On Graceful Line Graphs

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Abstract

A graph G with n vertices and q edges admits a graceful labeling if there is a one-to-one map f from the set of vertices of G to $\{0, 1, 2, \cdots, q\}$ such that when an edge xy is assigned the label $|f(x) - f(y)|$, the resulting set of edge labels is $\{1, 2, \cdots, q\}$. When such a labeling exists, G is called graceful. The line graph $L(G)$ of a graph G is well studied in the literature. The vertices of $L(G)$ correspond to the edges of G, with two vertices adjacent if the corresponding edges are adjacent in G. In this paper, we explore connections between graceful labelings and the line graph operation. We investigate questions such as which graphs have graceful line graphs, and when is the line graph of a graceful graph graceful. We present some classes of unicyclic graphs that have graceful line graphs.

1 Introduction

Unless stated otherwise, we assume that $G = (V, E)$ is a simple graph with $|V| = n$ vertices and $|E| = q$ edges. A graceful labeling f of G is a one-to-one map from V to the set $\{0, 1, 2, \cdots, q\}$ such that when an edge xy is assigned the label $|f(x) - f(y)|$, the resulting set of edge labels is $\{1, 2, \cdots, q\}$. A graph is graceful if it admits at least one graceful labeling. The concept of gracefulness was introduced by Rosa [27] in an effort to resolve a conjecture of Ringel [26] that for any tree T with m edges, there is a decomposition of the complete graph K_{2m+1} into $2m + 1$ subgraphs, each isomorphic to T. Indeed, Rosa [27] showed if T is graceful, then the conjecture holds. The well-known *graceful tree conjecture* states that all trees are graceful. Although many classes of trees have been shown to be graceful, the general problem is still open.

Besides trees, gracefulness of many other classes of graphs has been investigated, including the class of unicyclic graphs. Rosa [27] showed that the simplest unicyclic graphs, cycles C_n, are graceful if and only if $n \equiv 0$ or $n \equiv 3 \pmod 4$. Truszczyński [29] conjectured that unicyclic graphs that are not cycles are graceful. Again, many classes of unicyclic graphs have been proven to be graceful, but the general problem is open. Biatch', Bagga, and Arumugam [23] present a survey of results related to Truszczyński conjecture. We refer the reader to Gallian [15] for a general survey of graceful graphs and a large set of references on this topic.

The line graph transformation is one of the most studied and well-known graph transformations. The vertices of of the line graph $L(G)$ of a graph G correspond to the edges of G, with two vertices adjacent if the corresponding edges are adjacent in G. The book *Line Graphs and Line Digraphs* by Beineke and Bagga [8] presents a comprehensive account of this topic.

In this paper, we are interested in studying gracefulness of unicyclic graphs and gracefulness of their line graphs. It is natural to ask questions such as which graphs have graceful line graphs, and when is the line graph of a graceful graph graceful. In 1991, Gnanajothi [16] called a graph G bigraceful if both G and $L(G)$ are graceful. In a paper published in 2002, Murugan [21] called such graphs bi-graceful. We discuss below the results obtained by Gnanajothi and Murugan. Interestingly, according to Gallian [15] the term bigraceful has also been used for multiple other unrelated concepts.

We conclude this section by describing some notation we will use in later sections. Let G and H be two vertex disjoint graphs. For a vertex x of G and a vertex y of H, $G(x) \oplus H(y)$ denotes the graph obtained by identifying x with y. More generally, for a graph G of order n, with a specified ordering of its vertices, and n graphs H_i with specified vertices y_i, the graph obtained by identifying each vertex of G in the specified order with y_i will be denoted $G \oplus \big(H_1(y_1);\, H_2(y_2); \cdots;\, H_n(y_n)\big)$. Figure 1.1 shows the graph obtained by identifying each of the two adjacent vertices of C_4 with an end vertex of a P_3, and we use the simplified notation $C_4 \oplus (P_3; P_3; P_1; P_1)$ for this graph.

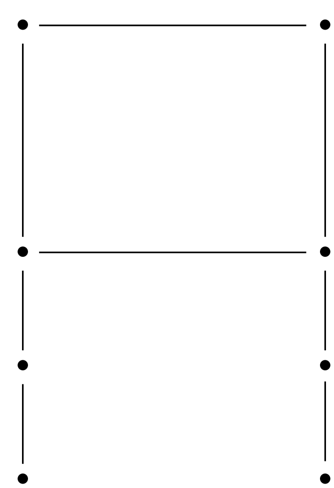


Figure 1.1: $C_4 \oplus (P_3; P_3; P_1; P_1)$

Bagga and Heinz [4] studied graceful labelings of a class of unicyclic graphs they called sun graphs.

For a pair (n, k) $n \geq 3, k \geq 1$, $Sun(n, k)$ is the unicyclic graph obtained by adding k pendant edges to each vertex of the cycle C_n. $(n \geq 3,\ k \geq 0)$. Thus $Sun(n, k) = C_n \oplus (K_{1,k}; K_{1,k}; \ldots; K_{1,k})$, where each vertex of C_n is identified with the central vertex of $K_{1,k}$.

For terms not defined here, we refer the reader to the books by Beineke and Bagga [8] and Gross and Yellen [9] and the survey paper of Gallian [15].

2 Known results

We begin by listing some known results about gracefulness and line graphs that will need below. For more details, we refer the reader to Gallian's survey [15] on graph labelings, and the aforementioned book on Line Graphs and Line Digraphs by Beineke and Bagga [8].

Observation 2.1 *If a graph G is graceful, then $n \leq q + 1$.*

Theorem 2.1 *The following graphs are graceful.*

- *Paths P_n $(n \geq 2)$*
- *Cycles C_n $(n \geq 3)$ if and only if $n \equiv 0$ or $n \equiv 3$ (mod 4).*
- *Complete graphs K_n if and only if $n \leq 3$*
- *Complete bipartite graphs $K_{m,n}$.*
- *Wheels $W_n = C_{n-1} + K_1$.*

Theorem 2.2 *If G is an Eulerien graph with q edges, and if $q \equiv 1$ or $q \equiv 2$ (mod 4), then G is not graceful.*

We recall that a *caterpillar* is a tree where the deletion of leaves results in a path. In [6] Barrientos studied graceful labelings of a special class of unicyclic graphs. He defined a *hairy cycle* to be a unicyclic graph in which the deletion of any edge in the cycle results in a caterpillar, and he showed that all hairy cycles are graceful. We observe that $Sun(n, k)$ are hairy cycles, and hence graceful. For more on graceful labelings of $Sun(n, k)$, please see Bagga and Heinz [4].

Theorem 2.3 *The sun graphs $Sun(n, k)$ $(k \geq 1)$are graceful.*

We next list some basic properties of line graphs. For more detail, we refer he reader to the book by Beineke and Bagga [8].

Theorem 2.4

- $L(P_n) = P_{n-1}$.
- $L(C_n) = C_n$.
- $L(K_{1,n}) = K_n$.
- *For a graph G with p vertices and q edges, $L(G)$ has q vertices and $\frac{1}{2}\sum_{i=1}^{p} d_i^2 - q$ edges.*
- *The degree of a vertex $e = uv$ in $L(G)$ is $d(e) = d(u) + d(v) - 2$.*

Based on the above results, both Gnanajothi [16] and Murugan [21] listed paths and cycles (with conditions as above) as classes of graceful graphs that have graceful line graphs. Similarly, both $K_{1,n}$ and $L(K_{1,n})$ are graceful if and only if $n \leq 3$.

Gnanajothi [16] proved the next two results. Also see Gallian [15].

Theorem 2.5 *The following graphs and their line graphs are graceful.*

- *K_n if and only if $n \leq 3$*
- *$P_m \times P_n$ $(m, n \geq 3)$*
- *$K_{1,n} \times P_2$ $(n \geq 3)$*
- *$K_{1,n} \times P_2$ if and only if $n \equiv 3 \pmod 4$*

Theorem 2.6 *$L(K_{m,n})$ is not graceful when $n > 3$ and $n \equiv 3 \pmod 4$*

The next three results were proved by Murugan [21].

Theorem 2.7 *For a graph G with n vertices, q edges and degree sequence $\{d_1, d_2, \ldots, d_n\}$, if $L(G)$ is graceful, then $\sum_{i=1}^{n} d_i^2 \geq 4q - 2$.*

Proof. Since $L(G)$ is graceful, it follows from Observation 2.1 and Theorem 2.4 that $q \leq \frac{1}{2}\sum d_i^2 - q + 1$. The result follows.

We observe that the converse of the above result does not hold. If G is the sun graph $Sun(n, 1)$, where $n \equiv 2 \pmod 4$, then it follows from Theorem 3.1 that $L(G)$ is not graceful. However, $\sum_{i=1}^{n} d_i^2 = 10n \geq 4q - 2 = 8n - 2$.

□

Theorem 2.8 *For the wheels W_n, $L(W_n)$ is not graceful when $n \equiv 0$ or $n \equiv 6 \pmod 8$.*

Theorem 2.9 *With notation as above, the graphs $G = C_4 \oplus (P_n; P_n; P_1; P_1)$ and $L(G)$ are graceful.*

In the next section we generalize some of the above results, and obtain some new classes of graceful line graphs.

3 Graceful Line Graphs of Some Unicyclic Graphs

In this section we present several new results about gracefulness of line graphs. We begin by proving that line graphs of some types of sun graphs are not graceful.

Theorem 3.1 *If k is odd, the line graph of the sun graph $Sun(n, k)$ is not graceful if*

- $n \equiv 1\ (mod\ 4)$ *and* $k \equiv 3$ *or* $5\ (mod\ 8)$;
- $n \equiv 2\ (mod\ 4)$ *and* $k \equiv 1$ *or* $5\ (mod\ 8)$;
- $n \equiv 3\ (mod\ 4)$ *and* $k \equiv 1$ *or* $3\ (mod\ 8)$;

Proof. For $G = Sun(n, k)$, we have $n + nk$ vertices and $n + nk$ edges. The n cycle vertices have degree $k + 2$ each, and there are nk vertices of degree 1 each. Hence $L(G)$ has $n + nk$ vertices, n vertices of degree $2k + 2$ each, and nk vertices of degree $k + 1$ each. Since k is odd, it follows that $L(G)$ is Eulerian. We use q to denote the number of edges in $L(G)$. It follows from Theorem 2.1 that $L(G)$ is not graceful if $q \equiv 1$ or $q \equiv 2$ (mod 4).

From Theorem 2.4 it follows that

$$\begin{aligned} q &= -n(1 + k) + \frac{1}{2}\sum(deg(v))^2 \\ &= -n(1 + k) + \frac{1}{2}[n(k + 2)^2 + nk] \\ &= \frac{1}{2}n(k^2 + 3k + 2) \\ &= \frac{1}{2}n(k + 1)(k + 2) \end{aligned}$$

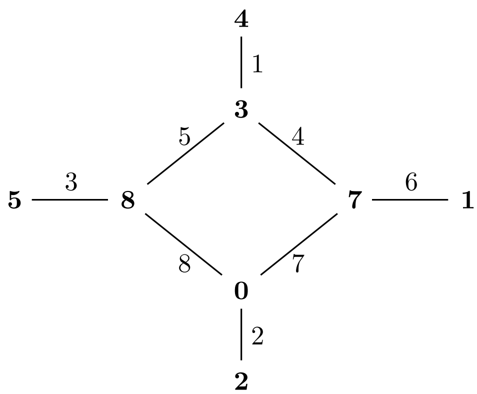
For the first case, we assume that $n = 4r+1$ and $k = 8s+3$. Then $q = \frac{1}{2}(4r+1)(8s+4)(8s+5) = (4r + 1)(4s + 2)(8s + 3) \equiv 2\ (mod\ 4)$. The other cases are similar, and we omit the proof.

□

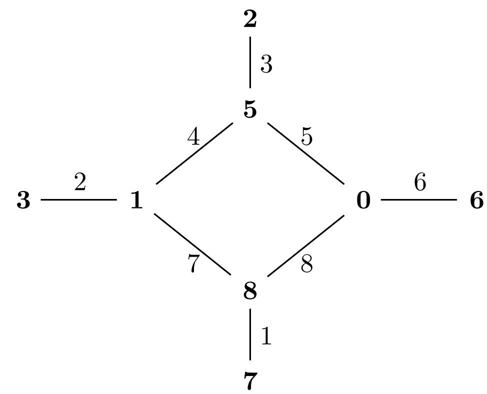
The sun graph $Sun(4, 1)$ is bigraceful. The figure 3.1 presents two graceful labelings of the sun graph $Sun(4, 1)$ and two graceful labelings of $L(Sun(4, 1))$, the line graph of $Sun(4, 1)$. With a naive enumeration algorithm, we find 672 graceful algorithms of $Sun(4, 1)$ and 2688 graceful labelings of $L(Sun(4, 1))$.

We leave the determination of the gracefulness of a general $Sun(n, k)$ as an open problem.

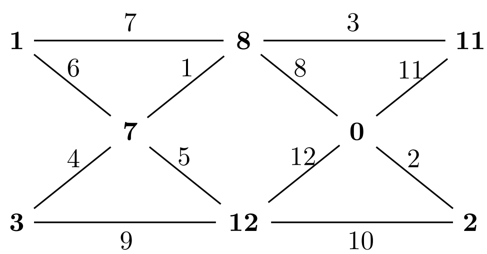
Now we study the gracefulness of the line graphs of graphs $C_n \oplus (P_t; P_s; P_1; \cdots ; P_1)$, unicyclic graphs with a cycle C_n and two paths P_t and P_s attached to two adjacent vertices of the cycle. We have shown the gracefulness of these unicyclic graphs in [2] by giving an algorithm to obtain a graceful labeling. We ameliorate that result by giving a one-to-one map giving a graceful labeling of these unicyclic graphs in the following lemma.



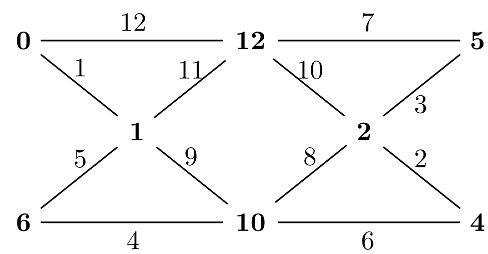
A graceful labeling of $Sun(4,1)$



A second graceful labeling of $Sun(4,1)$



A graceful labeling of $L(Sun(4,1))$



A second graceful labeling of $L(Sun(4,1))$

Figure 3.1: Graceful labelings of $Sun(4,1)$ and $L(Sun(4,1))$

Lemma 1 *$C_n \oplus (P_t; P_s; P_1; \cdots; P_1)$, n odd, $n = 2k+1$, $t, s \geq k-1$, $t \geq s$, are graceful.*

Proof. Let the vertex set of $C_n \oplus (P_t; P_s; P_1; \cdots; P_1)$ be $\{v_1, v_2, \cdots, v_{n+t+s-2}\}$ with

$$\begin{aligned} V(P_t) &= \{v_1, \cdots, v_t\} \\ V(C_n) &= \{v_t, \cdots, v_{t+4}\} \\ V(P_s) &= \{v_{t+4}, \cdots, v_{n+t+s-2}\} \end{aligned}$$

Now we give the graceful labeling for $C_n \oplus (P_t; P_s; P_1; \cdots; P_1)$ by the following map f

case 1. $k+t+s$ is even.

$$f(v_i) = n+t+s-2-\tfrac{i-1}{2}, \qquad i = 1, 3, \cdots, k+t+s-1.$$

If k is even, then

$f(v_i) = \frac{i}{2} - 1,$	$i = 2, 4, \cdots, n+t+s-3.$
$f(v_i) = n+t+s-2-\frac{i+1}{2},$	$i = k+t+s+1, k+t+s+3, \cdots, n+t+s-2.$

If k is odd, then

$f(v_i) = \frac{i}{2} - 1,$	$i = 2, 4, \cdots, k+t+s+2\left\lfloor \frac{k}{2} \right\rfloor.$
$f(v_i) = n+t+s-2-\frac{i+1}{2},$	$i = k+t+s+1, k+t+s+3, \cdots, n+t+s-3.$

case 2. $k+t+s$ is odd.

$$f(v_i) = \frac{i}{2} - 1, \qquad i = 2, 4, \cdots, k+t+s-1.$$

If k is even, then

$f(v_i) = n+t+s-2-\frac{i-1}{2},$	$i = 1, 3, \cdots, n+t+s-3.$
$f(v_i) = \frac{i}{2},$	$i = k+t+s+1, n+t+s-k+2, \cdots, n+t+s-2.$

If k is odd, then

$f(v_i) = n+t+s-2-\frac{i-1}{2},$	$i = 1, 3, \cdots, n+t+s-2.$
$f(v_i) = \frac{i}{2},$	$i = k+t+s+1, n+t+s-k+2, \cdots, n+t+s-3.$

The proof will be restricted to the case where $k+t+s$ is even. Then we could do similar with the other case. For the vertices $v_1, v_2, \cdots, v_{k+t+s}$, we have the following edge labels

$$\begin{aligned} |f(v_i) - f(v_{i+1})| &= |n+t+s-2-\frac{i-1}{2} - (\frac{i+1}{2} - 1)| \\ &= n+t+s-i-1 \end{aligned}$$

for $i \in \{1, 2, \cdots, k+t+s-1\}$, plus the edge label k produced automatically in the cycle. Thus the edge labels induced by the vertices $v_1, v_2, \cdots, v_{k+t+s}$ are $n+t+s-2, n+t+s-3, \cdots, k+1, k$. Now we have two cases :

k **is even.** For the vertices $v_{k+t+s+1}, \cdots, v_{n+t+s-2}$, we have the following edge labels

$$\begin{aligned} |f(v_i) - f(v_{i+1})| &= |n+t+s-2-\frac{i+1}{2} - (\frac{i+1}{2} - 1)| \\ &= |n+t+s-i-2| \end{aligned}$$

for $i \in \{k+t+s, k+t+s+1, \cdots, n+t+s-1\}$. The edge labels induced are $k-1, k-2, \cdots, 1$. Then we have a graceful labeling.

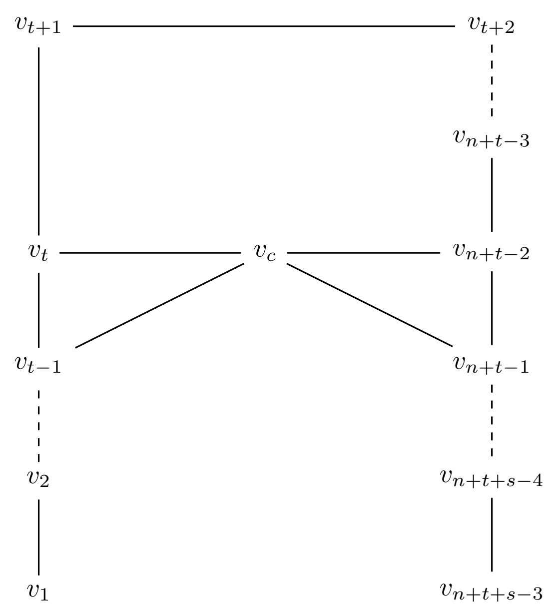


Figure 3.2: Line graph of $C_n \oplus (P_t; P_s; P_1; \cdots ; P_1)$

k **is odd.** For the vertices $v_{k+t+s+1}, \cdots , v_{n+t+s-2}$, we have the following edge labels

$$\begin{aligned} |f(v_i) - f(v_{i+1})| &= |n+t+s-2-\frac{i+1}{2} - (\frac{i+1}{2} - 1)| \\ &= |n+t+s-i-2| \end{aligned}$$

for $i \in \{k+t+s, k+t+s+1, \cdots , n+t+s-1\}$. The edge labels induced are $k-1, k-2, \cdots , 1$. Then we have a graceful labeling. Hence $C_n \oplus (P_t; P_s; P_1; \cdots ; P_1)$, n odd, $n = 2k+1$, $t, s \geq k-1$, $t \geq s$, is graceful. This ends the proof.

□

Now we study the gracefulness of the line graph of $C_n \oplus (P_t; P_s; P_1; \cdots ; P_1)$, denoted by $L(C_n \oplus (P_t; P_s; P_1; \cdots ; P_1))$, n odd, $n = 2k+1$, $t, s \geq k-1$, $t \geq s$. The figure 3.2 presents a scheme of this line graph. $L(C_n \oplus (P_t; P_s; P_1; \cdots ; P_1))$ is a graph with three cycles.

We begin by studying the gracefulness of a path. This result will be useful to show the gracefulness of $L(C_n \oplus (P_t; P_s; P_1; \cdots ; P_1))$, n odd, $n = 2k+1$, $t, s \geq k-1$, $t \geq s$.

Lemma 2 *Let P_{s-1} be a path with a peripheral vertex labeled with the label k,*

$$k = 2\left\lfloor \frac{n+t-1}{2} \right\rfloor - \left\lfloor \frac{t}{2} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor + 1,$$

V_a the set of $s+1$ available vertex labels,

$$\begin{aligned} V_a &= \left\{\left\lfloor\frac{n+t-1}{2}\right\rfloor, \cdots, n+t+s-\left\lfloor\frac{n+t-1}{2}\right\rfloor\right\} \setminus \left\{2\left\lfloor\frac{n+t-1}{2}\right\rfloor - \left\lfloor\frac{t}{2}\right\rfloor - \left\lfloor\frac{n}{2}\right\rfloor + 1, \right. \\ &\left. 2\left\lfloor\frac{n+t-1}{2}\right\rfloor - \left\lfloor t/2\right\rfloor\right\}, \end{aligned}$$

E_a the set of $s-2$ edge labels,

$$\begin{aligned} E_a &= \left\{1, 2, \cdots, n+t+s-3\left\lfloor\frac{n+t-1}{2}\right\rfloor + \left\lfloor\frac{t}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor - 1\right\} \setminus \left\{\left\lfloor\frac{n}{2}\right\rfloor - 1, 2\left\lfloor\frac{n+t-1}{2}\right\rfloor - 2\left\lfloor\frac{t}{2}\right\rfloor, \right. \\ &\left. n+t+s-3\left\lfloor\frac{n+t-1}{2}\right\rfloor + \left\lfloor\frac{t}{2}\right\rfloor + 1\right\}, \end{aligned}$$

with n odd, $n \in \{9, 11, 13\}$, t odd, and $t \geq s > n$. The vertices of P_{s-1} labeled with the labels in V_a produce the set of edge labels E_a.

Proof. We want to label the vertices of the path P_{s-1} with a peripheral vertex already labeled with k, looking for the edge labels in decreasing order. To find an edge label e in the path, we take a label v in the list of available vertex labels and put it next to the last vertex labeled with w in the path such that $|v-w| = e$. To start, we sort the list of available vertex labels in increasing order, hence we can check v starting at the beginning of the list (smaller values) or at the end of the list (greatest values). If w was found in one side of the list, we check v at the other beginning at the other side.

Let l a direct access list whose the elements are the edge labels we are looking for. When we find an edge label e, we update the list l by removing e from the list. We process as follows

1. for $i \in \{1, \cdots, \frac{s}{a}\}$ do
   (a) find the edge labels $l(1), \cdots, l(b)$,
   (b) find the edge labels $l(2), \cdots, l(c)$,
   (c) find the edge labels $l(1), \cdots, l(2)$,
   (d) find the edge label $l(3)$,
   (e) find the edge label $l(1)$,
2. find the edge labels in a specified sequence.

The values of n, a, b, c, and the specified sequence are given in Table 1.

□

Theorem 3.2 *The unicyclic graph $L(C_n \oplus (P_t; P_s; P_1; \cdots; P_1))$, line graph of the unicyclic graph $C_n \oplus (P_t; P_s; P_1; \cdots; P_1)$, $n \in \{9, 11, 13\}$, $t \geq s > n$, t odd, where*

$$n + s + 5\left\lfloor\frac{t}{2}\right\rfloor - 5\left\lfloor\frac{n+t-1}{2}\right\rfloor + 2 \neq 0$$

is graceful.

Table 1: Values of a, b, c, and the specified sequence

n	a	b	c	**specified sequence**	**value of** s
9	12	5	4	6-2-1-4	$s = 6(i-1)+18, \quad i \in \mathbb{N}^*$
				13-12-10-9-7-5-6-4-2-1	$s = 6(i-1)+24, \quad i \in \mathbb{N}^*$
11	13	5	5	11-9-7-6-5-3-2-1	$s = 26(i-1)+10, \quad i \in \mathbb{N}^*$
				12-11-8-7-6-5-3-1-2	$s = 26(i-1)+24, \quad i \in \mathbb{N}^*$
				16-15-14-12-11-9-8-7-6-5-3-2-1	$s = 26(i-1)+28, \quad i \in \mathbb{N}^*$
13	18	9	6	13-11-10-8-7-4-3-2-1-6	$s = 18(i-1)+12, \quad i \in \mathbb{N}^*$
				23-22-21-20-18-17-16-15-14-11-10-9-13-8-3-7-6-4-1-2	$s = 18(i-1)+22, \quad i \in \mathbb{N}^*$

Proof. $L(C_n \oplus (P_t; P_s; P_1; \cdots; P_1))$ is the line graph of $C_n \oplus (P_t; P_s; P_1; \cdots; P_1)$, unicyclic graph shown to be graceful in lemma 1. $L(C_n \oplus (P_t; P_s; P_1; \cdots; P_1))$ is illustrated in Figure 3.2. Suppose $n = 2k+1$, and let f be a map such that

$$\begin{aligned}
f(v_{2i+1}) &= i, && i = 0, 1, 2, \cdots, (n+t-4)/2.\\
f(v_j) &= n+t+s+1-(j/2), && j = 2, 4, \cdots, n+t-2.\\
f(v_c) &= 2\Big\lfloor (n+t-1)/2 \Big\rfloor - \Big\lfloor t/2 \Big\rfloor\\
f(v_{n+t-1}) &= 2\Big\lfloor (n+t-1)/2 \Big\rfloor - \Big\lfloor t/2 \Big\rfloor - \Big\lfloor n/2 \Big\rfloor + 1
\end{aligned}$$

The Figure 3.3 illustrates the labeling of the line graph of the Figure 3.2 when t is odd. The resulting edge labels set is

$$\Big\{n+t+s, \cdots, n+t+s-3\Big\lfloor (n+t-1)/2 \Big\rfloor + \Big\lfloor t/2 \Big\rfloor + \Big\lfloor n/2 \Big\rfloor\Big\}$$
$$\bigcup \Big\{n+t+s-3\Big\lfloor (n+t-1)/2 \Big\rfloor + \Big\lfloor t/2 \Big\rfloor + 1, 2\Big\lfloor (n+t-1)/2 \Big\rfloor - 2\Big\lfloor t/2 \Big\rfloor, \Big\lfloor n/2 \Big\rfloor - 1\Big\}$$

We then need to label the $s-1$ vertices of the path $< v_{n+t-1}, \cdots, v_{n+t+s-3} >$ with the remaining vertex labels in the following set

$$\Big\{\Big\lfloor \frac{n+t-1}{2} \Big\rfloor, \Big\lfloor \frac{n+t-1}{2} \Big\rfloor + 1, \cdots, n+t+s-\Big\lfloor \frac{n+t-1}{2} \Big\rfloor\Big\} \setminus \Big\{2\Big\lfloor \frac{n+t-1}{2} \Big\rfloor - \Big\lfloor \frac{t}{2} \Big\rfloor - \Big\lfloor \frac{n}{2} \Big\rfloor + 1,$$
$$2\Big\lfloor \frac{n+t-1}{2} \Big\rfloor - \Big\lfloor \frac{t}{2} \Big\rfloor\Big\}$$

to obtain the remaining edge labels in the following set

$$\Big\{1, 2, \cdots, n+t+s-3\Big\lfloor \frac{n+t-1}{2} \Big\rfloor + \Big\lfloor \frac{t}{2} \Big\rfloor + \Big\lfloor \frac{n}{2} \Big\rfloor\Big\} \setminus \Big\{\Big\lfloor \frac{n}{2} \Big\rfloor - 1, 2\Big\lfloor \frac{n+t-1}{2} \Big\rfloor - 2\Big\lfloor \frac{t}{2} \Big\rfloor,$$
$$n+t+s-3\Big\lfloor \frac{n+t-1}{2} \Big\rfloor + \Big\lfloor \frac{t}{2} \Big\rfloor + 1\Big\}$$

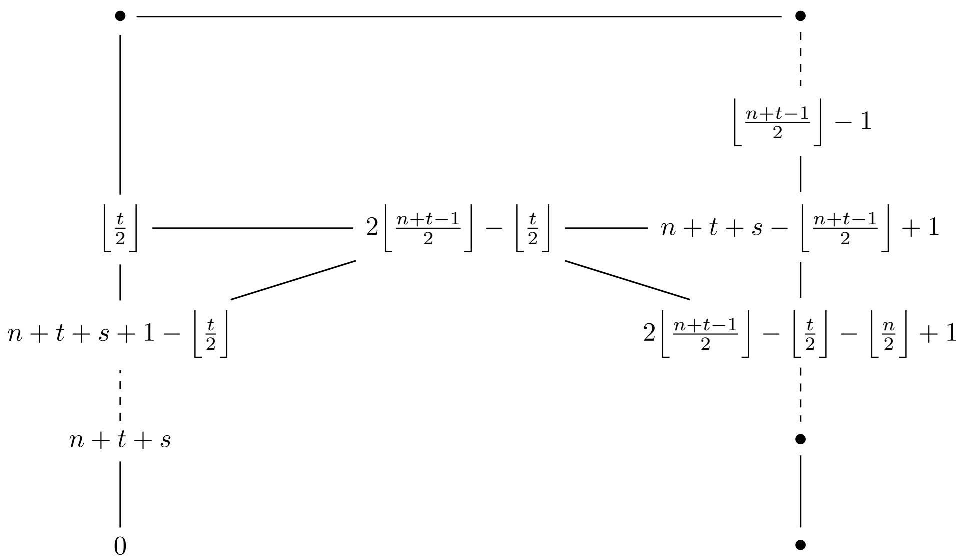


Figure 3.3: Labeling of $L(C_n \oplus (P_t; P_s; P_1; \cdots; P_1))$ when t is odd

This is done using lemma 2 with $n \in \{9, 11, 13\}$. We then obtain a graceful labeling.

$\square$

Corollary 1 *$L(C_n \oplus (P_t; P_s; P_1; \cdots; P_1))$, $n \in \{9, 11, 13\}$, $t \geq s > n$, t odd, where*

$$n + s + 5\left\lfloor \frac{t}{2} \right\rfloor - 5\left\lfloor \frac{n+t-1}{2} \right\rfloor + 2 \neq 0$$

is bigraceful.

The Figure 3.4 presents a graceful of the unicyclic graph $C_9 \oplus (P_{19}; P_{18}; P_1; \cdots; P_1)$ and a graceful labeling of its line graph, $L(C_9 \oplus (P_{19}; P_{18}; P_1; \cdots; P_1))$.

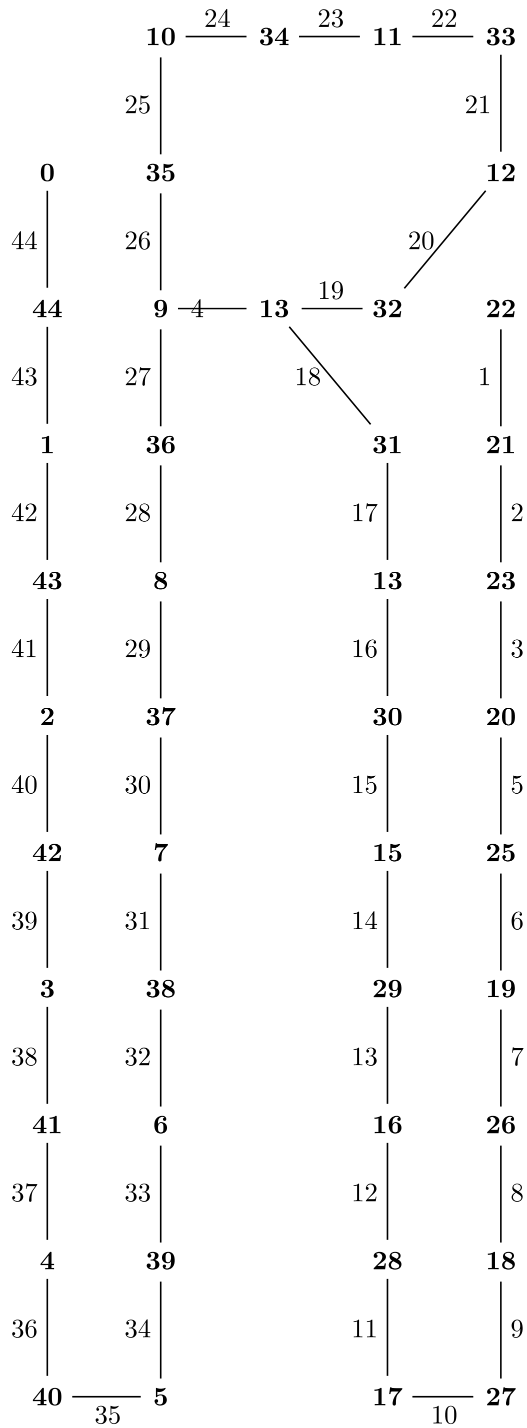
Conjecture 1 *$L(C_n \oplus (P_t; P_s; P_1; \cdots; P_1))$, n odd, $t \geq s > n$, t odd, where*

$$n + s + 5\left\lfloor \frac{t}{2} \right\rfloor - 5\left\lfloor \frac{n+t-1}{2} \right\rfloor + 2 \neq 0$$

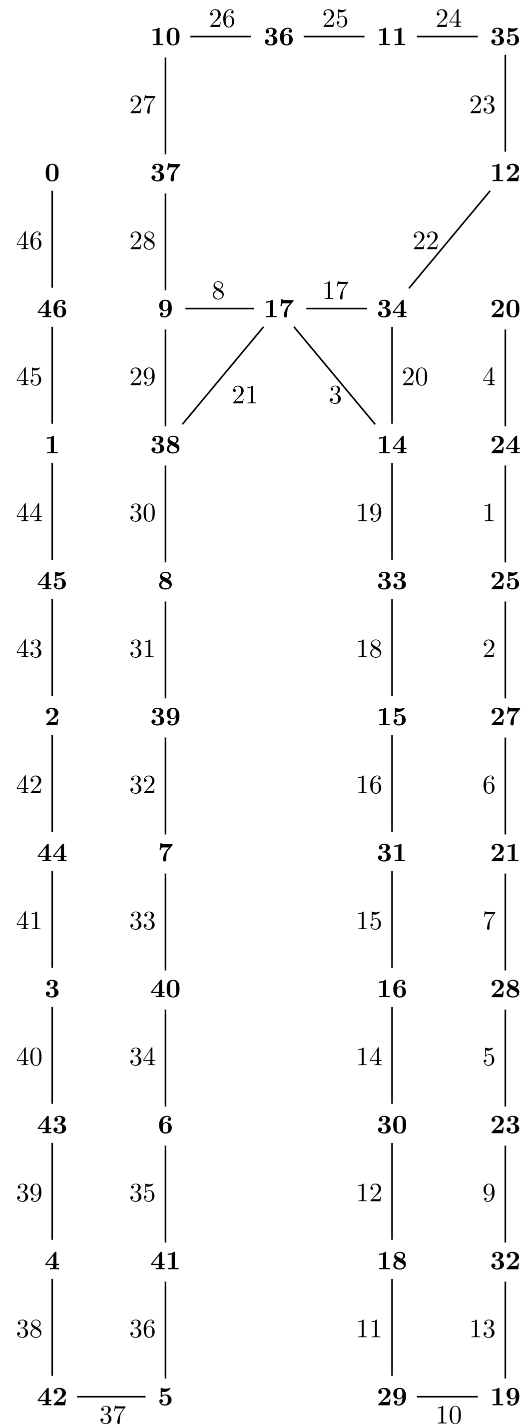
is bigraceful.

4 Summary

We have studied gracefulness of some graphs and their line graphs. Particularly, we prove non gracefulness of certain sun graphs with p edges attached to any vertex of the cycle. We also showed



A graceful labeling of
$C_9 \oplus (P_{19}; P_{18}; P_1; \cdots ; P_1)$



A graceful labeling of
$L(C_9 \oplus (P_{19}; P_{18}; P_1; \cdots ; P_1))$

Figure 3.4: Graceful labelings of $C_9 \oplus (P_{19}; P_{18}; P_1; \cdots ; P_1)$ and $L(C_9 \oplus (P_{19}; P_{18}; P_1; \cdots ; P_1))$

the bigracefulness of unicyclic graphs that have two paths pendant at two adjacent vertices of the cycle, and edges pendant at other vertices. For future work, we characterize the graphs that satisfy the condition of theorem 2.7, and prove conjecture 1.

References

[1] Abhyankar, Direct methods of gracefully labeling graphs, Ph. D. thesis, *University of Mumbai*, (2002)

[2] J. Bagga, L. P. Fotso, M. Pambe Biatch', S. Arumugam, New Classes of Graceful Unicyclic Graphs, *Electronic Notes of Discrete Mathematics*, **48** (2015), 27–32

[3] J. Bagga, L. P. Fotso, M. J. Pambe Biatch', Discovery of Some New Classes of Graceful Unicyclic Graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing (JCMCC)*, **109** (2019), 105–127

[4] J. Bagga, A. Heinz, Graceful Labelings of Unicyclic Graphs, *Proc. ICDM 2008,RMS-Lecture Notes Series*, No.13 (2010), 69–79

[5] J. Bagga, A. Heinz, M. M. Majumder, An Algorithm for Computing All Graceful labelings of Cycles, *Congressus Numerantium*, **186** (2007), 57–63

[6] C. Barrientos, Graceful graphs with pendant edges, *Australas. J. Combin.*, **33** (2005), 99–107

[7] C. Barrientos, Graceful labelings of chain and corona graphs, *Bull. Inst. Combin. Appl.*, **34** 2002, 17–26

[8] L. W. Beineke, J. S. Bagga, Line Graphs and Line Digraphs. *Springer*, (2021)

[9] J. L. Cross, J. Yellen, Graph Theory and its applications, *Chapman & Hall/CRC*, (2018)

[10] J. Doma, Unicyclic Graceful Graphs, M. S. thesis, *McMaster Univ.*, (1991)

[11] R. M. Figueroa-Centeno and R. Ichishima and F. A. Muntaner-Batle and A. Oshima, Gracefully Cultivating Trees on a cycle, *Electronic Notes in Discrete Math.*, **48** (2015), 143–148

[12] C.-M. Fu, Y.-F. Hsu, M.-F. Li, New Results of Graceful Unicyclic Graphs, *Ars Comb.*, **137** (2018), 273–282

[13] R. Frucht, Graceful numbering of wheels and related graphs, *Ann. N.Y. Acad. Sci.*, **319** (1979), 219–229

[14] J. A. Gallian, A survey: recent results, conjectures and open problems on labeling graphs, *J. Graph Theory*, **13** (1989), 491–504

[15] J. A. Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics, (http://www.combinatorics.org/) DS*, **6** 2019, 1–79

[16] R. B. Gnanajothi, Topics in Graph Theory, Ph.D. Thesis, Madurai Kamaraj University, 1991.

[17] W. F. Guo, Gracefulness of the graph B(m,n), *J. Inner Mongolia Normal Univ.*, (1994), 24–29

[18] Q. D. Kang, Z.-H. Liang, Y.-Z. Gao, G.-H. Yang, On the labeling of some graphs, *J. Combin. Math. Combin. Comput.*, **22** (1996), 193–210

[19] S.-R. Kim, J. Y. Park, On super edge-magic graphs, *Ars Combin.*, **81** (2006), 113–127

[20] K. M. Koh, D. G. Rogers, H. K. Teo, K. Y. Yap, Graceful graphs: some further results and problem, *Congr. Numer.*, **29** (1980), 559–571

[21] Murugan, M., Bi-graceful graphs.(English summary) Number theory and discrete mathematics (Chandigarh, 2000), 243–249. Trends Math. Birkhäuser Verlag, Basel, 2002.

[22] M. Murugan, G. Arumugan, Bi-graceful graphs, *Number theory and discrete mathematics* (Chandigarh, 2000) Trends Math., Birkhäuser 243–249.

[23] M. Pambe Biatch', J. Bagga, S. Arumugam, A survey and a new class of graceful unicylic graphs, *AKCE International Journal of Graphs and Combinatorics*, Taylor & Francis **0**, (2020), 1–6, doi 10.1080/09728600.2020.1832853, https://doi.org/10.1080/09728600.2020.1832853

[24] M. Pambe Biatch', J. Bagga, L. P. Fotso, An Algorithm for Graceful Labelings of Certain Unicyclic Graphs, *Vietnam National University Journal of Science: Comp. Science* & *Com. Eng.*, **30** (2014), 1–11

[25] J. Qian, On some conjectures and problems in graceful labelings graphs, *unpublished*

[26] G. Ringel, Problem 25, *Theory of Graphs and its Applications, Proc. Symposium Smolenice 1963, Prague*, (1964) 162

[27] A. Rosa, On certain valuations of the vertices of a graph, *Theory of Graphs (Internat. Symposium, Rome, July 1966), Gordon and Breach, N. Y. and Dunod Paris*, (1967), 349–355

[28] C. Sekar, Studies in Graph Theory, *Ph. D. thesis, Madurai Kamaraj University*, (2002)

[29] M. Truszczyński, Graceful unicyclic graphs, *Demonstatio Mathematica*, **17** (1984), 377–387