CHAPTER 24

Lecture No. 24

24.1 Eigenvalues and Eigenvectors

The matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$ may move vectors in a variety of directions, it often happens that there are special vectors on which the action of A is quite simple.

Example: Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$$A\mathbf{u} = \begin{bmatrix} -5 \\ 1 \end{bmatrix}, \qquad A\mathbf{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\mathbf{v}.$$

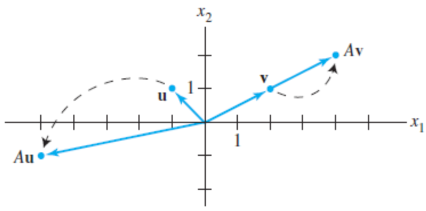


Figure 24.1: Matrix multiplication with a vector

Eigenvector and Eigenvector: An eigenvector of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an eigenvalue of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an eigenvector corresponding to λ.

Remark: It is easy to determine if a given vector is an eigenvector of a matrix. It is also easy to decide if a specified scalar is an eigenvalue.

Example: Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Are $\mathbf{u}$ and $\mathbf{v}$ eigenvectors of A?

Solution: We need to calculate

$$A\mathbf{u} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4\mathbf{u}.$$

which shows that $\mathbf{u}$ is an eigenvector of the given matrix and $\lambda = -4$ is the eigenvalue.

$$A\mathbf{v} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda \mathbf{v}.$$

Hence $\mathbf{v}$ is not an eigenvector of the given matrix.

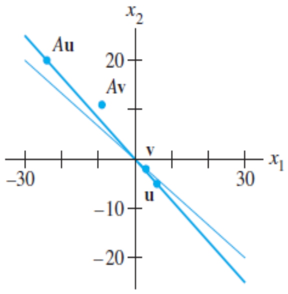


Figure 24.2: Eigenvector of a given matrix

Example: Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, show that 7 is an eigenvalue of the matrix A.

Solution: $\lambda = 7$ will be an eigenvalues of the given matrix if and only if the following equation has nontrivial solution

$$A\mathbf{x} = 7\mathbf{x}.$$

Which is equivalent to the following homogeneous system

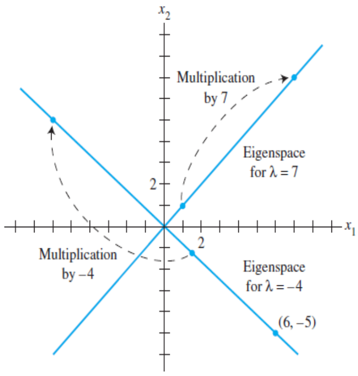
$$1\mathbf{x} - 7\mathbf{x} = \mathbf{0}.$$

If the above homogenous system has nontrivial solution then $\lambda = 7$ will be an eigenvalue of the given system.

The augmented matrix of the homogeneous system is

$$\begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus general solution has the form $x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. For each $x_2 \neq 0$ the vector $x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 7$.



Remark: Row reduction can be used to find eigenvectors, it cannot be used to find eigenvalues. An echelon form of a matrix A usually does not display the eigenvalues of A.

Example: Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigenvalue of A is 2. Find a basis for the corresponding eigenspace.

Solution: Eigenspace is the set of all vectors $\mathbf{x} \in \mathbb{R}^3$ such that $(A - 2I)\mathbf{x} = \mathbf{0}$. The matrix $A - 2I$ is

$$A - 2I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

and the echelon form of the above matrix is

$$\begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The general solution of the system $(A - 2I)\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

which shows that x_2 and x_3 are free and the set

$$\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

forms basis for the eigenspace of the given matrix corresponding to the eigenvalue 2.

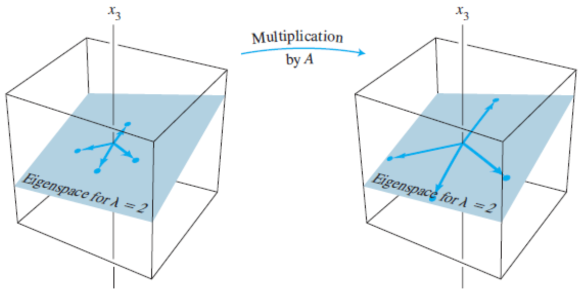


Figure 24.3: Eigenspace for $\lambda = 2$

Theorem: The eigenvalues of a triangular matrix are the entries on its main diagonal.

Example: Let $A = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$, and $B = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & -2 \end{bmatrix}$.

Solution: The eigenvalues of the matrix A are $\lambda = 3, 0, 2$ and of B are $\lambda = 4, 1, -2$.

Remark: When a matrix A has 0 as eigenvalue, it means that the system

$$A\mathbf{x} = 0\mathbf{x} \quad \rightarrow \quad A\mathbf{x} = \mathbf{0}$$

has non trivial solution.

By Invertible Matrix Theorem A in not invertible.

Theorem: If $\mathbf{v}_1, ..., \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, ..., \mathbf{v}_r\}$ is linearly independent.

Example: Is 5 an eigenvalue of the matrix $\begin{bmatrix} 6 & -3 & 1 \\ 3 & 0 & 5 \\ 2 & 2 & 6 \end{bmatrix}$?

Solution: $\lambda = 5$ will be an eigenvalue of the given matrix if and only if $(A - 5I)\mathbf{x} = \mathbf{0}$. The matrix $A - 5I$ is

$$A - 2I = \begin{bmatrix} 6 & -3 & 1 \\ 3 & 0 & 5 \\ 2 & 2 & 6 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 1 \\ 3 & -5 & 5 \\ 2 & 2 & 1 \end{bmatrix}$$

and the echelon form of the above matrix is

$$\begin{bmatrix} 1 & -3 & 1 \\ 3 & -5 & 5 \\ 2 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 1 \\ 0 & 4 & 2 \\ 0 & 8 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

which shows that the system $(A - 5I)\mathbf{x} = \mathbf{0}$ has only trivial solution and hence $\lambda = 5$ is not an eigenvalue of the given matrix.

Example: Without calculation, find one eigenvalue and two linearly independent eigenvectors of $A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$. Justify your answer.

Solution: Since matrix is singular so it must have one eigenvalue as $\lambda = 0$. Can you guess why?

Example: Find the eigenvalues of $\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$.

Solution: By definition we have

$$A\mathbf{x} = \lambda\mathbf{x} \quad \Leftrightarrow (A - \lambda I)\mathbf{x} = \mathbf{0}.$$

The system $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has no trivial solution.

By the Invertible Matrix Theorem, this problem is equivalent to finding all λ such that the matrix $A - \lambda I$ is not invertible, where

$$(A - \lambda I) = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}.$$

$$\det(A - \lambda I) = 0$$

So the eigenvectors of the given matrix are the roots of the polynomial

$$\lambda^2 + 4\lambda - 21 = 0.$$

Recall: Compute $\det A$ for $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$.

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Hence $\det A = -2$.

Theorem: The Invertible Matrix Theorem

Let A be an $n \times n$ matrix. Then A is invertible if and only if

1. The number 0 is not an eigenvalue of A.

2. The determinant of A is not zero.

Remark: When A is a 3×3 matrix, $|\det\ A|$ turns out to be the volume of the parallelepiped determined by the columns $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ of A. This volume is nonzero if and only if the vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are linearly independent, in which case the matrix A is invertible. (If the vectors are nonzero and linearly dependent, they lie in a plane or along a line.)

Theorem: Properties of Determinants

Let A and B be $n \times n$ matrices

1. A is invertible if and only if det $A \neq 0$.
2. det $AB =$ (det A)(det B).
3. det $A^T =$ det A.
4. If A is triangular, then det A is the product of the entries on the main diagonal of A.
5. A row replacement operation on A does not change the determinant. A row interchange changes the sign of the determinant. A row scaling also scales the determinant by the same scalar factor.

24.2 The Characteristic Equation

A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if λ satisfies the characteristic equation

$$\det(A - \lambda I) = 0.$$

Example: Find the characteristic equation of

$$\begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Solution:

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5_\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix}.$$

$\lambda^4 - 14\lambda^3 + 68\lambda^2 - 130\lambda + 75 = 0$. The roots of the polynomial is $\lambda = 5, 3, 5, 1$.

Remark: if A is an $n \times n$ matrix, then $\det(A - \lambda I)$ is a polynomial of degree n called the characteristic polynomial of A.

Example: Find the characteristic equation and eigenvalues of $\begin{bmatrix} 1 & -4 \\ 4 & 2 \end{bmatrix}$.

Solution: The characteristic polynomial is

$$\lambda^2 - 3\lambda + 20 = 0.$$

The roots of the polynomial are the eigenvalues of the given matrix.
Algebraic Multiplicity of an Eigenvalue: The algebraic multiplicity of an eigenvalue λ is its multiplicity as a root of the characteristic equation.

Example: The characteristic polynomial of a 6×6 matrix is $\lambda^6 - 4\lambda^5 - 12\lambda^4$. Find the eigenvalues and their multiplicities.

Solution: The roots of the given polynomials are $\lambda = 0, 3, 4$. The eigenvalue $\lambda = 0$ has algebraic multiplicity 4 and the eigenvalues $\lambda = 3, 4$ has multiplicity 1.

24.3 Similarity

If A and B are $n \times n$ matrices, then A is similar to B if there is an invertible matrix P such that $P^{-1}AP = B$, or, equivalently, $A = PBP^{1-}$.

Writing Q for P^{-1}, we have $Q^{-1}BQ = A$. So B is also similar to A, and we say simply that A and B are similar.

Changing A into $P^{-1}AP$ is called a similarity transformation.

Theorem: If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues with the same multiplicities.
Remark: The matrices

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

are not similar but have the same eigenvalues.
Remark: Similarity is not the same as row equivalence. (If A is row equivalent to B, then $B = EA$ for some invertible matrix E.) Row operations on a matrix usually change its eigenvalues.

24.4 Some Practice Problems

Question: List the real eigenvalues, repeated according to their multiplicities.

$$\begin{bmatrix} 5 & 5 & 0 & 2 \\ 0 & 2 & -3 & 6 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \qquad \begin{bmatrix} 3 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 \\ 3 & 0 & 3 & 0 \\ -1 & 0 & 0 & 3 \end{bmatrix}.$$

Question: It can be shown that the algebraic multiplicity of an eigenvalue λ is always greater than or equal to the dimension of the eigenspace corresponding to λ. Find h in the matrix A below such that the eigenspace for $\lambda = 4$ is two-dimensional

$$\begin{bmatrix} 4 & 2 & 3 & 3 \\ 0 & 2 & h & 3 \\ 0 & 0 & 4 & 14 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$