This print-out should have 18 questions. Multiple-choice questions may continue on the next column or page – find all choices before answering.

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001 10.0 points

When $\mathbf{x}$, $\mathbf{y}$, $\mathbf{w}$ and $\mathbf{z}$ are vectors in $\mathbb{R}^n$ such that $\mathbf{y}$, $\mathbf{z}$ are linearly independent and

$$\mathbf{z} = 5\mathbf{x} + 4\mathbf{y}, \quad \mathbf{w} = 10\mathbf{x} - 4\mathbf{y} - 2\mathbf{z},$$

which of the following

I: $\operatorname{Span}\{\mathbf{x}, \mathbf{y}\} = \operatorname{Span}\{\mathbf{x}, \mathbf{w}, \mathbf{z}\}$,

II: $\operatorname{Span}\{\mathbf{x}, \mathbf{z}\} = \operatorname{Span}\{\mathbf{y}, \mathbf{w}\}$,

III: $\operatorname{Span}\{\mathbf{y}\} = \operatorname{Span}\{\mathbf{w}\}$,

hold?

1. I only

2. II only

3. III only

4. II and III

5. I and II

6. I and III **correct**

Explanation:

To show that sets A, B satisfy $A = B$, it is enough to show that $A \subseteq B$ and $B \subseteq A$ both hold.

Now when

$$\mathbf{z} = 5\mathbf{x} + 4\mathbf{y}, \quad \mathbf{w} = 10\mathbf{x} - 4\mathbf{y} - 2\mathbf{z},$$

we see that

$$\mathbf{w} = 2(5\mathbf{x} - \mathbf{z}) - 4\mathbf{y} = 2(-4\mathbf{y}) - 4\mathbf{y} = -12\mathbf{y}.$$

Thus

I: $\mathbf{x}$, $\mathbf{y}$ belong to $\operatorname{Span}\{\mathbf{x}, \mathbf{w}, \mathbf{z}\}$, so

$$\operatorname{Span}\{\mathbf{x}, \mathbf{y}\} \subseteq \operatorname{Span}\{\mathbf{x}, \mathbf{w}, \mathbf{z}\},$$

while $\mathbf{x}$, $\mathbf{w}$, $\mathbf{z}$ belong to $\operatorname{Span}\{\mathbf{x}, \mathbf{y}\}$, so

$$\operatorname{Span}\{\mathbf{x}, \mathbf{w}, \mathbf{z}\} \subseteq \operatorname{Span}\{\mathbf{x}, \mathbf{y}\}.$$

II: Since $\mathbf{w} = -12\mathbf{y}$ and $4\mathbf{y} = \mathbf{z} - 5\mathbf{x}$,

$$\operatorname{Span}\{\mathbf{y}, \mathbf{w}\} = \operatorname{Span}\{\mathbf{y}\} \subseteq \operatorname{Span}\{\mathbf{x}, \mathbf{z}\},$$

but $\mathbf{z}$ is not in $\operatorname{Span}\{\mathbf{y}\}$ because $\mathbf{y}$ and $\mathbf{z}$ are linearly independent, so

$$\operatorname{Span}\{\mathbf{x}, \mathbf{z}\} \not\subseteq \operatorname{Span}\{\mathbf{y}\} = \operatorname{Span}\{\mathbf{y}, \mathbf{w}\}.$$

III: Since $\mathbf{w} = -12\mathbf{y}$,

$$\operatorname{Span}\{\mathbf{y}\} = \operatorname{Span}\{\mathbf{w}\}.$$

Consequently, only

$$\boxed{\text{I and III hold}}.$$

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002 10.0 points

If a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ spans a finite-dimensional vector space V, then any set of more than p vectors in V must be linearly dependent.

True or False?

1. TRUE **correct**

2. FALSE

Explanation:

Since

$$V = \operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\},$$

$\dim(V) \le p$. Thus no set of more than p vectors in V can be linearly independent.

Consequently, the statement is

$$\boxed{\text{TRUE}}.$$

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003 10.0 points

Find the vector $\mathbf{x}$ in $\mathbb{R}^3$ having coordinate vector

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -4 \\ 8 \\ -7 \end{bmatrix}$$

with respect to the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix} \right\}$$

for $\mathbb{R}^3$.

1. $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}$ **correct**

2. $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}$

3. $\mathbf{x} = \begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix}$

4. $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -5 \end{bmatrix}$

5. no such $\mathbf{x}$ exists

Explanation:

The coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ of a vector $\mathbf{x}$ in $\mathbb{R}^3$ with respect to a basis

$$\mathcal{B} = \{\mathbf{b}_1,\ \mathbf{b}_2,\ \mathbf{b}_3\}$$

for $\mathbb{R}^3$ satisfies the matrix equation

$$A[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}, \qquad A = [\,\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3\,]\,.$$

Consequently, when

$$\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix} \right\},$$

and

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -4 \\ 8 \\ -7 \end{bmatrix},$$

$$\boxed{\mathbf{x} = \begin{bmatrix} -1 & 3 & 4 \\ 2 & -5 & -7 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} -4 \\ 8 \\ -7 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}}\,.$$

004 10.0 points

Compute the Wronskian of the solenoidals

$$1,\ \cos t,\ \sin t,\ \cos 2t,\ \sin 2t$$

as indefinitely differentiable functions on $(-\infty, \infty)$.

Correct answer: 72.

Explanation:

By definition,

$$W(f_0, f_1, f_2, f_3, f_4)(x)$$
$$= \det \begin{vmatrix} f_0(x) & f_1(x) & f_2(x) & f_3(x) & f_4(x) \\ f_0'(x) & f_1'(x) & f_2'(x) & f_3'(x) & f_4'(x) \\ f_0''(x) & f_1''(x) & f_2''(x) & f_3''(x) & f_4''(x) \\ f_0'''(x) & f_1'''(x) & f_2'''(x) & f_3'''(x) & f_4'''(x) \\ f_0^{iv}(x) & f_1^{iv}(x) & f_2^{iv}(x) & f_3^{iv}(x) & f_4^{iv}(x) \end{vmatrix}$$

For the given solenoidals, therefore,

$$W(1, \cos t, \sin t, \cos 2t, \sin 2t)$$
$$= \det \begin{vmatrix} 1 & \cos t & \sin t & \cos 2t & \sin 2t \\ 0 & -\sin t & \cos t & -2\sin 2t & 2\cos 2t \\ 0 & -\cos t & -\sin t & -4\cos 2t & -4\sin 2t \\ 0 & \sin t & -\cos t & 8\sin 2t & -8\cos 4t \\ 0 & \cos t & \sin t & 16\cos 2t & 16\sin 2t \end{vmatrix}.$$

But then by properties of determinants,

$$W(t)$$
$$= \det \begin{vmatrix} -\sin t & \cos t & -2\sin 2t & 2\cos 2t \\ -\cos t & -\sin t & -4\cos 2t & -4\sin 2t \\ \sin t & -\cos t & 8\sin 2t & -8\cos 4t \\ \cos t & \sin t & 16\cos 2t & 16\sin 2t \end{vmatrix}$$

$$= \det \begin{vmatrix} -\sin t & \cos t & -2\sin 2t & 2\cos 2t \\ -\cos t & -\sin t & -4\cos 2t & -4\sin 2t \\ 0 & 0 & 6\sin 2t & -6\cos 4t \\ 0 & 0 & 12\cos 2t & 12\sin 2t \end{vmatrix}$$

$$= -\sin t\ \det \begin{vmatrix} -\sin t & -4\cos 2t & -4\sin 2t \\ 0 & 6\sin 2t & -6\cos 4t \\ 0 & 12\cos 2t & 12\sin 2t \end{vmatrix}$$

$$+\cos t\ \det\begin{vmatrix} \cos t & -2\sin 2t & 2\cos 2t \\ 0 & 6\sin 2t & -6\cos 4t \\ 0 & 12\cos 2t & 12\sin 2t \end{vmatrix}.$$

Consequently,

$$\boxed{W(t) \;=\; 72}.$$

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005 10.0 points

The columns of an invertible $n \times n$ matrix form a basis for $\mathbb{R}^n$.

True or False?

1. TRUE **correct**

2. FALSE

Explanation:

If an $n \times n$ matrix

$$A \;=\; [\mathbf{a}_1\ \ \mathbf{a}_2\ \ \ldots\ \ \mathbf{a}_n]$$

is invertible, then the set

$$[\mathbf{a}_1,\ \mathbf{a}_2,\ \ldots,\ \mathbf{a}_n]$$

of its columns form a linearly independent set in $\mathbb{R}^n$. But any linearly independent set of vectors in $\mathbb{R}^n$ is a basis for $\mathbb{R}^n$. Thus the columns of an invertible $n \times n$ matrix form a basis for $\mathbb{R}^n$.

Consequently, the statement is

$$\boxed{\text{TRUE}}.$$

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006 10.0 points

Find the polynomial $\mathbf{q}$ in $\mathbb{P}^2$ having coordinate vector

$$[\mathbf{q}]_{\mathcal{B}} \;=\; \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

with respect to the basis $\mathcal{B} = \{\,\mathbf{p}_1,\ \mathbf{p}_2,\ \mathbf{p}_3\,\}$ for $\mathbb{P}_2$ when

$$\mathbf{p}_1 \;=\; 1 + t^2, \quad \mathbf{p}_2 \;=\; t - 3t^2,$$

and

$$\mathbf{p}_3 \;=\; 1 + t - 3t^2.$$

1. $\mathbf{q} \;=\; 1 - 3t + 10t^2$

2. $\mathbf{q} \;=\; 1 + 3t - 10t^2$ **correct**

3. no such $\mathbf{q}$ exists

4. $\mathbf{q} \;=\; 1 - 3t - 10t^2$

5. $\mathbf{q} \;=\; 1 + 3t + 10t^2$

Explanation:

If

$$[\mathbf{q}]_{\mathcal{B}} \;=\; \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

is the coordinate vector with respect the basis

$$\mathcal{B} \;=\; \{\mathbf{p}_1,\ \mathbf{p}_2,\ \mathbf{p}_3\},$$

then

$$\begin{aligned} \mathbf{q}(t) \;&=\; -\mathbf{p}_1(t) + \mathbf{p}_2(t) - 2\mathbf{p}_3(t) \\ &=\; -(1+t^2) + (t - 3t^2) + 2(1 + t - 3t^2)\,. \end{aligned}$$

Consequently,

$$\boxed{\mathbf{q} \;=\; 1 + 3t - 10t^2}.$$

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007 (part 1 of 2) 10.0 points

Compute the Wronskian of the monomials

$$1,\ x,\ x^2,\ x^3$$

as indefinitely differentiable functions on $(-\infty, \infty)$.

Correct answer: 12.

Explanation:

By definition,

$$W(f_0, f_1, f_2, f_3)(x) = \begin{vmatrix} f_0(x) & f_1(x) & f_2(x) & f_3(x) \\ f_0'(x) & f_1'(x) & f_2'(x) & f_3'(x) \\ f_0''(x) & f_1''(x) & f_2''(x) & f_3''(x) \\ f_0'''(x) & f_1'''(x) & f_2'''(x) & f_3'''(x) \end{vmatrix}$$

For the given monomials, therefore,

$$W\left(1, x, x^2, x^3\right)(x) = \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & 0 & 2 & 6x \\ 0 & 0 & 0 & 6 \end{vmatrix} = 12$$

for all x.

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008 (part 2 of 2) 10.0 points

The value of the Wronskian of the monomials

$$1,\ x,\ x^2,\ x^3,\ x^4$$

is the product $1!2!3!4!5!$ for all x.

1. False **correct**

2. True

Explanation:

For the given monomials,

$$W\left(1, x, x^2, x^3, x^4\right)(x) = \begin{vmatrix} 1 & x & x^2 & x^3 & x^4 \\ 0 & 1 & 2x & 3x^2 & 4x^3 \\ 0 & 0 & 2 & 6x & 12x^2 \\ 0 & 0 & 0 & 6 & 24x \\ 0 & 0 & 0 & 0 & 24 \end{vmatrix} = 1!2!3!4!$$

for all x.

Consequently, the statement is

$$\boxed{\text{FALSE}}\,.$$

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009 10.0 points

Find the area of the triangle ΔABC when

$$A = (1,\ 1,\ 0)\,, \quad B = (1,\ 0,\ 1)\,,$$

and

$$C = (0,\ 1,\ 2)\,.$$

Hint: *Check if ΔABC is right-angled.*

1. area $= \sqrt{3}$

2. area $= \dfrac{\sqrt{3}}{2}$

3. area $= \sqrt{6}$

4. area $= \dfrac{1}{2}$

5. area $= \dfrac{\sqrt{6}}{2}$ **correct**

Explanation:

Set

$$\mathbf{u} = \overrightarrow{BA} = (0,\ 1,\ -1)$$

and

$$\mathbf{v} = \overrightarrow{BC} = (-1,\ 1,\ 1)\,.$$

Then $\mathbf{u}\cdot\mathbf{v} = 0$, so ΔABC is a right-angled triangle with side $\overline{BA}$ perpendicular to side $\overline{BC}$. Thus

$$\text{area}\,\Delta ABC = \frac{1}{2}\text{base} \times \text{height} = \|\mathbf{u}\|\|\mathbf{v}\| = \frac{\sqrt{2}\sqrt{3}}{2}$$

Consequently, ΔABC has

$$\boxed{\text{area} = \frac{\sqrt{6}}{2}}\,.$$

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010 10.0 points

Which of the following statements are true for all vectors $\mathbf{a}$, $\mathbf{b}$ in an inner product space V?

A. $|<\mathbf{a}, \mathbf{b}>| = \|\mathbf{a}\|\,\|\mathbf{b}\|,\ \mathbf{a} \neq 0,\ \mathbf{b} \neq 0 \implies \mathbf{a}$ parallel to $\mathbf{b}$,

B. $< \mathbf{a}, \mathbf{b} >= 0 \implies \mathbf{a} = 0$ or $\mathbf{b} = 0$,

C. $\|\mathbf{a}+\mathbf{b}\|^2 = \|\mathbf{a}\|^2 + 2 < \mathbf{a}, \mathbf{b} > + \|\mathbf{b}\|^2$.

1. B only

2. B and C only

3. all of them

4. A and C only **correct**

5. none of them

6. A and B only

7. A only

8. C only

Explanation:

If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then

$$< \mathbf{a}, \mathbf{b} >= \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta \,.$$

A. TRUE: when

$$| < \mathbf{a}, \mathbf{b} > | = \|\mathbf{a}\| \, \|\mathbf{b}\| \,, \quad \mathbf{a} \neq 0, \quad \mathbf{b} \neq 0 \,,$$

then $|\cos\theta| = 1$, *i.e.*, $\theta = 0$ or π. In this case $\mathbf{a}$ is parallel to $\mathbf{b}$.

B. FALSE: if $\mathbf{a} \perp \mathbf{b}$, then $\theta = \pi/2$. But then $\cos\theta = 0$. So $< \mathbf{a}, \mathbf{b} >= 0$ when $\mathbf{a} \perp \mathbf{b}$, as well as when $\mathbf{a} = 0$ or $\mathbf{b} = 0$.

C. TRUE: since $\|\mathbf{a}\|^2 =< \mathbf{a}, \mathbf{b} >$,

$$\begin{aligned} \|\mathbf{a}+\mathbf{b}\|^2 &=< (\mathbf{a}+\mathbf{b}), (\mathbf{a}+\mathbf{b}) > \\ &= \|\mathbf{a}\|^2 + < \mathbf{a}, \mathbf{b} > + < \mathbf{a}, \mathbf{b} > + \|\mathbf{b}\|^2 \\ &= \|\mathbf{a}\|^2 + 2 < \mathbf{a}, \mathbf{b} > + \|\mathbf{b}\|^2 \end{aligned}$$

because $< \mathbf{a}, \mathbf{b} >=< \mathbf{b}, \mathbf{a} >$.

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011 10.0 points

Simplify the expression

$$< (2\mathbf{u}+\mathbf{v}), (\mathbf{u}+3\mathbf{v}) > - \|3\mathbf{u}-\mathbf{v}\|^2$$

for vectors $\mathbf{u}$, $\mathbf{v}$ in an inner product space V.

1. $7\|\mathbf{u}\|^2 - 13 < \mathbf{u}, \mathbf{v} > + 2\|\mathbf{v}\|^2$

2. $7\|\mathbf{u}\|^2 + 11 < \mathbf{u}, \mathbf{v} > - 2\|\mathbf{v}\|^2$

3. $-7\|\mathbf{u}\|^2 + 13 < \mathbf{u}, \mathbf{v} > + 2\|\mathbf{v}\|^2$ **correct**

4. $7\|\mathbf{u}\|^2 + 11 < \mathbf{u}, \mathbf{v} > + 2\|\mathbf{v}\|^2$

5. $-7\|\mathbf{u}\|^2 + 13 < \mathbf{u}, \mathbf{v} > - 2\|\mathbf{v}\|^2$

6. $-7\|\mathbf{u}\|^2 + 11 < \mathbf{u}, \mathbf{v} > - 2\|\mathbf{v}\|^2$

Explanation:

By linearity,

$$\begin{aligned} &< (2\mathbf{u}+\mathbf{v}), (\mathbf{u}+3\mathbf{v}) > \\ &\quad =< 2\mathbf{u}, (\mathbf{u}+3\mathbf{v}) > + < \mathbf{v}, (\mathbf{u}+3\mathbf{v}) > \\ &\quad =< 2\mathbf{u}, \mathbf{u} > +6 < \mathbf{u}, \mathbf{v} > \\ &\qquad + < \mathbf{v}, \mathbf{u} > +3 < \mathbf{v}, \mathbf{v} >, \end{aligned}$$

while

$$\begin{aligned} \|3\mathbf{u}-\mathbf{v}\|^2 &=< (3\mathbf{u}-\mathbf{v}), (3\mathbf{u}-\mathbf{v}) > \\ &=< 3\mathbf{u}, (3\mathbf{u}-\mathbf{v}) - \mathbf{v}, (3\mathbf{u}-\mathbf{v}) > \\ &=< 9\mathbf{u}, \mathbf{u} > -3 < \mathbf{u}, \mathbf{v} > \\ &\quad - 3 < \mathbf{v}, \mathbf{u} > + < \mathbf{v}, \mathbf{v} > . \end{aligned}$$

But

$$< \mathbf{v}, \mathbf{u} > = < \mathbf{u}, \mathbf{v} >,$$

$$< \mathbf{u}, \mathbf{u} > = \|\mathbf{u}\|^2, \quad < \mathbf{v}, \mathbf{v} > = \|\mathbf{v}\|^2.$$

So after expansion the expression becomes

$$\begin{aligned} &2\|\mathbf{u}\|^2 + 7 < \mathbf{u}, \mathbf{v} > + 3\|\mathbf{v}\|^2 \\ &\quad - (9\|\mathbf{u}\|^2 - 6 < \mathbf{u}, \mathbf{v} > + \|\mathbf{v}\|^2) \\ &= \boxed{-7\|\mathbf{u}\|^2 + 13 < \mathbf{u}, \mathbf{v} > + 2\|\mathbf{v}\|^2}. \end{aligned}$$

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012 10.0 points

The function

$$< \mathbf{u}, \mathbf{v} > = u_1 v_2 + v_1 u_2$$

for

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

defines an inner product on $\mathbb{R}^2$.

True or False?

1. FALSE **correct**

2. TRUE

Explanation:

An inner product has to satisfy three conditions: Bilinearity, Symmetry, and Positivity. The Positivity condition requires that

$$< \mathbf{v}, \mathbf{v} > > 0, \quad \text{for all } \mathbf{v} \neq \mathbf{0}$$

while $< \mathbf{0}, \mathbf{0} >= 0$.

But for the given function

$$< \mathbf{v}, \mathbf{v} > = 2v_1v_2, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

in which case

$$< \mathbf{v}, \mathbf{v} > = 0, \quad \text{when } \mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

hence the function is not an inner product.

Consequently, the statement is

$$\boxed{\text{FALSE}}.$$

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013 10.0 points

When $\mathbf{u}$, $\mathbf{v}$ are vectors in an inner product space V such that

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \text{dist}(\mathbf{u}, -\mathbf{v}),$$

then $\mathbf{u}$, $\mathbf{v}$ are orthogonal.

True or False?

1. TRUE **correct**

2. FALSE

Explanation:

By definition,

$$\begin{aligned}(\text{dist}(\mathbf{a}, \mathbf{b}))^2 &= \|\mathbf{a} - \mathbf{b}\|^2 \\ &= < (\mathbf{a} - \mathbf{b}), (\mathbf{a} - \mathbf{b}) > \\ &= \|\mathbf{a}\|^2 - < \mathbf{a}, \mathbf{b} > - < \mathbf{b}, \mathbf{a} > + \|\mathbf{b}\|^2 \\ &= \|\mathbf{a}\|^2 - 2 < \mathbf{a}, \mathbf{b} > + \|\mathbf{b}\|^2,\end{aligned}$$

for vectors $\mathbf{a}$, $\mathbf{b}$ in V. Thus

$$(\text{dist}(\mathbf{u}, \mathbf{v}))^2 = \|\mathbf{u}\|^2 - 2 < \mathbf{u}, \mathbf{v} > + \|\mathbf{v}\|^2,$$

while

$$(\text{dist}(\mathbf{u}, -\mathbf{v}))^2 = \|\mathbf{u}\|^2 + 2 < \mathbf{u}, \mathbf{v} > + \|\mathbf{v}\|^2.$$

But if

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \text{dist}(\mathbf{u}, -\mathbf{v}),$$

then

$$-2 < \mathbf{u}, \mathbf{v} > = 2 < \mathbf{u}, \mathbf{v} >,$$

i.e., $< \mathbf{u}, \mathbf{v} > = 0$, in which case $\mathbf{u}$, $\mathbf{v}$ are orthogonal.

Consequently, the statement is

$$\boxed{\text{TRUE}}.$$

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014 10.0 points

Determine the component of

$$\mathbf{x} = 3\mathbf{u} - 4\mathbf{v}$$

in the direction of $\mathbf{w}$ when

$$\mathbf{u} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

1. $\text{comp}_{\mathbf{w}}\, \mathbf{x} = -23$ **correct**

2. $\text{comp}_{\mathbf{w}}\, \mathbf{x} = -21$

3. $\text{comp}_{\mathbf{w}}\, \mathbf{x} = -24$

4. $\text{comp}_{\mathbf{w}}\, \mathbf{x} = -22$

5. $\text{comp}_{\mathbf{w}}\, \mathbf{x} = -25$

Explanation:

The component of $\mathbf{x}$ in the direction of $\mathbf{w}$ is given by

$$\text{comp}_{\mathbf{w}} = \frac{\mathbf{x}\cdot\mathbf{w}}{\|\mathbf{w}\|}.$$

Now by scalar multiplication and addition of vectors,

$$\mathbf{x} = 3\begin{bmatrix}-1\\3\\2\end{bmatrix} - 4\begin{bmatrix}4\\1\\-3\end{bmatrix} = \begin{bmatrix}-19\\5\\18\end{bmatrix},$$

while

$$\|\mathbf{w}\|^2 = \mathbf{w}\cdot\mathbf{w} = 9,$$

i.e., $\|\mathbf{w}\| = 3$. Consequently,

$$\mathbf{x}\cdot\mathbf{w} = [-19 \quad 5 \quad 18]\begin{bmatrix}2\\1\\-2\end{bmatrix} = -69,$$

and

$$\boxed{\text{comp}_{\mathbf{w}}\mathbf{x} = \frac{\mathbf{x}\cdot\mathbf{w}}{\|\mathbf{w}\|} = -23}.$$

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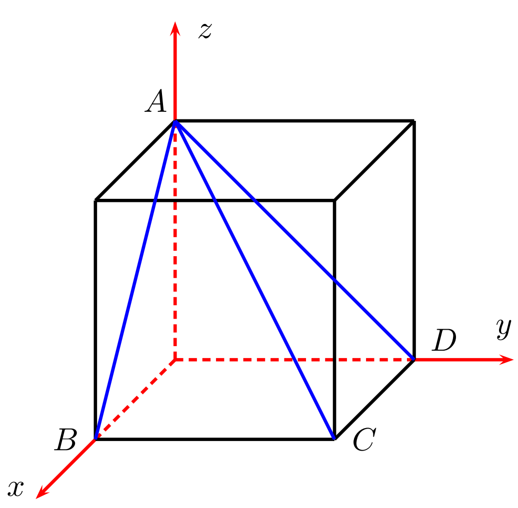
015 10.0 points

The box shown in



is the unit cube having one corner at the origin and the coordinate planes for three of its faces.

Find the cosine of the angle θ between $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

1. $\cos\theta = \sqrt{\frac{2}{3}}$ **correct**
2. $\cos\theta = \frac{1}{\sqrt{2}}$
3. $\cos\theta = 0$
4. $\cos\theta = \frac{\sqrt{3}}{2}$
5. $\cos\theta = \frac{1}{2}$
6. $\cos\theta = \frac{1}{\sqrt{3}}$

Explanation:

To use vectors we shall replace a line segment with the corresponding directed line segment.

Now the angle θ between any pair of vectors $\mathbf{u}$, $\mathbf{v}$ is given in terms of their dot product by

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}.$$

On the other hand, since the unit cube has sidelength 1,

$$A = (0, 0, 1), \quad B = (1, 0, 0),$$

while $C = (1, 1, 0)$. In this case $\overrightarrow{AB}$ is a directed line segment determining the vector

$$\mathbf{u} = \langle 1, 0, -1 \rangle,$$

while $\overrightarrow{AC}$ determines

$$\mathbf{v} = \langle 1, 1, -1 \rangle.$$

For these choices of $\mathbf{u}$ and $\mathbf{v}$,

$$\mathbf{u}\cdot\mathbf{v} = 2 = \sqrt{2}\sqrt{3}\cos\theta.$$

Consequently, the cosine of the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is given by

$$\boxed{\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \sqrt{\frac{2}{3}}}.$$

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keywords: vectors, dot product, unit cube, cosine, angle between vectors

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016 10.0 points

Determine the L^2-inner product of

$$f(x) = 4\cos 2x\,, \quad g(x) = 3\sin 2x\,,$$

as functions in $\mathcal{C}[0,\,2\pi]$.

1. $\langle\, f,\, g\,\rangle = 0$ **correct**

2. $\langle\, f,\, g\,\rangle = 1$

3. $\langle\, f,\, g\,\rangle = \frac{1}{2}$

4. $\langle\, f,\, g\,\rangle = 2$

5. $\langle\, f,\, g\,\rangle = -\frac{3}{2}$

Explanation:

The L^2-inner product of f, g in $\mathcal{C}[0,\,2\pi]$ is defined by

$$\langle\, f,\, g\,\rangle = \int_0^{2\pi} f(x)\,g(x)\,dx\,.$$

Now by double-angle formulas,

$$\int_0^{2\pi}(4\cos 2x)\,(3\sin 2x)\,dx$$

$$= 6\int_0^{2\pi}\sin 4x\,dx\,.$$

But

$$\int_0^{2\pi}\sin 4x\,dx = -\left[\frac{1}{4}\cos 4x\right]_0^{2\pi} = 0\,.$$

Consequently,

$$\boxed{\langle\, f,\, g\,\rangle = 0}\,.$$

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017 10.0 points

When $\mathbb{R}^3$ has the usual dot product as inner product, classify the vectors

$$\mathbf{v}_1 = \begin{bmatrix} -\frac{4}{13} \\ \frac{3}{5} \\ -\frac{48}{65} \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} \frac{12}{13} \\ 0 \\ -\frac{5}{13} \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} \frac{3}{13} \\ \frac{4}{5} \\ \frac{36}{65} \end{bmatrix},$$

as

(i) *basis only*,

(ii) *orthogonal not orthonormal basis*,

(iii) *orthonormal basis*,

for $\mathbb{R}^3$. (Recall that

$$3^2 + 4^2 = 5^2, \quad 5^2 + 12^2 = 13^2\,.)$$

1. none of these

2. orthonormal basis **correct**

3. basis only

4. orthogonal not orthonormal basis

Explanation:

Since

$$\begin{aligned}\|\mathbf{v}_1\|^2 &= <\mathbf{v}_1,\,\mathbf{v}_1> \\ &= \frac{4^2\times 5^2 + 3^2\times 13^2 + 4^2\times 12^2}{5^2\times 13^2} \\ &= \frac{4^2(5^2+12^2) + 3^2\times 13^2}{5^2\times 13^2} \\ &= \frac{(4^2+3^2)\times 13^2}{5^2\times 13^2} = 1\,,\end{aligned}$$

while

$$\begin{aligned}\|\mathbf{v}_2\|^2 &= <\mathbf{v}_2,\,\mathbf{v}_2> \\ &= \frac{12^2+5^2}{13^2} = 1\,,\end{aligned}$$

and

$$\begin{aligned}\|\mathbf{v}_3\|^2 &= <\mathbf{v}_3,\,\mathbf{v}_3> \\ &= \frac{3^2\times 5^2 + 4^2\times 13^2 + 3^2\times 12^2}{5^2\times 13^2} \\ &= \frac{(5^2+12^2)\times 3^2 + 4^2\times 13^2}{5^2\times 13^2} = 1\,.\end{aligned}$$

Thus $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ all have unit norm.

On the other hand,

$$< \mathbf{v}_1, \mathbf{v}_2 > \\ = \frac{-4 \times 12 \times 5 + 4 \times 12 \times 5}{5 \times 13^2} = 0,$$

while

$$< \mathbf{v}_1, \mathbf{v}_3 > \\ = \frac{4 \times 3(-5^2 + 13^2 - 12^2)}{5^2 \times 13^2} = 0,$$

and

$$< \mathbf{v}_2, \mathbf{v}_3 > \\ = \frac{12 \times 3 \times 5 - 5 \times 3 \times 12}{5^2 \times 13^2} = 0.$$

Consequently, $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are also orthogonal, and so together they form an

$$\boxed{\text{orthonormal basis}}$$

for $\mathbb{R}^3$.

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018 10.0 points

The polynomials

$$\mathbf{p}_0(t) = 1, \quad \mathbf{p}_1(t) = t,$$

$$\mathbf{p}_2(t) = t^2 - \frac{1}{3}, \quad \mathbf{p}_3(t) = t^3 - \frac{3}{5}t,$$

form an orthogonal basis with respect to the L^2-inner product

$$< f(t), g(t) > = \int_{-1}^{1} f(t)g(t)\, dt$$

for the vector space $\mathcal{P}^{(3)}$ of all polynomials of degree at most 3.

If

$$\mathbf{q}(t) = t^3 = \sum_{j=0}^{3} c_j\, \mathbf{p}_j(t),$$

determine c_3.

Correct answer: 1.

Explanation:

Since

$$\mathbf{p}_1, \quad \mathbf{p}_2, \quad \mathbf{p}_3, \quad \mathbf{p}_4$$

are orthogonal with respect to the L^2-inner product on $\mathcal{P}^{(3)}$, the coefficient c_2 in the expansion

$$\mathbf{q}(t) = t^3 = \sum_{j=0}^{3} c_j\, \mathbf{p}_j(t)$$

is given by

$$c_3 = \frac{< \mathbf{q}, \mathbf{p}_3 >}{\|\mathbf{p}_3\|^2}.$$

But then

$$< \mathbf{q}, \mathbf{p}_3 > = \int_{-1}^{1} t^3 \left(t^3 - \frac{3}{5}t\right) dt$$

$$= \int_{-1}^{1} \left(t^6 - \frac{3}{5}t^4\right) dt$$

$$= \left[\frac{t^7}{7} - \frac{3t^5}{25}\right]_{-1}^{1} = \frac{8}{175},$$

while

$$\|\mathbf{p}_3\|^2 = \int_{-1}^{1} \left(t^3 - \frac{3}{5}t\right)^2 dt$$

$$= \int_{-1}^{1} \left(t^6 - \frac{6}{5}t^4 + \frac{9}{25}t^2\right) dt$$

$$= \left[\frac{t^7}{7} - \frac{6}{5}t^5 + \frac{3}{25}t^3\right]_{-1}^{1} = \frac{8}{175}.$$

Consequently,

$$\boxed{c_3 = 1}.$$