\alpha(t)B_\beta(t-\tau)\hat{R}\right\rangle.$$

In terms of γ,

$$\int_0^\infty d\tau K_{\alpha\beta}(\tau)e^{i\omega\tau} = \frac{1}{2}\gamma_{\alpha\beta}(\omega) + i\,\text{p.v.}\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\frac{\gamma_{\alpha\beta}(\omega')}{\omega-\omega'}$$
$$\int_{-\infty}^0 d\tau K_{\alpha\beta}(\tau)e^{i\omega\tau} = \frac{1}{2}\gamma_{\alpha\beta}(\omega) - i\,\text{p.v.}\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\frac{\gamma_{\alpha\beta}(\omega')}{\omega-\omega'}.$$

Thus define

$$\delta H = \hbar\sum_\omega\sum_{\alpha\beta}\left(\text{p.v.}\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\frac{\gamma_{\alpha\beta}(\omega')}{\omega-\omega'}\right)A^\dagger_\alpha(\omega)A_\beta(\omega).$$

Switch back to the Schrodinger picture. The equation of motion for $\hat{\rho}(t)$ is now

$$\frac{d}{dt}\hat{\rho}(t) = -\frac{i}{\hbar}\left[\delta H,\hat{\rho}(t)\right] + \frac{1}{\hbar^2}\sum_{\omega\alpha\beta}\gamma_{\alpha\beta}(\omega)\left[A_\beta(\omega)\hat{\rho}(t)A^\dagger_\alpha(\omega) - \frac{1}{2}\left\{A^\dagger_\alpha(\omega)A_\beta(\omega),\hat{\rho}(t)\right\}\right].$$

2 Generalized Master equation by Projection Operators

Separate the Hilbert space into a sysetm and a bath, and take any arbitrary operator R that acts only on bath states and has trace equal to 1. Then define operators, $\mathcal{P}$ and $\mathcal{Q}$, such that for any operator acting on the *entire* Hilbert space,

$$\begin{aligned} \mathcal{P}A &\equiv R\mathrm{Tr}_R\left[A\right] \\ \mathcal{Q} &\equiv 1-\mathcal{P}. \end{aligned}$$

Evidently,

$$\mathcal{P}^2 A = R\mathrm{Tr}_R\left[R\right]\mathrm{Tr}_R\left[A\right] = R\mathrm{Tr}_R\left[A\right].$$

Denote the density matrix for system plus reservoir by W and let $\rho = \mathrm{Tr}_R[W]$ so that

$$\mathcal{P}W = \rho R$$

Now take the Schoredinger equation in the interaction picture for the *full* density matrix

$$\frac{d}{dt}W = -\frac{i}{\hbar}\left[V(t), W\right],$$

where $V(t)$ is the interaction Hamiltonian in the interaction picture. Apply $\mathcal{P}$ and $\mathcal{Q}$ to this equation.

$$\frac{d}{dt}\mathcal{P}W = -\frac{i}{\hbar}\mathcal{P}\left[V(t), \mathcal{P}W + \mathcal{Q}W\right] \tag{2.1}$$
$$\frac{d}{dt}\mathcal{Q}W = -\frac{i}{\hbar}\mathcal{Q}\left[V(t), \mathcal{P}W + \mathcal{Q}W\right] \tag{2.2}$$

Trace the first over the reservoir,

$$\begin{aligned} \frac{d}{dt}\rho(t) &= -\frac{i}{\hbar}\mathrm{Tr}_R\left[\mathcal{P}V(t)\mathcal{P}W + \mathcal{P}V(t)\mathcal{Q}W - \mathcal{P}\left(\mathcal{P}W\right)V(t) - \mathcal{P}\left(\mathcal{Q}W\right)V(t)\right] \\ &= -\frac{i}{\hbar}\mathrm{Tr}_R\left[R\mathrm{Tr}_R\left[V(t)R\rho\right] + R\mathrm{Tr}_R\left[V(t)\mathcal{Q}W\right] - R\mathrm{Tr}_R\left[\rho R V(t)\right] - R\mathrm{Tr}_R\left[\left(\mathcal{Q}W\right)V(t)\right]\right] \\ &= -\frac{i}{\hbar}\left[\langle V(t)R\rangle, \rho\right] - \frac{i}{\hbar}\langle V(t)\left(\mathcal{Q}W\right)\rangle + \frac{i}{\hbar}\langle\left(\mathcal{Q}W\right)V(t)\rangle. \end{aligned} \tag{2.3}$$

Expanding the $\mathcal{Q}W$ equation,

$$\begin{aligned} \frac{d}{dt}\mathcal{Q}W &= -\frac{i}{\hbar}\mathcal{Q}\left[V(t), \mathcal{P}W + \mathcal{Q}W\right] \\ &= -\frac{i}{\hbar}\mathcal{Q}\left[V(t), R\rho\right] - \frac{i}{\hbar}\mathcal{Q}\left[V(t), \mathcal{Q}W\right] \\ &= \mathcal{Q}\mathcal{L}(t)\mathcal{P}W + \mathcal{Q}\mathcal{L}(t)\mathcal{Q}W. \end{aligned}$$

In the last line I have re-written the equation in the Liouville form to formally integrate it using the time-ordered exponential of $\mathcal{L}(t)$ denoted symbolically by $\mathcal{U}(t_1,, t_2) = \mathcal{T}\left[e^{-\int_{t_1}^{t_2}\mathcal{Q}\mathcal{L}(t')}\right]$

$$\mathcal{Q}W(t) = \mathcal{U}(t,0)\mathcal{Q}W(0) - \int_0^t \mathcal{U}(t,t')\mathcal{Q}\mathcal{L}(t')R\rho(t')$$

and substitute it in 2.3,

$$\begin{aligned}
\frac{d}{dt}\rho(t) &= -\frac{i}{\hbar}\left\langle [V(t), \mathcal{Q}W(0)]\right\rangle - \frac{i}{\hbar}\left[\left\langle V(t)R\right\rangle, \rho(t)\right] \\
&-\frac{1}{\hbar^2}\int_0^t dt' \left\langle V(t)\mathcal{Q}\left[V(t'), R\rho(t')\right]\right\rangle + \left\langle V(t)\mathcal{Q}\left[V(t'), \mathcal{Q}W(t')\right]\right\rangle \\
&+\frac{1}{\hbar^2}\int_0^t dt' \left\langle \mathcal{Q}\left[V(t'), R\rho(t')\right]V(t)\right\rangle + \left\langle \mathcal{Q}\left[V(t'), \mathcal{Q}W(t')\right]V(t)\right\rangle
\end{aligned}$$

Re-arrange the terms

$$\begin{aligned}
\frac{d}{dt}\rho(t) &= -\frac{i}{\hbar}\left\langle [V(t), \mathcal{Q}W(0)]\right\rangle - \frac{i}{\hbar}\left[\left\langle V(t)R\right\rangle, \rho(t)\right] \\
&-\frac{1}{\hbar^2}\int_0^t dt' \left\langle V(t)\mathcal{Q}\left[V(t'), R\rho(t')\right]\right\rangle - \left\langle \mathcal{Q}\left[V(t'), R\rho(t')\right]V(t)\right\rangle \\
&+\frac{1}{\hbar^2}\int_0^t dt' \left\langle \mathcal{Q}\left[V(t'), \mathcal{Q}W(t')\right]V(t)\right\rangle - \left\langle V(t)\mathcal{Q}\left[V(t'), \mathcal{Q}W(t')\right]\right\rangle
\end{aligned} \tag{2.4}$$

Now write

$$V(t) = \sum_\alpha D_\alpha(t)B_\alpha(t),$$

where $A_\alpha(t)$ act on system space and B_α on the bath space. Define *two-time correlation functions*

$$C_{\alpha\beta}(t,t') = \left\langle B_\alpha(t)B_\beta(t')R\right\rangle - \left\langle B_\alpha(t)R\right\rangle\left\langle B_\beta(t')R\right\rangle = \left\langle \Delta B_\alpha(t)\Delta B_\beta(t')R\right\rangle, \tag{2.5}$$

$$\text{where} \qquad \Delta B \equiv B - \langle B\rangle .$$

Thus the correlation functions that naturally arise in this formalism are describe *fluctuations*. In a Green function treatment, they also arise naturally by constructing the theory using functional derivatives, which generate the exchange correlation part of multi-time Green functions. Thus suppose the operators B_α could be written as a sum over products of creation annihilation operators for some elementary excitations,

$$B_\alpha(t) = \sum_{\{i\},\{j\}} \mathcal{B}_\alpha(i_1\ldots i_n, j_1\ldots j_m)a^\dagger_{i_1}(t)a^\dagger_{i_2}(t)\cdots a^\dagger_{i_n}(t)a_{j_1}(t)a_{j_2}(t)\cdots a_{j_m}(t).$$

Then

$$
\begin{aligned}
& \langle B_\alpha(t)B_\beta(t')R\rangle - \langle B_\alpha(t)R\rangle \langle B_\beta(t')R\rangle \\
= & \sum_{\{i\},\{j\}} \mathcal{B}_\alpha(i_1 \dots i_n, j_1 \dots j_m)\mathcal{B}_\alpha(i'_1 \dots i'_n, j'_1 \dots j'_m) \\
& \left\langle \mathcal{T} a^\dagger_{i_1}(t)\cdots a^\dagger_{i_n}(t) a^\dagger_{i'_1}(t')\cdots a^\dagger_{i'_{n'}}(t') a_{j_1}(t)\cdots a_{j_{n'}}(t) a_{j'_1}(t')\cdots a_{j'_{m'}}(t')\right\rangle \\
& -\left\langle \mathcal{T} a^\dagger_{i_1}(t)\cdots a^\dagger_{i_n}(t) a_{j_1}(t)\cdots a_{j_m}(t)\right\rangle \left\langle \mathcal{T} a^\dagger_{i'_1}(t')\cdots a^\dagger_{i'_{n'}}(t') a_{j'_1}(t')\cdots a_{j'_{m'}}(t')\right\rangle \\
= & \sum_{\{i\},\{j\}} \mathcal{B}_\alpha(i_1 \dots i_n, j_1 \dots j_m)\mathcal{B}_\alpha(i'_1 \dots i'_n, j'_1 \dots j'_m) \Big[\\
& \mathcal{G}^{(m+m';n+n')}(j_1t,\dots,j_nt,j'_1t',\dots j'_mt';i_1t,\dots,i_nt,i'_1t',\dots i'_{m'}t') \\
& -\mathcal{G}^{(n,m)}(j_1t,\dots,j_nt;i_1t,\dots,i_mt)\mathcal{G}^{(n',m')}(j'_1t',\dots j'_{n'}t';i'_1t',\dots i'_{m'}t') \Big] \\
= & -\sum_{\{i\},\{j\}} \mathcal{B}_\alpha(i_1 \dots i_n, j_1 \dots j_m)\mathcal{B}_\alpha(i'_1 \dots i'_n, j'_1 \dots j'_m) \left[\frac{\delta^{m'}}{\delta U(1')\dots\delta U(m')}\mathcal{G}^{(n)}(1,,\dots n;\bar{1},\dots\bar{n})\right]_{t_j,\bar{t}_j=t\,;t_{1'}\dots t_{m'}=t'} ,
\end{aligned}
$$

where I have used the symbol $\mathcal{G}^{(m,n)}(1\dots m;\bar{1}\dots\bar{n})$ to represent the Green function in which m particles are destroyed and n particles are created at times $t_1\dots t_m$ and $\bar{t}_1\dots\bar{t}_n$ respectively. The notation of using numbers in the argument is the usual one for compactly representing indices and time arguments of the creation, annihilation operators. The symbol $\mathcal{T}$ in the first equality above represents the time ordering operator, so that the order of operators on the left hand side of the equations can be reproduced by an appropriate choice times on the Keldysh contour.

In most cases, where the particle number is conserved, $m = n$. For $m = n$, these are normal m-particle Green functions. When $m \neq n$, one obtains an anomalous Green function as in the case of a BEC or in the electron-hole picture for semiconductors. Thus the correlation functions $C_{\alpha,\beta}(t,t')$ are a restriction of some single or multi-particle Green function to two times in the way shown above. It is always the case that the functional derivative above corresponds to removal of factorization into the equal time correlations (as in the middle line of 2.5) because the lowest (factorized) term generated by the derivative have *both* the t and t' arguments. As usual, casting the theory in Green function form would allow a more efficient way of generating higher orders in perturbation expansion via diagrams. This may be particularly useful when partial summations beyond second order in the interaction between quantum dot and lead are desired. However, I am focusing on second order for now, and proceed with the density matrix formalism.

The functional derivative in the last line is with respect to an external perturbation, which may be physical or a mathematical device such that calculation is done at $U = 0$. The

As shown in Appendix B, the middle term of (2.4) takes

$$
\begin{aligned}
& -\frac{1}{\hbar^2}\int_0^t dt' \sum_{\alpha\beta} C_{\alpha\beta}(t,t')D_\alpha(t)D_\beta(t'),\rho(t') + C_{\beta\alpha}(t',t)\rho(t')D_\beta(t')D_\alpha(t) \\
& +\frac{1}{\hbar^2}\int_0^t dt' \sum_{\alpha\beta} C_{\beta\alpha}(t',t)D_\alpha(t)\rho(t')D_\beta(t') + C_{\alpha\beta}(t,t')D_\beta(t')\rho(t')D_\alpha(t).
\end{aligned}
$$

Change the dummy indices, and re-write the EOM in the following form

$$\begin{aligned}
\frac{d}{dt}\rho(t) &= -\frac{i}{\hbar}\left\langle [V(t), \mathcal{Q}W(0)] \right\rangle - \frac{i}{\hbar}\left[\langle V(t)R\rangle, \rho(t)\right] \\
&\quad -\frac{1}{\hbar^2}\int_0^t dt' \sum_{\alpha\beta}\left\{C_{\alpha\beta}(t,t')D_\alpha(t)D_\beta(t')\rho(t') + C_{\alpha\beta}(t',t)\rho(t')D_\alpha(t')D_\beta(t)\right\} \\
&\quad +\frac{1}{\hbar^2}\int_0^t dt' \sum_{\alpha\beta}\left\{C_{\alpha\beta}(t',t)D_\beta(t)\rho(t')D_\alpha(t') + C_{\alpha\beta}(t,t')D_\beta(t')\rho(t')D_\alpha(t)\right\} \\
&\quad +\frac{1}{\hbar^2}\int_0^t dt' \left\langle \mathcal{Q}\left[V(t'), \mathcal{Q}W(t')\right]V(t)\right\rangle - \left\langle V(t)\mathcal{Q}\left[V(t'), \mathcal{Q}W(t')\right]\right\rangle . \\
\mathcal{Q}W(t) &= \mathcal{Q}W(0) - \frac{i}{\hbar}\int_0^t dt'\, \mathcal{Q}\left[V(t'), R\rho\right] - \frac{i}{\hbar}\int_0^t dt' \mathcal{Q}\left[V(t'), \mathcal{Q}W(t)\right] \\
&= \mathcal{U}(t,0)\mathcal{Q}W(0) - \int_0^t \mathcal{U}(t,t')\mathcal{Q}\mathcal{L}(t')R\rho(t')
\end{aligned}$$

Another form of this equation may be obtained by using symmetrized and antisymmetrized correlation functions.

$$\begin{aligned}
-i\mathcal{A}_{\alpha\beta}(t,t') &\equiv \left[C_{\alpha\beta}(t,t') - C_{\alpha\beta}(t',t)\right] &= \left\langle \left[\Delta B_\alpha(t)\Delta B_\beta(t') - \Delta B_\alpha(t')\Delta B_\beta(t)\right] R\right\rangle \\
\mathcal{S}_{\alpha\beta}(t,t') &\equiv \left[C_{\alpha\beta}(t,t') + C_{\alpha\beta}(t',t)\right] &= \left\langle \left\{\Delta B_\alpha(t)\Delta B_\beta(t') + \Delta B_\alpha(t')\Delta B_\beta(t)\right\} R\right\rangle ,
\end{aligned}$$

Then I get

$$\begin{aligned}
\frac{d}{dt}\rho(t) &= -\frac{i}{\hbar}\left\langle [V(t), \mathcal{Q}W(0)] \right\rangle - \frac{i}{\hbar}\left[\langle V(t)R\rangle, \rho(t)\right] \\
&\quad -\frac{i}{2\hbar^2}\int_0^t dt' \sum_{\alpha\beta}\mathcal{A}_{\alpha\beta}(t,t')\left[D_\alpha(t)D_\beta(t')\rho(t') - \rho(t')D_\alpha(t')D_\beta(t)\right] \\
&\quad -\frac{i}{\hbar^2}\int_0^t dt' \sum_{\alpha\beta}\mathcal{A}_{\alpha\beta}(t,t')\left[D_\beta(t)\rho(t')D_\alpha(t') - D_\beta(t')\rho(t')D_\alpha(t)\right] \\
&\quad -\frac{1}{2\hbar^2}\int_0^t dt' \sum_{\alpha\beta}\mathcal{S}_{\alpha\beta}(t,t')\left\{D_\alpha(t)D_\beta(t')\rho(t') + \rho(t')D_\alpha(t')D_\beta(t)\right\} \\
&\quad +\frac{1}{\hbar^2}\int_0^t dt' \sum_{\alpha\beta}\mathcal{S}_{\alpha\beta}(t,t')\left\{D_\beta(t)\rho(t')D_\alpha(t') + D_\beta(t')\rho(t')D_\alpha(t)\right\} \\
&\quad +\frac{1}{\hbar^2}\int_0^t dt' \left[\langle RV(t)\rangle, \left\langle [V(t'), \mathcal{Q}W(t')]\right\rangle\right] - \left\langle [V(t), [V(t'), \mathcal{Q}W(t')]]\right\rangle .
\end{aligned}$$

A Manipulations for Markov equation

$$
\begin{aligned}
& \frac{1}{t_c}\int dt'\theta\left(|t|-\frac{t_c}{2}\right)\int_0^{t'} d\tau\left\langle\left[\hat{H}_I(t'),\left[\hat{H}_I(t''),\hat{\varrho}(t'')\otimes\hat{R}\right]\right]\right\rangle \\
= \; & \frac{1}{t_c}\int dt'\theta\left(|t|-\frac{t_c}{2}\right)\int_0^{t'} d\tau\sum_{nm}\sum_{lj} \\
& A^\dagger_\alpha(\omega')A_\beta(\omega)\hat{\rho}(t'-\tau)\left\langle B^\dagger_\alpha(t')B_\beta(t'-\tau)\hat{R}\right\rangle e^{i(\omega'-\omega)t'+i\omega\tau} \\
& +\hat{\rho}(t'-\tau)A^\dagger_\alpha(\omega')A_\beta(\omega)\left\langle \hat{R}B^\dagger_\alpha(t'-\tau)B_\beta(t')\right\rangle e^{i(\omega'-\omega)t'-i\omega'\tau} \\
& -A_\beta(\omega)\hat{\rho}(t'-\tau)A^\dagger_\alpha(\omega')\left\langle B_\beta(t')\hat{R}B^\dagger_\alpha(t'-\tau)\right\rangle e^{i(\omega'-\omega)t'-i\omega'\tau} \\
& -A_\beta(\omega)\hat{\rho}(t'-\tau)A^\dagger_\alpha(\omega')\left\langle B_\beta(t'-\tau)\hat{R}B^\dagger_\alpha(t')\right\rangle e^{i(\omega'-\omega)t'+i\omega\tau}.
\end{aligned}
$$

Now do the integral over t' assuming stationarity of the correlation functions so that they do not depend on t'.

$$
\begin{aligned}
= \; & e^{i(\omega'-\omega)t}\frac{\sin\left[(\omega-\omega')\frac{t_c}{2}\right]}{(\omega-\omega')t_c}\int_0^{t'} d\tau\sum_{\alpha\beta}\sum_{lj} \\
& A^\dagger_\alpha(\omega')A_\beta(\omega)\hat{\rho}(t'-\tau)\left\langle B^\dagger_\alpha(t')B_\beta(t'-\tau)\hat{R}\right\rangle e^{i\omega\tau}+\hat{\rho}(t'-\tau)A^\dagger_\alpha(\omega')A_\beta(\omega)\left\langle \hat{R}B^\dagger_\alpha(t'-\tau)B_\beta(t')\right\rangle e^{-i\omega'\tau} \\
& -A_\beta(\omega)\hat{\rho}(t'-\tau)A^\dagger_\alpha(\omega')\left\langle B_\beta(t')\hat{R}B^\dagger_\alpha(t'-\tau)\right\rangle e^{-i\omega'\tau}-A_\beta(\omega)\hat{\rho}(t'-\tau)A^\dagger_\alpha(\omega')\left\langle B_\beta(t'-\tau)\hat{R}B^\dagger_\alpha(t')\right\rangle e^{i\omega\tau}.
\end{aligned}
$$

Now let t_c be large enough that for any $\omega \neq \omega'$ (recall that ω are discrete natural frequencies of the finite system), the sinc function has a negligible magnitude.

B GME manipulations

Manipulate the integrand terms on the right hand side of (2.4)

$$
\begin{aligned}
\langle V(t)\mathcal{Q}\left[V(t'),R\rho(t')\right]\rangle &= \langle V(t)\mathcal{Q}\left[V(t')R\rho(t')\right]\rangle-\langle V(t)\mathcal{Q}\left[R\rho(t')V(t')\right]\rangle \\
&= \langle V(t)V(t')R\rho(t')\rangle-\langle V(t)R\rho(t')V(t')\rangle \\
&\quad -\langle V(t)\mathcal{P}\left[V(t')R\rho(t')\right]\rangle+\langle V(t)\mathcal{P}\left[R\rho(t')V(t')\right]\rangle \\
&= \langle V(t)V(t')R\rangle\rho(t')-\langle V(t)R\rho(t')V(t')\rangle-\langle V(t)R\rangle\left[\langle V(t')R\rangle\rho(t')-\rho(t')\langle V(t')R\rangle\right] \\
\langle \mathcal{Q}\left[V(t'),R\rho(t')\right]V(t)\rangle &= \langle V(t')R\rho(t')V(t)\rangle-\rho(t')\langle RV(t')V(t)\rangle-\langle V(t')R\rangle\rho(t')\langle V(t)R\rangle+\rho(t')\langle RV(t')\rangle\langle RV(t)\rangle
\end{aligned}
$$

$$
\begin{aligned}
\text{Subtract:}\quad & \langle V(t)V(t')R\rangle\rho(t')-\langle V(t)R\rho(t')V(t')\rangle-\langle V(t)R\rangle\langle V(t')R\rangle\rho(t')+\langle V(t)R\rangle\rho(t')\langle V(t')R\rangle \\
& -\langle V(t')R\rho(t')V(t)\rangle+\rho(t')\langle RV(t')V(t)\rangle+\langle V(t')R\rangle\rho(t')\langle V(t)R\rangle-\rho(t')\langle RV(t')\rangle\langle RV(t)\rangle.
\end{aligned}
$$

This yields

$$
\begin{aligned}
& \langle V(t)\mathcal{Q}\left[V(t'), R\rho(t')\right]\rangle - \langle \mathcal{Q}\left[V(t'), R\rho(t')\right] V(t)\rangle \\
= \; & \left(\langle V(t)V(t')R\rangle - \langle V(t)R\rangle \langle V(t')R\rangle\right)\rho(t') + \rho(t')\left(\langle RV(t')V(t)\rangle - \langle RV(t')\rangle \langle RV(t)\rangle\right) \\
& - \left(\langle V(t)R\rho(t')V(t')\rangle - \langle V(t)R\rangle \rho(t') \langle V(t')R\rangle\right) - \left(\langle V(t')R\rho(t')V(t)\rangle - \langle V(t')R\rangle \rho(t') \langle V(t)R\rangle\right).
\end{aligned}
$$

The correlation functions defined in (2.5) allow us to write the various parts of the above terms as follows,

$$
\begin{aligned}
\langle V(t)V(t')R\rangle \rho(t') - \langle V(t)R\rangle \langle V(t')R\rangle \rho(t') &= \sum_{\alpha\beta} C_{\alpha\beta}(t,t') D_\alpha(t) D_\beta(t')\rho(t') \\
\langle V(t)R\rho(t')V(t')\rangle - \langle V(t)R\rangle \rho(t') \langle V(t')R\rangle &= \sum_{\alpha\beta} \left[\langle B_\alpha(t)RB_\beta(t')\rangle - \langle B_\alpha(t)R\rangle \langle B_\beta(t')R\rangle\right] D_\alpha(t)\rho(t')D_\beta(t') \\
&= \sum_{\alpha\beta} \left[\langle B_\beta(t')B_\alpha(t)R\rangle - \langle B_\beta(t')R\rangle \langle B_\alpha(t)R\rangle\right] D_\alpha(t)\rho(t')D_\beta(t') \\
&= \sum_{\alpha\beta} C_{\beta\alpha}(t',t) D_\alpha(t)\rho(t')D_\beta(t')
\end{aligned}
$$

The last term of (2.4) can be re-written as

$$
\begin{aligned}
& \langle \left[V(t'), \mathcal{Q}W(t')\right] V(t)\rangle - \langle V(t)\left[V(t'), \mathcal{Q}W(t')\right]\rangle \\
& - \langle \mathcal{P}\left[V(t'), \mathcal{Q}W(t')\right] V(t)\rangle + \langle V(t)\mathcal{P}\left[V(t'), \mathcal{Q}W(t')\right]\rangle \\
= \; & \langle \left[\left[V(t'), \mathcal{Q}W(t')\right], V(t)\right]\rangle - \langle \left[V(t'), \mathcal{Q}W(t')\right]\rangle \langle RV(t)\rangle + \langle V(t)R\rangle \langle \left[V(t'), \mathcal{Q}W(t')\right]\rangle \\
= \; & \langle \left[\left[V(t'), \mathcal{Q}W(t')\right], V(t)\right]\rangle - \left[\langle \left[V(t'), \mathcal{Q}W(t')\right]\rangle, \langle RV(t)\rangle\right] \\
= \; & \left[\langle RV(t)\rangle, \langle \left[V(t'), \mathcal{Q}W(t')\right]\rangle\right] - \langle \left[V(t), \left[V(t'), \mathcal{Q}W(t')\right]\right]\rangle
\end{aligned}
$$