COMSATS Institute of Information Technology, Virtual Campus

Linear Algebra (MTH231)

Handouts

by

Salman Amin MALIK

To my unknown students

About the author

Dr. Salman Amin Malik did his MS and PhD (Mathematics) from University of La Rochelle, La Rochelle, France in 2009 and 2012, respectively. Prior to MS and PhD, Dr. Malik completed his MPhil and MSc (Mathematics) from University of the Punjab, Lahore, Pakistan. He has been affiliated with several universities in Pakistan and abroad. He has the experience of teaching a wide range mathematics courses at undergraduate and graduate level.

Dr. Malik has published several research articles in international journals and conferences. His area of research includes the study of fractional differential equations and their applications to image processing. He is also interested in inverse source problem related to fractional diffusion equations which has numerous applications in biomedical imaging and non-destructive testing.

About the handouts

For the course Linear Algebra the text book is **David C. Lay**, Linear Algebra and Its Applications, Fourth Edition, Addison-Wesley, ISBN-13: 978-1408280560. Consequently, the most of the examples considered in these notes are from the above mentioned book and its exercises, but not restricted to that book only. If you find any typing error in the text kindly report to me by writing an email to salman.amin.malik@gmail.com.

Course Information

Title and Course Code: Linear Algebra (MTH231)

Number of Credit Hours: 3 credits

Course Objective: The objective of the course is to provide a rigorous approach towards the solutions of linear models which involves more than one variable. The techniques discussed in this course can be implemented on a wide range of applications from physical world. The matrix algebra will be helpful in performing and understanding of matrix computations on a machine. The eigenvalues, eigenvectors, inner product spaces, orthogonality are useful concepts for the analysis of dynamical systems.

Prerequisites: None

Course Learning Outcomes: Upon completion of the course, students will be able to:

1. Understand the linear models, solve the System of Linear Equations and understand the necessity of matrix operations for a machine algorithm.
2. Understand the matrix operations and use these operations for the solution of linear models from different applications.
3. Understand the abstract concept of spaces and apply these concepts on spaces related to a matrix of linear model.
4. To construct the orthonormal basis for the finite dimensional spaces and apply the concept on the spaces obtained from linear models.

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CHAPTER 1

Lecture No. 01

The Linear Algebra course can be divided into three major parts as described

Part I:

- Systems of Linear Equations, Row Reduction and Echelon Forms.
- Vector Equations, The Matrix Equation.
- Solution Sets of Linear Systems, Linear Independence.
- Introduction to Linear Transformations, The Matrix of a Linear Transformation, Applications.

Part II:

- Matrix Operations
- The Inverse of a Matrix, Characterization of Invertible Matrices
- Matrix Factorizations, Applications.
- Introduction to Determinants and Properties of Determinants.

Part III:

- Vector Spaces and Subspaces, Bases, Null Spaces, Column Spaces.
- Coordinate Systems.
- The Dimension of a Vector Space Rank, Applications.
- Eigenvectors and Eigenvalues, The Characteristic Equation, Cayley Hamilton Theorem.
- Diagonalization, Applications, Inner Product, Length and Orthogonality.
- Orthogonal sets, Orthogonal Projections
- The Gram-Schmidt Process Applications

The text book:

David C. Lay, Linear Algebra and Its Applications, Fourth Edition, Addison-Wesley, ISBN-13: 978-1408280560.

Reference books:

- **Gilbert Strang**, Introduction to Linear Algebra, Fourth Edition, Wellesley-Cambridge Press, ISBN: 9780980232714.

- **Lee W. Johnson, R. Dean Riess and Jimmy T. Arnold**, Introduction to Linear Algebra, Fifth Edition, Addison-Wesley, ISBN-13: 9780201658590.

Assessment Plan for the Course:

- Four Assignments 10%.
- Four Quiz 15%.
- First Sessional Exam 10%.
- Second Sessional Exam 15%.
- Final Exam 50%.

1.1 Motivation

Mass Balance: Although there are several examples from real world which can used as motivation for the study of system of linear equations. But here I choose a simple example of finding the unknown masses of two objects namely h and c. The three masses are placed on a rod and rod is balanced on a wedge as show in the figure.

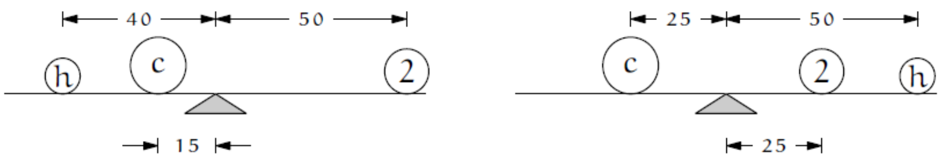


Figure 1.1: Mass balance

The turning effect counterclockwise = Turning effect clockwise

Then two equilibrium positions give rise to the following system of linear equations

$$\begin{aligned} 40h + 15c &= 100 \\ 25c &= 50 + 50h \end{aligned}$$

Some Other Applications:
Linear Programming: The airline industry, for instance, employs linear programs that **schedule flight crews**, **monitor the locations of aircraft**, or plan the varied schedules of support services such as maintenance and **terminal operations**.

Electrical Networks: Engineers use simulation software to design electrical circuits and microchips involving millions of transistors. Such software relies on linear algebra techniques and systems of linear equations.

1.2 System of Linear Equations

Example: The equations

$$3x_1 - 5x_2 = 4x_1 \quad \text{and} \quad x_1 - \sqrt{5}x_2 = 4x_2 + 5\sqrt{5}$$

are linear equations and can be simplified to

$$x_1 + 5x_2 = 0 \quad \text{and} \quad x_1 - (4+\sqrt{5})x_2 = 5\sqrt{5}.$$

The equations

$$x_1 - x_2 + x_2x_1 = 0 \quad \text{and} \quad x_1 - \sqrt{5}x_2 = 4x_2 + \sqrt{x_2}$$

are not linear equations due to the terms x_2x_1 and $\sqrt{x_2}$, respectively.

A linear equation in the variables $x_1, \ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = d$$

where $a_1, \cdots, a_n$ are real or complex numbers (**usually known**) $d \in \mathbb{R}$ is the constant.
Examples:

$$\begin{aligned} 2x_1 - x_2 + 3x_3 &= 10 \\ -x_1 + 5x_2 + x_3 &= 5 \end{aligned}$$

is a system of linear equations with two equations and three unknowns.

The following system

$$
\begin{aligned}
-x_1 + 5x_2 + 3x_3 + x_4 &= 10 \\
2x_1 + 5x_2 + 2x_3 - 2x_4 &= 5 \\
9x_1 - 10x_2 + x_3 - 3x_4 &= 5
\end{aligned}
$$

has three equation and four variables (unknowns).

A system of linear equations with m equations and n variables

$$
\begin{array}{ccccccccc}
a_{1,1}x_1 & + & a_{1,2}x_2 & + & \cdots & + & a_{1,n}x_n & = & d_1 \\
a_{2,1}x_1 & + & a_{2,2}x_2 & + & \cdots & + & a_{2,n}x_n & = & d_2 \\
 & & & & & & & & \vdots \\
a_{m,1}x_1 & + & a_{m,2}x_2 & + & \cdots & + & a_{m,n}x_n & = & d_m
\end{array}
$$

1.3 Solution of system of Linear Equations

Example: Unique Solution

$$
\begin{aligned}
x_1 - 2x_2 &= -1 \qquad l_1 \\
-x_1 + 3x_2 &= 3 \qquad l_2
\end{aligned}
$$

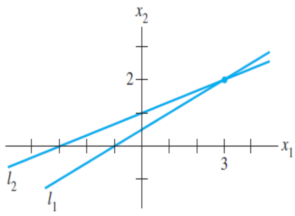


Figure 1.2: Unique solution

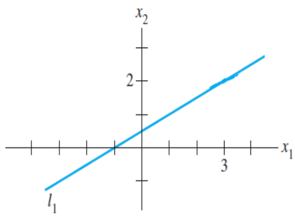


Figure 1.3: Infinite many solutions

Example: No solution and Infinite many solutions

$$
\begin{aligned}
(a) \qquad x_1 - 2x_2 &= -1 \qquad l_1 \\
-x_1 + 2x_2 &= 3 \qquad l_2
\end{aligned}
$$

$$(b) \qquad \begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 1 \end{aligned}$$

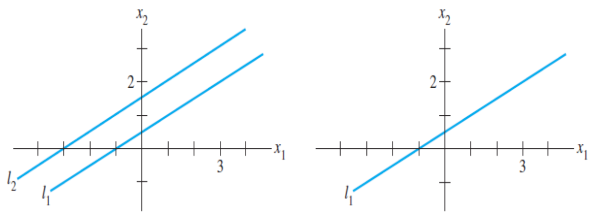


Figure 1.4: No solution and Infinite many solutions

Remark: For a system of linear equation with two variables and two unknown we have three possibilities, (i) system has unique solution, (ii) Infinite many solution, (iii) No solution.
Example: The ordered pair $(-1, 5)$ is a solution of this system. In contrast, $(5, -1)$ is not a solution.

$$\begin{array}{rcrcr} 3x_1 & + & 2x_2 & = & 7 \\ -x_1 & + & x_2 & = & 6 \end{array}$$

Solution: For $(-1, 5)$, we have $x_1 = -1$ and $x_2 = 5$ and the equations becomes

$$-3 + 10 = 7, \qquad 1 + 5 = 6$$

both equation are satisfied and hence the order pair $(-1, 5)$ is a solution of the system of linear equations. For the order pair $(5, -1)$, we have $x_1 = 5$ and $x_2 = -1$ then

$$15 - 2 = 7, \qquad -5 - 1 = 6$$

which shows that none of the equation is satisfied and hence order pair $(5, -1)$ is not a solution.
Example: Is $(3, 4, -2)$ a solution of the following system?

$$\begin{array}{rcrcrcr} 5x_1 & - & x_2 & + & 2x_3 & = & 7 \\ -2x_1 & + & 6x_2 & + & 9x_3 & = & 0 \\ -7x_1 & + & 5x_2 & - & 3x_3 & = & -7 \end{array}$$

Solution: We have $x_1 = 3, x_2 = 4$ and $x_3 = -2$ then the equation becomes

$$15 - 4 - 4 = 7, \quad -6 + 24 - 18 = 0, \quad -21 + 20 + 6 = -7$$

clearly the third equation is not satisfied. Consequently, $(3, 4, -2)$ is not a solution os the system of linear equations.

A system of linear equations with m equations and n variables

$$\begin{array}{ccccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \cdots & + & a_{1,n}x_n & = & d_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \cdots & + & a_{2,n}x_n & = & d_2 \\ & & & & & & & & \vdots \\ a_{m,1}x_1 & + & a_{m,2}x_2 & + & \cdots & + & a_{m,n}x_n & = & d_m \end{array}$$

has the solution $(s_1, s_2, \ldots, s_n)$ if that n-tuple is a solution of all of the equations in the system.
Recall: For a system of linear equation with two variables and two unknown we have three possibilities;

- System has a unique solution,
- Infinite many solution,
- No solution.

1.4 Consistent and Inconsistent System

A system of linear equations is said to be **consistent** if it has either one solution or infinitely many solutions; a system is inconsistent if it has no solution.
Question: Can a system of linear equations has only two solutions or only three solution or only 100 solutions?
How to find all solutions of a given system of linear equations?
Matrix: A matrix is a rectangular array of numbers. For example

$$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 6 & 1 & 0 & 2 \\ 3 & 1 & 0 & 0 \end{bmatrix}$$

is a matrix having three row and three columns.
The order of a matrix is defined as

$$\text{order} = \text{The number of rows} \times \text{the number of columns}.$$

The order of the matrix $\begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 6 & 1 & 0 & 2 \\ 3 & 1 & 0 & 0 \end{bmatrix}$ is 3×3.

Examples: $\begin{bmatrix} 1 \\ 1/3 \\ 1 \end{bmatrix}_{3\times 1}$ is called a columns matrix or vector. $\begin{bmatrix} -4 & 12 & 4 \\ 2 & -6 & -7 \end{bmatrix}_{2\times 3}$

For the system of linear equations

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & 3x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}$$

the matrix $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$ is known as **matrix of coefficients** of the system of linear equations.

The matrix $\begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$ is called **augmented matrix**.

Example:

$$\begin{array}{rcrcrcl} & & & & 3x_3 & = & 9 \\ x_1 & + & 5x_2 & - & 2x_3 & = & 2 \\ \frac{1}{3}x_1 & + & 2x_2 & & & = & 3 \end{array}$$

For the above system the **matrix of coefficients** is $\begin{bmatrix} 0 & 0 & 3 \\ 1 & 5 & -2 \\ \frac{1}{3} & 2 & 0 \end{bmatrix}$.

The matrix of **augmented matrix** is $\begin{bmatrix} 0 & 0 & 3 & 9 \\ 1 & 5 & -2 & 2 \\ \frac{1}{3} & 2 & 0 & 3 \end{bmatrix}$.

Example: If the matrix is the augmented matrix of a system of linear equations write down the system of linear equations. $\begin{bmatrix} 2 & 0 & -2 & 5 \\ 7 & 2 & 5 & 0 \\ 1 & 4 & 5 & 10 \end{bmatrix}$. The system of linear equations is

$$\begin{array}{rcrcrcl} 2x_1 & & & - & 2x_3 & = & 5 \\ 7x_1 & + & 2x_2 & + & 5x_3 & = & 0 \\ x_1 & + & 4x_2 & + & 5x_3 & = & 10 \end{array}$$

1.5 The Elimination Method

Solve the system of linear equations

$$\begin{array}{rcrcrcl} & & & & 3x_3 & = & 9 \\ x_1 & + & 5x_2 & - & 2x_3 & = & 2 \\ \frac{1}{3}x_1 & + & 2x_2 & & & = & 3 \end{array}$$

Solution: The augmented matrix of the system is $\begin{bmatrix} 0 & 0 & 3 & 9 \\ 1 & 5 & -2 & 2 \\ \frac{1}{3} & 2 & 0 & 3 \end{bmatrix}$.

The first transformation rewrites the system by **interchanging the first and third row**.

$$\text{swap row 1 with row 3} \quad \begin{array}{rcrcrcl} \frac{1}{3}x_1 & + & 2x_2 & & & = & 3 \\ x_1 & + & 5x_2 & - & 2x_3 & = & 2 \\ & & & & 3x_3 & = & 9 \end{array}$$

and augmented matrix becomes $\begin{bmatrix} \frac{1}{3} & 2 & 0 & 3 \\ 1 & 5 & -2 & 2 \\ 0 & 0 & 3 & 9 \end{bmatrix}$. The second transformation

rescales the first row by multiplying both sides of the equation by 3.

$$\begin{array}{lrcrcrcl} & x_1 & + & 6x_2 & & & = & 9 \\ \text{multiply row 1 by 3} & x_1 & + & 5x_2 & - & 2x_3 & = & 2 \\ & & & & & 3x_3 & = & 9 \end{array}$$

the corresponding change in the augmented matrix gives $\begin{bmatrix} 1 & 6 & 0 & 9 \\ 1 & 5 & -2 & 2 \\ 0 & 0 & 3 & 9 \end{bmatrix}$.

We **multiply both sides of the first row** by -1, and **add that to the second row**, and write the result in as the **new second row**.

$$\begin{array}{lrcrcrcl} & x_1 & + & 6x_2 & & & = & 9 \\ \text{add } -1 \text{ times row 1 to row 2} & & & -x_2 & - & 2x_3 & = & -7 \\ & & & & & 3x_3 & = & 9 \end{array}$$

$\begin{bmatrix} 1 & 6 & 0 & 9 \\ 0 & -1 & -2 & -7 \\ 0 & 0 & 3 & 9 \end{bmatrix}$.

The bottom equation shows that $x_3 = 3$. Substituting 3 for x_3 in the middle equation shows that $x_2 = 1$. Substituting those two into the top equation gives that $x_1 = 3$.

Thus the system has a **unique solution**; the solution set is $\{\, (3, 1, 3) \,\}$.
Verification that the vector $\{\, (3, 1, 3) \,\}$ is a solution set for the system of linear equations

$$\begin{array}{rcrcrcl} & & & & 3x_3 & = & 9 \\ x_1 & + & 5x_2 & - & 2x_3 & = & 2 \\ \frac{1}{3}x_1 & + & 2x_2 & & & = & 3 \end{array}$$

$$\begin{array}{rcrcrcl} & & & + & 3(3) & = & 9 \\ 3 & + & 5(1) & - & 2(3) & = & 2 \\ \frac{1}{3}(3) & + & 2(1) & & & = & 3 \end{array}$$

All equations of the system of linear equations are satisfied, hence the set $\{\, (3, 1, 3) \,\}$ is a solution set of the system of linear equations.
Example: Solve the system of linear equations

$$\begin{array}{rcrcrcl} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}$$

Solution: The augmented matrix of the given system is $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$.

Keep x_1 in the first equation and eliminate it from the other equations.

$$\begin{array}{lrcrcrcr} & x_1 & - & 2x_2 & + & x_3 & = & 0 \\ \text{4[Equation 1] +[Equation 3]} & & & 2x_2 & - & 8x_3 & = & 8 \\ & & - & 3x_2 & + & 13x_3 & = & -9 \end{array}$$

The corresponding change in the augmented matrix lead to the following matrix $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$.

Multiply equation 2 by 1/2 in order to obtain 1 as the coefficient for x_2.

$$\begin{array}{lrcrcrcr} & x_1 & - & 2x_2 & + & x_3 & = & 0 \\ \text{1/2[Equation 2]} & & & x_2 & - & 4x_3 & = & 4 \\ & & - & 3x_2 & + & 13x_3 & = & -9 \end{array}$$

The augmented matrix becomes $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$.

Use the x_2 in equation 2 to eliminate the $-3x_2$ in equation 3.

$$\begin{array}{lrcrcrcr} & x_1 & - & 2x_2 & + & x_3 & = & 0 \\ \text{3[Equation 2] +[Equation 3]} & & & x_2 & - & 4x_3 & = & 4 \\ & & & & & x_3 & = & 3 \end{array}$$

The corresponding augmented matrix takes the following form $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$.

$4[Equation3] + [Equation2]$ and $-1[Equation3] + [Equation1]$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & & & = & -3 \\ & & x_2 & & & = & 16 \\ & & & & x_3 & = & 3 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

2[Equation 2]+[Equation 1] leads to

$$\begin{array}{rcrcrcr} x_1 & - & & & & = & 29 \\ & & x_2 & & & = & 16 \\ & & & & x_3 & = & 3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

Solution of the system is $29, 16, 3$.

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}$$

$$\begin{array}{rcrcrcr} (29) & - & 2(16) & + & 3 & = & 0 \\ & & 2(16) & - & 8(3) & = & 8 \\ -4(29) & + & 5(16) & + & 9(3) & = & -9 \end{array}$$

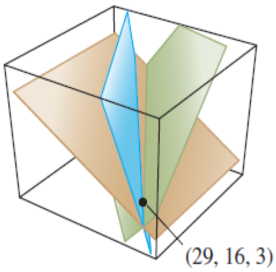


Figure 1.5: Solution of the system with three variables

Geometrically, unique solution for the system of linear equations with three variables is the point of intersection of the planes formed by these three equations.
Example: Determine if the following system is consistent.

$$\begin{array}{rcrcrcr} & & x_2 & - & 4x_3 & = & 8 \\ 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ 5x_1 & - & 8x_2 & + & 7x_3 & = & 1 \end{array}$$

Solution: The augmented matrix of the given system is $\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$.

To obtain an x_1 in the first equation, interchange rows 1 and 2:

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

To eliminate the $5x_1$ term in the third equation, add $-5/2$ times row 1 to row 3:

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -1/2 & 2 & -3/2 \end{bmatrix}$$

To eliminate the $-1/2x_2$ term from the third equation. Add $1/2$ times row 2 to row 3:

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}$$

The augmented matrix is in triangular form and we transform into equation notation

$$\begin{array}{rcrcrcl} 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ & & x_2 & - & 4x_3 & = & 8 \\ & & & & 0 & = & 5/2 \end{array}$$

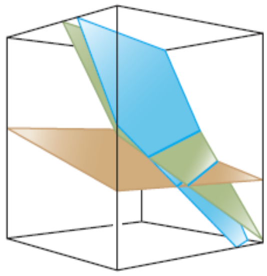


Figure 1.6: Inconsistent system with three variables

The planes formed by three equation don't have a common point of intersection as show in the figure and the system os inconsistent. **Example**: For what values of h and $k$is the following system consistent? $\begin{array}{rcrl} 2x_1 & - & x_2 = & h \\ -6x_1 & + & 3x_2 = & k \end{array}$

Solution: The augmented matrix of the system is $\begin{bmatrix} 2 & -1 & h \\ -6 & 3 & k \end{bmatrix}$.

3[Equation 1] + [Equation 2] or 3[Row 1] + [Row 2] $\begin{bmatrix} 2 & -1 & h \\ 0 & 0 & k+3h \end{bmatrix}$.

If $k + 3h \neq 0$ then we have $0 = k + 3h \neq 0$ implies the system is inconsistent.

So the system will be consistent if we have $k + 3h = 0$ or $k = -3h$.

For example: take $h = 2$ then $k = -9$ is one possibility. There are infinite many values of h and k satisfying $k + 3h = 0$.

1.6 Some Practice Problems

Question: Do the lines $2x_1 + 3x_2 = -1$, $6x_1 + 5x_2 = 0$, and $2x_1 - 5x_2 = 7$ have a common point of intersection?Justify your answer.

Question: Determine the value(s) of h such that the matrix is the augmented matrix of a consistent system

(a) $\begin{bmatrix} -4 & 12 & h \\ 2 & -6 & -3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 4 & -2 \\ 2 & h & -6 \end{bmatrix}$

Question: Determine whether the given system of linear equations are consistent or inconsistent.

$$\begin{array}{rcrcrcr} 2x_1 & & & - & 3x_3 & = & -8 \\ & & x_2 & - & 2x_3 & = & 3 \\ 3x_1 & + & 6x_2 & - & 2x_3 & = & -4 \end{array}$$

and

$$\begin{array}{rcrcrcr} x_1 & - & 5x_2 & + & 4x_3 & = & -3 \\ 2x_1 & - & 7x_2 & + & 3x_3 & = & -2 \\ -2x_1 & + & x_2 & + & 7x_3 & = & -1 \end{array}$$

Question: Find an equation involving g, h, and k that makes the augmented matrix correspond to a consistent system $\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{bmatrix}$

CHAPTER 2

Lecture No. 02

Question: Find an equation involving g, h, and k that makes the augmented matrix correspond to a consistent system $\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{bmatrix}$

Solution: After the elimination of the variable x_1 from the third equation and from the resulting matrix elimination of the x_2 variable from third row of the matrix gives us the following matrix

$$\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & 0 & 0 & h+k+2g \end{bmatrix}.$$

The system will be consistent if $h + k + 2g = 0$ otherwise we will have one equation of the form $0 = h + k + 2g \neq 0$, which is absurd.

2.1 Elementary Row Operations

From the examples considered in the previous lecture we have seen that if a linear system is changed to another by one of these operations

1. An equation is **swapped with another**.
2. An equation has **both sides multiplied** by a nonzero constant.
3. An equation is **replaced** by the **sum of itself and a multiple of another**.

then the **two systems have the same set of solutions**.
Elementary Row Operations: The row operations corresponding to the above mentioned changes on a matrix are know as elementary row operations. These are

1. (**Interchange**) Interchange two rows.
2. (**Scaling**) Multiply all entries in a row by a nonzero constant.
3. (**Replacement**) Replace one row by the sum of itself and a multiple of another row.

2.2 Echelon and Reduced Echelon form of a Matrix

Let us define the **Leading entry** of a row refers to the leftmost nonzero entry (in a nonzero row).

Echelon Form:

A rectangular matrix is in echelon form (or row echelon form) if it has the following three properties:

- All nonzero rows are above any rows of all zeros.
- Each leading entry of a row is in a column to the right of the leading entry of the row above it.
- All entries in a column below a leading entry are zeros.

Reduced Echelon Form:

If a matrix in echelon form satisfies the following additional conditions, then it is in reduced echelon form (or reduced row echelon form):

- The leading entry in each nonzero row is 1.
- Each leading 1 is the only nonzero entry in its column.

Example: The matrices in the first two matrices are in echelon form where as the third and fourth matrices are in reduced echelon form.

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Remark:

- Any nonzero matrix may be reduced to more than one echelon form, i.e., echelon form of a matrix is not unique.
- Each matrix is row equivalent to one and only one reduced echelon matrix, i.e., reduced echelon matrix of a matrix is unique.

Pivot Positions: A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the **reduced echelon** form of A. A **pivot column** is a column of A that contains a **pivot position**.

The matrix

$$\begin{bmatrix} 1 & 6 & 0 & 9 \\ 0 & -1 & -2 & -7 \\ 0 & 0 & 3 & 9 \end{bmatrix}$$

is in echelon form, with the pivot element in first, second and third columns.

The matrix

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

is in echelon form and it has three pivot elements in first, second and third columns. These pivot elements are 1. The matrix is not in reduced echelon form as in pivot columns pivot elements are not the only nonzero elements. We can transform the above matrix into reduced echelon form by applying following row operations on the matrix $R_1 - R_3$, $R_2 + 4R_3$ gives

$$\begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

and then by the row operation $R_1 + 2R_2$ leads to the following reduced echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

Example: Row reduce the given matrix A into echelon form and locate the pivot positions. $\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$

Solution:

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Pivot column

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Next pivot column

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns

2.3 The Row Reduction Algorithm

Transform the following matrix into echelon form and then in reduced echelon form by using elementary row operations.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}.$$

Step I: Choose the pivot column $\begin{bmatrix} \mathbf{0} & 3 & -6 & 6 & 4 & -5 \\ \mathbf{3} & -7 & 8 & -5 & 8 & 9 \\ \mathbf{3} & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$

Step II: Interchange rows 1 and 3 $\begin{bmatrix} \mathbf{3} & -9 & 12 & -9 & 6 & 15 \\ \mathbf{3} & -7 & 8 & -5 & 8 & 9 \\ \mathbf{0} & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$

Step III: add -1 times row 1 to row 2 $\begin{bmatrix} \mathbf{3} & -9 & 12 & -9 & 6 & 15 \\ \mathbf{0} & 2 & -4 & 4 & 2 & -6 \\ \mathbf{0} & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$

Step IV: Choose the pivot column $\begin{bmatrix} 3 & \mathbf{-9} & 12 & -9 & 6 & 15 \\ 0 & \mathbf{2} & -4 & 4 & 2 & -6 \\ 0 & \mathbf{3} & -6 & 6 & 4 & -5 \end{bmatrix}$

Step V: add $-3/2$ times row 2 to row 3 $\begin{bmatrix} 3 & -9 & 12 & -9 & \mathbf{6} & 15 \\ 0 & 2 & -4 & 4 & \mathbf{2} & -6 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 4 \end{bmatrix}$

Step VI: $\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

$Row1 + (-6)row3$ and $row2 + (-2)row3$ $\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

Row 2 scaled by $1/2$ $\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

Row 1 + (9) row 2 $\begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

1/3 Row 1 $\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$ which is the reduced echelon form of the given matrix.

Example: The augmented matrix of a linear system is $\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ write down the solution of the system.
Solution: The given augmented matrix is in echelon form and we get the following solution of the system

$$x_2 = 4 - x_3, \qquad x_1 = 1 + 5x_3.$$

This solution shows that the values of the variables x_1 and x_2 can be obtained by taking some value of the variable x_3.
The variables x_1, x_2 are known as **Basic variables** and the variable x_3, is called **Free variable**.
Example: Find the solution of the linear system whose augmented matrix has been reduced to

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}.$$

Solution: The reduced Echelon form of the given matrix is

$$\begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}.$$

Then we have from third row $x_5 = 7$, from second row we have $x_3 = 5 + 4x_4$ and from first row we have $x_1 = -6x_2 - 3x_4$.

This shows that x_2 and x_4 are free variables where as x_1, x_3, x_5 are basic variables.
Theorem: A linear system is **consistent** if and only if the **rightmost column of the augmented matrix is not a pivot column** that is, if and only if an **echelon form of the augmented matrix** has no row of the form

$$[0 \ldots 0 \;\; b]$$

with b **nonzero**.
If a linear system is **consistent**, then the solution set contains either

- a unique solution, when there are **no free variables**, or

- infinitely many solutions, when there is at **least one free variable**.

Example: Determine the existence and uniqueness of the solutions to the system

$$\begin{array}{rcrcrcrcrcr} & & 3x_2 & - & 6x_3 & + & 6x_4 & + & 4x_5 & = & -5 \\ 3x_1 & - & 7x_2 & + & 8x_3 & - & 5x_4 & + & 8x_5 & = & 9 \\ 3x_1 & - & 9x_2 & + & 12x_3 & - & 9x_4 & + & 6x_5 & = & 15 \end{array}$$

Solution: The augmented matrix of the system was row reduced to

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

The basic variables are x_1, x_2, and x_5; the free variables are x_3 and x_4. There is no equation such as $0 = 1$ that would indicate an inconsistent system, so we could use back substitution to find a solution.

Since there are free variables so the solution of the system is not unique, indeed we have infinite many solutions.

Example: Determine whether the system whose augmented matrix is row reduced to the following matrix is consistent or inconsistent

$$\begin{bmatrix} 1 & 0 & -9 & 0 & 4 \\ 0 & 1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution: The augmented matrix is in echelon, indeed reduced echelon form and one of the row (4th row) is of the form $[0...0\ b]$ where $b \neq 0$ and the fourth row leads to 0=1 which is absurd. hence the system of linear equations is inconsistent.

Example: Determine the value(s) of h such that the matrix is the augmented matrix of a consistent linear system

$(a) \begin{bmatrix} 1 & -1 & 4 \\ -2 & 3 & h \end{bmatrix}$,

$(b) \begin{bmatrix} 1 & -3 & 1 \\ h & 6 & -2 \end{bmatrix}$.

Solution: (a) The echelon form of the matrix is

$$\begin{bmatrix} 1 & -1 & 4 \\ 0 & 1 & h+8 \end{bmatrix}$$

We will never have a row of the type $[0...0\ b]$ whatever the value of h we take. Hence the system is consistent for all values of h.

(b) Similarly, get the echelon form of the matrix

$$\begin{bmatrix} 1 & -3 & 1 \\ h & 6 & -2 \end{bmatrix}$$

and try to apply the theorem for deciding the value of h to have consistent system.

2.4 Steps to Solve a Linear System

1. Write the augmented matrix of the system.

2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.

3. Continue row reduction to obtain the reduced echelon form.

4. Write the system of equations corresponding to the matrix obtained in step 3.

5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

2.5 Some Practice Problems

Question: choose h and k such that the system

$$\begin{array}{rcrcl} x_1 & - & 3x_2 & = & 1 \\ 2x_1 & + & hx_2 & = & k \end{array}$$

has (a) no solution, (b) a unique solution, and (c) many solutions.

Question: Find the general solutions of the systems whose augmented matrices are given by $\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & -5 & 0 & -8 & 3 \\ 0 & 1 & 4 & -1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Question: Find the interpolating polynomial $P(t) = a_0 + a_1 t + a_2 t^2$ for the data $(1, 6), (2, 15), (3, 28)$.

CHAPTER 3

Lecture No. 03

Question: Find the interpolating polynomial $P(t) = a_0 + a_1t + a_2t^2$ for the data $(1,6),(2,15),(3,28)$.
Solution: Interpolating polynomial means that $P(1) = 6, P(2) = 15$ and $P(3) = 28$. The leads to the following system of linear equations

$$\begin{aligned} a_0 + a_1 + a_2 &= 6 \\ a_0 + 2a_1 + 4a_2 &= 15 \\ a_0 + 3a_1 + 9a_2 &= 28 \end{aligned}$$

Solve the above system of linear equations and get the values of a_0, a_1 and a_3 and get the interpolating polynomial.

3.1 Vectors

A matrix with one columns is called a **vector**. vectors are usually denoted by bold letters. $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 5 \\ -6 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Sum of vectors: Let $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 8 \\ 7 \end{bmatrix}$ then $\mathbf{u}+\mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 8 \\ 7 \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$.

Scalar multiplication:
Let $\mathbf{v} = \begin{bmatrix} 8 \\ 7 \end{bmatrix}$ then $c\mathbf{v} = \begin{bmatrix} 8c \\ 7c \end{bmatrix}$, if $c = 3$ then $c\mathbf{v} = \begin{bmatrix} 24 \\ 21 \end{bmatrix}$.

Example: If $\mathbf{u} = \begin{bmatrix} 8 \\ 7 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, find $4\mathbf{u}, -3\mathbf{v}$ and $4\mathbf{u} + (-3)\mathbf{v}$.

$\mathbb{R}^2$ could be considered as set of all points in the plane.

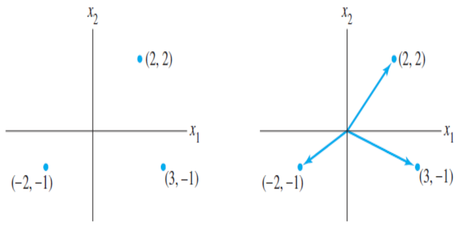


Figure 3.1: $\mathbb{R}^2$ the plane

Parallelogram Rule for Addition: If $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$ are represented as points in the plane, then $\mathbf{u}+\mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are $\mathbf{u}$, $\mathbf{0}$, and $\mathbf{v}$.

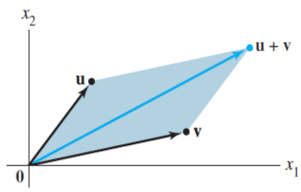


Figure 3.2: Parallelogram Rule for Addition

Example: The vectors $\mathbf{u}=\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix} -6 \\ 1 \end{bmatrix}$, and $\mathbf{u}+\mathbf{v}=\begin{bmatrix} -4 \\ 3 \end{bmatrix}$ as shown in figure.

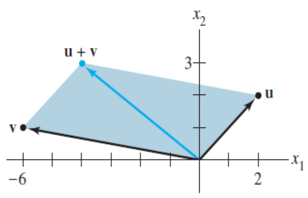
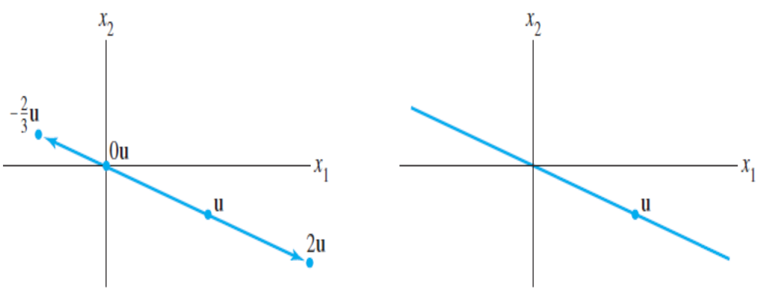


Figure 3.3: Parallelogram Rule for Addition

Example: Let $\mathbf{u}=\begin{bmatrix} 3 \\ -1 \end{bmatrix}$. Display the vectors $\mathbf{u}$, $2\mathbf{u}$, and $-\frac{2}{3}\mathbf{u}$.



Vectors in $\mathbb{R}^3$: Vectors in $\mathbb{R}^3$ are 3×1 matrices. For example $\mathbf{a} = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$.

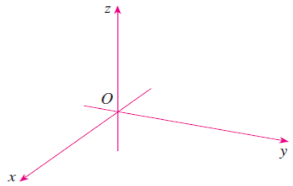


Figure 3.4: The three dimensional space

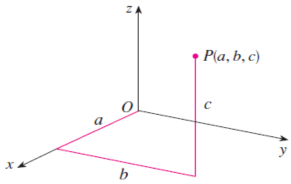


Figure 3.5: A point in $\mathbb{R}^3$

Vectors in $\mathbb{R}^n$, where n is a natural, the vectors in $\mathbb{R}^n$ are denoted by a matrix of order $n \times 1$,i.e., $\mathbf{u} = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}$. A zero vector in $\mathbb{R}^n$ is denoted by $\mathbf{0}$.

Algebraic Properties of $\mathbb{R}^n$: For all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^n$ and all scalars c and d:

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$ where $-\mathbf{u} = (-1)\mathbf{u}$
5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{v}$
7. c(d$\mathbf{u}$) = (cd)($\mathbf{u}$)
8. 1$\mathbf{u}$=$\mathbf{u}$

3.2 Linear Combination

Given vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, ..., c_p$, the vector $\mathbf{y}$ defined

$$\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_p\mathbf{v}_p$$

is called a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p$ with weights $c_1, c_2, ..., c_p$.

Example: Selected linear combinations of $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

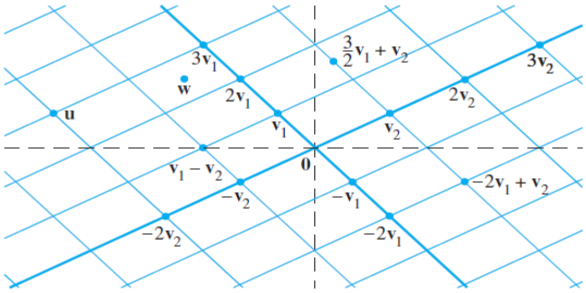


Figure 3.6: Some linear combinations of two vectors

Example: Write three different linear combinations of the the vectors

1. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$,

2. $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}$.

Solution: For $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, we can write the following three linear combinations, for example

$2\mathbf{u} + \mathbf{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$, $\quad \mathbf{u} + 2\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 10 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$ and $\mathbf{u} - \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$.

Indeed, you can write many linear combinations of these vectors. Try other linear combinations and do the other parts.

Example: Let $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$. Determine whether $\mathbf{b}$ can be written as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

Solution: We want to check that $\mathbf{b}$ is linear combination of the given vectors or not, i.e., we will check the solution of the system corresponding to equation $c_1\mathbf{u}+c_2\mathbf{v} = \mathbf{b}$.

The augmented matrix is $\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix}$ and the reduced echelon form is

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus the solution of the equation $c_1\mathbf{u}+c_2\mathbf{v} = \mathbf{b}$ is $c_1 = 3$ and $c_2 = 2$. Consequently, $\mathbf{b}$ is a liner combination of the vectors $\mathbf{u}$ and $\mathbf{v}$.

A vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_n\mathbf{v}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$[\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; ... \;\; \mathbf{v}_n \;\; \mathbf{b}]. \tag{3.1}$$

In particular, $\mathbf{b}$ can be generated by a linear combination of $\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_n$ if and only if there exists a solution to the linear system corresponding to the matrix (3.1).

Example: Write the system of linear equations corresponding to each vector equation

1. $x_1 \begin{bmatrix} -2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

2. $x_1 \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 0 \end{bmatrix}$.

Solution:

1. $$-2x_1 + 7x_2 = 0$$
$$3x_1 + 5x_2 = 1$$

   and

2. $$x_1 - x_2 = 7$$
$$5x_1 + 6x_2 = 5$$
$$-2x_1 + 3x_2 = 0$$

3.3 Spanning Set of Given Vectors

If $\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_p$ is denoted by $Span\{\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_p\}$ and is called the subset of $\mathbb{R}^n$ spanned (or generated) by $\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_p$.

That is, $Span\{\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_p\mathbf{v}_p$$

where $c_1, c_2, ..., c_p$ are scalars.
Remark: A given vector $\mathbf{b}$ is in $Span\{\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_p\}$ is the same as asking whether the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_p\mathbf{v}_p = \mathbf{b}$$

has a solution, or equivalently asking whether the linear system with augmented matrix $[\mathbf{v}_1 \ \ \mathbf{v}_2 ... \mathbf{v}_p \ \ \mathbf{b}]$ has a solution.
A Geometric Description of $Span\{\mathbf{v}\}$ **and** $Span\{\mathbf{v}, \mathbf{u}\}$

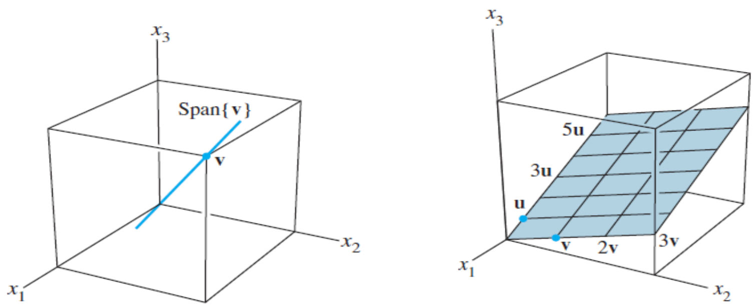


Figure 3.7: Geometry of spanning set of one and two vectors

Example: Let $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$. Then $Span\{\mathbf{u}, \mathbf{v}\}$ is a plane through the origin in $\mathbb{R}^3$. Is $\mathbf{b}$ in that plane?
Solution: We have to check whether the vector equation $c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{b}$ is consistent or inconsistent.

The augmented matrix is

$$\begin{bmatrix} 1 & 5 & -3 \\ -2 & -13 & 8 \\ 3 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & -18 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

The third equation is $0 = -2$, which shows that that the system is inconsistent.

Conclusion: The vector equation $c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{b}$ has no solution, and $\mathbf{b}$ is not in the $Span\{\mathbf{u}, \mathbf{v}\}$.

Example: Let $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} h \\ -3 \\ -5 \end{bmatrix}$, Find the values of h such that $\mathbf{b}$ is in the plane generated by $\mathbf{u}$ and $\mathbf{v}$.

Solution: The augmented matrix of the vector equation $c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{b}$ is

$$\begin{bmatrix} 1 & -2 & h \\ 0 & 1 & -3 \\ -2 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & h \\ 0 & 1 & -3 \\ 0 & 3 & -5+2h \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & h \\ 0 & 1 & -3 \\ 0 & 0 & 4+2h \end{bmatrix}$$

For the system to be consistent we must have $h + 2 = 0$, which gives $h = -2$. For $h = 2$, $\mathbf{b}$ is in the plane generated by $\mathbf{u}$ and $\mathbf{v}$.

3.4 Some Practice Problems

Question: Let $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} h \\ k \end{bmatrix}$, show that $\begin{bmatrix} h \\ k \end{bmatrix}$ is in the $Span\{\mathbf{u}, \mathbf{v}\}$ for all values of h and k.

Question: Let $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} h \\ k \end{bmatrix}$, show that $\begin{bmatrix} h \\ k \end{bmatrix}$ is in the $Span\{\mathbf{u}, \mathbf{v}\}$ for all values of h and k.

Question: Let $A = \begin{bmatrix} 2 & 0 & 6 \\ -1 & 8 & 5 \\ 1 & -2 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 10 \\ 3 \\ 7 \end{bmatrix}$. Is $\mathbf{b}$ in the span of columns of A? Show that the second column of A is in the span of columns of A.

Question: Construct a 3×3 matrix A, with nonzero entries, and a vector $\mathbf{b}$ in $\mathbb{R}^3$ such that $\mathbf{b}$ is not in the set spanned by the columns of A.

Question: Write the vector equations corresponding to the linear systems

$$\begin{array}{rcrcrcr} 5x_1 & - & x_2 & + & 2x_3 & = & 7 \\ -2x_1 & + & 6x_2 & + & 9x_3 & = & 0 \\ -7x_1 & + & 5x_2 & - & 3x_3 & = & -7 \end{array}$$

and

$$\begin{array}{rcrcr} x_1 & - & 3x_2 & = & 0 \\ -x_1 & + & 6x_2 & = & 0 \\ x_1 & + & 4x_2 & = & 0 \end{array}.$$

CHAPTER 4

Lecture No. 04

Question: Let $A = \begin{bmatrix} 2 & 0 & 6 \\ -1 & 8 & 5 \\ 1 & -2 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 10 \\ 3 \\ 7 \end{bmatrix}$. Is $\mathbf{b}$ in the span of columns of A? Show that the second column of A is in the span of columns of A.

Solution: Augmented matrix of the vector equation is

$$\begin{bmatrix} 2 & 0 & 6 & 10 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 7 \end{bmatrix},$$

after doing the row operations $1/2R_1$, $R_2 + R_1$ and $R_3 - R_1$

$$\begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 8 & 8 & 8 \\ 0 & -2 & -2 & 2 \end{bmatrix},$$

with row operations $R_3 + 1/4R_2$ $\begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 8 & 8 & 8 \\ 0 & 0 & 0 & 3 \end{bmatrix}$, $0 = 3$ hence system is inconsistent, $\mathbf{b}$ is not in the span of columns of the matrix A. Notice that we can write the second column of A as

$$\begin{bmatrix} 0 \\ 8 \\ -2 \end{bmatrix} = 0 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 8 \\ -2 \end{bmatrix} + 0 \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}.$$

4.1 The Matrix Equation

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, ..., \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the product of A and $\mathbf{x}$, denoted by $A\mathbf{x}$, is the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights; i.e.,

$$A\mathbf{x} = [\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; ... \;\; \mathbf{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + ... + x_n\mathbf{a}_n.$$

Remark: $A\mathbf{x}$ is defined only if the number of columns of A equals the number of entries in $\mathbf{x}$.
Example:

$$\begin{aligned}\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} &= 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ -15 \end{bmatrix} + 7\begin{bmatrix} -7 \\ 21 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}\end{aligned}$$

Example:

$$\begin{aligned}\begin{bmatrix} 2 & 2 \\ 7 & -5 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} &= 4\begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -5 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} 8 \\ 28 \\ 12 \end{bmatrix} + \begin{bmatrix} 6 \\ -15 \\ 18 \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \\ 30 \end{bmatrix}\end{aligned}$$

Example: For $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in $\mathbb{R}^m$, write the linear combination $3\mathbf{v}_1 - 5\mathbf{v}_2 + 7\mathbf{v}_3$ as a matrix times a vector.

Solution: Let $A = [\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3]$ and $\mathbf{b} = \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix}$ then the linear combination can be written as

$$A\mathbf{b}.$$

Example: Write the system of linear equation in the matrix for $A\mathbf{x} = \mathbf{b}$

$$\begin{array}{rcrcrl} 2x_1 & - & x_2 & + & x_3 = & 10 \\ -6x_1 & + & 3x_2 & - & 2x_3 = & 4 \end{array}$$

Solution: The matrix of coefficient of the given system is

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -6 & 3 & -2 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 10 \\ 4 \end{bmatrix}.$$

Then the matrix equation corresponding to the given system of linear equation is $A\mathbf{x} = \mathbf{b}$.
Theorem: If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, ..., \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbb{R}^m$, the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + ... + x_n\mathbf{a}_n = \mathbf{b}$$

which, in turn has the same solution set as the system of linear equations whose augmented matrix is

$$[\mathbf{a}_1 \ \ \mathbf{a}_2 ... \ \mathbf{a}_n \ \ \mathbf{b}].$$

Existence of Solutions:

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.

Example: Compute $A\mathbf{x}$, where $A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

Solution: The matrix product

$$A\mathbf{b} = \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ -x_1 + 5x_2 + -3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{bmatrix},$$

whose order is 3×1 and hence a vector of $\mathbb{R}^3$.

Example: What should be the order of the vector $\mathbf{x}$? If $A = \begin{bmatrix} 2 & 3 \\ -1 & 5 \\ 6 & -2 \end{bmatrix}$ and we need to compute $A\mathbf{x}$.

Solution: Since for the matrix multiplication $A\mathbf{x}$ the number of columns of A should be equal to the number of columns of $\mathbf{x}$. This means the the vector should be of order 2×1. The answer of the matrix product $A\mathbf{x}$ will be a vector from $\mathbb{R}^3$.

Remark: Row-Vector Rule for Computing $A\mathbf{x}$;

If the product $A\mathbf{x}$ is defined, then the ith entry in $A\mathbf{x}$ is the sum of the products of corresponding entries from row i of A and from the vector $\mathbf{x}$.

Example: We have $\begin{bmatrix} 4 & -3 & 1 \\ 5 & -2 & 5 \\ -6 & 2 & -3 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix}$. Use this fact (and no row operations) to find the scalars c_1, c_2, c_3 such that

$$c_1 \begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix}.$$

Solution: The given matrix equation $\begin{bmatrix} 4 & -3 & 1 \\ 5 & -2 & 5 \\ -6 & 2 & -3 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix}$. is equivalent to the following vector equation

$$-3 \begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix} - 1 \begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix}.$$

Comparing the above equation with

$$c_1 \begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix},$$

we get $c_1 = -3, c_2 = -1$ and $c_3 = 2$.

Example:Let $\mathbf{u} = \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$. It can be shown that $2\mathbf{u} - 3\mathbf{v} - \mathbf{w} = \mathbf{0}$. Use this fact to find the values of x_1, x_2 that satisfy the equation $\begin{bmatrix} 7 & 3 \\ 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$.

Solution: The given matrix equation $\begin{bmatrix} 7 & 3 \\ 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$. is equivalent to

$$x_1 \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix},$$

which can be written as

$$x_1 \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Comparing with the given relation $2\mathbf{u} - 3\mathbf{v} - \mathbf{w} = \mathbf{0}$ gives $x_1 = 2, x_2 = -3$.

Example: Let $\begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all possible b_1, b_2, b_3?

Solution: The augmented matrix of the system $A\mathbf{x} = \mathbf{b}$ is

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix}$$

and reduce that matrix to echelon form and decide about the system is consistent or not.

Theorem: Let A be an $m \times n$ matrix. Then the following statements are **logically equivalent**. That is, for a particular A, either they are all true statements or they are all false.

- For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

- Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

- The columns of A span $\mathbb{R}^m$.

- A has a pivot position in every row.

Remark: The above theorem is about a coefficient matrix, not an augmented matrix. If an augmented matrix $[A \;\; \mathbf{b}]$ has a pivot position in every row, then the equation $A\mathbf{x} = \mathbf{b}$ may or may not be consistent.
Theorem: If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar, then:

1. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$,

2. $A(c\mathbf{u}) = c(A\mathbf{u})$.

Example: Let $\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 \\ -3 \\ 9 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 4 \\ -2 \\ -6 \end{bmatrix}$. Does $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ span $\mathbb{R}^3$? Justify your answer.

Solution: Let $\mathbf{b} \in \mathbb{R}^3$ be an arbitrary element of $\mathbb{R}^3$, i.e., $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ where $b_1, b_2, b_3 \in \mathbb{R}$ then we have to check the system $A\mathbf{x} = \mathbf{b}$ is consistent or inconsistent, where $A = [\mathbf{u} \;\; \mathbf{v} \;\; \mathbf{w}]$.

If the system $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b}$ then we can conclude that $span\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} = \mathbb{R}^3$. Thus augmented matrix of the system $A\mathbf{x} = \mathbf{b}$ is

$$\begin{bmatrix} 0 & 0 & 4 & b_1 \\ 0 & -3 & -2 & b_2 \\ -3 & 9 & -6 & b_3 \end{bmatrix}$$

interchanging first and third rows we have

$$\begin{bmatrix} -3 & 9 & -6 & b_3 \\ 0 & -3 & -2 & b_2 \\ 0 & 0 & 4 & b_1 \end{bmatrix}.$$

The augmented matrix is in echelon form and has pivot element in every row, hence the system $A\mathbf{x} = \mathbf{b}$ is consistent for all values of b_1, b_2, b_3. Hence

$$Span\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} = \mathbb{R}^3.$$

4.2 Some Practice Problems

Question: (a) Let $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$. Does $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ span $\mathbb{R}^4$? Justify your answer.

(b) Are columns of the matrix span $\mathbb{R}^4$ $\begin{bmatrix} 7 & 2 & -5 & 8 \\ -5 & -3 & 4 & -9 \\ 6 & 10 & -2 & 7 \\ -7 & 9 & 2 & 15 \end{bmatrix}$.

Question: Suppose A is a 4×4 matrix and $\mathbf{b}$ is a vector in $\mathbb{R}^4$ with the property that $A\mathbf{x} = \mathbf{b}$ has a unique solution. Explain why the columns of A must span $\mathbb{R}^4$.

Question: Rewrite the (numerical) matrix equation below in symbolic form as a vector equation. $\begin{bmatrix} -3 & 5 & -4 & 9 & 7 \\ 5 & 8 & 1 & -2 & -4 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ -11 \end{bmatrix}$.

Chapter 5

Lecture No. 05

Question: Suppose A is a 4×4 matrix and $\mathbf{b}$ is a vector in $\mathbb{R}^4$ with the property that $A\mathbf{x} = \mathbf{b}$ has a unique solution. Explain why the columns of A must span $\mathbb{R}^4$.

Solution: We are given that the system $A\mathbf{x} = \mathbf{b}$ has unique solution for $\mathbf{b} \in \mathbb{R}^4$, which means that the augmented matrix of the system in echelon form has a pivot element in every row and the columns of A are linearly independent and span $\mathbb{R}^3$.

5.1 Homogeneous Linear System

A system of linear equations is said to be homogeneous if it can be written in the form $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$ matrix and $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$.

Trivial and Nontrivial Solution: The zero solution is usually called the trivial solution. For a given equation $A\mathbf{x} = \mathbf{0}$; the important question is whether there exists a nontrivial solution, that is, a nonzero vector $\mathbf{x}$ that satisfies $A\mathbf{x} = \mathbf{0}$.

Remark: The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

Example: Determine if the following homogeneous system has a nontrivial solution.

$$\begin{array}{rcrcrcl} 3x_1 & + & 5x_2 & - & 4x_3 & = & 0 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ 6x_1 & + & x_2 & - & 8x_3 & = & 0 \end{array}$$

Solution: The augmented matrix is

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The reduced echelon form is

$$\begin{bmatrix} 3 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{cccl} x_1 & & -\frac{4}{3}x_3 & = 0 \\ & x_2 & & = 0 \\ & & 0 & = 0 \end{array}$$

which gives x_1, x_2 are basic variables and x_3 is free variable.

Example: A **single linear equation** can be treated as a very simple system of equations. Describe all solutions of the homogeneous system

$$10x_1 - 3x_2 - 2x_3 = 0.$$

Solution: From the given equation $10x_1 - 3x_2 - 2x_3 = 0$ we have

$$x_1 = \frac{1}{10}(3x_2 + 2x_3)$$

the solution of the system can be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 3/10 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1/5 \\ 0 \\ 1 \end{bmatrix}.$$

The above form of the solution is known as parametric vector form of the solution of homogeneous system.

Parametric Vector Form:

$$\mathbf{x} = s\mathbf{u} + t\mathbf{v}, \qquad (s, t \in \mathbb{R}).$$

Example: If possible write the nontrivial solution of the following system in parametric form.

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & 3x_3 & = & 0 \\ 2x_1 & + & x_2 & - & 3x_3 & = & 0 \\ -x_1 & + & x_2 & & & = & 0 \end{array}$$

Solution: The augmented matrix of the given homogeneous system of linear equations is

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & 1 & -3 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus we have $-3x_2 + 3x_3 = 0$ and $x_1 + 2x_2 - 3x_3 = 0$, if we let $x_3 = t$ where $t \in \mathbb{R}^3$ then the parametric vector from of the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{x} = t\mathbf{v}$$

where $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Example: Solve the nonhomogeneous system of linear equations

$$\begin{array}{rcrcrcr} 3x_1 & + & 5x_2 & - & 4x_3 & = & 7 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & -1 \\ 6x_1 & + & x_2 & - & 8x_3 & = & -4 \end{array}$$

Solution: The augmented matrix is

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{rrrcr} x_1 & & -\frac{4}{3}x_3 & = & -1 \\ & x_2 & & = & 2 \\ & & 0 & = & 0 \end{array}$$

Thus $x_1 = -1 + \frac{4}{3}x_3, x_2 = 2$, and x_3 is free. In parametric form we can write the solution as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}.$$

or

$$\mathbf{x} = \mathbf{p} + x_3\mathbf{v}$$

where

$$\mathbf{p} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}.$$

Notice that $\mathbf{p}$ is a particular solution of the nonhomogeneous system and the second part is the solution of the associated homogenous system of the given system, i.e., $x_3\mathbf{v}$ is the solution of the homogenous system

$$\begin{array}{rcrcrcr} 3x_1 & + & 5x_2 & - & 4x_3 & = & 0 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ 6x_1 & + & x_2 & - & 8x_3 & = & 0 \end{array}$$

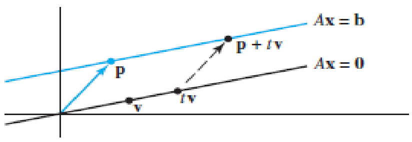


Figure 5.1: Relation between solution of nonhomogenous and homogenous systems

Example: Describe and compare the solution sets of $x_1 + 5x_2 - 3x_3 = 0$ and $x_1 + 5x_2 - 3x_3 = -2$.
Solution: The solution of the nonhomogenous equation i.e., $x_1 + 5x_2 - 3x_3 = -2$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.$$

Where as the solution of the homogenous equation $x_1 + 5x_2 - 3x_3 = 0$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.$$

Theorem: Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

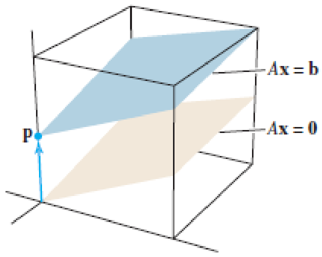


Figure 5.2: Solution of nonhomogenous system

Remark: The above result apply only to an equation $A\mathbf{x} = \mathbf{b}$ that has **at least one solution**. If the linear system is **inconsistent** the off course the solution set is empty.

Writing a solution set (of a consistent system) in parametric vector form:

1. Row reduce the augmented matrix to reduced echelon form.

2. Express each basic variable in terms of any free variables appearing in an equation.

3. Write a typical solution $\mathbf{x}$ as a vector whose entries depend on the free variables, if any.

4. Decompose $\mathbf{x}$ into a linear combination of vectors (with numeric entries) using the free variables as parameters.

Example: Describe the solutions of the following system in parametric vector form. Also, give a geometric description of the solution set and compare it to the homogeneous system of linear system

$$\begin{array}{rcrcrcr} 2x_1 & + & 2x_2 & + & 4x_3 & = & 8 \\ -4x_1 & - & 4x_2 & - & 8x_3 & = & -16 \\ & - & 3x_2 & - & 3x_3 & = & 12 \end{array}$$

Solution: The augmented matrix is

$$\begin{bmatrix} 2 & 2 & 4 & 8 \\ -4 & -4 & -8 & -16 \\ 0 & -3 & -3 & 12 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 4 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & -3 & 12 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 4 & 8 \\ 0 & -3 & -3 & 12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The solution of the system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

The associated homogeneous system has solution $x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ which is a line passing through origin and the the solution of nonhomogenous system is a line parallel to this line but passing through the point $\begin{bmatrix} 8 \\ -4 \\ 0 \end{bmatrix}$.

Example: Describe the solutions of the following system in parametric vector form. Also, give a geometric description of the solution set and compare it to the homogeneous system of linear system

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & 3x_3 & = & 5 \\ 2x_1 & + & x_2 & - & 3x_3 & = & 13 \\ -x_1 & + & x_2 & & & = & -8 \end{array}$$

Solution: The augmented matrix is

$$\begin{bmatrix} 1 & 2 & -3 & 5 \\ 2 & 1 & -3 & 13 \\ -1 & 1 & 0 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & -3 & 3 & 3 \\ 0 & 3 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & -3 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and repeat the same procedure as we did in he above example.

Example: Describe the solutions of the following system in parametric vector form (if possible). Also, give a geometric description of the solution set and compare it to the homogeneous system of linear system

$$\begin{array}{rcrcrcrcr} x_1 & & & + & x_3 & + & x_4 & = & -1 \\ 2x_1 & - & x_2 & & & + & x_4 & = & 3 \\ x_1 & + & x_2 & + & 3x_3 & + & 2x_4 & = & 1 \end{array}$$

Solution: $[A \quad \mathbf{b}] = \begin{bmatrix} 1 & 0 & 1 & 1 & -1 \\ 2 & -1 & 0 & 1 & 3 \\ 1 & 1 & 3 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 & -1 \\ 0 & -1 & -2 & -1 & 5 \\ 0 & 1 & 2 & 1 & 2 \end{bmatrix} \sim$ $\begin{bmatrix} 1 & 0 & 1 & 1 & -1 \\ 0 & -1 & -2 & -1 & 5 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}$ The third row gives us $0 = 7$ which is absurd and hence system is inconsistent.

Example: Describe the solutions of the following system in parametric vector form (if possible). Also, give a geometric description of the solution set and compare it to the homogeneous system of linear system

$$\begin{array}{rcrcrcrcr} x_1 & & & + & x_3 & + & x_4 & = & 0 \\ 2x_1 & - & x_2 & & & + & x_4 & = & 0 \\ x_1 & + & x_2 & + & 3x_3 & + & 2x_4 & = & 0 \end{array}$$

Solution: The augmented matrix is

$$[A \quad \mathbf{0}] = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 2 & -1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 \\ 0 & 1 & 2 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The solution set is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

5.2 Some Practice Problems

Question: If possible write the nontrivial solution of the following system in parametric form. $\begin{array}{rcrcrcl} 2x_1 & + & 2x_2 & + & 4x_3 & = & 0 \\ -4x_1 & - & 4x_2 & - & 8x_3 & = & 0 \\ & - & 3x_2 & - & 3x_3 & = & 0 \end{array}$

Question: Use the relationship between the solution of nonhomogeneous system of linear equation and homogeneous linear system to find the solution of the linear system $\begin{array}{rcrcrcr} 2x_1 & + & 2x_2 & + & 4x_3 & = & 8 \\ -4x_1 & - & 4x_2 & - & 8x_3 & = & -16 \\ & - & 3x_2 & - & 3x_3 & = & 12 \end{array}$.

Question: Describe all the solutions of homogeneous linear system $A\mathbf{x} = \mathbf{0}$ in parametric vector form where A is equivalent to the following matrix $\begin{bmatrix} 1 & -2 & 3 & -6 & 5 & 0 \\ 0 & 0 & 0 & 1 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Question: Does the equation $A\mathbf{x} = \mathbf{0}$ have a nontrivial solution and does the equation $A\mathbf{x} = \mathbf{b}$ have at least one solution for every possible $\mathbf{b}$. If A is a 3×3 matrix with three pivot positions.

CHAPTER 6

Lecture No. 06

Question: Describe all the solutions of homogeneous linear system $A\mathbf{x} = \mathbf{0}$ in parametric vector form where A is equivalent to the following matrix

$$\begin{bmatrix} 1 & -2 & 3 & -6 & 5 & 0 \\ 0 & 0 & 0 & 1 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution: The equations from the given matrix are

$$\begin{aligned} x_1 - 2x_2 + 3x_3 - 6x_4 + 5x_5 &= 0 \\ x_4 + 4x_5 - 6x_6 &= 0 \\ x_6 &= 0 \end{aligned}$$

and the parametric form of the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -29 \\ 0 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}.$$

Concept: Consider the vector equation $c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 7 \\ -1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.
What is the solution of the above vector equation? There are two possibilities

1. The vector equation has only trivial solution (the vectors are linearly independent)

2. The vector equation has nontrivial solution (the vectors are linearly dependent)

Solve the system and decide the vectors are linearly independent or dependent.

6.1 Linearly Independent set of vectors

An indexed set of vectors $\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_p\mathbf{v}_p = 0$$

has only the **trivial solution**.

The set $\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $\{c_1, ..., c_p\}$, **not all zero**, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_p\mathbf{v}_p = 0.$$

Example: Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

- Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is **linearly independent**.
- If possible, find a linear dependence relation among the vectors $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3\}$

Solution: The augmented matrix corresponding to the vector equation $c_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ is

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So we have the following equations

$$\begin{array}{rcrcr} x_1 & & - & 2x_2 = & 0 \\ & x_2 & + & x_3 = & 0 \\ & & & 0 = & 0 \end{array}$$

and say $x_3 = 2$, then we have the following linear combination

$$4\mathbf{v}_1 - 2\mathbf{v}_2 + 2\mathbf{v}_3 = \mathbf{0}.$$

Example: Find the value(s) of h for which the vectors are linearly dependent.

$$\begin{bmatrix} 3 \\ -6 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} 9 \\ h \\ 3 \end{bmatrix}.$$

Solution: We have to choose the the values of h such that the vector equation

$$c_1 \begin{bmatrix} 3 \\ -6 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix} + c_3 \begin{bmatrix} 9 \\ h \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has nontrivial solution. The augmented matrix of the system is

$$\begin{bmatrix} 3 & -6 & 9 & 0 \\ -6 & 4 & h & 0 \\ 1 & -3 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 & 0 \\ -6 & 4 & h & 0 \\ 3 & -6 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 & 0 \\ 0 & -14 & h+18 & 0 \\ 0 & 3 & 0 & 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & -3 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -14 & h+18 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & h+32 & 0 \end{bmatrix}.$$

In order to system has nontrivial solution $h+32=0$, which gives the required value of the h.

Linear Independence of Matrix Columns: Suppose we have a matrix $A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad ... \quad \mathbf{a}_n]$. We can consider the columns of this matrix as a set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n\}$ and then we can talk about the **linearly independence** or **linearly dependence** of the vectors of this set, i.e.,

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + ... + c_n\mathbf{a}_n = \mathbf{0}.$$

Each **linear dependence** relation among the columns of A corresponds to a non-trivial solution of $A\mathbf{x} = \mathbf{0}$. Thus we have the following important fact.

The columns of a matrix A are **linearly independent** if and only if the **equation** $A\mathrm{x} = 0$ **has only the trivial solution**.

Example: Check whether the columns of the matrix $A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$ are linearly independent.

Solution: We should check whether the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only trivial solution? For this we take the augmented matrix

$$\begin{bmatrix} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & -2 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 13 & 0 \end{bmatrix}$$

It is clear that the homogeneous system has only trivial solution, hence the columns of the matrix are linearly independent.

Example: Given $\begin{bmatrix} 4 & 3 & -5 \\ -2 & -2 & 4 \\ -2 & -3 & 7 \end{bmatrix}$, observe that the first column minus three times the second column equals the third column. Find a nontrivial solution of $A\mathbf{x} = \mathbf{0}$.

Solution: The one nontrivial solution of the system is

$$\begin{bmatrix} 4 \\ -2 \\ -2 \end{bmatrix} - 3\begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix} - 1\begin{bmatrix} -5 \\ 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Hence $c_1 = 1, c_2 = -3$ and $c_3 = -1$.
Sets of one or two vectors: The set of one non zero vector is always linearly independent.

Example: Determine if the following sets of vectors are linearly independent

1. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -9 \\ -3 \end{bmatrix}$ Linearly dependent.

2. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -9 \\ 2 \end{bmatrix}$ Linearly independent.

Remark: A set of two vectors $\{\mathbf{v}_1; \mathbf{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other. The set is linearly independent if and only if neither of the vectors is a multiple of the other.
Theorem: **Characterization of Linearly Dependent Sets**

An indexed set $S = \{\mathbf{v}_1, ..., \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others.

In fact, if S is linearly dependent and $\mathbf{v}_1 \neq 0$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, ..., \mathbf{v}_{j-1}$.

Remark: The above result does not say that every vector in a linearly dependent set is a linear combination of the preceding vectors.

A vector in a linearly dependent set may fail to be a linear combination of the other vectors.

Example: Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -9 \\ 2 \\ 0 \end{bmatrix}$. Describe the set spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$, and explain why a vector $\mathbf{w}$ is in $Span\{\mathbf{v}_1, \mathbf{v}_2\}$ if and only if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}\}$ is linearly dependent. See the figure below.

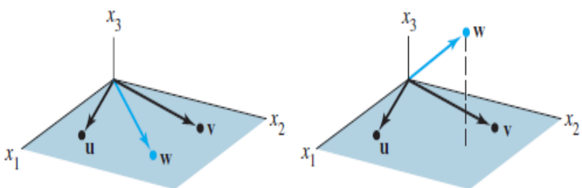


Figure 6.1: Span of two vectors

Theorem: If a set contains more vectors than there are entries in each vector, then the set is **linearly dependent**. That is, any set $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

Remark: The above result says nothing about the case in which the number of vectors in the set does not exceed the number of entries in each vector.

Theorem: If a set $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.
Example: Determine whether the given vectors are linearly dependent or linearly independent.

1. $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix}$ Linearly dependent.

2. $\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -9 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Linearly dependent.

3. $\begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -9 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ Linearly dependent.

4. $\begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -9 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ We can't say anything we have to check the vector equation.

6.2 Some Practice Problems

Question: Find a non trivial solution $A\mathbf{x} = \mathbf{0}$ where A is equivalent to the following matrix $\begin{bmatrix} 2 & 3 & 5 \\ -5 & 1 & -4 \\ -3 & -1 & -4 \\ 1 & 0 & 1 \end{bmatrix}$. Notice that the third column is the sum of the first two columns. (You don't need any computations for the answer of this question)
Question: Without any computations decide whether the given vectors are linearly dependent or linearly independent.

1. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$.

2. $\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -9 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

3. $\begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -9 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$

Chapter 7

Lecture No. 07

Concept: Consider the following matrix equations $\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$ and $\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

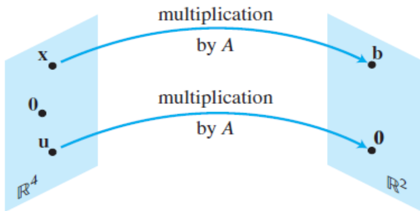


Figure 7.1: Matrix Transformation

7.1 Transformation

A transformation (or function or mapping) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$. The set $\mathbb{R}^n$ is called the **domain** of T, and $\mathbb{R}^m$ is called the **codomain** of T. The notation $T : \mathbb{R}^n \to \mathbb{R}^m$ indicates that the domain of T is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$. For $\mathbf{x}$ in $\mathbb{R}^n$, the vector $T(\mathbf{x})$ in $\mathbb{R}^m$ is called the image of $\mathbf{x}$ (under the action of T). The set of all images $T(\mathbf{x})$ is called the range of T.

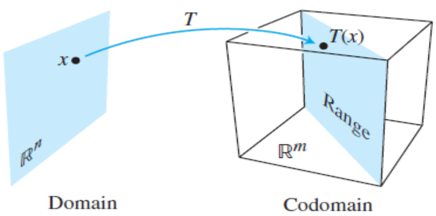


Figure 7.2: Domain and codomain

Matrix Transformation: For each $\mathbf{x}$ in $\mathbb{R}^n$, $T(\mathbf{x})$ is computed as $A\mathbf{x}$, where A is an $m \times n$ matrix.

For simplicity, we sometimes denote such a matrix transformation by $\mathbf{x} \mapsto A\mathbf{x}$. Observe that the domain of T is $\mathbb{R}^n$ when A has n columns and the codomain of T is $\mathbb{R}^m$ when each column of A has m entries. The range of T is the set of all linear combinations of the columns of A, because each image $T(\mathbf{x})$ is of the form $A\mathbf{x}$.

Example: $A = \begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}$ then find the transformation defined by the given matrix.
Solution: The order of the matric is 2×4, thus matrix gives a transformation from $\mathbb{R}^4$ to $\mathbb{R}^2$.
Example: Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ and define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$, so that

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$$

1. Find $T(\mathbf{u})$, the image of $\mathbf{u}$ under the transformation.
2. Find an $\mathbf{x}$ in $\mathbb{R}^2$ whose image under T is $\mathbf{b}$.
3. Is there more than one $\mathbf{x}$ whose image under T is $\mathbf{b}$?
4. Determine if $\mathbf{c}$ is in the range of the transformation T.

Solution: 1. $T(\mathbf{u}) = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$.

2. Solve $T(\mathbf{x}) = \mathbf{b}$ for $\mathbf{x}$, i.e., solve $A\mathbf{x} = \mathbf{b}$, the augmented matrix of the system is

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{bmatrix}$$

Which shows that the system has a unique solution and there is one unique vector in $\mathbb{R}^2$, i.e., $x_1 = 1.5$ and $x_2 = -0.5$ such that $\mathbf{b}$ is the under the given transformation.

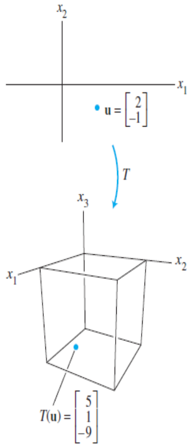


Figure 7.3: Image of $\mathbf{b}$ under matrix transformation

3. No there is one and only one $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{b}$ holds.

4. Solve $T(\mathbf{x}) = \mathbf{c}$ for $\mathbf{x}$, i.e., solve $A\mathbf{x} = \mathbf{c}$, the augmented matrix of the system is $\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 14 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -35 \end{bmatrix}$ which shows that the system is inconsistent, hence there doesn't exist any $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{c}$ holds.

Example: Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -3 \\ 2 & -5 & 6 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -6 \\ -4 \\ -5 \end{bmatrix}$. Find a vector $\mathbf{x}$ such that $T(\mathbf{x}) = A\mathbf{x} = \mathbf{b}$ satisfy. Is such an $\mathbf{x}$ is unique?
Solution: The augmented matrix is

$$\begin{bmatrix} 1 & -2 & 3 & -6 \\ 0 & 1 & -3 & -4 \\ 2 & -5 & 6 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & -6 \\ 0 & 1 & -3 & -4 \\ 0 & -9 & 12 & -17 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & -6 \\ 0 & 1 & -3 & -4 \\ 0 & 0 & -15 & -53 \end{bmatrix}.$$

The above augmented matrix in echelon form shows that the system has a unique solution.

Example: If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ projects points in $\mathbb{R}^3$ onto x_1x_2-plane because

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}.$$

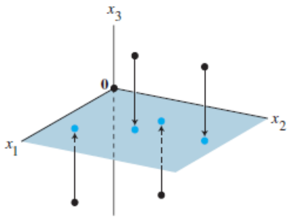


Figure 7.4: Projection transformation

Example: The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$.

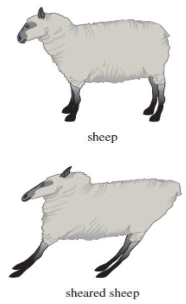


Figure 7.5: Shear transformation

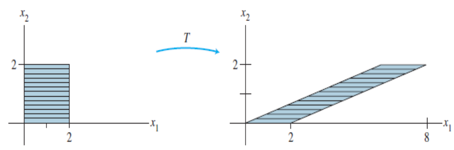


Figure 7.6: Another example of shear transformation

Shear transformations appear in physics, geology, and crystallography.

7.2 Linear Transformation

A transformation (or mapping) T is linear if

1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}$, $\mathbf{v}$ in the domain of T;
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all $\mathbf{u}$ in the domain of T.

Remark: We know that a given matrix A of order $m \times n$ satisfies the following properties

1. $A(\mathbf{u}+\mathbf{v}) = A(\mathbf{u}) + A(\mathbf{v})$ and
2. $T(c\mathbf{u}) = cT(\mathbf{u})$.

Every matrix transformation is a linear transformation.
Remark: If T is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0}, \quad \text{and,} \quad T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

for all vectors $\mathbf{u}, \mathbf{v}$ in the domain of T and scalars c, d.

Example: Let T be a linear transformation such that $T(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix}$ and $T(\begin{bmatrix} 0 \\ 2 \end{bmatrix}) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. Find the image of $\begin{bmatrix} 5 \\ 9 \end{bmatrix}$ under the linear transformation T.

Example: Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{x}$ into $x_1\mathbf{u} + x_2\mathbf{v}$ where $\mathbf{u} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$. Find a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for each $\mathbf{x}$.

Contraction and Dilation: Given a scalar r, the mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = r\mathbf{x}$. The mapping T is called a contraction when $0 \leq r \leq 1$ and a dilation if $r > 1$.

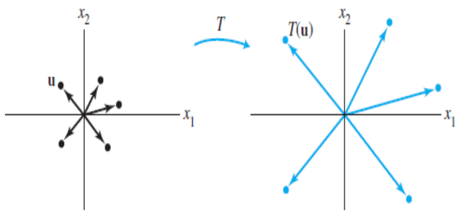


Figure 7.7: Contraction and dilation

Example: Define a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(\mathbf{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}.$$

Find the images under T of $\mathbf{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$.

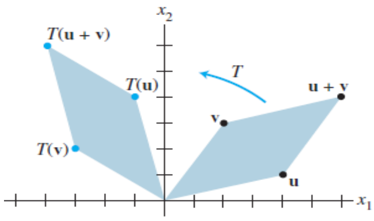


Figure 7.8: Image under linear transformation

7.3 Some Practice Problems

Question: Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -3 \\ 2 & -5 & 6 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -6 \\ -4 \\ -5 \end{bmatrix}$. Find a vector $\mathbf{x}$ such that $T(\mathbf{x}) = A\mathbf{x}$ satisfy. Is such an $\mathbf{x}$ is unique?

Question: Define $T : \mathbb{R} \to \mathbb{R}$ by $T(x) = mx + b$.

1. Show that T is a linear transformation when $b = 0$.
2. Find a property of a linear transformation that is violated when $b \neq 0$.

Question: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$(i) \qquad T(\mathbf{x}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad (ii) \qquad T(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Describe geometrically what T does to each vector in $\mathbb{R}^2$.

CHAPTER 8

Lecture No. 08

Question: Define $T : \mathbb{R} \to \mathbb{R}$ by $T(x) = mx + b$.

1. Show that T is a linear transformation when $b = 0$.
2. Find a property of a linear transformation that is violated when $b \neq 0$.

Solution: 1. Check the two conditions of linear transformation.
2. The first property of the linear transformation is not satisfied. Verify

Identity Matrix: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times4}$.

Example: The columns of $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ such that

$$T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} \qquad \text{and} \qquad T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}.$$

With no additional information, find a formula for the image of an arbitrary $\mathbf{x}$ in $\mathbb{R}^2$.

Solution: Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ then we can write

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Since T is a linear transformation then we have

$$T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = x_T(1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}) + x_2 T(\begin{bmatrix} 0 \\ 1 \end{bmatrix})$$

and thus

$$T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = x_1 \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ -7 & 8 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Consequently, we can say that $A = [T(\mathbf{e}_1) \;\; T(\mathbf{e}_2)]$.

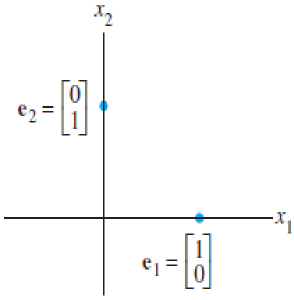


Figure 8.1: The I_2 columns

Theorem: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x}, \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n.$$

In fact, A is the $m \times n$ matrix whose jth column is he vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the jth column of the identity matrix in $\mathbb{R}^n$:

$$A = [T(\mathbf{e}_1)\ \ T(\mathbf{e}_2)\ \ \ ...\ \ \ T(\mathbf{e}_n)].$$

Remark: The matrix A such that $T(\mathbf{x}) = A\mathbf{x}$, is called a matrix of transformation for T.

Example: Assume that T is a linear transformation. Find the standard matrix of T

1. $T : \mathbb{R}^2 \to \mathbb{R}^4$, $T(\mathbf{e}_1) = (3, 1, 3, 1)$, and $T(\mathbf{e}_2) = (-5, 2, 0, 0)$, where $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$.

2. $T : \mathbb{R}^3 \to \mathbb{R}^2$, $T(\mathbf{e}_1) = (1, 4)$, and $T(\mathbf{e}_2) = (-2, 9)$, and $T(\mathbf{e}_3) = (3, -8)$ where $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are the columns of the 3×3 identity matrix.

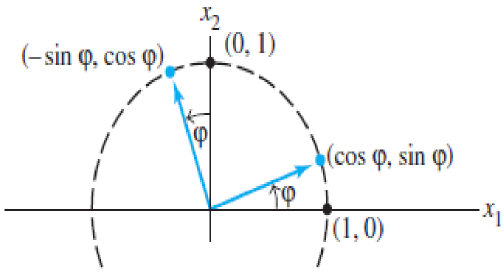
Solution: 1. The standard matrix of linear transformation is

$$A = [T(\mathbf{e}_1)\ \ T(\mathbf{e}_2)] = \begin{bmatrix} 3 & -5 \\ 1 & 2 \\ 3 & 0 \\ 1 & 0 \end{bmatrix}$$

2. The standard matrix of linear transformation is

$$A = [T(\mathbf{e}_1)\ \ T(\mathbf{e}_2)\ \ T(\mathbf{e}_3)] = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 9 & -8 \end{bmatrix}.$$

Example: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin through an angle φ, with counterclockwise rotation for a positive angle. We could show geometrically that such a transformation is linear. Find the standard matrix A of this transformation.



8.1 Finding the matrix of transformation of a linear transformation defined geometrically

:

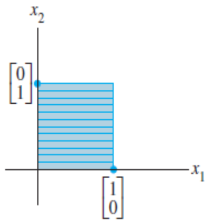


Figure 8.2: The I_2 columns and rectangle

Reflection about a line: You have to understand reflection about a line see the lecture in which I have explained this concept.
Reflection through the x_1-axis:

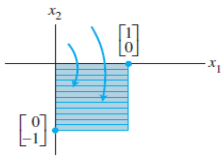


Figure 8.3: Reflection about x_1-axis

The matrix of Transformation: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection through the x_2-axis:

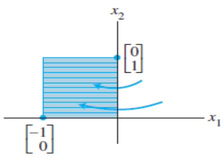


Figure 8.4: Reflection about x_2-axis

The matrix of Transformation: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Reflection through the the line $x_2 = x_1$:

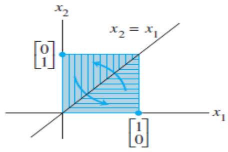


Figure 8.5: Reflection about the line $x_2 = x_1$

The matrix of Transformation: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Reflection through the line $x_2 = -x_1$:

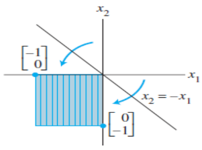
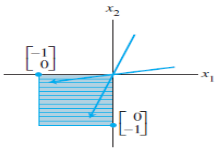


Figure 8.6: Reflection about the line $x_2 = -x_1$

The matrix of Transformation: $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

Reflection through the origin:



The matrix of Transformation: $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Horizontal / Vertical contraction and expansion:

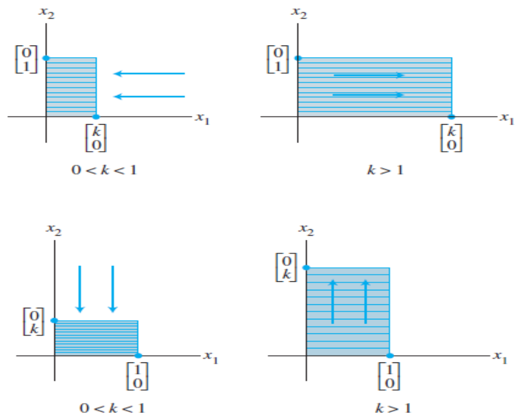


Figure 8.7: Horizontal and vertical contraction and expansion

The matrix of Transformations: $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

Horizontal / Vertical Shears:

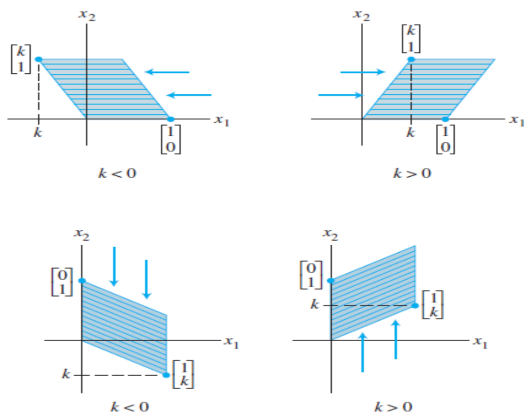


Figure 8.8: Horizontal and vertical shears

The matrix of Transformations: $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

Projections:

Projection onto x_1-axis:

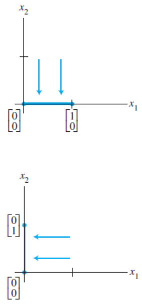


Figure 8.9: Projection on x_1-axis and x_2-axis

Projection onto x_2-axis:

The matrix of Transformations: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Example: Assume that T is a linear transformation. Find the standard matrix of T when

1. T is a vertical shear transformation that maps $\mathbf{e}_1$ into $\mathbf{e}_1 - 3\mathbf{e}_2$ and leaves $\mathbf{e}_2$ unchanged.

2. T is a horizontal shear transformation that maps $\mathbf{e}_2$ into $\mathbf{e}_2 + 2\mathbf{e}_1$ and leaves $\mathbf{e}_1$ unchanged.

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at least one $\mathbf{x}$ in $\mathbb{R}^n$.

Remark: T is onto if codomain of $T = \mathbb{R}^m$.

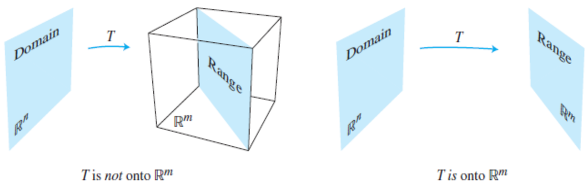


Figure 8.10: Onto mapping

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one to one** if each $\mathbf{b}$ is the image of at most one $\mathbf{x}$ in $\mathbb{R}^n$.

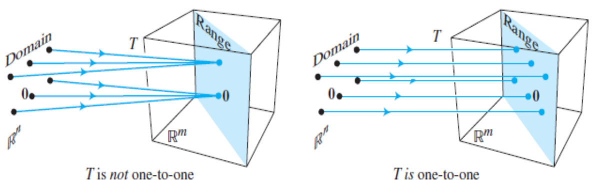


Figure 8.11: One to one transformation

Example: Let T be the linear transformation whose standard matrix is

$$A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T a one to one mapping?
Solution: Let $\mathbf{b} \in \mathbb{R}^3$ be arbitrary element then we have to check the system of linear equations $A\mathbf{x} = \mathbf{b}$. For given A the augmented matrix is

$$\begin{bmatrix} 1 & -4 & 8 & 1 & b_1 \\ 0 & 2 & -1 & 3 & b_2 \\ 0 & 0 & 0 & 5 & b_3 \end{bmatrix}.$$

The augmented matrix is in echelon form and for every $\mathbf{b}$ the system $A\mathbf{x} = \mathbf{b}$ has solution (indeed not unique, verify).

T is not one to one map. Can you justify.
Theorem: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

Theorem: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T, then

1. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$;

2. T is one to one if and only if the columns of A are linearly independent.

Example: Let $T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$. Show that T is a one-to-one linear transformation. Does T map $\mathbb{R}^2$ onto $\mathbb{R}^3$?
Solution: The matrix of transformation is

$$[T(\mathbf{e}_1 \;\; T(\mathbf{e}_2))] = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix}$$

clearly the columns of the matrix are linearly independent and hence the transformation is one to one.

The transformation T is not onto. Can you justify?

8.2 Some Practice Problems

Question: Assume that T is a linear transformation. Find the standard matrix of T

1. $T : \mathbb{R}^2 \to \mathbb{R}^3$, $T(\mathbf{e}_1) = (3, 0, 3)$, and $T(\mathbf{e}_2) = (2, 1, 5)$, where $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$.

2. $T : \mathbb{R}^3 \to \mathbb{R}^3$, $T(\mathbf{e}_1) = (1, 4, 0)$, and $T(\mathbf{e}_2) = (-2, 0, 2)$, and $T(\mathbf{e}_3) = (0, 3, 5)$ where $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are the columns of the 3×3 identity matrix.

3. $T : \mathbb{R}^2 \to \mathbb{R}^2$ first performs a horizontal shear that transforms $\mathbf{e}_2$ into $\mathbf{e}_2 + 2\mathbf{e}_1$ and leaves $\mathbf{e}_1$ unchanged and then reflects points through the line $x_2 = -x_1$

4. $T : \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points through the horizontal x_1-axis and then reflects points through the line $x_2 = x_1$.

Question: Show that T is a linear transformation by finding a matrix that implements the mapping

1. $T(x_1, x_2) = (x_1 + 4x_2, 0, x_1 - 3x_2, x_1)$.

2. $T(x_1, x_2, x_3, x_4) = 3x_1 + 4x_3 - 2x_4$.

CHAPTER 9

Lecture No. 09

9.1 A Homogeneous System in Economics

Suppose a nation's economy is divided into many sectors, such as various manufacturing, communication, entertainment, and service industries.

Suppose that for each sector we know its total output for one year and we know exactly how this output is divided or "exchanged" among the other sectors of the economy. Let the total dollar value of a sector's output be called the price of that output.

Leontief proved the following result:
There exist equilibrium prices that can be assigned to the total outputs of the various sectors in such a way that the income of each sector exactly balances its expenses.
Example: Suppose an economy consists of the Coal, Electric (power), and Steel sectors, and the output of each sector is distributed among the various sectors as shown in Table, where the entries in a column represent the fractional parts of a sector's total output.

TABLE 1 A Simple Economy

Distribution of Output from:			
Coal	Electric	Steel	Purchased by:
.0	.4	.6	Coal
.6	.1	.2	Electric
.4	.5	.2	Steel

The second column of Table 1, for instance, says that the total output of the Electric sector is divided as follows: 40% to Coal, 50% to Steel, and the remaining 10% to Electric. (Electric treats this 10% as an expense it incurs in order to operate its business.) Since all output must be taken into account, the decimal fractions in each column must sum to 1.

Denote the prices (i.e., dollar values) of the total annual outputs of the Coal, Electric, and Steel sectors by p_C, p_E, and p_S, respectively. If possible, find equilibrium prices that make each sector's income match its expenditures.

TABLE 1 A Simple Economy

Distribution of Output from:			
Coal	Electric	Steel	Purchased by:
.0	.4	.6	Coal
.6	.1	.2	Electric
.4	.5	.2	Steel

We have the following system of linear equations

$$\begin{array}{rcrcrcl} & & 0.6p_S & + & 0.4p_E & = & p_C \\ 0.6p_C & + & 0.1p_E & + & 0.2p_S & = & p_E \\ 0.4p_C & + & 0.5p_E & + & 0.2p_S & = & p_S \end{array}$$

the above system can be reduced to

$$\begin{array}{rcrcrcl} p_C & - & 0.4p_E & - & 0.6p_S & = & 0 \\ -0.6p_C & + & 0.9p_E & - & 0.2p_S & = & 0 \\ -0.4p_C & - & 0.5p_E & + & 0.8p_S & = & 0 \end{array}$$

$$\begin{array}{rcrcrcl} p_C & - & 0.4p_E & - & 0.6p_S & = & 0 \\ -0.6p_C & + & 0.9p_E & - & 0.2p_S & = & 0 \\ -0.4p_C & - & 0.5p_E & + & 0.8p_S & = & 0 \end{array}$$

The augmented matrix is

$$\begin{bmatrix} 1 & -.4 & -.6 & 0 \\ -.6 & .9 & -.2 & 0 \\ -.4 & -.5 & .8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -.4 & -.6 & 0 \\ 0 & .66 & -.56 & 0 \\ 0 & -.66 & .56 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -.4 & -.6 & 0 \\ 0 & .66 & -.56 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -.4 & -.6 & 0 \\ 0 & 1 & -.85 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -.94 & 0 \\ 0 & 1 & -.86 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The equilibrium price vector is

$$\mathbf{p} = \begin{bmatrix} p_C \\ p_E \\ p_S \end{bmatrix} = \begin{bmatrix} .94p_S \\ .85p_S \\ p_S \end{bmatrix} = p_S \begin{bmatrix} .94 \\ .85 \\ 1 \end{bmatrix}.$$

9.2 Balancing Chemical Equations

Chemical equations describe the quantities of substances consumed and produced by chemical reactions. For instance, when propane gas burns, the propane (C_3H_8) combines with oxygen (O_2) to form carbon dioxide (CO_2) and water (H_2O), according to an equation of the form

$$(x_1)C_3H_8 + (x_2)O_2 \to (x_3)CO_2 + (x_4)H_2O.$$

To balance the above equation, we need to find the coefficients x_1, x_2 and x_3 such that

$$x_1 \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$

To solve we rewrite the system

$$x_1 \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

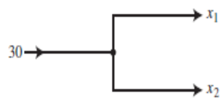
9.3 Balancing Chemical Equations

To solve we rewrite the system

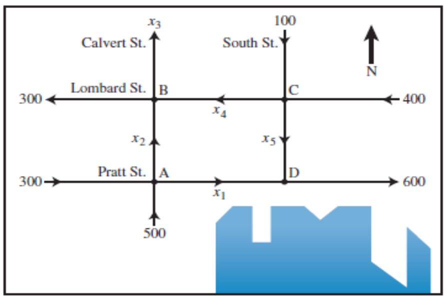
$$x_1 \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Network Flow: A network consists of a set of points called junctions, or nodes, with lines or arcs called branches connecting some or all of the junctions. The direction of flow in each branch is indicated, and the flow amount (or rate) is either shown or is denoted by a variable.

Assumption: The basic assumption of network flow is that the total flow into the network equals the total flow out of the network and that the total flow into a junction equals the total flow out of the junction.



Network Flow:The network shows the traffic flow (in vehicles per hour) over several one-way streets in downtown Baltimore during a typical early afternoon. Determine the general flow pattern for the network.



Intersection	Flow in		Flow out
A	$300+500$	=	x_1+x_2
B	x_2+x_4	=	$300+x_3$
C	$100+400$	=	x_4+x_5
D	x_1+x_5	=	600

Network Flow:

Intersection	Flow in		Flow out
A	$300+500$	=	x_1+x_2
B	x_2+x_4	=	$300+x_3$
C	$100+400$	=	x_4+x_5
D	x_1+x_5	=	600

$$\begin{array}{rcrcrcrcrl} x_1 & + & x_2 & & & & & & & = 800 \\ & & x_2 & - & x_3 & + & x_4 & & & = 300 \\ & & & & & & x_4 & + & x_5 & = 500 \\ x_1 & & & & & & & + & x_5 & = 600 \\ & & & & x_3 & & & & & = 400 \end{array}$$

Row reduction of the associated augmented matrix leads to

$$\begin{array}{rcrcrcrcrl} x_1 & & & & & & & + & x_5 & = 600 \\ & & x_2 & & & & & - & x_5 & = 200 \\ & & & & x_3 & & & & & = 400 \\ & & & & & & x_4 & + & x_5 & = 500 \end{array}$$

Network Flow: The general flow pattern for the network is described by

$$x_1 = 600 - x_5, \quad x_2 = 200 + x_5, \quad x_3 = 400, \quad x_4 = 500 - x_5, \quad x_5 \text{ is free}$$

Remark:

A negative flow in a network branch corresponds to flow in the direction opposite to that shown on the model. Since the streets in this problem are one way, none of the variables here can be negative. This fact leads to certain limitations on the possible values of the variables. For instance, $x_5 \leq 500$ because x_4 cannot be negative.

9.4 Some Practice Problems

Question: Consider an economy with three sectors: Fuels and Power, Manufacturing, and Services. Fuels and Power sells 80% of its output to Manufacturing, 10% to Services, and retains the rest. Manufacturing sells 10% of its output to Fuels and Power, 80% to Services, and retains the rest. Services sells 20% to Fuels and Power, 40% to Manufacturing, and retains the rest.

(i) Construct the exchange table for this economy.

(ii) Develop a system of equations that leads to prices at which each sector's income matches its expenses. Then write the augmented matrix that can be row reduced to find these prices.

Question: Balance the chemical equations. (a) Aluminum oxide and carbon react to create elemental aluminum and carbon dioxide: $Al_2O_3 + C \to Al + CO_2$.

(b) Alka-Seltzer contains sodium bicarbonate ($NaHCO_3$) and citric acid ($H_3C_6H_5O_7$). When a tablet is dissolved in water, the following reaction produces sodium citrate, water, and carbon dioxide (gas):

$$NaHCO_3 + H_3C_6H_5O_7 \to Na_3C_6H_5O_7 + H_2O + CO_2.$$

Balance the chemical equation.

Chapter 10

Lecture No. 10 (Revision)

In lecture number 10, I revised the important concepts in this lecture. I advise you to do the practice problems given in the handouts and from the exercises of your recommended text books.

CHAPTER 11

Lecture No. 11

11.1 Matrix Algebra

A matrix is an arrangement of numbers, for example a matrix A of order $m \times n$ often written as

$$A = [\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; ... \;\; \mathbf{a}_n]$$

where each column $\mathbf{a}_i \in \mathbb{R}^m$.

The diagonal entries in an $m \times n$ matrix $A = a_{ij}$ are $a_{11}, a_{22}, a_{33}, ...$ and they form the main diagonal of A.

A diagonal matrix is a square $n \times n$ matrix whose nondiagonal entries are zero.

An example is the $n \times n$ identity matrix, I_n.

An $m \times n$ matrix whose entries are all zero is a zero matrix and is written as 0. The size of a zero matrix is usually clear from the context.

Sums and Scalar Multiples: The two matrices A and B could be added, i.e., $A + B$ is possible if and only if the have the same size (order).

Example: If $A = \begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ then

$$A + B = \begin{bmatrix} 2 & 1 & -4 & 8 \\ 2 & -2 & 2 & 1 \\ 5 & -8 & 8 & 1 \end{bmatrix}.$$

Can we add the matrices $C = \begin{bmatrix} 0 & 1 & -4 \\ 2 & -3 & 2 \\ 5 & -8 & 7 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 1 \\ 2 & -3 \\ 5 & -8 \end{bmatrix}$?

No both matrices C and D have different order.

Example: For A and B as in the above example then $3A = \begin{bmatrix} 0 & 3 & -12 & 24 \\ 6 & -9 & 6 & 3 \\ 15 & -24 & 21 & 3 \end{bmatrix}$

Theorem: Let A, B, and C be matrices of the same size, and let r and s be scalars.

1. $A + B = B + A$.

2. $(A + B) + C = A + (B + C)$

3. $A + 0 = 0 + A = A$

4. $r(A+B) = rA + rB$

5. $(r+s)A = rA + sA$

6. $r(sA) = (rs)A$

Matrix Multiplication: If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $\mathbf{b}_1, ..., \mathbf{b}_p$, then the product AB is the $m \times p$ matrix whose columns are $A\mathbf{b}_1, ..., A\mathbf{b}_p$. That is,

$$AB = A[\mathbf{b}_1 \ \mathbf{b}_2 \ ... \ \mathbf{b}_p] = [A\mathbf{b}_1 \ A\mathbf{b}_2 \ ... \ A\mathbf{b}_p].$$

Example: If $A = \begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ then

$$A\mathbf{b}_1 = \begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 10 \end{bmatrix}, \quad A\mathbf{b}_2 = \begin{bmatrix} 1 \\ -3 \\ -8 \end{bmatrix}.$$

$$A\mathbf{b}_3 = \begin{bmatrix} -4 \\ 2 \\ 7 \end{bmatrix}, \quad A\mathbf{b}_4 = \begin{bmatrix} 24 \\ 3 \\ 3 \end{bmatrix}, \qquad AB = \begin{bmatrix} 0 & 1 & -4 & 24 \\ 4 & -3 & 2 & 3 \\ 10 & -8 & 7 & 3 \end{bmatrix}$$

Remark: Each column of AB is a linear combination of the columns of A using weights from the corresponding column of B.

Example: If A is a 3×5 matrix and B is a 5×2 matrix, what are the sizes of AB and BA, if they are defined?

Row-Column Rule for Computing AB: If the product AB is defined, then the entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B. If $(AB)_{ij}$ denotes the (i,j)-entry in AB, and if A is an $m \times n$ matrix, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{in}b_{nj}.$$

Example: Let $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$. Use the row-column rule to compute $(AB)_{13}$ and $(AB)_{22}$ entries in AB.

Remark:

$$row_i(AB) = row_i(A).B$$

Example: Let $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$. Find all the rows of AB by the above formula.

Properties of Matrix Multiplication:

Theorem: Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

1. $A(BC) = (AB)C$ (associative law of multiplication)
2. $A(B + C) = AB + AC$ (Left distributive law)
3. $(B + C)A = BA + CA$ (Right distributive law)
4. $r(AB) = (rA)B = A(rB)$ for any scalar r
5. $I_m A = A = AI_n$ (identity for matrix multiplication)

Example: Let $A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$. Check whether the commutative law holds or not.

Remark:

1. In general, $AB \neq BA$.
2. The cancelation laws do not hold for matrix multiplication. That is, if $AB = AC$, then it is not true in general that $B = C$.
3. If a product AB is the zero matrix, you cannot conclude in general that either $A = 0$ or $B = 0$.

Powers of a Matrix:

If A is an $n \times n$ matrix and if k is a positive integer, then A^k denotes the product of k copies of A

$$A^k = A...A \quad \text{(k times)}.$$

Transpose of a matrix: Given an $m \times n$ matrix A, the transpose of A is the $n \times m$ matrix, denoted by A^T , whose columns are formed from the corresponding rows of A.

Example: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} 5 & 2 \\ 3 & -2 \\ 0 & 5 \end{bmatrix}$, and $C = \begin{bmatrix} 5 & 1 & 1 & 1 \\ 3 & -2 & 2 & 3 \end{bmatrix}$ then

$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $B^T = \begin{bmatrix} 5 & 3 & 0 \\ 2 & -2 & 5 \end{bmatrix}$, and $C^T = \begin{bmatrix} 5 & 3 \\ 1 & -2 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$

Theorem: Let A and B denote matrices whose sizes are appropriate for the following sums and products.

1. $(A^T)^T = A$

2. $(A+B)^T = A^T + B^T$

3. For any scalar r, $(rA)^T = rA^T$

$(AB)^T = B^T A^T$

Remark: The transpose of a product of matrices equals the product of their transposes in the reverse order.

Example: Let $A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 2 \\ 3 & -2 \end{bmatrix}$, then verify $(AB)^T = B^T A^T$ and $(A+B)^T = A^T + B^T$.

Invertible Matrix: An $n \times n$ matrix A is said to be invertible if there is an $n \times n$ matrix C such that

$$CA = I \qquad \text{and} \qquad AC = I$$

where $I = I_n$, the $n \times n$ identity matrix. Usually we denote $C = A^{-1}$.

Remark: The inverse of a matrix is unique (If it exists).

A matrix that is not invertible is sometimes called a **singular matrix**, and an invertible matrix is called a **nonsingular matrix**.

Example: If $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ and $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$ then $AC = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $CA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Thus $C = A^{-1}$.

Recall that :

Theorem: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is not invertible.

Example: Find the inverse of $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$.

Theorem: If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Example: Use inverse of the matrix of coefficients of the system

$$\begin{array}{rcrl} 3x_1 & - & 4x_2 = & 3 \\ 5x_1 & + & 6x_2 = & 7 \end{array}.$$

Theorem:

- If A is an invertible matrix, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A.$$

- If A and B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is the product of the inverses of A and B in the reverse order. That is,

$$(AB)^{-1} = B^{-1}A^{-1}.$$

- If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. That is,

$$(A^T)^{-1} = (A^{-1})^T.$$

Elementary Matrices: An **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix.

Example: Let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 5 \end{bmatrix}$, $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

Compute E_1A, E_2A, and E_3A, and describe how these products can be obtained by elementary row operations on A.

$E_1A = \begin{bmatrix} a & b & c \\ d & e & f \\ g+4a & h+4b & i+4c \end{bmatrix}$, $E_2A = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}$ and $E_3A = \begin{bmatrix} d & e & f \\ a & b & c \\ 5g & 5h & 5i \end{bmatrix}$.

Remark: If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA, where the $m \times m$ matrix E is created by performing the same row operation on I_m.

Remark: Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I.

Example: Find the inverse of $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$

To transform E_1 into I , add +4 times row 1 to row 3. The elementary matrix that does this is

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

Theorem: An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

Example: Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$, if it exists.

Solution:

$$\begin{aligned}
[A \quad I] &= \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}.
\end{aligned}$$

Consequently, the inverse of the matrix is

$$A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}.$$

11.2 Some Practice Problems

Question: Let $A = \begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $\mathbf{b}_4 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

1. Find A^{-1}, and use it to solve the four equations

$$A\mathbf{x} = \mathbf{b}_1, \quad A\mathbf{x} = \mathbf{b}_2, \quad A\mathbf{x} = \mathbf{b}_3, \quad A\mathbf{x} = \mathbf{b}_4.$$

2. The four equations in part (1) can be solved by the same set of row operations, since the coefficient matrix is the same in each case. Solve the four equations in part (1) by row reducing the augmented matrix $[A \quad \mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \quad \mathbf{b}_4]$.

Question: Find the inverse the matrices $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$, if it exists.

CHAPTER 12

Lecture No. 12

Question: Let $A = \begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $\mathbf{b}_4 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

1. Find A^{-1}, and use it to solve the four equations

$$A\mathbf{x} = \mathbf{b}_1, \quad A\mathbf{x} = \mathbf{b}_2, \quad A\mathbf{x} = \mathbf{b}_3, \quad A\mathbf{x} = \mathbf{b}_4.$$

2. The four equations in part (1) can be solved by the same set of row operations, since the coefficient matrix is the same in each case. Solve the four equations in part (1) by row reducing the augmented matrix $[A \quad \mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \quad \mathbf{b}_4]$.

Solution:

12.1 Characterization of Invertible Matrices

Theorem: (The Invertible Matrix theorem) Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

1. A is an invertible matrix.
2. A is row equivalent to the $n \times n$ identity matrix.
3. A has n pivot positions.
4. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
5. The columns of A form a linearly independent set.
6. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
7. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in Rn.
8. The columns of A span $\mathbb{R}^n$.
9. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
10. There is an $n \times n$ matrix C such that $CA = I$.

11. There is an $n \times n$ matrix D such that $AD = I$.

12. A^T is an invertible matrix.

Example: Use the Invertible Matrix Theorem to decide if A is invertible:

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}.$$

Solution:

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}.$$

So A has three pivot positions and thus A is invertible by the Invertible Matrix Theorem.

Recall: One of the equivalent statement in the Invertible Matrix Theorem is that A has n pivot positions.
Invertible Linear Transformation: A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be invertible if there exists a function $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$S(T(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n$$

$$T(S(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n$$

We call S the inverse of T and write it as T^{-1}.

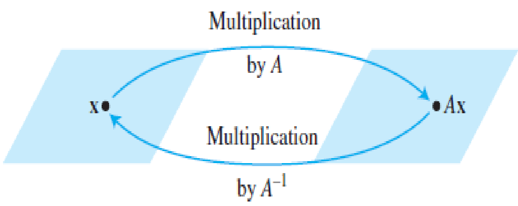


Figure 12.1: Invertible transformation

Theorem Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S : \mathbb{R}^n \to \mathbb{R}^n$ is the unique function satisfying the equations

$$S(T(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n$$

$$T(S(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n.$$

Example: What can you say about a one-to-one linear transformation T from $\mathbb{R}^n$ into $\mathbb{R}^n$?

The columns of the standard matrix A of T are linearly independent, because the system $A\mathbf{x} = \mathbf{0}$ has only trivial solution. So A is invertible, by the Invertible Matrix Theorem.

12.2 Partitioned Matrix

The matrix

$$A = \left[\begin{array}{ccc|cc|c} 3 & 0 & 1 & 2 & 2 & 2 \\ -5 & 3 & 4 & -1 & 5 & 1 \\ \hline -8 & -6 & 7 & 3 & 7 & 0 \end{array}\right]$$

can also be written into the 2×3 partitioned (or block) matrix

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

where

$$A_{11} = \begin{bmatrix} 3 & 0 & 1 \\ -5 & 3 & 4 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} -8 & -6 & 7 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 3 & 7 \end{bmatrix}, \quad A_{23} = \begin{bmatrix} 0 \end{bmatrix}$$

Addition of Partitioned Matrices:

If matrices A and B are the same size and are partitioned in exactly the same way, then it is natural to make the same partition of the ordinary matrix sum $A+B$. In this case, each block of $A+B$ is the (matrix) sum of the corresponding blocks of A and B. Multiplication of a partitioned matrix by a scalar is also computed block by block.

Example: The matrix $A = \left[\begin{array}{ccc|cc|c} 3 & 0 & 1 & 2 & 2 & 2 \\ -5 & 3 & 4 & -1 & 5 & 1 \\ \hline -8 & -6 & 7 & 3 & 7 & 0 \end{array}\right]$ can also be written into the 2×3 partitioned (or block) matrix

Multiplications of Partitioned Matrices: Partitioned matrices can be multiplied by the usual row-column rule as if the block entries were scalars, provided that for a product AB, the column partition of A matches the row partition of B.

Example: The matrix $A = \left[\begin{array}{ccc|cc} 2 & -3 & 1 & 0 & -4 \\ 1 & 5 & -2 & 3 & -1 \\ \hline 0 & -4 & -2 & 7 & -1 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $B = \left[\begin{array}{cc} 6 & 4 \\ -2 & 1 \\ -3 & 7 \\ \hline -1 & 3 \\ 5 & 2 \end{array}\right] = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ then $AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} A_{11}B_1 + A_{12}B_2 \\ A_{21}B_1 + A_{22}B_2 \end{bmatrix} =$

$$\left[\begin{array}{cc} -5 & 4 \\ -6 & 2 \\ \hline 2 & 1 \end{array}\right]$$

Column Row Expansion of AB:

Example: Let $A = \begin{bmatrix} -3 & 1 & 2 \\ 1 & -4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$. Verify that

$$AB = col_1(A)row_1(B) + col_2(A)row_2(B) + col_3(A)row_3(B).$$

Theorem: Column Row Expansion of AB

If A is $m \times n$ and B is $n \times p$, then

$$\begin{aligned} AB &= [col_1(A) \quad col_2(A) \quad \ldots \quad col_n(A)] \begin{bmatrix} row_1(B) \\ row_2(B) \\ . \\ . \\ . \\ row_n(B) \end{bmatrix} \\ &= col_1(A)row_1(B) + \ldots + col_n(A)row_n(B) \end{aligned}$$

Block Upper Triangular Matrix: The matrix $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ is said to be block upper triangular matrix.

Remark: A block diagonal matrix is a partitioned matrix with zero blocks off the main diagonal (of blocks). Such a matrix is invertible if and only if each block on the diagonal is invertible.

A factorization of a matrix A is an equation that expresses A as a product of two or more matrices.

12.3 The LU Factorizations

At first, assume that A is an $m \times n$ matrix that can be row reduced to echelon form, without row interchanges.

Then A can be written in the form $A = LU$, where L is an $m \times m$ lower triangular matrix with 1's on the diagonal and U is an $m \times n$ echelon form of A.

Such a factorization is called an LU factorization of A.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix} \begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

When $A = LU$, the equation $A\mathbf{x} = \mathbf{b}$ can be written as $L(U\mathbf{x}) = \mathbf{b}$. Writing $\mathbf{y}$ for $U\mathbf{x}$, we can find $\mathbf{x}$ by solving the pair of equations

$$L\mathbf{y} = \mathbf{b}, \qquad \text{and} \quad U\mathbf{x} = \mathbf{y}.$$

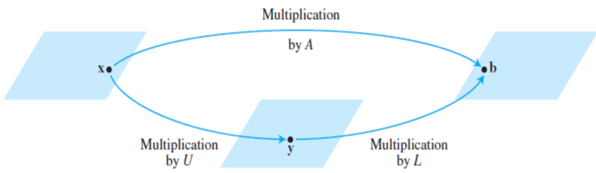


Figure 12.2: Solution of system by LU factorzation

Example: Verify that

$$A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Use this LU factorization of A to solve $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} -9 \\ 5 \\ 7 \\ 11 \end{bmatrix}$.

Solution: $[L \quad \mathbf{b}] = \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ -1 & 1 & 0 & 0 & 5 \\ 2 & -5 & 1 & 0 & 7 \\ -3 & 8 & 3 & 1 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$.

Remark: The solution of $L\mathbf{y} = \mathbf{b}$ needs only 6 multiplications and 6 additions, because the arithmetic takes place only in column 5.

$$[U \quad \mathbf{y}] = \begin{bmatrix} 3 & -7 & -2 & 2 & -9 \\ 0 & -2 & -1 & 2 & -4 \\ 0 & 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

Hence $\mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ -6 \\ -1 \end{bmatrix}$.

Remark: For $U\mathbf{x} = \mathbf{y}$, the "backward" phase of row reduction requires 4 divisions, 6 multiplications, and 6 additions.

Remark: To find $\mathbf{x}$ requires 28 arithmetic operations, or flops (floating point operations), excluding the cost of finding L and U. In contrast, row reduction of $[A \quad \mathbf{b}]$ to $[I \quad \mathbf{x}]$ takes 62 operations.

12.4 An LU Factorization Algorithm

1. Reduce A to an echelon form U by a sequence of row replacement operations, if possible.
2. Place entries in L such that the same sequence of row operations reduces L to I.

Example: Find an LU factorization of the matrix

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}.$$

Solution: L should be of order 4×4. The first column of L is the first column of A divided by the top pivot entry:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix}.$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & * & 1 & 0 \\ 3 & * & * & 1 \end{bmatrix}.$$

Then row reduce the matrix A into echelon form

$$\begin{aligned} A &= \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix}, \\ &\sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 7 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} = U \end{aligned}$$

and L take the form

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ 3 & 4 & 2 & 1 \end{bmatrix}.$$

12.5 Some Practice Problems

Question: Find the solution of the system $A\mathbf{x} = \mathbf{b}$ by using given LU factorization of A

$$A = \begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} -7 \\ 5 \\ 2 \end{bmatrix}.$$

Question: Find LU factorization of the matrix $A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix}$.

Question: Can a square matrix with two identical rows be invertible? Justify your answer.

Question: Can a square matrix with two identical columns be invertible? Justify your answer.

Question If an $n \times n$ matrix G cannot be row reduced to I_n; what can you say about the columns of G? Justify your answer.

Question: Find LU factorization of the matrix $A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix}$.

CHAPTER 13

Lecture No. 13

13.1 Subspace of $\mathbb{R}^n$

A subspace of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has three properties:

1. The zero vector is in H.

2. For each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u}+\mathbf{v}$ is in H.

3. For each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

Example: If $\mathbf{v}_1$ and $\mathbf{v}_2$ are in $\mathbb{R}^n$ and $H = Span\{\mathbf{v}_1, \mathbf{v}_2\}$, then H is a subspace of $\mathbb{R}^n$.

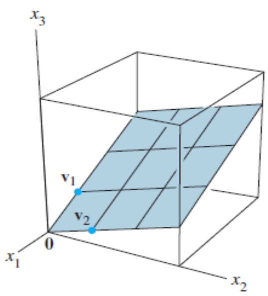


Figure 13.1: Subspace spanned by two vectors

Example: A line L not through the origin is not a subspace, because it does not contain the origin, as required. L is not closed under addition or scalar multiplication.

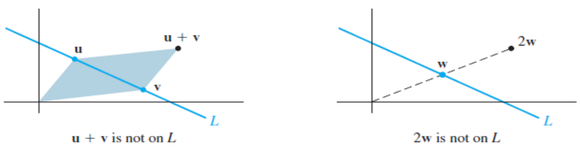


Figure 13.2: Line not passing through origin

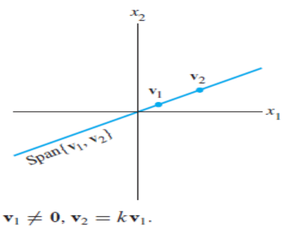


Figure 13.3: A line passing through origin is span of a nonzero vector

Example: For $\mathbf{v}_1, ..., \mathbf{v}_p$ in $\mathbb{R}^n$, the set of all linear combinations of $\mathbf{v}_1, ..., \mathbf{v}_p$ is a subspace of $\mathbb{R}^n$. The verification of this statement is similar to the argument given in previous Example.

Remark: We shall now refer to $Span\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ as the subspace spanned (or generated) by $\mathbf{v}_1, ..., \mathbf{v}_p$.

Remark: Note that $\mathbb{R}^n$ is a subspace of itself because it has the three properties required for a subspace. Another special subspace is the set consisting of only the zero vector in $\mathbb{R}^n$. This set, called the zero subspace, also satisfies the conditions for a subspace.

13.2 Column Space and Null Space of a Matrix

Column Space: The column space of a matrix A is the set $Col\ \ A$ of all linear combinations of the columns of A.

Remark: If $A = [\mathbf{v}_1 \quad ... \quad \mathbf{v}_n]$, with the columns in $\mathbb{R}^m$, then $Col\ \ A$ is the same as $Span\{\mathbf{v}_1, ..., \mathbf{v}_n\}$.

The column space of an $m \times n$ matrix is a subspace of $\mathbb{R}^m$.

Note that $Col\ \ A$ equals $\mathbb{R}^m$ only when the columns of A span $\mathbb{R}^m$. Otherwise, $Col\ \ A$ is only part of $\mathbb{R}^m$.

Example: Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$. Determine whether $\mathbf{b}$ is in the column space of A.

Solution: Let the columns of A be denoted by $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ then we have to check whether the system

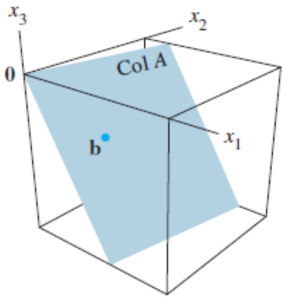
$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{b}$$

is consistent or inconsistent.

The augmented matrix of the above system and echelon form is

$$\begin{bmatrix} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & -2 & -6 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which shows that $\mathbf{b}$ is in the columns space of the given matrix.



Null Space: The null space of a matrix A is the set $Nul\ \ A$ of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Remark: When A has n columns, the solutions of $A\mathbf{x} = \mathbf{0}$ belong to $\mathbb{R}^n$, and the null space of A is a subset of $\mathbb{R}^n$.

In fact, $Nul\ \ A$ has the properties of a subspace of $\mathbb{R}^n$.

Theorem: The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions of a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

Example: Let $A = \begin{bmatrix} 1 & -1 & 5 \\ 2 & 0 & 7 \\ -3 & -5 & -3 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} -7 \\ 3 \\ 2 \end{bmatrix}$. Is $\mathbf{u}$ in Nul A? Is $\mathbf{u}$ in Col A? Justify your answer.

Solution: In order to check whether the given vector $\mathbf{u}$ is in the Null space of A or not we will check the matrix product

$$\begin{bmatrix} 1 & -1 & 5 \\ 2 & 0 & 7 \\ -3 & -5 & -3 \end{bmatrix} \begin{bmatrix} -7 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -7-3+10 \\ -14+0+14 \\ 21-15-6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which shows that $\mathbf{u}$ is in the Null space of the matrix A. For checking whether the vector is in column space of A or not, we will solve the system $A\mathbf{x} = \mathbf{u}$. If the system is consistent then $\mathbf{u}$ is in the column space of A otherwise not. The augmented matrix is

$$\begin{bmatrix} 1 & -1 & 5 & -7 \\ 2 & 0 & 7 & 3 \\ -3 & -5 & -3 & 2 \end{bmatrix}.$$

Check yourself or see the lecture.
Basis for a Subspace: A basis for a subspace H of $\mathbb{R}^n$ is a linearly independent set in H that spans H.

Example: The columns of an invertible $n \times n$ matrix form a basis for all of $\mathbb{R}^n$ because they are linearly independent and span $\mathbb{R}^n$, by the Invertible Matrix Theorem.

One such matrix is the $n \times n$ identity matrix. Its columns are denoted by $\mathbf{e}_1, ..., \mathbf{e}_n$.

The set $\{\mathbf{e}_1, ..., \mathbf{e}_n\}$ is called a standard basis for $\mathbb{R}^n$.
Example: Find a basis for the Null Space of the matrix

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

Solution:The echelon form of the augmented matrix of the system $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and the parametric vector form of the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

Which show that the basis for the Null space of the given matrix is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Basis for the column space of a matrix: Finding a basis for the column space of a matrix is actually less work than finding a basis for the null space.

Example: Find a basis for the column space of the matrix

$$\begin{bmatrix} 1 & 0 & 2 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution: Firs of all notice that $\mathbf{b}_3 = -3\mathbf{b}_1 + 2\mathbf{b}_2$ and $\mathbf{b}_4 = 5\mathbf{b}_1 - \mathbf{b}_2$.
This means that

$$Span\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4, \mathbf{b}_5\} = Span\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_5\}$$

and the vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_5$ are linearly independent and hence form a basis for the column space of the given matrix.
Theorem: The pivot columns of a matrix A form a basis for the column space of A.

Remark: Be careful to use pivot columns of A itself for the basis of $Col \quad A$. The columns of an echelon form B are often not in the column space of A.

Example: Suppose an $n \times n$ matrix A is invertible. What you can say about $Col \quad A$? About $Null \quad A$?

13.3 Some Practice Problems

Question Let

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 0 \\ 6 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 3 \\ 3 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ -5 \\ 5 \end{bmatrix} \quad \text{and} \quad \mathbf{p} = \begin{bmatrix} -6 \\ 1 \\ 17 \end{bmatrix}$$

Determine if $\mathbf{p}$ is in the columns space of $A[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]$? Is $\mathbf{p}$ is in the Null space of A?

Question: Give integers p and q such that $Nul \quad A$ is a subspace of $\mathbb{R}^p$ and $Col \quad A$ is a subspace of $\mathbb{R}^q$.

1. $A = \begin{bmatrix} 3 & 2 & 1 & -5 \\ -9 & -4 & 1 & 7 \\ 9 & 2 & -5 & 1 \end{bmatrix}$.

2. $A = \begin{bmatrix} 3 & 2 & 1 \\ -9 & -4 & 1 \\ 9 & 2 & -5 \end{bmatrix}$.

CHAPTER 14

Lecture No. 14

Question: Give integers p and q such that $Nul \quad A$ is a subspace of $\mathbb{R}^p$ and $Col \quad A$ is a subspace of $\mathbb{R}^q$.

1. $A = \begin{bmatrix} 3 & 2 & 1 & -5 \\ -9 & -4 & 1 & 7 \\ 9 & 2 & -5 & 1 \end{bmatrix}.$

2. $A = \begin{bmatrix} 3 & 2 & 1 \\ -9 & -4 & 1 \\ 9 & 2 & -5 \end{bmatrix}.$

Solution: 1. Null space of A will be a subspace of $\mathbb{R}^4$ and column space will be a subspace of $\mathbb{R}^3$.
2. Null space of A will be a subspace of $\mathbb{R}^3$ and column space also will be a subspace of $\mathbb{R}^3$.
Concept: Let H be a subspace of $\mathbb{R}^n$, then each vector in H can be written in only one way as a linear combination of the basis vectors.

Suppose $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_p\}$ is a basis for H, and suppose that a vector $\mathbf{x}$ in H can be generated in two ways

$$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + ... + c_p\mathbf{b}_p \quad \text{and} \quad \mathbf{x} = d_1\mathbf{b}_1 + d_2\mathbf{b}_2 + ... + d_p\mathbf{b}_p.$$

Then by subtracting these two equation we have

$$\mathbf{x} = (c_1 - d_1)\mathbf{b}_1 + (c_2 - d_2)\mathbf{b}_2 + ... + (c_p - d_p)\mathbf{b}_p.$$

Since $\mathcal{B}$ is a basis so the vectors are linearly independent. Which gives

$$(c_1 - d_1) = (c_2 - d_2) = ... = (c_p - d_p) = 0 \rightarrow c_1 = d_1, c_2 = d_2....c_p = d_p.$$

14.1 Coordinates System

Suppose the set $\mathcal{B} = \{\mathbf{b}_1, ..., \mathbf{b}_p\}$ is a basis for a subspace H. For each $\mathbf{x}$ in H, the coordinates of $\mathbf{x}$ relative to the basis $\mathcal{B}$ are the weights $c_1, ..., c_p$ such that $\mathbf{x} =$

$c_1\mathbf{b}_1 + ... + c_p\mathbf{b}_p$ and the vector in $\mathbb{R}^p$

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ . \\ . \\ . \\ c_p \end{bmatrix}$$

is called the coordinate vector of $\mathbf{x}$(relative to $\mathcal{B}$) or the $\mathcal{B}$-coordinate vector of $\mathbf{x}$.

Example: Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$ and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$. Then $\mathcal{B}$ is a basis for $H = Span\{\mathbf{v}_1, \mathbf{v}_2\}$ because $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. Determine if $\mathbf{x}$ is in H, and if it is, find the coordinate vector of $\mathbf{x}$ relative to $\mathcal{B}$.

Solution: For the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{x}$ the augmented matrix is

$$\begin{bmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus we have

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

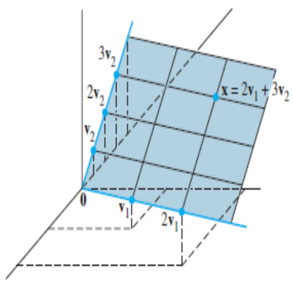


Figure 14.1: Coordinate map

Remark: If $\mathcal{B} = \{\mathbf{b}_1, ..., \mathbf{b}_p\}$ is a basis for H, then the mapping $\mathbf{x} \mapsto \mathbf{x}_{\mathcal{B}}$ is a one-to-one correspondence that makes H look and act the same as $\mathbb{R}^p$

Example: Consider the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\} \right.$$

for $\mathbb{R}^2$. If $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, what is $\mathbf{x}$?

Solution: By definition of coordinate vector with respect to a basis we have

$$\mathbf{x} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

The Dimension of a Subspace: The dimension of a nonzero subspace H, denoted by $dim\ H$, is the number of vectors in any basis for H. The dimension of the zero subspace $\{\mathbf{0}\}$ is defined to be zero.

Examples:

1. The space $\mathbb{R}^n$ has dimension n. Every basis for $\mathbb{R}^n$ consists of n vectors.
2. A plane through $\mathbf{0}$ in $\mathbb{R}^3$ is two-dimensional.
3. A line through $\mathbf{0}$ is one-dimensional.

Example: Find a basis for the Null Space of the matrix

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Solution: the matrix in echelon form is

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The parametric vector form of the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

Hence dimension of the Null space is 3.

Example: Determine the dimension of the subspace H of $\mathbb{R}^3$ spanned by the vectors $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$, (First, find a basis for H.)

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -8 \\ 6 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ -7 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 6 \\ -7 \end{bmatrix}.$$

Solution: We will check whether the vectors are linearly independent or not for which we consider the matrix

$$\begin{bmatrix} 2 & 3 & -1 \\ -8 & -7 & 6 \\ 6 & -1 & -7 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & -1 \\ 0 & 5 & 2 \\ 0 & -7 & -4 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 2/5 \\ 0 & 0 & -34/5 \end{bmatrix}.$$

Thus the columns of the matrix are linearly independent and dimension of the space H is 3.
Rank of a Matrix: The rank of a matrix A, denoted by rank A, is the dimension of the column space of A.

Example: Determine the rank of the matrix

$$A = \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix}.$$

Solution: Reduce A to echelon form

$$A \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & -5 & -7 \\ 0 & -6 & 4 & 14 & -20 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix} \sim ... \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & -5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The basis for the column space is the set

$$\left\{ \begin{bmatrix} 2 \\ 4 \\ 6 \\ 0 \end{bmatrix}, \left\{ \begin{bmatrix} 5 \\ 7 \\ 9 \\ -9 \end{bmatrix}, \left\{ \begin{bmatrix} -4 \\ -3 \\ 2 \\ 5 \end{bmatrix} \right\}.$$

The above set contains the columns corresponding to the pivot columns and hence dimension of the column space is 3, which is also rank of the matrix.
The Rank Theorem: If a matrix A has n columns, then

$$\text{rank } A + \text{dim Nul } A = n.$$

The Basis Theorem: Let H be a p-dimensional subspace of $\mathbb{R}^n$. Any linearly independent set of exactly p elements in H is automatically a basis for H. Also, any set of p elements of H that spans H is automatically a basis for H.

The Invertible Matrix Theorem: Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

1. The columns of A form a basis of $\mathbb{R}^n$.

2. Col $A = \mathbb{R}^n$.

3. dim Col $A = n$.

4. rank $A = n$.

5. Nul $A = \{\mathbf{0}\}$.

6. dim Nul $A = 0$.

Example:

1. Suppose a 4×7 matrix A has three pivot columns. Is Col $A = \mathbb{R}^3$. What is the dimension of Nul A? Explain your answers

2. Suppose a 4×6 matrix A has four pivot columns. Is Col $A = \mathbb{R}^4$. Is Nul $A = \mathbb{R}^2$? Explain your answers

Solution: 1. Yes, Col $A = \mathbb{R}^3$. The dimension of Null space will be 4 because we know that

$$rank + Nul \;\; A = 7.$$

Remember that Rank = dimension of the column space of the matrix.
2. Yes, Col $A = \mathbb{R}^4$ as dimension of the column space of the matrix will be 4. Null space of the matrix will be two dimensional space and Nul $A = \mathbb{R}^2$.

14.2 Some Practice Problems

Question: Consider the basis $\mathcal{B} = \left\{ \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\} \right.$ for $\mathbb{R}^2$. If $[\mathbf{x}]_{\mathcal{B}} = \left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right.$, what is $\mathbf{x}$?
Question: Determine the dimension of the subspace H of $\mathbb{R}^4$ spanned by the vectors $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$, (First, find a basis for H.)

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \\ -4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 9 \\ -6 \\ 12 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} -4 \\ 5 \\ -3 \\ 7 \end{bmatrix}.$$

Question: Find bases for Col A and Nul A, and then state the dimensions of these subspaces for the matrix

$$A = \begin{bmatrix} 1 & -2 & -1 & 5 & 4 \\ 2 & -1 & 1 & 5 & 6 \\ -2 & 0 & -2 & 1 & -6 \\ 3 & 1 & 4 & 1 & 5 \end{bmatrix}.$$

CHAPTER 15

Lecture No. 15

Recall: For a 2×2, $A = [a_{ij}]$, then determinant is the number det $A = a_{11}a_{22} - a_{12}a_{21}$.

Notion: For instance, if $A = \begin{bmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{bmatrix}$ then A_{32} is obtained by crossing out row 3 and column 2,

$$\begin{bmatrix} 1 & \mathbf{-2} & 5 & 0 \\ 2 & \mathbf{0} & 4 & -1 \\ \mathbf{3} & \mathbf{1} & \mathbf{0} & \mathbf{7} \\ 0 & \mathbf{4} & -2 & 0 \end{bmatrix} \quad \text{so that} \quad A_{32} = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}.$$

Remark: We can now give a recursive definition of a determinant. When $n = 3$, det A is defined using determinants of the 2×2 submatrices A_{1j}.

When $n = 4$, det A uses determinants of the 3×3 submatrices A_{1j}. In general, an $n \times n$ determinant is defined by determinants of $(n-1) \times (n-1)$ submatrices.

15.1 Determinant

For $n \geq 2$, the determinant of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j}$ detA_{1j}, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, ..., a_{1n}$ are from the first row of A. In symbols

$$\begin{aligned} \det A &= a_{11}\det A_{11} - a_{12}\det A_{12} + ... + (-1)^{1+n} a_{1n}\det A_{1n} \\ &= \sum_{j=1}^{n} a_{1j}(-1)1 + j\ A_{1j} \end{aligned}$$

Example: Compute the determinant of $\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$.

Solution: The determinant of the matrix is -2.

Notion: Given $A = [a_{ij}]$ then (i, j)-Cofactor of A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} \det A_{ij}.$$

Then

$$\det A = a_{11}C_{11} + a_{12}C_{12} + ... + a_{1n}C_{1n}.$$

This formula is called a cofactor expansion across the first row of A.

Theorem: The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the ith row using the cofactors

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + ... + a_{in}C_{in}.$$

The cofactor expansion down the jth column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + ... + a_{nj}C_{nj}.$$

Example: Compute the determinant of $\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$.

Solution: Use cofactor notion to show that the determinant of the matrix is -2.

Example: Compute the determinant of

$$A = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}.$$

Solution: Expanding along the first column gives us

$$\begin{aligned} det\ A &= 3\begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix} = (3)(2)\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} \\ &= = (3)(2)(1)\begin{vmatrix} 1 & 5 \\ 0 & -2 \end{vmatrix} = (3)(2)(1)(-2) = -12. \end{aligned}$$

Theorem: If A is a triangular matrix, then det A is the product of the entries on the main diagonal of A.

Example: Compute the determinant of

$$A = \begin{bmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{bmatrix}.$$

Example: Compute the determinants of the elementary matrices given in

$$E_1 = \begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{bmatrix}.$$

Solution: These are very important examples and the determinants are

$$det\ E_1 = k, \qquad det\ E_2 = -1 \qquad det\ E_3 = 1.$$

Theorem: Let A be a square matrix.

1. If a multiple of one row of A is added to another row to produce a matrix B, then det B =det A.

2. If two rows of A are interchanged to produce B, then det$B = -$det A.

3. If one row of A is multiplied by k to produce B, then det$B = k$ det$A$.

Example: Compute the determinant $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$.

Solution: The strategy is to reduce A to echelon form and then to use the fact that the determinant of a triangular matrix is the product of the diagonal entries.

$$\det A = \begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix}.$$

An interchange of rows 2 and 3 reverses the sign of the determinant

$$\det A = -\begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} = -(1)(3)(-5) = 15.$$

Remark: A common use of 3 axiom of Theorem in hand calculations is to factor out a common multiple of one row of a matrix. For instance

$$\begin{vmatrix} * & * & * \\ * & * & * \\ 5k & -3k & 7k \end{vmatrix} = k \begin{vmatrix} * & * & * \\ * & * & * \\ 5 & -3 & 7 \end{vmatrix}$$

where the starred entries are unchanged.

Example: Compute the determinant of A, where $A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$.

Solution:

$$
\begin{aligned}
\det A &= 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{vmatrix} \\
&= 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} \\
&= 2(1)(3)(-6)(1) = -36.
\end{aligned}
$$

Remark: Suppose a square matrix A has been reduced to an echelon form U by row replacements and row interchanges then If there are r interchanges

$$\det A = (-1)^r \det U$$

and det U is just the multiplication of the diagonal elements of U. Thus we have

$$\det A = \begin{cases} (-1)^r(\text{product of pivots in} U), & \text{when } A \text{ is invertible,} \\ 0 & \text{when } A \text{ is not invertible.} \end{cases}$$

Theorem: A square matrix A is invertible if and only if det $A \neq 0$.

Example: Compute det A, where $A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$.

Solution: Add 2 times row 1 to row 3 to obtain $A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 5 & -3 & -6 \\ -5 & -8 & 0 & 9 \end{bmatrix}$.
because the second and third rows of the second matrix are equal.

Example: Compute det A, where $A = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{bmatrix}$.

Solution:

$$
\begin{aligned}
\det A &= \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{vmatrix} = -2\begin{vmatrix} 1 & 2 & -1 \\ 3 & 6 & 2 \\ 0 & -3 & 1 \end{vmatrix} \\
&= -2\begin{vmatrix} 1 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & -3 & 1 \end{vmatrix} = -2\begin{vmatrix} 1 & 2 & -1 \\ 0 & -3 & 1 \\ 0 & 0 & 5 \end{vmatrix} \\
&= -2(1)(-3)(5) = -30
\end{aligned}
$$

Theorem: If A is an $n \times n$ matrix, then det $A^T =$ det A.

Theorem: If A and B are $n \times n$ matrices, then det $(AB) =$ (det A)(det B).

Example: Use a determinant to decide if $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly independent, when $\mathbf{v}_1 = \begin{bmatrix} 5 \\ -7 \\ 9 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -3 \\ 3 \\ -5 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 2 \\ -7 \\ 5 \end{bmatrix}$.

Solution: Consider the matrix $A = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]$, the vectors will be linearly independent if det $A \neq 0$. Otherwise vectors will be linearly dependent.

15.2 Some Practice Problems

Question: Find the determinant of the matrices, where $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5$.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{vmatrix}, \quad \begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix}, \quad \begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix}.$$

Question: Use determinants to find out the matrices are invertible or not

$$\begin{vmatrix} 2 & 3 & 0 \\ 1 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix}, \quad \begin{vmatrix} 2 & 0 & 0 & 8 \\ 1 & -7 & -5 & 0 \\ 3 & 8 & 6 & 0 \\ 0 & 7 & 5 & 4 \end{vmatrix}.$$

Question: Use a determinant to decide if $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly independent, when $\mathbf{v}_1 = \begin{bmatrix} 4 \\ 6 \\ -7 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -7 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -3 \\ -5 \\ 6 \end{bmatrix}$.

CHAPTER 16

Lecture No. 16

Question: Use determinants to find out the matrices are invertible or not

$$\begin{vmatrix} 2 & 3 & 0 \\ 1 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix}, \quad \begin{vmatrix} 2 & 0 & 0 & 8 \\ 1 & -7 & -5 & 0 \\ 3 & 8 & 6 & 0 \\ 0 & 7 & 5 & 4 \end{vmatrix}.$$

Solution: Use the result that a matrix is invertible if and only if det $A = 0$.

16.1 Cramer's Rule

Notion: For any $n \times n$ matrix A and any $\mathbf{b}$ in $\mathbb{R}^n$, let $A_i(\mathbf{b})$ be the matrix obtained from A by replacing column i by the vector $\mathbf{b}$, i.e.,

$$A_i(\mathbf{b}) = [\mathbf{a}_1 \;\; ...\mathbf{b} \;\; ...\mathbf{a}_n].$$

Theorem: Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}.$$

Remark: The Cramer's rule formula is inefficient for hand calculations, except for 2×2 and 3×3 systems.
Example: Use Cramer's rule to solve the system

$$\begin{array}{rcrcl} 3x_1 & - & 2x_2 & = & 6 \\ -5x_1 & + & 4x_2 & = & 8 \end{array}$$

Solution: Using the notation introduced

$$A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}, \quad A_1(\mathbf{b}) = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix}, \quad A_2(\mathbf{b}) = \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix}.$$

Since det A=2, the system has a unique solution. By Cramer's rule

$$\begin{aligned} x_1 &= \frac{\det A_1(\mathbf{b})}{\det A} = \frac{24+16}{2} = 20 \\ x_2 &= \frac{\det A_2(\mathbf{b})}{\det A} = \frac{24+30}{2} = 27 \end{aligned}$$

16.2 Applications to Engineering

Laplace transformation.
Example:Determine the values of s for which the system has a unique solution, and use Cramer's rule to describe the solution.

$$\begin{array}{rcrcl} 3sx_1 & - & 2x_2 & = & 4 \\ -6x_1 & + & sx_2 & = & 1 \end{array}$$

Solution: Using the notation introduced

$$A = \begin{bmatrix} 3s & -2 \\ -6 & s \end{bmatrix}, \quad A_1(\mathbf{b}) = \begin{bmatrix} 4 & -2 \\ 1 & s \end{bmatrix}, \quad A_2(\mathbf{b}) = \begin{bmatrix} 3s & 4 \\ -6 & 1 \end{bmatrix}.$$

Since det $A = 3s^2 - 12 = 3(s+2)(s-2)$, the system has a unique solution when $s \neq \pm 2$. By Cramer's rule

$$\begin{aligned} x_1 &= \frac{\det A_1(\mathbf{b})}{\det A} = \frac{4s+2}{3(s+2)(s-2)} \\ x_2 &= \frac{\det A_2(\mathbf{b})}{\det A} = \frac{3s+24}{3(s+2)(s-2)} \end{aligned}$$

A Formula for A^{-1}: Cramer's rule leads easily to a general formula for the inverse of an $n \times n$ matrix A. The jth column of A^{-1} is a vector $\mathbf{x}$ that satisfies

$$A\mathbf{x} = \mathbf{e}_j$$

where $\mathbf{e}_j$ is the jth column of the identity matrix, and the ith entry of $\mathbf{x}$ is the (i, j)-entry of A^{-1}. By Cramer's rule

$$\{(i, j) - \text{entry of} A^{-1}\} = x_i = \frac{\det A_i(\mathbf{e}_j)}{\det A}.$$

Recall that A_{ji} denotes the submatrix of A formed by deleting row j and column i. A cofactor expansion down column i of $A_i(\mathbf{e}_j)$ shows that

$$\det A_i(\mathbf{e}_j) = (-1)^{i+j} \det A_{ji} = C_{ji}$$

where C_{ji} is a cofactor of A. Thus

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & . \; . \; . & C_{n1} \\ C_{21} & C_{22} & . \; . \; . & C_{n2} \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ C_{1n} & C_{2n} & . \; . \; . & C_{nn} \end{bmatrix}$$

The matrix of cofactors on the right side of is called the adjugate (or classical adjoint) of A, denoted by adj A.

Theorem: An Inverse Formula

Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A.$$

Example: Compute the adjugate of the given matrix $\begin{bmatrix} 1 & 1 & 3 \\ 2 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Example: Find the inverse of the matrix $\begin{bmatrix} 2 & 1 & -3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$.

Solution: The cofactors are

$$\begin{aligned} C_{11} &= +\begin{vmatrix} -1 & 1 \\ 4 & -2 \end{vmatrix} = -2, \quad C_{12} = -\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = 3, \quad C_{13} = +\begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} = 5 \\ C_{21} &= -\begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix} = 14, \quad C_{22} = +\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -7, \quad C_{23} = -\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = -7 \\ C_{31} &= +\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = -2, \quad C_{32} = -\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 3, \quad C_{33} = +\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3 \end{aligned}$$

Thus we have

$$\operatorname{adj} A = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix}.$$

Notice that $(\operatorname{adj} A).A = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 & -3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix} = 14I$, consequently

$$A^{-1} = \frac{1}{14} \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix}.$$

Theorem: If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$.

If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.

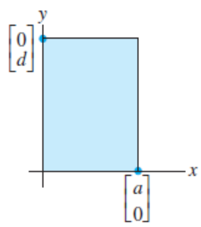


Figure 16.1: Determinant as area

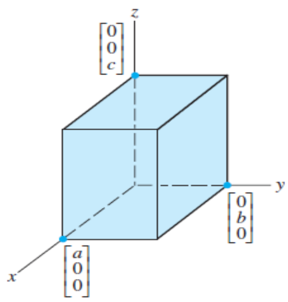


Figure 16.2: Determinant as volume

Remark: Let $\mathbf{a}_1$ and $\mathbf{a}_2$ be nonzero vectors. Then for any scalar c, the area of the parallelogram determined by $\mathbf{a}_1$ and $\mathbf{a}_2$ equals the area of the parallelogram determined by $\mathbf{a}_1$ and $\mathbf{a}_2 + c\mathbf{a}_1$.

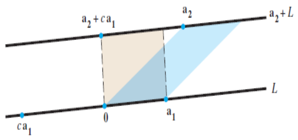


Figure 16.3: Area preservation

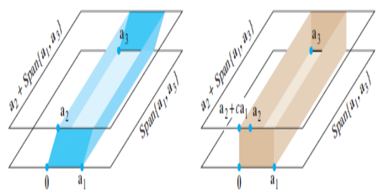


Figure 16.4: Volume preservation

Example: Calculate the area of the parallelogram determined by the points $(-2,-2), (0,3), (4,-1)$, and $(6,4)$.

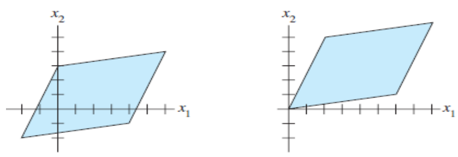


Figure 16.5: Area of parallelogram

Solution: First we translate the parallelogram and got the new coordinates as show in the figure then the are is given by the absolute value of the determinant of the following matrix

$$A = \begin{bmatrix} 2 & 6 \\ 5 & 1 \end{bmatrix}$$

which is $28m^2$.

Example: Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(1, 0, -2)$, $(1, 2, 4)$, and $(7, 1, 0)$.

Solution: The absolute value of the determinant

$$\begin{vmatrix} 1 & 1 & 7 \\ 0 & 2 & 1 \\ -2 & 4 & 0 \end{vmatrix}$$

gives the volume of the parallelepiped, which is $22m^3$.

Theorem: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then

$$\{\text{area of } T(S) = |\text{det}A|.\{\text{area of } S\}\}$$

If $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation determined by a 3×3 matrix A. If S is a parallelepiped in $\mathbb{R}^3$, then

$$\{\text{volume of } T(S) = |\text{det}A|.\{\text{volume of } S\}\}.$$

Remark: The conclusions of above result hold whenever S is a region in $\mathbb{R}^2$ with finite area or a region in $\mathbb{R}^3$ with finite volume.

Example: Let a and b be positive numbers. Find the area of the region E bounded by the ellipse whose equation is

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1.$$

Solution: We claim that E is the image of the unit disk D under the linear transformation T determined by the matrix $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, because if $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{x} = A\mathbf{u}$.

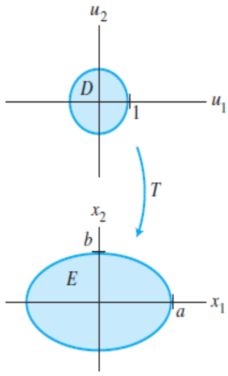


Figure 16.6: Area of ellipse

Then

$$\text{Area of ellipse} = (\text{det A})\ (\text{area of unit circle})$$

which gives area of the ellipse $= \pi ab$.

Example: Let S be the parallelogram determined by the vectors $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and let $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$. Compute the area of the image of S under the mapping $\mathbf{x} \mapsto A\mathbf{x}$.

Solution: The area of S is given by the determinant of the matrix $\begin{bmatrix} 1 & 5 \\ 3 & 1 \end{bmatrix}$, i.e., 14. Then the required area is

$$\text{Area of T(S)} = (\text{det A})\ (\text{area of S})$$

which is 28.

16.3 Some Practice Problems

Question: Let S be the parallelogram determined by the vectors $\mathbf{b}_1 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$ and let $A = \begin{bmatrix} 6 & -2 \\ -3 & 2 \end{bmatrix}$. Compute the area of the image of S under the mapping $\mathbf{x} \mapsto A\mathbf{x}$.

Question: Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation determined by the matrix $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ where a, b and c are positive numbers. Let S be the unit ball, whose bounding surface has the equation $x_1^2 + x_2^2 + x_3^2 = 1$.

1. Show that $T(S)$ is bounded by the ellipsoid with the equation $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1$.

2. Use the fact that the volume of the unit ball is $4\pi/3$ to determine the volume of the region bounded by ellipsoid in part (1).

Question: Find the area of the parallelogram whose vertices are $(-1, 0), (0, 5), (1, -4)$ and $(2, 1)$.

CHAPTER 17

Lecture No. 17

17.1 The Leontief Input-Output Model

Suppose a nation's economy is divided into n sectors that produce goods or services, and let x be a production vector in $\mathbb{R}^n$ that lists the output of each sector for one year.

Also, suppose another part of the economy (called the open sector) does not produce goods or services but only consumes them, and let $\mathbf{d}$ be a final demand vector (or bill of final demands) that lists the values of the goods and services demanded from the various sectors by the nonproductive part of the economy.

The vector $\mathbf{d}$ can represent consumer demand, government consumption, surplus production, exports, or other external demands.

As the various sectors produce goods to meet consumer demand, the producers themselves create additional **intermediate demand** for goods they need as inputs for their own production.
Leontief **asked if** there is a production level $\mathbf{x}$ such that the amounts produced (or "supplied") will exactly balance the total demand for that production.
Assumption: The basic assumption of Leontief's input-output model is that for each sector, there is a unit consumption vector in $\mathbb{R}^n$ that lists the inputs needed per unit of output of the sector.

All input and output units are measured in millions of dollars, rather than in quantities such as tons or bushels
Example: Suppose the economy consists of three sectors-manufacturing, agriculture, and service-with unit consumption vectors $\mathbf{c}_1$, $\mathbf{c}_2$, and $\mathbf{c}_3$, as shown in the table that follows. What amounts will be consumed by the manufacturing sector if it decides to produce 100 units?

	Inputs Consumed per Unit of Output		
Purchased from:	**Manufacturing**	**Agriculture**	**Services**
Manufacturing	.50	.40	.20
Agriculture	.20	.30	.10
Services	.10	.10	.30
	↑	↑	↑
	$\mathbf{c}_1$	$\mathbf{c}_2$	$\mathbf{c}_3$

Figure 17.1: Input consumed per unit of output

If manufacturing decides to produce x_1 units of output, then $x_1\mathbf{c}_1$ represents the intermediate demands of manufacturing, because the amounts in $x_1\mathbf{c}_1$ will be consumed in the process of creating the x_1 units of output.

Likewise, if x_2 and x_3 denote the planned outputs of the agriculture and services sectors, $x_2\mathbf{c}_2$ and $x_3\mathbf{c}_3$ list their corresponding intermediate demands.

The total intermediate demand from all three sectors is given by

$$\{\text{intermediate demand}\} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + x_3\mathbf{c}_3 = C\mathbf{x}$$

where C is the consumption matrix $[\mathbf{c}_1 \ \ \mathbf{c}_2 \ \ \mathbf{c}_3]$,

$$C = \begin{bmatrix} .50 & .40 & .20 \\ .20 & .30 & .10 \\ .10 & .10 & .30 \end{bmatrix}.$$

Then Leontief input-output model is

$$\mathbf{x} = C\mathbf{x} + \mathbf{d}$$

or

$$\begin{aligned} I\mathbf{x} - C\mathbf{x} &= \mathbf{d} \\ (I - C)\mathbf{x} &= \mathbf{d}. \end{aligned}$$

Example: Consider the economy whose consumption matrix is given by $C = \begin{bmatrix} .50 & .40 & .20 \\ .20 & .30 & .10 \\ .10 & .10 & .30 \end{bmatrix}$. Suppose the final demand is 50 units for manufacturing, 30 units for agriculture, and 20 units for services. Find the production level $\mathbf{x}$ that will satisfy this demand.

Solution: The coefficient matrix for the Leontief's input-output model $(I-C)\mathbf{x} = \mathbf{d}$ is

$$I - C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} .50 & .40 & .20 \\ .20 & .30 & .10 \\ .10 & .10 & .30 \end{bmatrix} = \begin{bmatrix} .50 & -.40 & -.20 \\ -.20 & .70 & -.10 \\ -.10 & -.10 & .70 \end{bmatrix}$$

To solve $(I - C)\mathbf{x} = \mathbf{d}$, the augmented matrix is

$$\begin{bmatrix} .50 & -.40 & -.20 & 50 \\ -.20 & .70 & -.10 & 30 \\ -.10 & -.10 & .70 & 20 \end{bmatrix}$$

To solve $(I - C)\mathbf{x} = \mathbf{d}$, the augmented matrix is

$$\begin{aligned} \begin{bmatrix} .50 & -.40 & -.20 & 50 \\ -.20 & .70 & -.10 & 30 \\ -.10 & -.10 & .70 & 20 \end{bmatrix} &\sim \begin{bmatrix} 5 & -4 & -2 & 500 \\ -2 & 7 & -1 & 300 \\ -1 & -1 & 7 & 200 \end{bmatrix} \\ &\sim \ ... \sim \begin{bmatrix} 1 & 0 & 0 & 226 \\ 0 & 1 & 0 & 119 \\ 0 & 0 & 1 & 78 \end{bmatrix} \end{aligned}$$

Manufacturing must produce approximately 226 units, agriculture 119 units, and services only 78 units.

Theorem: Let C be the consumption matrix for an economy, and let $\mathbf{d}$ be the final demand.

If C and $\mathbf{d}$ have nonnegative entries and if each column sum of C is less than 1, then $(I - C)^{-1}$ exists and the production vector

$$\mathbf{x} = (I - C)^{-1}\mathbf{d}$$

has nonnegative entries and is the unique solution of

$$\mathbf{x} = C\mathbf{x} + \mathbf{d}.$$

A Formula for $(I - C)^{-1}$:

$$(I - C)^{-1} \simeq I + C + C^2 + ... + C^m$$

when the columns sums of C are less than 1.

The entries in $(I - C)^{-1}$ are significant because they can be used to predict how the production $\mathbf{x}$ will have to change when the final demand $\mathbf{d}$ changes.

In fact, the entries in column j of $(I - C)^{-1}$ are the increased amounts the various sectors will have to produce in order to satisfy an increase of 1 unit in the final demand for output from sector j.

17.2 Some Practice Problems

Question: Consider the production model $\mathbf{x} = C\mathbf{x} + \mathbf{d}$ for an economy with two sectors, where $\begin{bmatrix} .2 & .5 \\ .6 & .1 \end{bmatrix}$, $\begin{bmatrix} 16 \\ 12 \end{bmatrix}$ Use an inverse matrix to determine the production level necessary to satisfy the final demand.

Question: Solve the Leontief production equation for an economy with three sectors, given that $\begin{bmatrix} .2 & .2 & .0 \\ .3 & .1 & 0.3 \\ .1 & .0 & .2 \end{bmatrix}$, $\begin{bmatrix} 40 \\ 60 \\ 80 \end{bmatrix}$ Use an inverse matrix to determine the production level necessary to satisfy the final demand.

Question: Suppose an economy has two sectors: goods and services. One unit of output from goods requires inputs of .2 unit from goods and .5 unit from services. One unit of output from services requires inputs of .4 unit from goods and .3 unit from services.

There is a final demand of 20 units of goods and 30 units of services. Set up the Leontief input-output model for this situation.

CHAPTER 18

Lecture No. 18

18.1 Vector Space

A vector space is a nonempty set V of objects, called vectors, on which are defined two operations, called addition and multiplication by scalars (real numbers), subject to the ten axioms (or rules). The axioms must hold for all vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$, is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$
4. There is a zero vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$.

Remark: Using only these axioms, one can show that the zero vector in Axiom 4 is unique, and the vector $-\mathbf{u}$, called the negative of $\mathbf{u}$, in Axiom 5 is unique for each $\mathbf{u}$ in V.

Example: The spaces $\mathbb{R}^n$, where $n \geq 1$, are the premier examples of vector spaces.

Remark: The geometric intuition developed for $\mathbb{R}^3$ will help you understand and visualize many concepts.
Example: Let V be the set of all arrows (directed line segments) in three-dimensional space, with two arrows regarded as equal if they have the same length and point in the same direction.

Define **addition** by the parallelogram rule, and for each $\mathbf{v}$ in V , define $c\mathbf{v}$ to be the arrow whose length is $|c|$ times the length of $\mathbf{v}$, pointing in the same direction as $\mathbf{v}$ if $c \geq 0$ and otherwise pointing in the opposite direction.

Show that V is a vector space.

Remark: This space is a common model in physical problems for various forces.

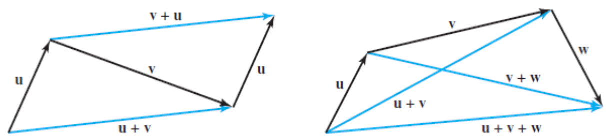


Figure 18.1: Addition of two vectors

Example (The space of discrete signals): Let $\mathbb{S}$ be the space of all doubly infinite sequences of numbers (usually written in a row rather than a column):

$$\{y_k\} = (...y_{-2}, y_{-1}, y_0, y_1, ...).$$

If $\{z_k\}$ is another element of the space the addition is defined as $\{y_k\} + \{z_k\}$ is the sequence $\{y_k+z_k\}$ formed by adding the corresponding components of the sequences $\{y_k\}$ and $\{z_k\}$.

The scalar multiplication $c\{y_k\}$ is defined as the sequence $\{cy_k\}$ obtained by scalar multiplication of each element of the sequence.

Show that $\mathbb{S}$ is a vector space.

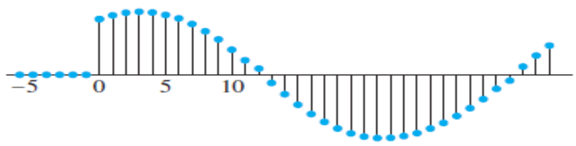


Figure 18.2: Discrete signal

Example (The space of polynomials): For $n \geq 0$, set $\mathbb{P}_n$ be the set of all polynomials of degree at most n consists of all polynomials of the form

$$\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + ... + a_n t^n$$

where the coefficients $a_0, a_1, ..., a_n$ and the variable t are real numbers.

The The degree of $\mathbf{p}$ is the highest power of t in above expression whose coefficient is not zero.

If $\mathbf{p}(t) = a_0 \neq 0$, the degree of $\mathbf{p}$ is zero. If all coefficients of a polynomial are zero then $\mathbf{p}(t)$ is a zero polynomial.

The addition of polynomials $\mathbf{p}(t)$ and $\mathbf{q}(t) = b_0 + b_1 t + b_2 t^2 + ... + b_n t^n$ is defined as

$$(\mathbf{p}+\mathbf{q})(t) = \mathbf{p}(t) + \mathbf{q}(t) = (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 + ... + (a_n + b_n)t^n$$

The scalar multiplication for c a scalar is defined by

$$(c\mathbf{p})(t) = c\mathbf{p}(t) = ca_0 + ca_1 t + ca_2 t^2 + ... + ca_n t^n$$

Show that $\mathbb{P}_n$ is a vector space.
Example (The space of real valued functions defined on a set): Let V be the set of all real-valued functions defined on a set $\mathbb{D}$.

The functions are added in a usual way:

$$(\mathbf{f} + \mathbf{g})(t) = \mathbf{f}(t) + \mathbf{g}(t).$$

The scalar multiplication is defined as

$$(c\mathbf{f})(t) = c\mathbf{f}(t).$$

Show that V is a vector space.

Remark: It is important to think of each function in the vector space V as a single object, as just one "point" or vector in the vector space.

18.2 Subspace of a Vector Space

A subspace of a vector space V is a subset H of V that has three properties:

1. The zero vector of V is in H.

2. H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.

3. H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

Trivial Subspaces: The set consisting of only the zero vector in a vector space V is a subspace of V, called the zero subspace and written as $\{\mathbf{0}\}$.

Since $V \subset V$ and this is also a subspace of any vector space V.
Example: Let $\mathbb{P}$ be the set of all polynomials with real coefficients, with operations in $\mathbb{P}$ defined as for functions.

Then $\mathbb{P}$ is a subspace of the space of all real-valued functions defined on $\mathbb{R}$.

Example: For each $n \geq 0$, Is $\mathbb{P}_n$ is a subspace of $\mathbb{P}$?
Example: Is the vector space $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$? Why or why not? Justify your answer.

Example: The set

$$H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s \text{ and } t \text{are real} \right\}$$

is a subset of $\mathbb{R}^3$ and looks and act like $\mathbb{R}^2$.

Show that H is a subspace of $\mathbb{R}^3$.

Example: A plane in $\mathbb{R}^3$ not through the origin is not a subspace of $\mathbb{R}^3$, because the plane does not contain the zero vector of $\mathbb{R}^3$.

Similarly, a line in $\mathbb{R}^2$ not through the origin is not a subspace of $\mathbb{R}^2$.

A Subspace Spanned by a Set: Given $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p$ vectors from a vector space V then $Span\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\}$ is a subspace of the vector space V.

Example: Given $\mathbf{v}_1, \mathbf{v}_2$ in a vector space V, let $H = Span\{\mathbf{v}_1, \mathbf{v}_2\}$. Show that H is a subspace of V.

Theorem: If $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p$ are in a vector space V, then $Span\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\}$ is a subspace of V.

Example: Let $H = \{(a - 3b, b - a, a, b) : a \text{ and } b \text{in } \mathbb{R}\}$. Show that H is a subspace of $\mathbb{R}^4$.

Example: For what value(s) of h will $\mathbf{y}$ be in the subspace of $\mathbb{R}^3$ spanned by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}.$$

18.3 Some Practice Problems

Question: Show that the set H of all points in $\mathbb{R}^2$ of the form $(3s, 2 + 5s)$ is not a vector space, by showing that it is not closed under scalar multiplication.

Question: The set $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 \leq 1 \right\}$ is a subset of $\mathbb{R}^2$. Show that H is not a subspace of $\mathbb{R}^2$.

Question: Let H be the set of all vectors of the form $\begin{bmatrix} -2t \\ 5t \\ 3t \end{bmatrix}$. Find a vector $\mathbf{v} \in \mathbb{R}^3$ such that $H = Span\{\mathbf{v}\}$. Why does this show that H is a subspace of $\mathbb{R}^3$.

Question: Let H be the set of all vectors of the form $\begin{bmatrix} 4t + 2s \\ 5s \\ 3t \\ 2s - 3t \end{bmatrix}$. Find two vector $\mathbf{v}, \mathbf{u} \in \mathbb{R}^3$ such that $H = Span\{\mathbf{v}, \mathbf{u}\}$. Why does this show that H is a subspace of $\mathbb{R}^4$.

CHAPTER 19

Lecture No. 19

19.1 Null Space

The null space of an $m \times n$ matrix A, written as Nul A, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation,

$$Nul\ A = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}.$$

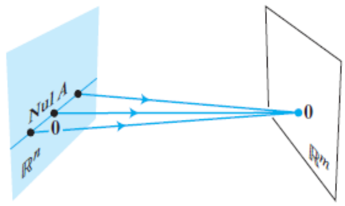


Figure 19.1: Null space

Example: Let $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$ and let $\mathbf{u} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$. Determine if $\mathbf{u}$ belongs to the Null space of A.

Solution: The given vector will be in Null space of the matrix if

$$A\mathbf{u} = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{0}.$$

Theorem: The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

Example: Let H be the set of all vectors in $\mathbb{R}^4$ whose coordinates a, b, c, d satisfy the equations $a - 2b + 5c = d$ and $c - a = b$. Show that H is a subspace of $\mathbb{R}^4$.

Solution: The given equations can be written as $a - 2b + 5c - d = 0$ and $c - a - b = 0$ and these equations can be written in matrix form as

$$\begin{bmatrix} 1 & -2 & 5 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus the given equations indeed represents the Null space of the matrix

$$\begin{bmatrix} 1 & -2 & 5 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix}$$

which is a subspace of $\mathbb{R}^4$.

An Explicit Description of Nul A: Find a spanning set for the Null Space of the matrix

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Solution:The echelon form of the given matrix is

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The solution of the homogeneous system in parametric vector form is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

The spanning set for the Null space is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

19.2 The Column Space of a Matrix

The column space of an $m \times n$ matrix A, written as Col A, is the set of all linear combinations of the columns of A. If $A = [\mathbf{a}_1, ..., \mathbf{a}_n]$, then

$$Col\ A = Span\{\mathbf{a}_1, ..., \mathbf{a}_n\}$$

or

$$Col\ A = \{\mathbf{b} : A\mathbf{x} = \mathbf{b} \text{ for some } \mathbf{x} \in \mathbb{R}^n\}.$$

Theorem: The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

Example: Find a matrix A such that $W = Col\ A$.

$$W = \left\{ \begin{bmatrix} 6a-b \\ a+b \\ -7a \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

Solution: The defining element of W can be written as

$$\begin{bmatrix} 6a-b \\ a+b \\ -7a \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Thus W is the column space of the matrix $\begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix}$ and is a subspace of $\mathbb{R}^3$.

Remark: The column space of an $m \times n$ matrix A is all of $\mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^m$.

The Contrast Between Nul A **and Col** A: Let

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$$

1. If the column space of A is a subspace of $\mathbb{R}^k$, what is k?
2. If the null space of A is a subspace of $\mathbb{R}^k$, what is k?

Solution: 1. Since the column space is the linear combination of columns of the given matrix and each column of the given matrix is a vector from $\mathbb{R}^3$. Hence, column space of the given matrix will be a subspace of $\mathbb{R}^3$, which in turns give $k = 3$.

2. Null space will be a subspace of $\mathbb{R}^4$, so $k = 4$. Why (can you justify)?

Example: Let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$, find a nonzero vector in Col A and a nonzero vector in Nul A.

Solution: For getting a nonzero vector belonging to Null space we have to take the augmented matrix of the system $A\mathbf{x} = \mathbf{0}$. The echelon form of the augmented matrix is

$$[A\ \ \mathbf{0}] \sim \begin{bmatrix} 1 & 0 & 9 & 0 & 0 \\ 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

From above form we can get a vector which belongs to Null space of the given matrix.

2. Since the column space is the linear combination of columns of the given matrix, so each nonzero column could be answer. Also there are many other possibilities.

Example: Let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$.

1. Determine if $\mathbf{u}$ is in Nul A. Could $\mathbf{u}$ be in Col A.

2. Determine if $\mathbf{v}$ is in Col A. Could $\mathbf{v}$ be in Nul A

Solution: For $\mathbf{u}$ to be in Null space of the matrix, it must satisfy the matrix equation $A\mathbf{x} = \mathbf{0}$. So we take

$$A\mathbf{u} = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus $\mathbf{u}$ is not in the Null space of the given matrix.

For the column space we will check the consistency of the system $A\mathbf{x} = \mathbf{v}$, the augmented matrix of the system is

$$[A \quad \mathbf{v}] = \begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ -2 & -5 & 7 & 3 & -1 \\ 3 & 7 & -8 & 6 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ 0 & 1 & -5 & -4 & -2 \\ 0 & 0 & 0 & 17 & 1 \end{bmatrix}$$

which shows that the system is consistent and hence $\mathbf{v}$ is in the column space of the matrix A.

Contrast Between Nul A and Col A for an $m \times n$ Matrix A

Nul A	Col A
1. Nul A is a subspace of $\mathbb{R}^n$.	**1.** Col A is a subspace of $\mathbb{R}^m$.
2. Nul A is implicitly defined; that is, you are given only a condition ($A\mathbf{x} = \mathbf{0}$) that vectors in Nul A must satisfy.	**2.** Col A is explicitly defined; that is, you are told how to build vectors in Col A.
3. It takes time to find vectors in Nul A. Row operations on $[\,A \quad \mathbf{0}\,]$ are required.	**3.** It is easy to find vectors in Col A. The columns of A are displayed; others are formed from them.
4. There is no obvious relation between Nul A and the entries in A.	**4.** There is an obvious relation between Col A and the entries in A, since each column of A is in Col A.
5. A typical vector $\mathbf{v}$ in Nul A has the property that $A\mathbf{v} = \mathbf{0}$.	**5.** A typical vector $\mathbf{v}$ in Col A has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in Nul A. Just compute $A\mathbf{v}$.	**6.** Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in Col A. Row operations on $[\,A \quad \mathbf{v}\,]$ are required.
7. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	**7.** Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
8. Nul $A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	**8.** Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.

Figure 19.2: Contrast between Null and Column spaces

19.3 Kernel and Range of a Linear Transformation

A linear transformation T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x}) \in W$, such that

1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in V.
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in V and al scalars c.

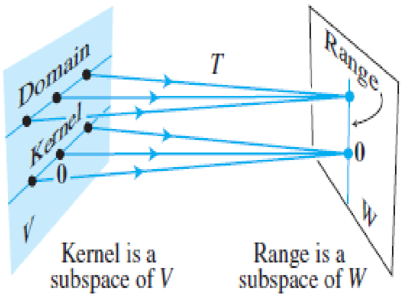


Figure 19.3: Kernel and Range of Linear Transformation

Example: Let V be the vector space of all real-valued functions f defined on an interval $[a, b]$ with the property that they are differentiable and their derivatives are continuous functions on $[a, b]$.

Let W be the vector space $C[a, b]$ of all continuous functions on $[a, b]$, and let $D : V \to W$ be the transformation that changes f in V into its derivative f'.

In calculus, two simple differentiation rules are

$$D(f + g) = D(f) + D(g) \quad \text{and} \quad D(cf) = cD(f).$$

Consequently, D is a linear transformation.

The Kernel of D is the set of all constant functions.

An example describing vibration of a weighted spring, the movement of a pendulum, and the voltage in an inductance-capacitance electrical circuit:

The differential equation

$$y'' + \omega^2 y = 0.$$

19.4 Some Practice Problems

Question: Let $A = \begin{bmatrix} 7 & -3 & 5 \\ -4 & 1 & -5 \\ -5 & 2 & -4 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 7 \\ 6 \\ -3 \end{bmatrix}$. Suppose you know that the equations $A\mathbf{x} = \mathbf{v}$ and $A\mathbf{x} = \mathbf{w}$ are both consistent. What can you say about the equation $A\mathbf{x} = \mathbf{v} + \mathbf{W}$?

Question: Either show that the given set is a vector space or give counter example

1. $\left\{ \begin{bmatrix} 3p-5q \\ 4q \\ p \\ q+1 \end{bmatrix} ; p, q \in \mathbb{R} \right\}.$

2. $\left\{ \begin{bmatrix} s-2t \\ 4s \\ s+t \\ -t \end{bmatrix} ; s, t \in \mathbb{R} \right\}$

Chapter 20

Lecture No. 20

Question: Either show that the given set is a vector space or give counter example

1. $\left\{ \begin{bmatrix} 3p-5q \\ 4q \\ p \\ q+1 \end{bmatrix} ; p, q \in \mathbb{R} \right\}.$

2. $\left\{ \begin{bmatrix} s-2t \\ 4s \\ s+t \\ -t \end{bmatrix} ; s, t \in \mathbb{R} \right\}$

Solution: 1. The set is not a vector space as the set doesn't contain the null vector.
2. Notice that the generating element of the the set can be written as

$$\begin{bmatrix} s-2t \\ 4s \\ s+t \\ -t \end{bmatrix} = s \begin{bmatrix} 1 \\ 4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

Thus the given set

$$\left\{ \begin{bmatrix} s-2t \\ 4s \\ s+t \\ -t \end{bmatrix} ; s, t \in \mathbb{R} \right\} = Span \left\{ \begin{bmatrix} 1 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

Since the subspace of a vector space is itself vector space and we know that spanning set of a given vectors is a subspace. Consequently, the given set is a vector space.
Recall: An indexed set of vectors $\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ in V is said to be linearly independent if the vector equation

$$c_1\mathbf{v}_1 + ... + c_p\mathbf{v}_p = \mathbf{0}$$

has only trivial solution $c_1 = 0, ..., c_p = 0$.

The set $\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ is said to be linearly dependent if the above equation has a nontrivial solution, that is if there are some weights, $c_1, ..., c_p$, not all zero such that the equation

$$c_1\mathbf{v}_1 + ... + c_p\mathbf{v}_p = \mathbf{0}$$

and the relation is called linear dependence relation.

Theorem: An indexed set $\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ of two or more vectors, with $\mathbf{v}_1 \neq 0$, is dependent if and only if some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of thee preceding vectors, $\mathbf{v}_1, ..., \mathbf{v}_{j-1}$.
Example: Determine whether the polynomials $\mathbf{p}_1(t) = 1, \mathbf{p}_2(t) = t$ and $\mathbf{p}_3(t) = 5 - 2t$ are linearly independent or linearly dependent in $\mathbb{P}$. Justify your answer.
Solution: We have to check the solution os the equation

$$c_1\mathbf{p}_1(t) + c_2\mathbf{p}_2(t) + c_3\mathbf{p}_3(t) = \mathbf{0}$$

where $\mathbf{0}$ is the null polynomial. The above equation lead to

$$(c_1 + 5c_3) + (c_2 - 2c_3)t = 0 + 0t$$

which gives us the following system of linear equations

$$c_1 + 5c_3 = 0$$

$$c_2 - 2c_3 = 0$$

and the solution of this system is $c_1 = -5c_3$ and $c_2 = 2c_3$ where as c_3 is free variable. Thus by definition the set of vectors $\{\mathbf{p}_1(t), \mathbf{p}_2(t), \mathbf{p}_3(t)\}$ is linearly dependent.

You can see that we have $\mathbf{p}_3(t) = 5\mathbf{p}_1(t) - 2\mathbf{p}_2(t)$.

Example: Determine whether the set $\{\cos t, \sin t\}$ is linearly independent or linearly dependent in the vector space $C([0, 1])$.

The vectors are linearly independent. can you figure out why?

20.1 Basis of a Vector Space

Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, ..., \mathbf{b}_p\}$ in V is a basis for H if

1. $\mathcal{B}$ is a linearly independent set, and

2. the subspace spanned by $\mathcal{B}$ coincide with H; that is,

$$H = Span\{\mathbf{b}_1, ..., \mathbf{b}_p\}.$$

Remark: The definition of a basis applies to the case when $H = V$, because any vector space is a subspace of itself.

Remark: When $H \neq V$, the second condition also includes that the requirement that the vectors $\mathbf{b}_1, ..., \mathbf{b}_p$ must belong to H.
Example: Let A be an invertible $n \times n$ matrix say, $A = [\mathbf{a}_1 \ \ ... \ \mathbf{a}_n]$. Then the columns of A form a basis for $\mathbb{R}^n$ because they are linearly independent and they span $\mathbb{R}^n$, by the Invertible Matrix Theorem.

Example: Let $\mathbf{e}_1, ..., \mathbf{e}_n$ be the columns of the $n \times n$ identity matrix, I_n. That is

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ . \\ . \\ . \\ 0 \end{bmatrix}, \quad ..., \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 1 \end{bmatrix}$$

then the set $\{\mathbf{e}_1, ..., \mathbf{e}_n\}$ forms a basis for $\mathbb{R}^n$ and is called a standard basis of $\mathbb{R}^n$.
Example: Let

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, \text{ and } \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}.$$

Determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$.
Solution: Consider the matrix $A = [\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3]$ and check whether the matrix is invertible and use Invertible Matrix Theorem to decide the given vectors form basis for $\mathbb{R}^3$ or not.
Example: Let $S = \{1, t, t^2, ..., t^n\}$. Verify that S is basis for $\mathbb{P}_n$. This basis is called the standard basis for $\mathbb{P}_n$.
Solution: Any nth order polynomial can be generated from the polynomials $\{1, t, t^2, ..., t^n\}$ and the set forms linearly independent set of vectors. Hence a basis for the vector space $\mathbb{P}_n$.
Example: Let

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \text{ and } \mathbf{v}_3 = \begin{bmatrix} 6 \\ 16 \\ -5 \end{bmatrix}$$

and $H = Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Note that $\mathbf{v}_3 = 5\mathbf{v}_1 + 3\mathbf{v}_2$, and show that $Span\{\mathbf{v}_1, \mathbf{v}_2\} = Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Find a basis for the subspace H.
Solution: It is clear that $Span\{\mathbf{v}_1, \mathbf{v}_2\} \subseteq Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Now any vector $\mathbf{u} \in Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ will have the form

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3.$$

We are also given $\mathbf{v}_3 = 5\mathbf{v}_1 + 3\mathbf{v}_2$ put the expression of $\mathbf{v}_3$ in the above expression we have

$$\mathbf{u} = (c_1 + 5c_3)\mathbf{v}_1 + (c_2 + 3c_3)\mathbf{v}_2.$$

which shows that $\mathbf{u} \in Span\{\mathbf{v}_1, \mathbf{v}_2\}$. Hence $Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subseteq Span\{\mathbf{v}_1, \mathbf{v}_2\}$

Consequently, we have

$$Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = Span\{\mathbf{v}_1, \mathbf{v}_2\}.$$

As the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are not multiple of each other, so the vectors are linearly independent and the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ forms a basis for the space spanned by these two vectors.

Theorem: Let $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\}$ be a set in V, and let $H = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\}$.

1. If one of the vectors in S say $\mathbf{v}_k$ is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.

2. If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.

Example: Find a basis for the Col A, where

$$A = [\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; ... \;\; \mathbf{a}_5] = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution: We know that the pivot columns are the linearly independent columns and $Span\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5\} = Span\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\}$ Thus basis for the column space of the given matrix is the set

$$S = \{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5\}.$$

Remark: What about a matrix A that is not in reduced echelon form?

Remark: Recall that any linear dependence relationship among the columns of A can be expressed in the form $A\mathbf{x} = 0$, where $\mathbf{x}$ is a column of weights. (If some columns are not involved in a particular dependence relation, then their weights are zero.)

When A is row reduced to a matrix B, the columns of B are often totally different from the columns of A.

However, the equations $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have exactly the same set of solutions. If $A = [\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; ... \;\; \mathbf{a}_n]$ and $B = [\mathbf{b}_1 \;\; \mathbf{b}_2 \; ; ... \; ; \mathbf{b}_n]$, then the vector equations

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + ... + x_n\mathbf{a}_n = \mathbf{0} \quad \text{and} \quad x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + ... + x_n\mathbf{b}_n = \mathbf{0}$$

also have the same set of solutions.

That is, the columns of A have exactly the same linear dependence relationships as the columns of B.

Example: The matrix B

$$B = [\mathbf{b}_1 \;\; \mathbf{b}_2 \;\; ... \;\; \mathbf{b}_5] = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$$

is row equivalent to the matrix A

$$A = [\mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \ldots \ \ \mathbf{a}_5] = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a basis for the Col B.

Solution: Basis for the column space of B is the set

$$S = \{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5\}.$$

The columns corresponding to the pivot columns in the echelon form.
Theorem: The pivot columns of a matrix A form a basis for Col A.
Remark: The pivot columns of a matrix A are evident when A has been reduced only to echelon form.

But, be careful to use the pivot columns of A itself for the basis of Col A. Row operations can change the column space of a matrix. The columns of an echelon form B of A are often not in the column space of A.

Two views of a basis: Basis of a vector space is the smallest spanning set and largest linearly independent set.
Example: The following three sets in $\mathbb{R}^3$ show how a linearly independent set can be enlarged to a basis and how further enlargement destroys the linear independence of the set.

Also, a spanning set can be shrunk to a basis, but further shrinking destroys the spanning property.

Linearly independent set but does not span $\mathbb{R}^3$: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \right\}$.

A basis for $\mathbb{R}^3$: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$.

Spans $\mathbb{R}^3$ **but is linearly dependent**: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$.

Example: Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 7 \\ -9 \end{bmatrix}$. Determine if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is basis for $\mathbb{R}^3$. Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ a basis for $\mathbb{R}^2$?

Example: Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $H = \left\{ \begin{bmatrix} s \\ s \\ 0 \end{bmatrix} : s \in \mathbb{R} \right\}$ Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ a basis for H.

Solution: The given vectors didn't form a basis for the space $\mathbb{R}^3$ as the vectors didn't span the space $\mathbb{R}^3$.

The vectors can't be basis for $\mathbb{R}^2$ as vectors are not vectors of $\mathbb{R}^2$.

Solution: Yes, the given vectors form a basis for the space H, because the vectors are linearly independent and

$$H = \left\{ \begin{bmatrix} s \\ s \\ 0 \end{bmatrix} : s \in \mathbb{R} \right\} = Span\{\mathbf{v}_1, \mathbf{v}_2\}.$$

Example: Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix} -4 \\ -8 \\ 9 \end{bmatrix}$.
Find a basis for the subspace W spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

Solution: Check whether the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ form linearly independent set or not. If the set is linearly independent then the set forms basis for W, otherwise one of the vector can be written as linear combination of other vectors. Delete that vector from the set and check the remaining vectors form a linearly independent set or not keep on doing this process unless you reach a linearly independent set of vector which will form a basis for the space W.

20.2 Some Practice Problems

Question:

1. Find a basis for the set of vectors in $\mathbb{R}^3$ in the plane $x - 3y + 2z = 0$.
2. Find a basis for the set of vectors in $\mathbb{R}^2$ on the line $y = -3x$.

Question: Determine whether the sets

1. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$
2. $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ 6 \end{bmatrix} \right\}.$

are basis for $\mathbb{R}^3$. Of the sets that are not basis, determine which ones are linearly independent and which ones span $\mathbb{R}^3$. Justify your answers.

Question: In the vector space of all real-valued functions, find a basis for the subspace spanned by $\{\sin t, \sin 2t, \sin t \cos t\}$.

CHAPTER 21

Lecture No. 21

Question:

1. Find a basis for the set of vectors in $\mathbb{R}^3$ in the plane $x - 3y + 2z = 0$.
2. Find a basis for the set of vectors in $\mathbb{R}^2$ on the line $y = -3x$.

Solution: 1. The parametric vector form of the solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, the given plane in $\mathbb{R}^3$ can be written as

$$Span\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \}.$$

Consequently, the basis for the given plane is

$$\mathcal{B} = \{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \}$$

and the dimension of the plane is 2.
2. Do yourself in the same way as we did in part 1.
Question: In the vector space of all real-valued functions, find a basis for the subspace spanned by $\{\sin t, \sin 2t, \sin t \cos t\}$.
Solution: Notice that $\sin 2t = 2 \sin t \cos t$. Now it is evident that the set of vectors $\{\sin t, \sin 2t, \sin t \cos t\}$ is linearly dependent. We will delete the vector $\sin 2t$ from the given set. Then the set

$$\{\sin t, \sin t \cos t\}.$$

Which is linearly independent set and forma basis for the space spanned by the given set.
Recall: The coordinates of each $\mathbf{x} \in \mathbb{R}^n$ with respect to a given basis $\mathcal{B}$ of $\mathbb{R}^n$.
Remark: An important reason for specifying a basis $\mathcal{B}$ for a vector space V is to impose a "coordinate system" on V.
Theorem: The Unique Representation Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, ..., \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, ..., c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1, ..., c_n\mathbf{b}_n.$$

21.1 Coordinate of x relative to the basis $\mathcal{B}$

Suppose $\mathcal{B} = \{\mathbf{b}_1, ..., \mathbf{b}_n\}$ is a basis for V and $\mathbf{x}$ is in V. The coordinates of $\mathbf{x}$ relative to the basis $\mathcal{B}$ (or the $\mathcal{B}$-coordinates of $\mathbf{x}$) are the weights $c_1, ..., c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1, ..., c_n\mathbf{b}_n.$$

The mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is the coordinate mapping (determined by $\mathcal{B}$)

Recall: Consider a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ for $\mathbb{R}^2$, where $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Suppose an $\mathbf{x} \in \mathbb{R}^2$ has the coordinates $\mathbf{x}_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Find $\mathbf{x}$.

Example: The entries in the vector $\mathbf{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ are the coordinates of $\mathbf{x}$ relative to the standard basis of $\mathbb{R}^2$. Why?

Example: In crystallography, the description of a crystal lattice is aided by choosing a basis $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ for $\mathbb{R}^3$ that corresponds to three adjacent edges of one "unit cell" of the crystal.

An entire lattice is constructed by stacking together many copies of one cell. There are fourteen basic types of unit cells; three are displayed in Fig

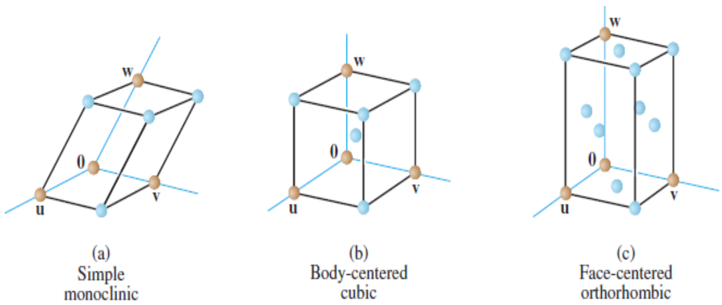


Figure 21.1: Coordinate map in crystallography

Remark: When a basis $\mathcal{B}$ for $\mathbb{R}^n$ is fixed, the $\mathcal{B}$-coordinate vector of a specified $\mathbf{x}$ is easily found.

Example: Let $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ of $\mathbf{x}$ relative to $\mathcal{B}$.

Remark: An analogous change of coordinates can be carried out in $\mathbb{R}^n$ for a basis $\mathcal{B} = \{\mathbf{b}_1, ..., \mathbf{b}_n\}$. Let

$$P_{\mathcal{B}} = [\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n].$$

Then the vector equation $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + ... + c_n\mathbf{b}_n$ is equivalent to

$$\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}.$$

We call $P_{\mathcal{B}}$ the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$.

Since the columns of $P_{\mathcal{B}}$ form a basis for $\mathbb{R}^n$, $P_{\mathcal{B}}$ is invertible (by the Invertible Matrix Theorem).

Left-multiplication by $P_{\mathcal{B}}^{-1}$ converts $\mathbf{x}$ into its $\mathcal{B}$-coordinate vector, $P_{\mathcal{B}}^{-1}\mathbf{x} = [\mathbf{x}]_{\mathcal{B}}$.

The correspondence $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$, produced here by $P_{\mathcal{B}}^{-1}$, is the coordinate mapping mentioned earlier.

Since $P_{\mathcal{B}}^{-1}$ is an invertible matrix, the coordinate mapping is a one-to-one linear transformation from $\mathbb{R}^n$ onto $\mathbb{R}^n$, by the Invertible Matrix Theorem.

21.2 The Coordinate Mapping

Theorem: Let $\mathcal{B} = \{\mathbf{b}_1, ..., \mathbf{b}_n\}$. be a basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a one to one linear transformation from V onto $\mathbb{R}^n$.

Remark: In general, a one-to-one linear transformation from a vector space V onto a vector space W is called an **isomorphism** from V onto W (iso from the Greek for "the same," and morph from the Greek for "form" or "structure").

The notation and terminology for V and W may differ, but the two spaces are indistinguishable as vector spaces.

Every vector space calculation in V is accurately reproduced in W, and vice versa.

In particular, any real vector space with a basis of n vectors is indistinguishable from $\mathbb{R}^n$, i.e., isomorphic to $\mathbb{R}^n$.

Example: Let $\mathcal{B}$ be the standard basis of the space $\mathbb{P}_3$ of polynomials; that is, let $\mathcal{B} = \{1, t, t^2, t^3\}$. Find the coordinate of arbitrary polynomial in the space $\mathbb{P}_3$ and define the coordinate mapping and decide the vector space $\mathbb{P}_3$ is isomorphic to which Euclidean space.

Solution: The arbitrary vector of $\mathbb{P}_3$ is $\mathbf{p}(t) = a_0 + a_1t + a_2t^2 + a_3t^3$, where $a_0, a_1, a_2, a_3 \in \mathbb{R}$ then the coordinate vector of that polynomial is

$$[\mathbf{p}(t)]_{\mathcal{B}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Since the dimension of $\mathbb{P}_3$ is 4 and its is isomorphic to $\mathbb{R}^4$.

Example: Use coordinate vectors to verify that the polynomials $1+2t^2$, $4+t+5t^2$, and $3+2t$ are linearly dependent in $\mathbb{P}_2$.

Solution: The standard basis for $\mathbb{P}_2$ is $\mathcal{B} = \{1, t, t^2\}$ and the coordinates of the given vectors with respect to basis $\mathcal{B}$ is

$$[1+2t^2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \qquad [4+t+5t^2]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}, \qquad [3+2t]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}.$$

The given vectors are linearly independent if and only if the coordinate vectors of these polynomials are linearly independent. So the augmented matrix is

$$\begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & 5 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which shows that the given polynomials are linearly dependent.

Example: Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$ and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$. Then $\mathcal{B}$ is basis for $H = Span\{\mathbf{v}_1, \mathbf{v}_2\}$. Determine if $\mathbf{x}$ is in H, and if it is, find the coordinate vector of $\mathbf{x}$ relative to $\mathcal{B}$.

Solution: We will solve the following vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{x}$. The augmented matrix is

$$\begin{bmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $c_1 = 3$ and $c_2 = 2$ and the coordinate of $\mathbf{x}$ is

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Example: Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}$,and $\mathbf{b}_3 = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} -8 \\ 2 \\ 3 \end{bmatrix}$.

1. Show that the set $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$. is basis for $\mathbb{R}^3$.
2. Find the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis.
3. Write the equation that relates $\mathbf{x} \in \mathbb{R}^3$ to $[\mathbf{x}]_{\mathcal{B}}$.
4. Find $[\mathbf{x}]_{\mathcal{B}}$, for the $\mathbf{x}$ given above.

Solution: 1. The matrix $A = [\mathbf{b}_1 \ \ \mathbf{b}_2 \ \ \mathbf{b}_3]$ has pivot in every row hence the set $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is linearly independent and forms a basis for the space $\mathbb{R}^3$.
2. The change of coordinate matrix from $\mathbb{B}$ to the standard basis of $\mathbb{R}^3$ is

$$\begin{bmatrix} 1 & -3 & 3 \\ 0 & 4 & -6 \\ 0 & 0 & 3 \end{bmatrix}.$$

3. Can you write the equation?
4. For $[\mathbf{x}]_{\mathcal{B}}$, we will solve the system

$$\begin{bmatrix} 1 & -3 & 3 & -8 \\ 0 & 4 & -6 & 2 \\ 0 & 0 & 3 & 3 \end{bmatrix}.$$

Example: The set $\mathcal{B} = \{1+t, 1+t^2, t+t^2\}$. is a basis for $\mathbb{P}_2$. Find the coordinate vector of $\mathbf{p}(t) = 6+3t-t^2$ relative to $\mathcal{B}$.
Solution: The one way of getting the answer is to write down the coordinate of the polynomials with respect to standard basis and then solve the system.

$$[1+t]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad [1+t^2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad [t+t^2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad [6+3t-t^2]_{\mathcal{B}} = \begin{bmatrix} 6 \\ 3 \\ -1 \end{bmatrix}.$$

For coordinate vector of $\mathbf{p}(t)$ we solve the system whose augmented matrix is

$$\begin{bmatrix} 1 & 1 & 0 & 6 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 6 \\ 0 & -1 & 1 & -3 \\ 0 & 1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 6 \\ 0 & -1 & 1 & -3 \\ 0 & 0 & 2 & -4 \end{bmatrix}.$$

21.3 Some Practice Problems

Question: The set $\mathcal{B} = \{1+t^2, t+t^2, 1+2t+t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $\mathbf{p}(t) = 1+4t+7t^2$.

Question: Determine whether the sets of polynomials form a basis for $\mathbb{P}_3$. justify your answer

1. $3+7t,\ 5+t-2t^3,\ t_2t^2,\ 1+16t-6t^2+2t^3$.
2. $5-3t+4t^2+2t^3,\ 9+t+8t^2-6t^3,\ 6-2t+5t^2,\ t^3$.

Question: Find the vector $\mathbf{x}$ determined by the given coordinate $[\mathbf{x}]_{\mathcal{B}}$ and the given basis $\mathcal{B}$.

1. $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} \right\}$, $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$.
2. $\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} \right\}$, $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}$.

CHAPTER 22

Lecture No. 22

Theorem: If a vector space V has a basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$, then any set in V containing more than n vectors must be linearly dependent.

Theorem: If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

22.1 Infinite Dimensional and Finite Dimensional Vector Spaces

If V is spanned by a finite set, then V is said to be finite dimensional, and the dimension of V, written as dim V, is the number of vectors in a basis for V.

The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero.

If V is not spanned by a finite set, then V is said to be infinite dimensional.
Example: The standard basis for $\mathbb{R}^n$ contains n vectors, so dim $\mathbb{R}^n = n$.

The standard polynomial basis $\{1, t, t^2\}$ show that the dim $\mathbb{P}_2 = 3$. In general dim $\mathbb{P}_n = n + 1$. The space of all polynomials $\mathbb{P}$ is infinite dimensional space.

Example: Let $H = Span\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. Find the dimension of the space H.
Solution: Since the two vectors are not multiple of each other hence the se $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent and form a basis for the space H. Hence the dimension of the space H is 2.

Example: Find the dimension of the subspace

$$H = \left\{ \begin{bmatrix} a - 3b + 6c \\ 5a + 4d \\ b - 2c - d \\ 5d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}.$$

Solution: The defining vector of the subspace H can be written as

$$\begin{bmatrix} a-3b+6c \\ 5a+4d \\ b-2c-d \\ 5d \end{bmatrix} = a\begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 6 \\ 0 \\ -1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix}.$$

The set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix} \right\}$$

is linearly independent and hence the dimension of H is 4.
Example: The subspaces of $\mathbb{R}^3$ can be classified by dimension

1. 0 dimensional subspace, i.e., subspace containing origin only.
2. 1 dimensional subspaces, i.e., subspace spanned a nonzero vector.
3. 2 dimensional subspaces, i.e., subspace spanned by two non parallel vectors.

22.2 Subspaces of a Finite Dimensional Space

Theorem: Let H be a subspace of a finite dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite dimensional and

$$\dim\ H \leq \dim\ V.$$

The Basis Theorem: Let V be a p dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V.

Any set of exactly p elements that spans V is automatically a basis for V.

The Dimensions of Nul A and Col A: The dimension of Nul A is the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$, and the dimension of Col A is the number of pivot columns in A.
Example: Find the dimensions of the null space and the column space of

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

Solution: Row reduce the augmented matrix $[A \quad \mathbf{0}]$ to echelon form

$$A \sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The pivot columns in the echelon form of the given matrix form the basis for the column space of the given matrix. Since there are two pivot columns so the dimension of the column space of the matrix is 2.

According to the rank theorem the dimension of the null space of the matrix is 3.

You can get the dimension of the Null space by writing the parametric vector form of the above matrix.

Example: Find the dimension of the subspace spanned by the given vectors

1. $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}$

2. $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 5 \end{bmatrix}$

Solution: In both parts the given vectors are linearly dependent, find a vector which is linear combination of the other vectors, delete that matrix from the set and check the remaining vectors are linearly independent or not.

Example: The first four Laguerre polynomials are $1, 1-t, 2-4t+t^2$, and $6-18t+t^2-t^3$. Show that these polynomials form a basis of $\mathbb{P}_3$.

Solution: The given vectors, i.e., polynomials are 4 and the dimension of the vector space $\mathbb{P}_3$ is also 4. Hence we have to show that the given polynomials are linearly independent, then these polynomials automatically span the vector space $\mathbb{P}_3$. For checking the linearly independence we write the coordinate vectors of the given matrices with respect to standard basis $\mathcal{B} = \{1, t, t^2, t^3\}$ of vector space $\mathbb{P}_3$.

$$[1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [1-t]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad [2-4t+t^2]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -4 \\ 1 \\ 0 \end{bmatrix},$$

$$[6-18t+t^2-t^3]_{\mathcal{B}} = \begin{bmatrix} 6 \\ -18 \\ 1 \\ -1 \end{bmatrix}.$$

Then the augmented matrix is

$$\begin{bmatrix} 1 & 1 & 2 & 6 & 0 \\ 0 & -1 & -4 & -18 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

Which shows that the vectors are linearly independent and hence the Lagurre polynomials are Linearly independent hence forma basis for vector space $\mathbb{P}_3$.

22.3 The Row Space

If A is an $m \times n$ matrix, each row of A has n entries and thus can be identified with a vector in $\mathbb{R}^n$. The set of all linear combinations of the row vectors is called the row space of A and is denoted by Row A.

Notice that Row $A =$ Col A^T.

Example: Let $A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$. Then Row A will be space of which Euclidean space? Find the basis for the Row A.

Solution: The matrix A is row equivalent to the following matrix

$$A \sim B = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The nonzero rows form basis for the row space. hence dimension of the row space is 3 and row space is subspace of $\mathbb{R}^5$.

Theorem: If two matrices A and B are row equivalent, then their row spaces are the same.

If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.

Theorem: The rank of A is the dimension of the column space of A.

Theorem The Rank Theorem

The dimensions of the column space and the row space of an $m \times n$ matrix A are equal. This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation

$$\text{rank}A + \dim \text{Nul}A = n.$$

number of pivot columns +number of non pivot columns = number of columns

Example:

1. If A is a 7×9 matrix with a two dimensional null space, what is the rank of A?

2. Could a 6×9 matrix have a two-dimensional null space?

Solution: Use the above Theorem to answer the questions.

Example: Let $A = \begin{bmatrix} 3 & 0 & -1 \\ 3 & 0 & -1 \\ 4 & 0 & 5 \end{bmatrix}$

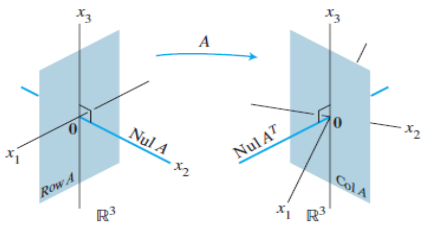


Figure 22.1: Row, column and null space of a matrix

Theorem The Invertible Matrix Theorem (Contd) Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

1. The columns of A form a basis of $\mathbb{R}^n$.

2. Col $A = \mathbb{R}^n$.

3. dim Col $A = n$.

4. rank $A = n$.

5. Nul $A = \{\mathbf{0}\}$.

6. dim Nul $A = 0$.

Example: The matrices below are row equivalent

$$A = \begin{bmatrix} 2 & -1 & 1 & -6 & 8 \\ 1 & -2 & -4 & 3 & -2 \\ -7 & 8 & 10 & 3 & -10 \\ 4 & -5 & -7 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 & -4 & 3 & -2 \\ 0 & 3 & 9 & -12 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1. Find rank A and dim Nul A.

2. Find basis for Col A and Row A.

3. What is the next step to perform to find a basis for Nul A?

4. How many pivot columns are in a row echelon form of A^T?

Solution: 1. We know that "rank + dim Null space =n", where n is number of columns of the matrix. Also, we know that the number of nonzero rows form basis for the row space of the matrix. So the given matrix has dimension of the row space as 2.

Null space of the dimension 3.

2. The columns in A corresponding to the pivot columns in B form the basis for the column space of A. The non zero rows in B form the basis of row space of A and B as well. Remember that row spaces of a matrix and its echelon form are same.
3. The next step is to write down the parametric vector form of the solution of the system $A\mathbf{x} = \mathbf{0}$ or reduce the matrix to the reduced echelon form and decide about the free and basic variables.
4. Two columns.

22.4 Some Practice Problems

Question: Find the dimension of the subspace of all vectors in $\mathbb{R}^3$ whose first and third entries are equal.

Question: Explain why the space $\mathbb{P}$ of all polynomials is an infinite dimensional space.

Question: Let H be an n-dimensional subspace of an n-dimensional vector space V. Show that $H = V$.

Question: If the null space of an 8×7 matrix A is 5-dimensional, what is the dimension of the column space of A? If the null space of an 8×7 matrix A is 3-dimensional, what is the dimension of the row space of A?
Question: Find the Null, Row and Column spaces of the matrix $A = \begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 2 & 6 & 6 & 0 & -3 \\ 3 & 9 & 3 & 6 & -3 \\ 3 & 9 & 0 & 9 & 0 \end{bmatrix}$.

CHAPTER 23

Lecture No. 23

Question: Find the dimension of the subspace of all vectors in $\mathbb{R}^3$ whose first and third entries are equal.
Solution: The given set can be written as

$$H = \left\{ \begin{bmatrix} s \\ t \\ s \end{bmatrix} : s, t \in \mathbb{R} \right\}.$$

The generating element of the above set can be written as

$$\begin{bmatrix} s \\ t \\ s \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Which gives

$$H = \left\{ \begin{bmatrix} s \\ t \\ s \end{bmatrix} : s, t \in \mathbb{R} \right\} = Span\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \}.$$

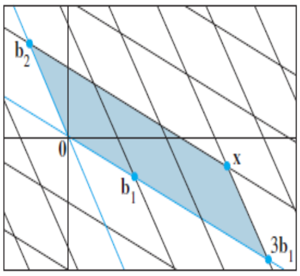
The dimension of the space is 2.

Question: Explain why the space $\mathbb{P}$ of all polynomials is an infinite dimensional space.

Solution: The set $\{1, t, t^2, ..., t^n, ...\}$ forms basis for the space $\mathbb{P}$ and hence dimension of ı $= \infty$.

Remark: When a basis $\mathcal{B}$ is chosen for an n-dimensional vector space V, the associated coordinate mapping onto $\mathbb{R}^n$ provides a coordinate system for V.

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad [\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$



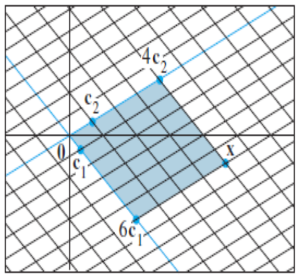


Figure 23.1: Coordinate system with respect to two different basis of the same space

Remark: Our problem is to find the connection between the two coordinate vectors.

23.1 Change of Coordinate Matirx

Example: Consider two bases $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ for a vector space V, such that

$$\mathbf{b}_1 = 4\mathbf{c}_1 + \mathbf{c}_2, \quad \text{and} \quad \mathbf{b}_2 = -6\mathbf{c}_1 + \mathbf{c}_2.$$

Suppose $\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2$, that is, suppose $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Find $[\mathbf{x}]_{\mathcal{C}}$.

Solution: The coordinate mapping is a linear transformation. So we can have

$$[\mathbf{x}]_{\mathcal{C}} = 3[\mathbf{b}_1]_{\mathcal{C}} + [\mathbf{b}_2]_{\mathcal{C}}.$$

We are also given

$$\mathbf{b}_1 = 4\mathbf{c}_1 + \mathbf{c}_2, \quad \text{and} \quad \mathbf{b}_2 = -6\mathbf{c}_1 + \mathbf{c}_2,$$

that is

$$[\mathbf{b}_2]_{\mathcal{C}} \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \qquad [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}.$$

Consequently, we have

$$[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$

The above coordinate vector $[\mathbf{x}]_{\mathcal{C}}$ can be obtained as

$$[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} [\mathbf{x}]_{\mathcal{B}}.$$

The matrix

$$\begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix}$$

is know as change of coordinate matrix from basis $\mathcal{B}$ to the basis $\mathcal{C}$.

Theorem: Let $\mathcal{B} = \{\mathbf{b}_1, ..., \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, ..., \mathbf{c}_n\}$ be basis of a vector space V. The there is a unique $n \times n$ matrix $P_{\mathcal{C} \leftarrow \mathcal{B}}$such that

$$[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C} \leftarrow \mathcal{B}}[\mathbf{x}]_{\mathcal{B}}.$$

The columns of $P_{\mathcal{C} \leftarrow \mathcal{B}}$ are the $\mathcal{C}$-coordinate vectors of the vectors in the basis $\mathcal{B}$. That is,

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = [[\mathbf{b}_1]_{\mathcal{C}} \;\; [\mathbf{b}_2]_{\mathcal{C}} \;\; ... \;\; [\mathbf{b}_n]_{\mathcal{C}}].$$

Remark: The matrix $P_{\mathcal{C} \leftarrow \mathcal{B}}$ is called the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

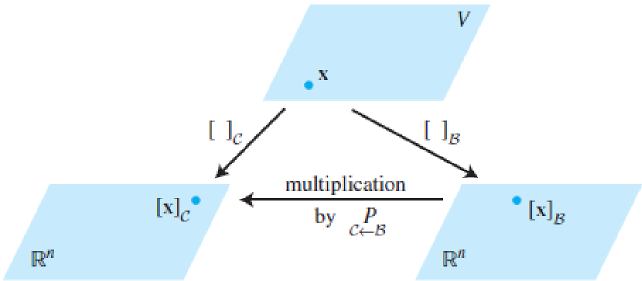


Figure 23.2: Change of coordinate matrix

Remark: The columns of $P_{\mathcal{C} \leftarrow \mathcal{B}}$ are linearly independent because they are the coordinate vectors of the linearly independent set $\mathcal{B}$.

Remark: The matrix of $P_{\mathcal{C} \leftarrow \mathcal{B}}$ is invertible (by invertible matrix theorem) and we have

$$(P_{\mathcal{C} \leftarrow \mathcal{B}})^{-1}[\mathbf{x}_{\mathcal{C}}] = [\mathbf{x}_{\mathcal{B}}].$$

Thus $(P_{\mathcal{C} \leftarrow \mathcal{B}})^{-1}$ is the matrix that converts $\mathcal{C}$-coordinates into $\mathcal{B}$-coordinates. That is,

$$(P_{\mathcal{C} \leftarrow \mathcal{B}})^{-1} = P_{\mathcal{B} \leftarrow \mathcal{C}}.$$

Change of Basis in $\mathbb{R}^n$: If $\mathcal{B} = \{\mathbf{b}_1, ..., \mathbf{b}_n\}$ and $\mathcal{E} = \{\mathbf{e}_1, ..., \mathbf{e}_n\}$ be the standard basis of $\mathbb{R}^n$ then change of coordinate matrix is

$$P_{\mathcal{B}} = [\mathbf{b}_1 \;\; ... \;\; \mathbf{b}_n].$$

Example: Let $\mathbf{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, $\mathbf{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$, and consider the basis for $\mathbb{R}^2$ given by $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$. Find the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

Solution: The matrix $P_{\mathcal{C} \leftarrow \mathcal{B}}$ involves the $\mathcal{C}$-Coordinate of the vectors $\mathbf{b}_1$ and $\mathbf{b}_2$. Let $[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $[\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. By definition, we have

$$[\mathbf{c}_1 \quad \mathbf{c}_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{b}_1 \quad \text{and} \quad [\mathbf{c}_1 \quad \mathbf{c}_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{b}_2.$$

$$\left[\begin{array}{cc|cc} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{array}\right].$$

Thus we have

$$[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} 6 \\ -5 \end{bmatrix} \qquad [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}.$$

Consequently, the change of coordinate matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = [[\mathbf{b}_1]_{\mathcal{C}} \quad [\mathbf{b}_2]_{\mathcal{C}}] = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}.$$

Example: Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, $\mathbf{c}_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$, and consider the basis for $\mathbb{R}^2$ given by $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$.

1. Find the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$.
2. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

Solution: 1. Notice that we have to find out the matrix $P_{\mathcal{B}\leftarrow\mathcal{C}}$ involves the $\mathcal{B}$-Coordinate of the vectors $\mathbf{c}_1$ and $\mathbf{c}_2$. By definition, we have

$$[\mathbf{b}_1 \quad \mathbf{b}_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{c}_1 \quad \text{and} \quad [\mathbf{b}_1 \quad \mathbf{b}_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{c}_2.$$

which gives

$$\left[\begin{array}{cc|cc} 1 & -2 & -7 & -5 \\ -3 & 4 & 9 & 7 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 5 & 3 \\ 0 & 1 & 6 & 4 \end{array}\right].$$

Thus we have

$$[\mathbf{c}_1]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \qquad [\mathbf{c}_2]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Consequently, the change of coordinate matrix $P_{\mathcal{B}\leftarrow\mathcal{C}}$ is

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = [[\mathbf{c}_1]_{\mathcal{B}} \quad [\mathbf{c}_2]_{\mathcal{B}}] = \begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}.$$

2. $P_{\mathcal{C}\leftarrow\mathcal{B}} = (P_{\mathcal{B}\leftarrow\mathcal{C}})^{-1}$ and is given by

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 2 & -3/2 \\ -3 & 5/2 \end{bmatrix}.$$

Example: Let $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ and $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ be basis for a vector space V, and suppose $\mathbf{f}_1 = 2\mathbf{d}_1 - \mathbf{d}_2 + \mathbf{d}_3$, $\mathbf{f}_2 = 3\mathbf{d}_2 + \mathbf{d}_3$, $\mathbf{f}_3 = -3\mathbf{d}_1 + 2\mathbf{d}_3 + \mathbf{d}_3$.

1. Find the change-of-coordinates matrix from $\mathcal{F}$ to $\mathcal{D}$.
2. Find $[\mathbf{x}]_{\mathcal{D}}$ for $\mathbf{x} = \mathbf{f}_1 - 2\mathbf{f}_2 + 2\mathbf{f}_3$.

Solution: 1. The change of coordinate matrix $P_{\mathcal{D}\leftarrow\mathcal{F}}$ is

$$P_{\mathcal{D}\leftarrow\mathcal{F}} = [[\mathbf{f}_1]_{\mathcal{D}} \quad [\mathbf{f}_2]_{\mathcal{D}} \quad [\mathbf{f}_3]_{\mathcal{D}}] = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

2.

$$[\mathbf{x}]_{\mathcal{D}} = P_{\mathcal{D}\leftarrow\mathcal{F}}[\mathbf{x}]_{\mathcal{F}} = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \\ 1 \end{bmatrix}.$$

Example: In $\mathbb{P}_2$, find the change of coordinates matrix from the basis $\mathcal{B} = \{1 - 2t + t^2, 3 - 5t + 4t^2, 2t + 3t^2\}$ to the standard basis $\mathcal{C} = \{1, t, t^2\}$. Then find $\mathcal{B}$-coordinate vector for $-1 + 2t$.

Solution: The change of coordinate matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = [[1 - 2t + t^2]_{\mathcal{C}} \quad [3 - 5t + 4t^2]_{\mathcal{C}} \quad [2t + 3t^2]_{\mathcal{C}}] = \begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix}.$$

Now we have

$$[-1 + 2t]_{\mathcal{B}} = (P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1}[\mathbf{x}]_{\mathcal{C}}.$$

The inverse of the matrix $(P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1}$ is

$$\begin{bmatrix} -23 & -9 & 6 \\ 8 & 3 & -2 \\ -3 & -1 & 1 \end{bmatrix}.$$

Consequently, we have

$$[-1 + 2t]_{\mathcal{B}} = (P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1}[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} -23 & -9 & 6 \\ 8 & 3 & -2 \\ -3 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}.$$

Can you verify the above calculations?

Example: Let

$$P = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -5 & 0 \\ 4 & 6 & 1 \end{bmatrix}, \qquad \mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -8 \\ 5 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -7 \\ 2 \\ 6 \end{bmatrix}.$$

Find a basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ for $\mathbb{R}^3$ such that P is the change of coordinate matrix from $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ to the basis $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Solution: If P is the change of coordinate matrix from $\mathcal{U}$ to $\mathcal{V}$ then the columns of P must satisfy

$$[\mathbf{u}_1]_{\mathcal{V}} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}, \quad [\mathbf{u}_2]_{\mathcal{V}} = \begin{bmatrix} 2 \\ -5 \\ 6 \end{bmatrix}, \quad [\mathbf{u}_3]_{\mathcal{V}} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Consequently, the required vectors are

$$\mathbf{u}_1 = 1\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} - 3\begin{bmatrix} -8 \\ 5 \\ 2 \end{bmatrix} + 4\begin{bmatrix} -7 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \\ 21 \end{bmatrix}$$

$$\mathbf{u}_2 = 2\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} - 5\begin{bmatrix} -8 \\ 5 \\ 2 \end{bmatrix} + 6\begin{bmatrix} -7 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ -9 \\ 32 \end{bmatrix}$$

$$\mathbf{u}_3 = -1\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} + 0\begin{bmatrix} -8 \\ 5 \\ 2 \end{bmatrix} + 1\begin{bmatrix} -7 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} -8 \\ 0 \\ 3 \end{bmatrix}.$$

23.2 Some Practice Problems

Question: In $\mathbb{P}_2$, find the change of coordinates matrix from the basis $\mathcal{B} = \{1 - 3t^2, 2 + t - 5t^2, 1 + 2t\}$ to the standard basis. Then write t^2 as linear combination of the polynomials in $\mathcal{B}$.

Question: Mark each statement as true or false

1. The columns of the change of coordinates matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ are $\mathcal{B}$-coordinate vector of the vectors in $\mathcal{C}$.

2. The columns of $P_{\mathcal{C}\leftarrow\mathcal{B}}$ are linearly independent.

3. If $V = \mathbb{R}^2$, $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$, then row reduction of $[\mathbf{c}_1 \ \ \mathbf{c}_2 \ \ \mathbf{b}_1 \ \ \mathbf{b}_2 \ ]$ to $[I \ \ P]$ produces a matrix P that satisfies $[\mathbf{x}]_{\mathcal{B}} = P[\mathbf{x}]_{\mathcal{C}}$ for all $\mathbf{x}$ in V.

Question: Let $P = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -5 & 0 \\ 4 & 6 & 1 \end{bmatrix}$ $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. Find a basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ for $\mathbb{R}^3$ such that P is the change of coordinate matrix from $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

CHAPTER 24

Lecture No. 24

24.1 Eigenvalues and Eigenvectors

The matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$ may move vectors in a variety of directions, it often happens that there are special vectors on which the action of A is quite simple.

Example: Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$$A\mathbf{u} = \begin{bmatrix} -5 \\ 1 \end{bmatrix}, \qquad A\mathbf{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\mathbf{v}.$$

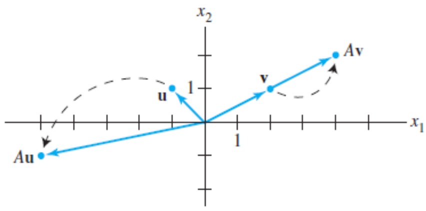


Figure 24.1: Matrix multiplication with a vector

Eigenvector and Eigenvector: An eigenvector of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an eigenvalue of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an eigenvector corresponding to λ.

Remark: It is easy to determine if a given vector is an eigenvector of a matrix. It is also easy to decide if a specified scalar is an eigenvalue.

Example: Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Are $\mathbf{u}$ and $\mathbf{v}$ eigenvectors of A?

Solution: We need to calculate

$$A\mathbf{u} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4\mathbf{u}.$$

which shows that $\mathbf{u}$ is an eigenvector of the given matrix and $\lambda = -4$ is the eigenvalue.

$$A\mathbf{v} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda \mathbf{v}.$$

Hence $\mathbf{v}$ is not an eigenvector of the given matrix.

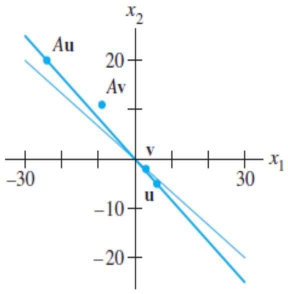


Figure 24.2: Eigenvector of a given matrix

Example: Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, show that 7 is an eigenvalue of the matrix A.

Solution: $\lambda = 7$ will be an eigenvalues of the given matrix if and only if the following equation has nontrivial solution

$$A\mathbf{x} = 7\mathbf{x}.$$

Which is equivalent to the following homogeneous system

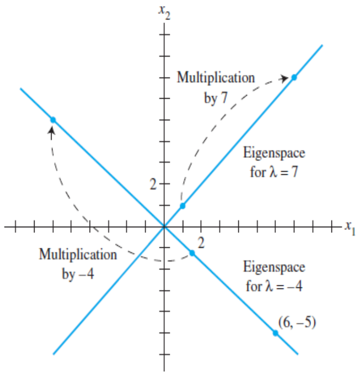
$$1\mathbf{x} - 7\mathbf{x} = \mathbf{0}.$$

If the above homogenous system has nontrivial solution then $\lambda = 7$ will be an eigenvalue of the given system.

The augmented matrix of the homogeneous system is

$$\begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus general solution has the form $x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. For each $x_2 \neq 0$ the vector $x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 7$.



Remark: Row reduction can be used to find eigenvectors, it cannot be used to find eigenvalues. An echelon form of a matrix A usually does not display the eigenvalues of A.

Example: Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigenvalue of A is 2. Find a basis for the corresponding eigenspace.

Solution: Eigenspace is the set of all vectors $\mathbf{x} \in \mathbb{R}^3$ such that $(A - 2I)\mathbf{x} = \mathbf{0}$. The matrix $A - 2I$ is

$$A - 2I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

and the echelon form of the above matrix is

$$\begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The general solution of the system $(A - 2I)\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

which shows that x_2 and x_3 are free and the set

$$\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

forms basis for the eigenspace of the given matrix corresponding to the eigenvalue 2.

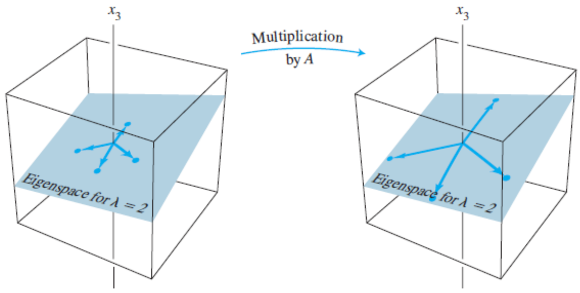


Figure 24.3: Eigenspace for $\lambda = 2$

Theorem: The eigenvalues of a triangular matrix are the entries on its main diagonal.

Example: Let $A = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$, and $B = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & -2 \end{bmatrix}$.

Solution: The eigenvalues of the matrix A are $\lambda = 3, 0, 2$ and of B are $\lambda = 4, 1, -2$.

Remark: When a matrix A has 0 as eigenvalue, it means that the system

$$A\mathbf{x} = 0\mathbf{x} \quad \rightarrow \quad A\mathbf{x} = \mathbf{0}$$

has non trivial solution.

By Invertible Matrix Theorem A in not invertible.

Theorem: If $\mathbf{v}_1, ..., \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, ..., \mathbf{v}_r\}$ is linearly independent.

Example: Is 5 an eigenvalue of the matrix $\begin{bmatrix} 6 & -3 & 1 \\ 3 & 0 & 5 \\ 2 & 2 & 6 \end{bmatrix}$?

Solution: $\lambda = 5$ will be an eigenvalue of the given matrix if and only if $(A - 5I)\mathbf{x} = \mathbf{0}$. The matrix $A - 5I$ is

$$A - 2I = \begin{bmatrix} 6 & -3 & 1 \\ 3 & 0 & 5 \\ 2 & 2 & 6 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 1 \\ 3 & -5 & 5 \\ 2 & 2 & 1 \end{bmatrix}$$

and the echelon form of the above matrix is

$$\begin{bmatrix} 1 & -3 & 1 \\ 3 & -5 & 5 \\ 2 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 1 \\ 0 & 4 & 2 \\ 0 & 8 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

which shows that the system $(A - 5I)\mathbf{x} = \mathbf{0}$ has only trivial solution and hence $\lambda = 5$ is not an eigenvalue of the given matrix.

Example: Without calculation, find one eigenvalue and two linearly independent eigenvectors of $A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$. Justify your answer.

Solution: Since matrix is singular so it must have one eigenvalue as $\lambda = 0$. Can you guess why?

Example: Find the eigenvalues of $\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$.
Solution: By definition we have

$$A\mathbf{x} = \lambda\mathbf{x} \quad \Leftrightarrow (A - \lambda I)\mathbf{x} = \mathbf{0}.$$

The system $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has no trivial solution.

By the Invertible Matrix Theorem, this problem is equivalent to finding all λ such that the matrix $A - \lambda I$ is not invertible, where

$$(A - \lambda I) = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2 - \lambda & 3 \\ 3 & -6 - \lambda \end{bmatrix}.$$

$$\det(A - \lambda I) = 0$$

So the eigenvectors of the given matrix are the roots of the polynomial

$$\lambda^2 + 4\lambda - 21 = 0.$$

Recall: Compute $\det A$ for $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$.

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Hence $\det A = -2$.

Theorem: The Invertible Matrix Theorem

Let A be an $n \times n$ matrix. Then A is invertible if and only if

1. The number 0 is not an eigenvalue of A.
2. The determinant of A is not zero.

Remark: When A is a 3×3 matrix, $|\det A|$ turns out to be the volume of the parallelepiped determined by the columns $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ of A. This volume is nonzero if and only if the vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are linearly independent, in which case the matrix A is invertible. (If the vectors are nonzero and linearly dependent, they lie in a plane or along a line.)

Theorem: Properties of Determinants

Let A and B be $n \times n$ matrices

1. A is invertible if and only if det $A \neq 0$.
2. det $AB = (\det A)(\det B)$.
3. det $A^T = \det A$.
4. If A is triangular, then det A is the product of the entries on the main diagonal of A.
5. A row replacement operation on A does not change the determinant. A row interchange changes the sign of the determinant. A row scaling also scales the determinant by the same scalar factor.

24.2 The Characteristic Equation

A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if λ satisfies the characteristic equation

$$\det(A - \lambda I) = 0.$$

Example: Find the characteristic equation of

$$\begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Solution:

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5_\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix}.$$

$\lambda^4 - 14\lambda^3 + 68\lambda^2 - 130\lambda + 75 = 0$. The roots of the polynomial is $\lambda = 5, 3, 5, 1$.

Remark: if A is an $n \times n$ matrix, then $\det(A - \lambda I)$ is a polynomial of degree n called the characteristic polynomial of A.

Example: Find the characteristic equation and eigenvalues of $\begin{bmatrix} 1 & -4 \\ 4 & 2 \end{bmatrix}$.

Solution: The characteristic polynomial is

$$\lambda^2 - 3\lambda + 20 = 0.$$

The roots of the polynomial are the eigenvalues of the given matrix.
Algebraic Multiplicity of an Eigenvalue: The algebraic multiplicity of an eigenvalue λ is its multiplicity as a root of the characteristic equation.

Example: The characteristic polynomial of a 6×6 matrix is $\lambda^6 - 4\lambda^5 - 12\lambda^4$. Find the eigenvalues and their multiplicities.

Solution: The roots of the given polynomials are $\lambda = 0, 3, 4$. The eigenvalue $\lambda = 0$ has algebraic multiplicity 4 and the eigenvalues $\lambda = 3, 4$ has multiplicity 1.

24.3 Similarity

If A and B are $n \times n$ matrices, then A is similar to B if there is an invertible matrix P such that $P^{-1}AP = B$, or, equivalently, $A = PBP^{1-}$.

Writing Q for P^{-1}, we have $Q^{-1}BQ = A$. So B is also similar to A, and we say simply that A and B are similar.

Changing A into $P^{-1}AP$ is called a similarity transformation.

Theorem: If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues with the same multiplicities.
Remark: The matrices

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

are not similar but have the same eigenvalues.
Remark: Similarity is not the same as row equivalence. (If A is row equivalent to B, then $B = EA$ for some invertible matrix E.) Row operations on a matrix usually change its eigenvalues.

24.4 Some Practice Problems

Question: List the real eigenvalues, repeated according to their multiplicities.

$$\begin{bmatrix} 5 & 5 & 0 & 2 \\ 0 & 2 & -3 & 6 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \qquad \begin{bmatrix} 3 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 \\ 3 & 0 & 3 & 0 \\ -1 & 0 & 0 & 3 \end{bmatrix}.$$

Question: It can be shown that the algebraic multiplicity of an eigenvalue λ is always greater than or equal to the dimension of the eigenspace corresponding to λ. Find h in the matrix A below such that the eigenspace for $\lambda = 4$ is two-dimensional

$$\begin{bmatrix} 4 & 2 & 3 & 3 \\ 0 & 2 & h & 3 \\ 0 & 0 & 4 & 14 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

CHAPTER 25

Lecture No. 25

Question: It can be shown that the algebraic multiplicity of an eigenvalue λ is always greater than or equal to the dimension of the eigenspace corresponding to λ. Find h in the matrix A below such that the eigenspace for $\lambda = 4$ is two-dimensional

$$\begin{bmatrix} 4 & 2 & 3 & 3 \\ 0 & 2 & h & 3 \\ 0 & 0 & 4 & 14 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Solution: For the basis of eigenspace we will solve the system $(A - 4I)\mathbf{x} = \mathbf{0}$, i.e.,

$$A - 4I = \begin{bmatrix} 0 & 2 & 3 & 3 \\ 0 & -2 & h & 3 \\ 0 & 0 & 0 & 14 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

the augmented matrix of the homogeneous system is

$$\begin{bmatrix} 0 & 2 & 3 & 3 & 0 \\ 0 & -2 & h & 3 & 0 \\ 0 & 0 & 0 & 14 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 2 & 3 & 3 & 0 \\ 0 & -2 & h & 3 & 0 \\ 0 & 0 & 0 & 14 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}.$$

We have $x_4 = 0$, $x_2 = \frac{hx_3}{2}$ also we have $x_2 = \frac{3x_3}{2}$, which gives the value of $h = 3$. For $h = 3$ the solution of the system $(A - 4I)\mathbf{x} = \mathbf{0}$ in parametric form is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 3/2 \\ 1 \\ 0 \end{bmatrix}$$

which shows that basis for the eigenspace corresponding to $\lambda = 4$ is 2.

25.1 Diagonalization

Example: If $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$ then calculate D^2, D^3 and D^k where $k \geq 1$.

Solution: The expressions are

$$D^2 = \begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix}, \qquad D^3 = \begin{bmatrix} 5^3 & 0 \\ 0 & 3^3 \end{bmatrix}$$

and

$$D^k = \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}.$$

Remark: If $A = PDP^{-1}$ for some invertible P and diagonal D, then A^k is also easy to compute.

Example: Let $\begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Find a formula for A^k, given that $A = PDP^{-1}$ where

$$P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}.$$

Solution: The standard formula for the inverse of a 2×2 matrix yields

$$P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}.$$

Then we can check that $A = PDP^{-1}$, also

$$A^2 = PD^2P^{-1}, \qquad A^3 = PD^3P^{-1}$$

and

$$A^k = PD^kP^{-1}.$$

Diagonalizable Matrix: A square matrix A is said to be diagonalizable if A is similar to a diagonal matrix, that is, if $A = PDP^{-1}$ for some invertible matrix P and some diagonal matrix D.

Theorem: The Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

Example: Diagonalize the following matrix, if possible

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

Solution: **Step I.** Find all the eigenvalues of the matrix A

$$\begin{aligned} \det(A - \lambda I) = 0 \Rightarrow -\lambda^3 - 3\lambda^2 + 4 &= 0 \\ -(\lambda - 1)(\lambda + 2)^2 &= 0. \end{aligned}$$

Step II. Find three linearly independent eigenvectors of A.

$$\text{Basis for } \lambda = 1 : \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\text{Basis for} \lambda = -2 : \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

Step III. Construct P from the vectors in step II.

The order of the vectors is unimportant.

$$P = [\mathbf{v}_1 \ \mathbf{v}_3 \ \mathbf{v}_3] = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Step IV. Construct D from the corresponding eigenvalues

It is essential that the order of the eigenvalues matches the order chosen for the columns of P.

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Verification: It is a good idea to check that P and D really work.

$$AP = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

$$PD = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

Example: Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

Solution: The characteristic equation of the given matrix is

$$\begin{aligned} \det(A - \lambda I) = 0 \Rightarrow -\lambda^3 - 3\lambda^2 + 4 &= 0 \\ -(\lambda - 1)(\lambda + 2)^2 &= 0. \end{aligned}$$

The eigenvalues are $\lambda = 1$ and $\lambda = -2$. The eigenspace for each eigenvalue is one dimensional

$$\text{Basis for}\lambda = 1: \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \text{Basis for}\lambda = -2: \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

There are no other eigenvalues, and every eigenvector of A is a multiple of either $\mathbf{v}_1$ or $\mathbf{v}_2$.

Hence A is not diagonalizable.

Theorem: An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Example: Determine if the following matrix is diagonalizable

$$A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}.$$

25.2 Matrices Whose Eigenvalues Are Not Distinct

Theorem: Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, ..., \lambda_p$.

1. For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.

2. The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n, and this happens if and only if

   (i) the characteristic polynomial factors completely into linear factors and

   (ii) the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.

3. If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $\mathcal{B}_1, ..., \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.

Example: Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}.$$

Solution: Since A is a triangular matrix, the eigenvalues are 5 and -3, each with algebraic multiplicity 2

$$\text{Basis for}\lambda = 5: \quad \mathbf{v}_1 = \begin{bmatrix} -8 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -16 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Basis for}\lambda = -3: \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent. So the matrix $P = [\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3 \;\; \mathbf{v}_4]$ is invertible and $A = PDP^{-1}$, where

$$P = \begin{bmatrix} -8 & -16 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & -3 \end{bmatrix}.$$

25.3 Some Practice Problems

Question: Let $A = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Suppose you are told that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A. Use this information to diagonalize A.

Question: Let A be a 4×4 matrix with eigenvalues 5, 3, and -2, and suppose you know that the eigenspace for $\lambda = 3$ is two-dimensional. Do you have enough information to determine if A is diagonalizable?

Question: Diagonalize the given matrix, if possible

$$\begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & -2 & -2 \\ 3 & -3 & -2 \\ 2 & -2 & -2 \end{bmatrix}.$$

Question: A is a 7×7 matrix with three eigenvalues. One eigenspace is two-dimensional, and one of the other eigenspaces is three dimensional. Is it possible that A is not diagonalizable? Justify your answer.

Question: A is a 3×3 matrix with two eigenvalues. Each eigenspace is one-dimensional. Is A diagonalizable? Why?

CHAPTER 26

Lecture No. 26

Question: Let $A = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Suppose you are told that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A. Use this information to diagonalize A.
Solution: We need to find out the eigenvalues corresponding to the given eigenvectors.

$$A\mathbf{v}_1 = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 3 \\ 1 \end{bmatrix},$$

which gives $\lambda = 1$.

$$A\mathbf{v}_2 = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

which gives $\lambda = 3$. Thus we have

$$P = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

Thus $A = PDP^{-1}$.
Question: Let A be a 4×4 matrix with eigenvalues 5, 3, and -2, and suppose you know that the eigenspace for $\lambda = 3$ is two-dimensional. Do you have enough information to determine if A is diagonalizable?
Solution: Yes, the matrix A is diagonalizable.

26.1 Eigenvectors and Linear Transformation

Let V be an n-dimensional vector space, let W be an m-dimensional vector space, and let T be any linear transformation from V to W. To associate a matrix with T, choose (ordered) bases $\mathcal{B}$ and $\mathcal{C}$ for V and W, respectively.

Given any $\mathbf{x}$ in V, the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ is in $\mathbb{R}^n$ and the coordinate vector of its image, $[T(\mathbf{x})]_{\mathcal{C}}$, is in $\mathbb{R}^m$.

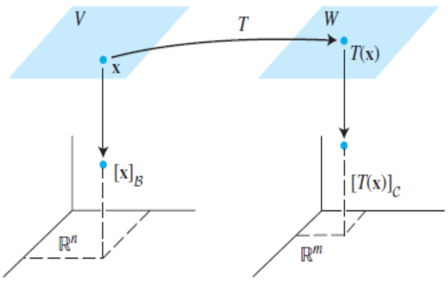


Figure 26.1: Coordinate map and linear transformation

The connection between $[\mathbf{x}]_{\mathcal{B}}$ and $[T(\mathbf{x})]_{\mathcal{C}}$, is easy to find. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n\}$ be the basis for B, if $\mathbf{x} = r_1\mathbf{b}_1 + r_2\mathbf{b}_2 + ... + r_n\mathbf{b}_n$ then

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} r_1 \\ r_2 \\ . \\ . \\ . \\ r_n \end{bmatrix}$$

and $\quad T(\mathbf{x}) = T(r_1\mathbf{b}_1 + r_2\mathbf{b}_2 + ... + r_n\mathbf{b}_n) = r_1T(\mathbf{b}_1) + r_2T(\mathbf{b}_2) + ... + r_nT(\mathbf{b}_n)$

because T is linear. Now, since the coordinate mapping from W to $\mathbb{R}^m$ is linear then we have

$$[T(\mathbf{x})]_{\mathcal{C}} = r_1[T(\mathbf{b}_1)]_{\mathcal{C}} + r_2[T(\mathbf{b}_2)]_{\mathcal{C}} + ... + r_n[T(\mathbf{b}_n)]_{\mathcal{C}}.$$

The vector equation can be written in the matrix form $[T(\mathbf{x})]_{\mathcal{C}} = M[\mathbf{x}]_{\mathcal{B}}$

$$\text{where} \qquad M = [[T(\mathbf{b}_1)]_{\mathcal{C}} \;\; [T(\mathbf{b}_2)]_{\mathcal{C}} \;\; ... \;\; [T(\mathbf{b}_n)]_{\mathcal{C}}].$$

Example: Suppose $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis for V and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ is a basis for W. Let $T : V \to W$ be a linear transformation with the property that

$$T(\mathbf{b}_1) = 3\mathbf{c}_1 - 2\mathbf{c}_2 + 5\mathbf{c}_3 \quad \text{and} \quad T(\mathbf{b}_2) = 4\mathbf{c}_1 + 7\mathbf{c}_2 - \mathbf{c}_3$$

Find the matrix M for T relative to $\mathcal{B}$ and $\mathcal{C}$.
Solution: The $\mathcal{C}$-coordinate vectors of the images of $\mathbf{b}_1$ and $\mathbf{b}_2$ are

$$[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \qquad [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} 4 \\ 7 \\ -1 \end{bmatrix}$$

and the required matrix is

$$M = \begin{bmatrix} 3 & 4 \\ -2 & 7 \\ 5 & -1 \end{bmatrix}.$$

Remark: If $\mathcal{B}$ and $\mathcal{C}$ are basis for the same space V and if T is the identity transformation $T(\mathbf{x}) = \mathbf{x}$ for $\mathbf{x}$ in V, then matrix $M = [[T(\mathbf{b}_1)]_{\mathcal{C}} \;\; [T(\mathbf{b}_2)]_{\mathcal{C}} \;\; \ldots \;\; [T(\mathbf{b}_n)]_{\mathcal{C}}]$ is just a change of coordinates matrix.

The matrix M is called the matrix for T relative to $\mathcal{B}$, or simply the $\mathcal{B}$-matrix for T.

Example: The mapping $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined by

$$T(a_0 + a_1 t + a_2 t^2) = a_1 + 2a_2 t$$

is a linear transformation.

1. Find $\mathcal{B}$-matrix for T, when $\mathcal{B} = \{1, t, t^2\}$.
2. Verify that $[T(\mathbf{p})]_{\mathcal{B}} = [T]_{\mathcal{B}}[\mathbf{p}]_{\mathcal{B}}$ for each $\mathbf{p}$ in $\mathbb{P}_2$.

Solution: Compute the images of the basis vector, that is, $T(1) = 0, \quad T(t) = 1, \quad T(t^2) = 2t$.

Then write the $\mathcal{B}$-coordinate vectors of $T(1), T(t), T(t^2)$ and place them together as the $\mathcal{B}$-matrix for T:

$$[T(1)]\mathcal{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad [T(t)]\mathcal{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad [T(t^2)]\mathcal{B} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

and $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

Example: The mapping $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined by

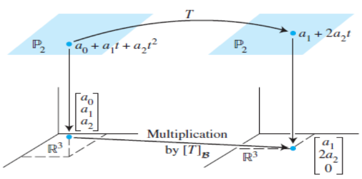
$$T(a_0 + a_1 t + a_2 t^2) = a_1 + 2a_2 t$$

is a linear transformation.

1. Find $\mathcal{B}$-matrix for T, when $\mathcal{B} = \{1, t, t^2\}$.
2. Verify that $[T(\mathbf{p})]_{\mathcal{B}} = [T]_{\mathcal{B}}[\mathbf{p}]_{\mathcal{B}}$ for each $\mathbf{p}$ in $\mathbb{P}_2$.

Solution: (2) For general $\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2$,

$$[T(\mathbf{p})]_{\mathcal{B}} = [a_1 + 2a_2 t]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ 2a_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = [T]_{\mathcal{B}}[\mathbf{p}]_{\mathcal{B}}$$



Theorem: Diagonal Matrix Representation

Suppose $A = PDP^{-1}$, where D is a diagonal $n \times n$ matrix. If B is the basis for $\mathbb{R}^n$ formed from the columns of P, then D is the $\mathcal{B}$-matrix for the transformation $\mathbf{x} \mapsto A\mathbf{x}$.

Example: Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Find a basis $\mathcal{B}$ for $\mathbb{R}^2$ with the property that the $\mathcal{B}$-matrix for T is a diagonal matrix.

Solution: $A = PDP^{-1}$ where

$$P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}.$$

The columns of P, call them $\mathbf{b}_1$ and $\mathbf{b}_2$, are eigenvectors of A. The $\mathcal{B}$-matrix for T when $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. The mappings $\mathbf{x} \mapsto A\mathbf{x}$ and $\mathbf{u} \mapsto D\mathbf{u}$ describe the same linear transformation, relative to different basis.

Remark: If A is similar to a matrix C, with $A = PCP^{-1}$, then C is the $\mathcal{B}$-matrix for the transformation $\mathbf{x} \mapsto A\mathbf{x}$ when the basis $\mathcal{B}$ is formed from the columns of P. It is not necessary that C is a diagonal matrix.

$$\begin{tikzcd}
\mathbf{x} \arrow[r, "\text{Multiplication}", "\text{by } A"'] \arrow[d, "\text{Multiplication}"', "\text{by } P^{-1}"] & A\mathbf{x} \\
{[\mathbf{x}]_{\mathcal{B}}} \arrow[r, "\text{Multiplication}", "\text{by } C"'] & {[A\mathbf{x}]_{\mathcal{B}}} \arrow[u, "\text{Multiplication}"', "\text{by } P"]
\end{tikzcd}$$

Conversely, if $T : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $T(\mathbf{x}) = A\mathbf{x}$, and if B is any basis for $\mathbb{R}^n$, then the $\mathcal{B}$-matrix for T is similar to A. In fact, if P is the matrix whose columns come from the vectors in $\mathcal{B}$, then $[T]_{\mathcal{B}} = P^{-1}AP$. Thus, the set of all matrices similar to a matrix A coincides with the set of all matrix representations of the transformation $\mathbf{x} \mapsto A\mathbf{x}$.

Example: Let $A = \begin{bmatrix} 4 & -9 \\ 4 & -8 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, and $\mathbf{b}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. The characteristic polynomial of A is $(\lambda + 2)^2$, but the eigenspace for the eigenvalue -2 is only onedimensional; so A is not diagonalizable. However, the basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ has the property that the $\mathcal{B}$-matrix for the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is a triangular matrix called the Jordan form of A. Find this $\mathcal{B}$-matrix.

Solution: If $P = [\mathbf{b}_1 \quad \mathbf{b}_2]$, then the $\mathcal{B}$-matrix is $P^{-1}AP$.

$$AP = \begin{bmatrix} 4 & -9 \\ 4 & -8 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -6 & -1 \\ -4 & 0 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} -6 & -1 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$$

Notice that the eigenvalue of A is on the diagonal.

Example: Find $T(a_0 + a_1t + a_2t^2)$, if T is the linear transformation from $\mathbb{P}_2$ to $\mathbb{P}_2$ whose matrix relative to $\mathcal{B} = \{1, t, t^2\}$ is

$$[T]_{\mathcal{B}} = \begin{bmatrix} 3 & 4 & 0 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{bmatrix}.$$

Example: Let A, B, and C be $n \times n$ matrices. Verify that

1. A is similar to A.

2. If A is similar to B and B is similar to C, then A is similar to C.

26.2 Some Practice Problems

Question: If T is transformation from $\mathbb{P}_2$ to $\mathbb{P}_3$ that maps a polynomial $\mathbf{p}(t)$ to $(t+3)\mathbf{p(t)}$

1. Find the image of $\mathbf{p}(t) = 3 - 2t + t^2$.
2. Show that T is a linear transformation.
3. Find the matrix for T relative to the bases $\{1, t, t^2\}$ and $\{1, t, t^2, t^3\}$.

Question: If T is transformation from $\mathbb{P}_3$ to $\mathbb{R}^4$ that maps a polynomial $\mathbf{p}(t)$ to $\begin{bmatrix} \mathbf{p}(-2) \\ \mathbf{p}(3) \\ \mathbf{p}(1) \\ \mathbf{p}(0) \end{bmatrix}$

1. Find the image of $\mathbf{p}(t) = 2t - t^2 + t^3$.
2. Show that T is a linear transformation.
3. Find the matrix for T relative to the basis $\{1, t, t^2, t^3\}$ and standard basis of $\mathbb{R}^4$.

Question: The trace of a square matrix A is the sum of the diagonal entries in A and is denoted by tr A. It can be verified that tr $(FG) = tr(GF)$ for any two $n \times n$ matrices F and G. Show that if A and B are similar, then tr A=tr B.

CHAPTER 27

Lecture No. 27

27.1 The Inner Product

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, then we regard $\mathbf{u}$ and $\mathbf{u}$ as $n \times 1$ matrices.

The transpose $\mathbf{u}^T$ is a $1 \times n$ matrix, and the matrix product $\mathbf{u}^T\mathbf{v}$ is a 1×1 matrix, which we write as a single real number (a scalar) without brackets.

The number $\mathbf{u}^T\mathbf{v}$ is called the inner product of $\mathbf{u}$ and $\mathbf{v}$, and often it is written as $\mathbf{u}.\mathbf{v}$.

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ . \\ . \\ . \\ u_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ . \\ . \\ . \\ v_n \end{bmatrix}$$

then the inner product of $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u}.\mathbf{v} = u_1v_1 + u_2v_2 + ... + u_nv_n.$$

Example: Compute $\mathbf{u}.\mathbf{v}$ and $\mathbf{v}.\mathbf{u}$ for $\mathbf{u}$ and $\mathbf{v}$ for $\mathbf{u} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$.

Theorem: Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

1. $\mathbf{v}.\mathbf{u} = \mathbf{u}.\mathbf{v}$
2. $(\mathbf{v} + \mathbf{u}).\mathbf{w} = \mathbf{v}.\mathbf{w} + \mathbf{u}.\mathbf{w}$
3. $(c\mathbf{u}).\mathbf{v} = \mathbf{u}.(c\mathbf{v}) = c(\mathbf{v}.\mathbf{u})$
4. $\mathbf{u}.\mathbf{u} \geq 0$, and $\mathbf{u}.\mathbf{u} = 0$ if and only if $\mathbf{u} = 0$

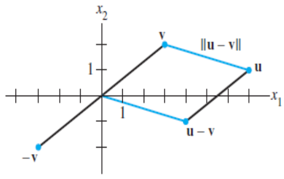
The Length of a Vector: The length (or norm) of $\mathbf{v}$ is the nonnegative scalar $\mathbf{v}$ defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v}.\mathbf{v}} = \sqrt{v_1^2 + v_2^2 + ... + v_n^2} = \mathbf{v}.\mathbf{v}, \quad \text{and} \quad \|\mathbf{v}\|^2 = \mathbf{v}.\mathbf{v}$$

Example: Let $\mathbf{v} = (1, -2, 2, 0)$. Find a unit vector $\mathbf{u}$ in the direction of $\mathbf{v}$.

Solution: the unit vector is $\frac{\mathbf{v}}{\sqrt{\mathbf{v}.\mathbf{v}}}$.
Example: Let W be the subspace of $\mathbb{R}^2$ spanned by $\mathbf{x} = (2/3, 1)$. Find a unit vector $\mathbf{z}$ that is a basis for W.
Distance in $\mathbb{R}^n$: For $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, the distance between $\mathbf{u}$ and $\mathbf{v}$,written as dist($\mathbf{u}$,$\mathbf{v}$), is the length of the vector $\mathbf{u} - \mathbf{v}$. That is

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|.$$



Example: Compute the distance between the vectors $\mathbf{u} = (2, -1, 2, 1)$, $\mathbf{v} = (1, -2, 2, 0)$.
Example: Compute the distance between the vectors $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$.

27.2 Orthogonal Vectors

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal (to each other) if $\mathbf{u}.\mathbf{v} = 0$.

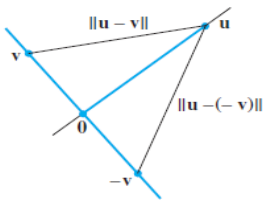


Figure 27.1: Orthogonal vectors

Theorem: Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2+\|\mathbf{v}\|^2$.

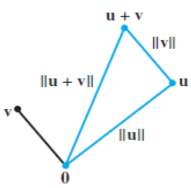
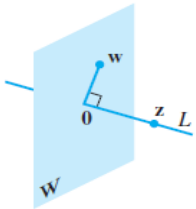


Figure 27.2: Orthogonal vectors

Orthogonal Complements: If a vector $\mathbf{z}$ is orthogonal to every vector in a subspace W of $\mathbb{R}^n$, then $\mathbf{z}$ is said to be orthogonal to W. The set of all vectors $\mathbf{z}$ that

are orthogonal to W is called the orthogonal complement of W and is denoted by $W^{\perp}$.
Example: Let W be a plane through the origin in $\mathbb{R}^3$, and let L be the line through the origin and perpendicular to W. If $\mathbf{z}$ and $\mathbf{w}$ are nonzero, $\mathbf{z}$ is on L, and $\mathbf{w}$ is in W then the line segment from $\mathbf{0}$ to $\mathbf{z}$ is perpendicular to the line segment from $\mathbf{0}$ to $\mathbf{w}$; that is, $\mathbf{z}.\mathbf{w} = \mathbf{0}$.



So each vector on L is orthogonal to every w in W.

In fact, L consists of all vectors that are orthogonal to the $w's$ in W, and W consists of all vectors orthogonal to the $z's$ in L. That is,

$$L = W\perp \quad \text{and} \quad W = L^{\perp}.$$

Orthogonal Complements:

1. A vector $\mathbf{x}$ is in $W^{\perp}$ if and only if $\mathbf{x}$ is orthogonal to every vector in a set that spans W.

2. W is a subspace of $\mathbb{R}^n$.

Theorem: Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T:

$$(\text{Row} \quad A)^{\perp} = \text{Nul} \quad A \quad \text{and} \quad (\text{Col} \quad A)^{\perp} = \text{Nul} \quad A^T.$$

Orthogonal Sets: A set of vectors $\{\mathbf{u}_1, ..., \mathbf{u}_p\}$ in $\mathbb{R}^n$ is said to be an orthogonal set if each pair of distinct vectors from the set is orthogonal, that is, if $\mathbf{u}_i.\mathbf{u}_j = 0$ whenever $i \neq j$.

Example: Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set, where

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$$

Theorem: If $S = \{\mathbf{u}_1, ..., \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent and hence is a basis for the subspace spanned by S.

Orthogonal Basis: An orthogonal basis for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.

Theorem: If $\{\mathbf{u}_1, ..., \mathbf{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$, For each $\mathbf{y}$ in W, the weights in the linear combination

$$\mathbf{y} = c_1\mathbf{u}_1 + ... + c_p\mathbf{u}_p$$

are given by

$$c_j = \frac{\mathbf{y}.\mathbf{u}_j}{\mathbf{u}_j.\mathbf{u}_j}, \quad (j = 1, ..., p).$$

Example: The set $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$ where

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}.$$

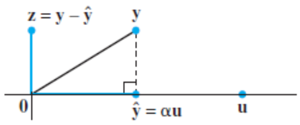
Express the vector $\mathbf{y} = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$ as a linear combination of the vectors in S.

An Orthogonal Projection: Given a nonzero vector $\mathbf{u}$ in $\mathbb{R}^n$, consider the problem of decomposing a vector $\mathbf{y}$ in $\mathbb{R}^n$ into the sum of two vectors, one a multiple of $\mathbf{u}$ and the other orthogonal to $\mathbf{u}$. We wish to write

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

where $\hat{\mathbf{y}} = \alpha\mathbf{u}$ for some scalar α and $\mathbf{z}$ is some vector orthogonal to $\mathbf{u}$.

$$\hat{\mathbf{y}} = \frac{\mathbf{y}.\mathbf{u}}{\mathbf{u}.\mathbf{u}}\mathbf{u}.$$



Example: Let $\mathbf{y} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Find the orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$. Then write $\mathbf{y}$ as the sum of two orthogonal vectors, one in Span $\{\mathbf{u}\}$ and one orthogonal to $\mathbf{u}$.

Solution: Compute

$$\mathbf{y}.\mathbf{u} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} . \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 40, \quad \mathbf{u}.\mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} . \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 20.$$

The orthogonal projection is

$$\hat{\mathbf{y}} = \frac{\mathbf{y}.\mathbf{u}}{\mathbf{u}.\mathbf{u}}\mathbf{u} = \frac{40}{20}\mathbf{u} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}.$$

27.3 Orthonormal Sets

A set of vectors $\{\mathbf{u}_1, ..., \mathbf{u}_p\}$ in $\mathbb{R}^n$ is said to be an orthogonal set if each pair of distinct vectors from the set is orthogonal, that is, if $\mathbf{u}_i.\mathbf{u}_j = 0$ whenever $i \neq j$ and $\mathbf{u}_i.\mathbf{u}_i = 1$.

Example: Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthonormal basis in $\mathbb{R}^3$, where

$$\mathbf{v}_1 = \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1/\sqrt{66} \\ -4/\sqrt{66} \\ 7/\sqrt{66} \end{bmatrix}.$$

Theorem: An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I$.
Theorem: Let U be an $m \times n$ matrix with orthonormal columns, and let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbb{R}^n$. Then

1. $\|U\mathbf{x}\| = \|\mathbf{x}\|$
2. $(U\mathbf{x}).(U\mathbf{y}) = \mathbf{x}.\mathbf{y}$
3. $(U\mathbf{x}).(U\mathbf{y}) = 0$ if and only if $\mathbf{x}.\mathbf{y} = 0$.

Example: Let $U = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}$. Show that U has orthonormal columns and verify that $\|U\mathbf{x}\| = \|\mathbf{x}\|$.

The matrix
$$U = \begin{bmatrix} 3/\sqrt{11} & -1/\sqrt{6} & -1/\sqrt{66} \\ 1/\sqrt{11} & 2/\sqrt{6} & -4/\sqrt{66} \\ 1/\sqrt{11} & 1/\sqrt{6} & 7/\sqrt{66} \end{bmatrix}$$
is an orthogonal matrix because it is square and because its columns are orthonormal. Verify that the rows are orthonormal, too.

27.4 Some Practice Problems

Question: Determine which set of vectors are orthogonal

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}.$$

Question: Compute orthogonal projection of $\begin{bmatrix} 1 \\ 7 \end{bmatrix}$ onto the line passing through $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ and the origin.

Question: Suppose W is a subspace of $\mathbb{R}^n$ spanned by n nonzero orthogonal vectors. Explain why $W = \mathbb{R}^n$.

Question: Let U be a square matrix with orthonormal columns. Explain why U is invertible.

Question: Let U be an $n \times n$ orthogonal matrix. how that the rows of U form an orthonormal basis of $\mathbb{R}^n$.

CHAPTER 28

Lecture No. 28

Question: Suppose W is a subspace of $\mathbb{R}^n$ spanned by n nonzero orthogonal vectors. Explain why $W = \mathbb{R}^n$.
Solution: We know that an orthogonal set of vectors is linearly independent. Hence form a basis for the space W. Consequently, dimension of W is n and $W = \mathbb{R}^n$.

28.1 Orthogonal Projections

The orthogonal projection of a point in $\mathbb{R}^2$ onto a line through the origin has an important analogue in $\mathbb{R}^n$.

Given a vector $\mathbf{y}$ and a subspace W in $\mathbb{R}^n$, there is a vector $\hat{\mathbf{y}}$ in W such that
(1) $\hat{\mathbf{y}}$ is the unique vector in W for which $\mathbf{y} - \hat{\mathbf{y}}$ is orthogonal to W, and
(2) $\hat{\mathbf{y}}$ is the unique vector in W closest to $\mathbf{y}$.

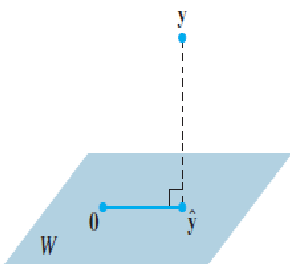


Figure 28.1: Shortest distance

These two properties of $\hat{\mathbf{y}}$ provide the key to finding least-squares solutions of linear systems
Example: Let $\{\mathbf{u}_1, ..., \mathbf{u}_5\}$ be an orthonormal basis for $\mathbb{R}^5$ and let

$$\mathbf{y} = c_1\mathbf{u}_1 + ... + c_5\mathbf{u}_5.$$

Consider the subspace $W = Span\{\mathbf{u}_1, \mathbf{u}_2\}$, and write $\mathbf{y}$ as the sum of a vector $\mathbf{z}_1$ in W and a vector $\mathbf{z}_2$ in $W^{\perp}$.
Solution: Write $\mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + ... + c_5\mathbf{u}_5$

where $\mathbf{z}_1 = c_1\mathbf{u}_1 + c_2\mathbf{u}_2$ is in $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ and $\mathbf{z}_2 = c_3\mathbf{u}_3 + c_4\mathbf{u}_4 + c_5\mathbf{u}_5$ is in $\text{Span}\{\mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5\}$.
Theorem: The Orthogonal Decomposition Theorem

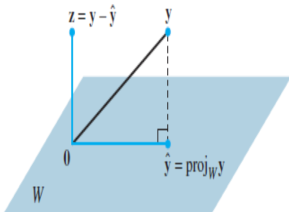
Let W be a subspace of $\mathbb{R}^n$. Then each $\mathbf{y}$ in $\mathbb{R}^n$ can be written uniquely in the form

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

where $\hat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^\perp$.

In fact, if $\{\mathbf{u}_1, ..., \mathbf{u}_p\}$ is any orthogonal basis of W, then

$$\hat{\mathbf{y}} = \frac{\mathbf{y}.\mathbf{u}_1}{\mathbf{u}_1.\mathbf{u}_1}\mathbf{u}_1 + ... + \frac{\mathbf{y}.\mathbf{u}_p}{\mathbf{u}_p.\mathbf{u}_p}\mathbf{u}_p.$$



The vector $\hat{\mathbf{y}}$ is called the orthogonal projection of $\mathbf{y}$ onto W and is often is written $proj_W$.

Example: Let $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Observe that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis for $W = Span\{\mathbf{u}_1, \mathbf{u}_2\}$. Write $\mathbf{y}$ as the sum of a vector in W and a vector orthogonal to W.

Solution: The orthogonal projection of $\mathbf{y}$ onto W is

$$\begin{aligned} \hat{\mathbf{y}} &= \frac{\mathbf{y}.\mathbf{u}_1}{\mathbf{u}_1.\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}.\mathbf{u}_2}{\mathbf{u}_2.\mathbf{u}_2}\mathbf{u}_2 \\ &= \frac{9}{30}\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{3}{6}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \frac{9}{30}\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{15}{30}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix}. \end{aligned}$$

Also

$$\mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix} = \begin{bmatrix} 7/5 \\ 0 \\ 14/5 \end{bmatrix}.$$

28.2 A Geometric Interpretation of the Orthogonal Projection

When W is a subspace of $\mathbb{R}^3$ spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$. Here $\hat{\mathbf{y}}_1$ and $\hat{\mathbf{y}}_2$ denote the projections of $\mathbf{y}$ onto the lines spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$, respectively.

The orthogonal projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ onto W is the sum of the projections of $\mathbf{y}$ onto one-dimensional subspaces that are orthogonal to each other.

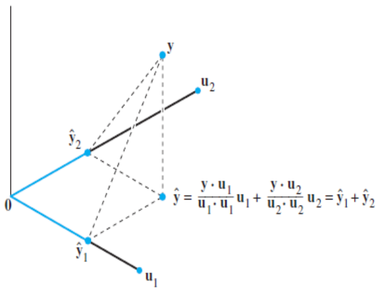


Figure 28.2: Geometric interpretation of orthogonal projection

Properties of Orthogonal Projections:

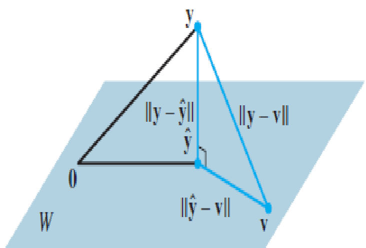
If $\mathbf{y}$ is in $W = Span\{\mathbf{u}_1, ..., \mathbf{u}_p\}$, then $proj_W\, \mathbf{y} = \mathbf{y}$.

28.3 The Best Approximation Theorem

Let W be a subspace of $\mathbb{R}^n$, let $\mathbf{y}$ be any vector in $\mathbb{R}^n$, and let $\hat{\mathbf{b}}$ be the orthogonal projection of $\mathbf{y}$ onto W. Then $\hat{\mathbf{b}}$ is the closest point in W to $\mathbf{y}$, in the sense that

$$\|\mathbf{y} - \hat{\mathbf{b}}\| < \|\mathbf{y} - \mathbf{v}\|$$

for all $\mathbf{v}$ in W distinct from $\hat{\mathbf{b}}$.



Remark: The vector $\hat{\mathbf{b}}$ is called the best approximation to $\mathbf{y}$ by elements of W.

Example: If $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and $W = Span\{\mathbf{u}_1, \mathbf{u}_2\}$ then the closest point in W to $\mathbf{y}$ is

$$\hat{\mathbf{y}} = \frac{\mathbf{y}.\mathbf{u}_1}{\mathbf{u}_1.\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}.\mathbf{u}_2}{\mathbf{u}_2.\mathbf{u}_2}\mathbf{u}_2 = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix}.$$

Example: The distance from a point $\mathbf{y}$ in $\mathbb{R}^n$ to a subspace W is defined as the distance from $\mathbf{y}$ to the nearest point in W. Find the distance from $\mathbf{y}$ to $W = Span\{\mathbf{u}_1, \mathbf{u}_2\}$, where $\mathbf{y} = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$, and $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

Solution: By the Best Approximation Theorem, the distance from $\mathbf{y}$ to W is $\|\mathbf{y}-\hat{\mathbf{y}}\|$, where $\hat{\mathbf{y}} = proj_W\mathbf{y}$. Since $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis for W,

$$\begin{aligned} \hat{\mathbf{y}} &= \frac{\mathbf{y}.\mathbf{u}_1}{\mathbf{u}_1.\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}.\mathbf{u}_2}{\mathbf{u}_2.\mathbf{u}_2}\mathbf{u}_2 = \frac{15}{30}\mathbf{u}_1 + \frac{-21}{6}\mathbf{u}_2 \\ &= \frac{1}{2}\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} - \frac{7}{2}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -8 \\ 4 \end{bmatrix}. \end{aligned}$$

$$\|\mathbf{y}-\hat{\mathbf{y}}\|^2 = 0^2 + 3^2 + -^2 = 45.$$

Theorem: If $\{\mathbf{u}_1, ..., \mathbf{u}_p\}$ is an orthonormal basis for a subspace W of $\mathbb{R}^n$, then

$$proj_W\ \mathbf{y} = (\mathbf{y}.\mathbf{u}_1)\mathbf{u}_1 + (\mathbf{y}.\mathbf{u}_2)\mathbf{u}_2 + ... + (\mathbf{y}.\mathbf{u}_p)\mathbf{u}_p.$$

If $U = [\mathbf{u}_1\ \ \mathbf{u}_1\ ...\ \mathbf{u}_p]$ then

$$proj_W\ \mathbf{y} = UU^T\mathbf{y} \quad \text{for all } \mathbf{y} \in \mathbb{R}^n.$$

Example: Let $\mathbf{y} = \begin{bmatrix} -9 \\ 1 \\ 6 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$, and $\mathbf{u}_2 = \begin{bmatrix} -9 \\ 1 \\ 6 \end{bmatrix}$, and $W = Span\{\mathbf{u}_1, \mathbf{u}_2\}$. Use this fact that $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal to compute $proj_W\ \mathbf{y}$.

28.4 Some Practice Problems

Question: Let $\mathbf{x} = \begin{bmatrix} 4 \\ 5 \\ -3 \\ 3 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}$. Write $\mathbf{x}$ as the sum of two vectors, one in Span$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and the other in Span$\{\mathbf{u}_4\}$.

Question: Find the closed point to $\mathbf{y}$ in the subspace W spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$ $\mathbf{y} = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$.

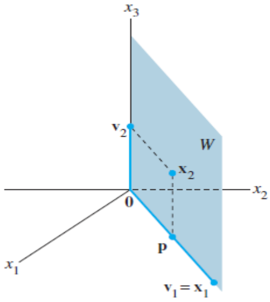
Question: Find find the best approximation to $\mathbf{x}$ by vectors of the form $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ as the vectors in previous example.

Chapter 29

Lecture No. 29

29.1 The Gram-Schmidt process

The Gram-Schmidt process is a simple algorithm for producing an orthogonal or orthonormal basis for any nonzero subspace of $\mathbb{R}^n$.



Example: Let $W = Span\{\mathbf{x}_1, \mathbf{x}_2\}$, where $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Construct an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ for W.

Solution: Let $\mathbf{v}_1 = \mathbf{x}_1$ and

$$\mathbf{v}_2 = \mathbf{x}_2 - \mathbf{p} = \mathbf{x}_2 - \frac{\mathbf{x}_2.\mathbf{x}_1}{\mathbf{x}_1.\mathbf{x}_1}\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{15}{45}\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

Example: Let $W = Span\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$, where $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. Construct an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for W which is subspace of $\mathbb{R}^4$.

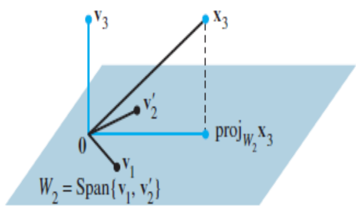
Solution: **Step I**: Let $\mathbf{v}_1 = \mathbf{x}_1$ and $W_1 = Span\{\mathbf{x}_1\} = Span\{\mathbf{v}_1\}$.
Step II: Let $\mathbf{v}_2$ be the vector produced by subtracting from $\mathbf{x}_2$ its projection onto

the subspace W_1. That is, let

$$\mathbf{v}_2 = \mathbf{x}_2 - proj_{W_1}\mathbf{x}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2.\mathbf{v}_1}{\mathbf{v}_1.\mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}.$$

$\mathbf{v}_2$ is the component of $\mathbf{x}_2$ orthogonal to $\mathbf{x}_1$ and $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for the subspace W_2 Spanned by $\mathbf{x}_1$ and $\mathbf{x}_2$.

Step II' Optional: Scale $\mathbf{v}_2$, we get $\mathbf{v}_2' = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.



Step III: Let $\mathbf{v}_3$ be the vector produced by subtracting from $\mathbf{x}_3$ its projection onto the subspace W_2. Use the orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2'\}$ to compute this projection onto W_2:

$$proj_{W_2}\mathbf{x}_3 = \frac{\mathbf{x}_3.\mathbf{v}_1}{\mathbf{v}_1.\mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{x}_3.\mathbf{v}_2'}{\mathbf{v}_2'.\mathbf{v}_2'}\mathbf{v}_2' = \frac{2}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{2}{12}\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

Then $\mathbf{v}_3$ is the component of $\mathbf{x}_3$ orthogonal to W_2, namely,

$$\mathbf{v}_3 = \mathbf{x}_3 - proj_{W_2}\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 2/3 \\ 2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}.$$

Theorem: The Gram-Schmidt Process

Given a basis $\{\mathbf{x}_1, ..., \mathbf{x}_p\}$ for a nonzero subspace W of $\mathbb{R}^n$, define

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2.\mathbf{v}_1}{\mathbf{v}_1.\mathbf{v}_1}\mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3.\mathbf{v}_1}{\mathbf{v}_1.\mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3.\mathbf{v}_2}{\mathbf{v}_2.\mathbf{v}_2}\mathbf{v}_2 \\
&. \\
&. \\
&. \\
\mathbf{v}_p &= \mathbf{x}_p - \frac{\mathbf{x}_p.\mathbf{v}_1}{\mathbf{v}_1.\mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_p.\mathbf{v}_2}{\mathbf{v}_2.\mathbf{v}_2}\mathbf{v}_2 - ... - \frac{\mathbf{x}_p.\mathbf{v}_{p-1}}{\mathbf{v}_{p-1}.\mathbf{v}_{p-1}}\mathbf{v}_{p-1}.
\end{aligned}$$

Then $\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ is an orthogonal basis for W. In addition, we have

$$Span\{\mathbf{v}_1, ..., \mathbf{v}_k\} = Span\{\mathbf{x}_1, ..., \mathbf{x}_k\} \qquad \text{for } 1 \le k \le p.$$

Example: Let $W = Span\{\mathbf{x}_1, \mathbf{x}_2\}$, where

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1/3 \\ 1/3 \\ -2/3 \end{bmatrix}.$$

Construct an orthonormal basis for W.

Orthonormal Basis: Construct an orthonormal basis for the subspace spanned by the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

Example: Find an orthogonal basis for the column space and null space of the matrix

$$\begin{bmatrix} 1 & 3 & 5 \\ -1 & -3 & 1 \\ 0 & 2 & 3 \\ 1 & 5 & 2 \\ 1 & 5 & 8 \end{bmatrix}.$$

29.2 Some Practice Problems

Question: Find an orthogonal basis for the column space and null space of each matrix

$$\begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 3 & 5 & 0 \\ -1 & -3 & 1 & 1 \\ 0 & 2 & 3 & 1 \\ 1 & 5 & 2 & 2 \end{bmatrix}.$$

Question: Mark each statement as true or false

1. If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for W , then multiplying $\mathbf{v}_3$ by a scalar c gives a new orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, c\mathbf{v}_3\}$.
2. If $W = Span\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ with $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ linearly independent, and if If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set in W, then If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for W.
3. If $\mathbf{x}$ is not in a subspace W, then $\mathbf{x} - proj_W\mathbf{x}$ is not zero.

Question: Construct an orthogonal basis using Gram-Schmidt process for $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix}, \qquad \mathbf{x}_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}.$

Chapter 30

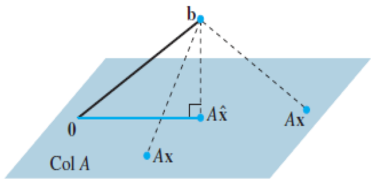
Lecture No. 30

30.1 Least Squares Solutions

If A is $m \times n$ and $\mathbf{b}$ is in $\mathbb{R}^m$, a least-squares solution of $A\mathbf{x} = \mathbf{b}$ is an $\hat{\mathbf{x}}$ in $\mathbb{R}^n$ such that

$$\|\mathbf{b} - A\hat{\mathbf{x}}\| \leq \|\mathbf{b} - A\mathbf{x}\|$$

for all $\mathbf{x}$ in $\mathbb{R}^n$.

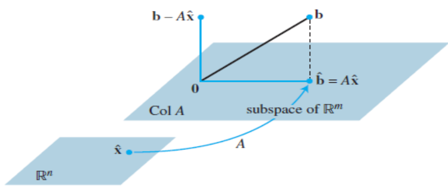


Remark: The most important aspect of the least-squares problem is that no matter what $\mathbf{x}$ we select, the vector $A\mathbf{x}$ will necessarily be in the column space, Col A. So we seek an $\mathbf{x}$ that makes $A\mathbf{x}$ the closest point in Col A to $\mathbf{b}$.
Solution of the General Least-Squares Problem: Given A and $\mathbf{b}$, apply the Best Approximation Theorem to the subspace Col A. Let

$$\hat{\mathbf{b}} = proj_{Col\ A}\mathbf{b}$$

Because $\hat{\mathbf{b}}$ is in the column space of A, the equation $A\mathbf{x} = \hat{\mathbf{b}}$ is consistent, and there is an $\hat{\mathbf{x}}$ in $\mathbb{R}^n$ such that

$$A\hat{\mathbf{x}} = \hat{\mathbf{b}}.$$



Since $\hat{\mathbf{x}}$ is the closest point in Col A to $\mathbf{b}$, a vector $\hat{\mathbf{x}}$ is a least-squares solution of $A\mathbf{x} = \mathbf{b}$ if and only if $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$. Suppose $\hat{\mathbf{x}}$ satisfies $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$. By the Orthogonal Decomposition Theorem the projection $\hat{\mathbf{b}}$ has the property that $\mathbf{b} - \hat{\mathbf{b}}$ is orthogonal to Col A, so $\mathbf{b} - A\hat{\mathbf{x}}$ is orthogonal to each column of A.

If $\mathbf{a}_j$ is any column of A, then $\mathbf{a}_j.(\mathbf{b} - A\hat{\mathbf{x}}) = 0$, and $\mathbf{a}_j^T(\mathbf{b} - A\hat{\mathbf{x}}) = 0$. Since each $\mathbf{a}_j^T$ is a row of A^T,

$$A^T(\mathbf{b} - A\hat{\mathbf{x}}) = 0 \Rightarrow A^T A\mathbf{x} = A^T\mathbf{b}.$$

Each least-squares solution of $A\mathbf{x} = \mathbf{b}$ satisfies the above equation.

Theorem: The set of least-squares solutions of $A\mathbf{x} = \mathbf{b}$ coincides with the nonempty set of solutions of the normal equations $A^T A\mathbf{x} = A^T\mathbf{b}$.
Example: Find a least-squares solution of the inconsistent system $A\mathbf{x} = \mathbf{b}$ for

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}.$$

Solution: Each least square solution satisfy the normal equation $A^T A\mathbf{x} = A^T\mathbf{b}$, thus we will compute:

$$A^T A = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix}$$

$$A^T\mathbf{b} = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}.$$

Then equation $A^T A\mathbf{x} = A^T\mathbf{b}$ becomes

$$\begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}.$$

Example: Find a least-squares solution of the inconsistent system $A\mathbf{x} = \mathbf{b}$ for

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}.$$

Solution:

$$(A^T A)^{-1} = \frac{1}{84} = \begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}$$

and then

$$\mathbf{x} = (A^T A)^{-1} A^T\mathbf{b} = \frac{1}{84} = \begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix} \begin{bmatrix} 19 \\ 11 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Example: Find a least-squares solution of the inconsistent system $A\mathbf{x} = \mathbf{b}$ for

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{bmatrix}.$$

Solution:
$$A^T A = \begin{bmatrix} 6 & 2 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}, \qquad A^T\mathbf{b} = \begin{bmatrix} 4 \\ -4 \\ 2 \\ 6 \end{bmatrix}.$$

The augmented matrix for $A^T A\mathbf{x} = A^T\mathbf{b}$ is

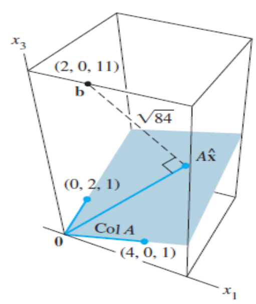
$$\begin{bmatrix} 6 & 2 & 2 & 2 & 4 \\ 2 & 2 & 0 & 0 & -4 \\ 2 & 0 & 2 & 0 & 2 \\ 2 & 0 & 0 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & -1 & -5 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Theorem: Let A be an $m \times n$ matrix. The following statements are logically equivalent:

1. The equation $A\mathbf{x} = \mathbf{b}$ has a unique least-squares solution for each $\mathbf{b}$ in $\mathbb{R}^m$.
2. The columns of A are linearly independent.
3. The matrix $A^T A$ is invertible.
4. When these statements are true, the least-squares solution $\hat{\mathbf{x}}$ is given by
$$\hat{\mathbf{x}} = (A^T A)^{-1} A^T\mathbf{b}.$$

Remark: When a least-squares solution $\hat{\mathbf{x}}$ is used to produce $A\hat{\mathbf{x}}$ as an approximation to $\mathbf{b}$, the distance from $\mathbf{b}$ to $A\hat{\mathbf{x}}$ is called the least-squares error of this approximation.
Example: Determine the least-squares error in the least-squares solution of $A\mathbf{x} = \mathbf{b}$ for $A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}.$

Solution: $A\hat{\mathbf{x}} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix}$. Then calculate $\|\mathbf{b} - A\hat{\mathbf{x}}\|$.



Alternative Calculations: When the columns of A are orthogonal.

Example: Find a least-squares solution of the system $A\mathbf{x} = \mathbf{b}$ for

$$A = \begin{bmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 6 \end{bmatrix}.$$

Solution: Because the columns $\mathbf{a}_1$ and $\mathbf{a}_2$ of A are orthogonal, the orthogonal projection of $\mathbf{b}$ onto Col A is given by

$$\begin{aligned} \hat{\mathbf{b}} &= \frac{\mathbf{b}.\mathbf{a}_1}{\mathbf{a}_1.\mathbf{a}_1}\mathbf{a}_1 + \frac{\mathbf{b}.\mathbf{a}_2}{\mathbf{a}_2.\mathbf{a}_2}\mathbf{a}_2 = \frac{8}{4}\mathbf{a}_1 + \frac{45}{90}\mathbf{a}_2 \\ &= \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ -1 \\ 1/2 \\ 7/2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 5/2 \\ 11/2 \end{bmatrix} \end{aligned}$$

We don't need to solve $A\hat{\mathbf{x}} = \mathbf{b}$. Why? We have

Remark: In some cases, the normal equations for a least-squares problem can be ill conditioned, that is, small errors in the calculations of the entries of A^TA can sometimes cause relatively large errors in the solution $\hat{\mathbf{x}}$.

Example: Find a least-squares solution of the system $A\mathbf{x} = \mathbf{b}$ for $A = \begin{bmatrix} 1 & -3 & -3 \\ 1 & 5 & 1 \\ 1 & 7 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 5 \\ -3 \\ -5 \end{bmatrix}$ compute the associated least-squares error.

Solution:

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ -3 & 5 & 7 \\ -3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -3 & -3 \\ 1 & 5 & 1 \\ 1 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 9 & 0 \\ 9 & 83 & 28 \\ 0 & 28 & 14 \end{bmatrix}$$

,

$$A^T\mathbf{b} = \begin{bmatrix} 1 & 1 & 1 \\ -3 & 5 & 7 \\ -3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ -5 \end{bmatrix} = \begin{bmatrix} -3 \\ -65 \\ -28 \end{bmatrix}.$$

Solution: Next, row reduce the augmented matrix for the normal equations, $A^TA\mathbf{x} = A^T\mathbf{b}$:

$$\begin{bmatrix} 3 & 9 & 0 & -3 \\ 9 & 83 & 28 & -65 \\ 0 & 28 & 14 & -28 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 56 & 28 & -56 \\ 0 & 28 & 14 & -28 \end{bmatrix} \sim .. \sim \begin{bmatrix} 1 & 0 & -3/2 & 2 \\ 0 & 1 & 1/2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

,

30.2 Some Practice Problems

Question: Find a least-squares solution of the system $A\mathbf{x} = \mathbf{b}$ and compute the associated least-squares error.

1. $A = \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

2. $A = \begin{bmatrix} 4 & 0 & 1 \\ 1 & -5 & 1 \\ 6 & 1 & 0 \\ 1 & -1 & -5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Question: Mark each statement true or false, justify your answer

1. The general least-squares problem is to find an $\mathbf{x}$ that makes $A\mathbf{x}$ as close as possible to $\mathbf{b}$.

2. A least-squares solution of $A\mathbf{x} = \mathbf{b}$ is a vector $\hat{\mathbf{x}}$ that satisfies $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$, where $\hat{\mathbf{b}}$is the orthogonal projection of $\mathbf{b}$ onto Col A.

3. If $\hat{\mathbf{x}}$ is a least-squares solution of $A\mathbf{x} = \mathbf{b}$, then $\hat{\mathbf{x}} = (A^TA)^{-1}A^T\hat{\mathbf{b}}$.

4. The normal equations always provide a reliable method for computing least-squares solutions.

Chapter 31

Lecture No. 31

31.1 Discrete Dynamical Systems

Eigenvalues and eigenvectors provide the key to understanding the long-term behavior, or evolution, of a dynamical system described by a difference equation

$$\mathbf{x}_{k+1} = A\mathbf{x}_k.$$

The vectors $\mathbf{x}_k$ give information about the system as time (denoted by k) passes.

The applications in this section focus on ecological problems because they are easier to state and explain than, say, problems in physics or engineering.

However, dynamical systems arise in many scientific fields. For instance, standard undergraduate courses in control systems discuss several aspects of dynamical systems. The modern statespace design method in such courses relies heavily on matrix algebra. The steady-state response of a control system is the engineering equivalent of what we call here the "long-term behavior" of the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$.

Assume that A is diagonalizable, with n linearly independent eigenvectors, $\mathbf{v}_1, ..., \mathbf{v}_n$, and corresponding eigenvalues, $\lambda_1, ..., \lambda_n$. For convenience, assume the eigenvectors are arranged so that $|\lambda_1| \geq |\lambda_2| \geq ... \geq |\lambda_n|$. Since $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$, any initial vector $\mathbf{x}_0$ can be written uniquely as

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + ... + c_n\mathbf{v}_n.$$

This eigenvector decomposition of $\mathbf{x}_0$ determines what happens to the sequence $\{\mathbf{x}_k\}$.

$$\begin{aligned} \mathbf{x}_1 = A\mathbf{x}_0 &= A(c_1\mathbf{v}_1 + ... + c_n\mathbf{v}_n) \\ &= c_1\lambda_1\mathbf{v}_1 + ... + c_n\lambda_n\mathbf{v}_n \end{aligned}$$

In general,

$$\mathbf{x}_k = A\mathbf{x}_{k-1} = c_1\lambda_1^k\mathbf{v}_1 + ... + c_n\lambda_n^k\mathbf{v}_n.$$

A Predator-Prey System: Deep in the redwood forests of California, dusky-footed wood rats provide up to 80% of the diet for the spotted owl, the main predator of the wood rat.

Example: Denote the owl and wood rat populations at time k by $\mathbf{x}_k = \begin{bmatrix} O_k \\ R_k \end{bmatrix}$, where k is the time in months, O_k is the number of owls in the region studied, and R_k is the number of rats (measured in thousands). Suppose

$$O_{k+1} = (.5)O_k + (.4)R_k, \qquad R_{k+1} = -pO_k + (1.1)R_k$$

where p is a positive parameter to be specified. The $(.5)O_k$ in the first equation says that with no wood rats for food, only half of the owls will survive each month, while the $(1.1)R_k$ in the second equation says that with no owls as predators, the rat population will grow by 10% per month. If rats are plentiful, the $(.4)R_k$ will tend to make the owl population rise, while the negative term $-pO_k$ measures the deaths of rats due to predation by owls. (In fact, $1000p$ is the average number of rats eaten by one owl in one month.) Determine the evolution of this system when the predation parameter p is .104.
Solution: When $p = .104$, the eigenvalues of the coefficient matrix

$$O_{k+1} = (.5)O_k + (.4)R_k, \qquad R_{k+1} = -pO_k + (1.1)R_k$$

turn out to be $\lambda_1 = 1.02$ and $\lambda_2 = .58$. The corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} 10 \\ 13 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

An initial $\mathbf{x}_0$ can be written as $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. Then, for $k \geq 0$,

$$\begin{aligned} \mathbf{x}_k &= c_1(1.02)^k\mathbf{v}_1 + c_2(.58)^k\mathbf{v}_2 \\ &= c_1(1.02)^k \begin{bmatrix} 10 \\ 13 \end{bmatrix} + c_2(.58)^k \begin{bmatrix} 5 \\ 1 \end{bmatrix} \end{aligned}$$

As $k \to \infty$, $(.58)^k$ rapidly approaches zero. Assume $c_1 > 0$. Then, for all sufficiently large k, $\mathbf{x}_k$ is approximately the same as $c_1(1.02)^k\mathbf{v}_1$, and we write

$$\mathbf{x}_k \approx c_1(1.02)^k \begin{bmatrix} 10 \\ 13 \end{bmatrix}.$$

Solution: we can have

$$\mathbf{x}_{k+1} \approx c_1(1.02)^{k+1} \begin{bmatrix} 10 \\ 13 \end{bmatrix} = (1.02)c_1(1.02)^k \begin{bmatrix} 10 \\ 13 \end{bmatrix} = 1.02\mathbf{x}_k.$$

Conclusion: The approximation says that eventually both entries of $\mathbf{x}_k$ (the numbers of owls and rats) grow by a factor of almost 1.02 each month, a 2% monthly growth rate.

The vector $\mathbf{x}_k$ is approximately a multiple of $(10, 13)$, so the entries in $\mathbf{x}_k$ are nearly in the same ratio as 10 to 13. That is, for every 10 owls there are about 13 thousand rats.

Facts about Dynamical Systems: A dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ in which A is $n \times n$, its eigenvalues satisfy $|\lambda_1| \geq 1$ and $1 \geq |\lambda_1|$ for $j = 2, ..., n$, and $\mathbf{v}_1$ is an eigenvector corresponding to λ_1. If $\mathbf{x}_0 = c_1\mathbf{v}_1 + ... + c_n\mathbf{v}_n$, with $c_1 \neq 0$, then we have the following equations for all sufficiently large k,

$$\mathbf{x}_{k+1} \approx \lambda_1 \mathbf{x}_k$$

and

$$\mathbf{x}_k \approx c_1(\lambda_1)^k \mathbf{v}_1.$$

The approximations in above equations can be made as close as desired by taking k sufficiently large.

By equation $\mathbf{x}_{k+1} \approx \lambda_1 \mathbf{x}_k$, the $\mathbf{x}_k$ eventually grow almost by a factor of λ_1 each time, so λ_1 determines the eventual growth rate of the system.

Also, by equation $\mathbf{x}_k \approx c_1(\lambda_1)^k \mathbf{v}_1$, the ratio of any two entries in $\mathbf{x}_k$ (for large k) is nearly the same as the ratio of the corresponding entries in $\mathbf{v}_1$.

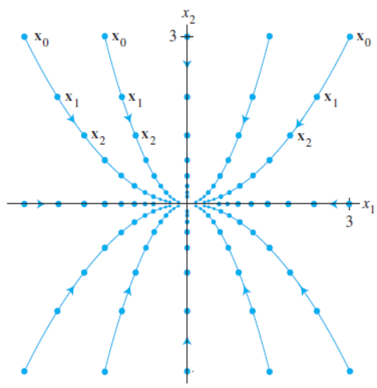
Graphical Description of Solutions: When A is 2×2, algebraic calculations can be supplemented by a geometric description of a system's evolution.

We can view the equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$ as a description of what happens to an initial point $\mathbf{x}_0$ in $\mathbb{R}^2$ as it is transformed repeatedly by the mapping $\mathbf{x} \mapsto A\mathbf{x}$. The graph of $\mathbf{x}_0, \mathbf{x}_1, ...$ is called a trajectory of the dynamical system.

Example: Plot several trajectories of the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$, when $A = \begin{bmatrix} .80 & 0 \\ 0 & .64 \end{bmatrix}$.

Solution: The eigenvalues of the matrix A are $\lambda_1 = .8$ and $\lambda_2 = .64$. The corresponding eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. An initial $\mathbf{x}_0$ can be written as $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ then

$$\mathbf{x}_k = c_1(.8)^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2(.64)^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$



The origin is called an attractor of the dynamical system because all trajectories tend toward $\mathbf{0}$. This occurs whenever both eigenvalues are less than 1 in magnitude. The direction of greatest attraction is along the line through $\mathbf{0}$ and the eigenvector $\mathbf{v}_2$ for the eigenvalue of smaller magnitude.

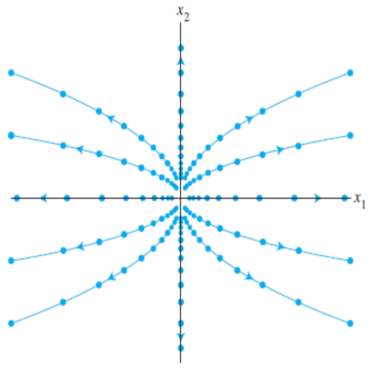
Graphical Description of Solutions:

Example: Plot several typical solutions of the equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$, when $A = \begin{bmatrix} 1.44 & 0 \\ 0 & 1.2 \end{bmatrix}$.

Solution: The eigenvalues of the matrix A are $\lambda_1 = 1.44$ and $\lambda_2 = 1.2$. The corresponding eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. An initial $\mathbf{x}_0$ can be written as $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ then

$$\mathbf{x}_k = c_1(1.44)^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2(1.2)^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Both terms grow in size, but the first term grows faster. So the direction of greatest repulsion is the line through $\mathbf{0}$ and the eigenvector for the eigenvalue of larger magnitude.



Both eigenvalues of A are larger than 1 in magnitude, and $\mathbf{0}$ is called a repeller of the dynamical system.

All solutions of $\mathbf{x}_{k+1} = A\mathbf{x}_k$ except the (constant) zero solution are unbounded and tend away from the origin.

Graphical Description of Solutions:

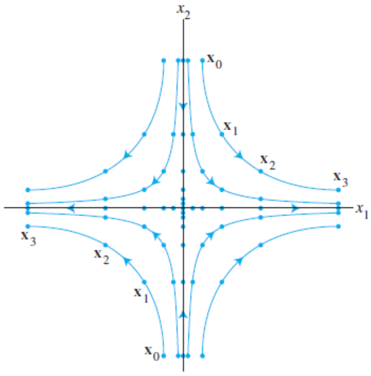
Example: Plot several typical solutions of the equation $\mathbf{x}_{k+1} = D\mathbf{x}_k$, when $D = \begin{bmatrix} 2.0 & 0 \\ 0 & 0.5 \end{bmatrix}$. Show that a solution $\{\mathbf{y}_k\}$ is unbounded if its initial point is not on the x_2-axis.

Solution: The eigenvalues of the matrix D are $\lambda_1 = 2$ and $\lambda_2 = .5$. The corresponding eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. An initial $\mathbf{x}_0$ can be written as $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ then

$$\mathbf{x}_k = c_1(2)^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2(.5)^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

If $\mathbf{y}_0$ is on the x_2-axis, then $c_1 = 0$ and $\mathbf{y}_k \to \mathbf{0}$ as $k \to \infty$.

But if $\mathbf{y}_0$ is not on the x_2-axis, then the first term in the sum for $\mathbf{y}_k$ becomes arbitrarily large, and so $\{\mathbf{y}_k\}$ is unbounded.



The $\mathbf{0}$ is called a saddle point because the origin attracts solutions from some directions and repels them in other directions. This occurs whenever one eigenvalue is greater than 1 in magnitude and the other is less than 1 in magnitude.

The direction of greatest attraction is determined by an eigenvector for the eigenvalue of smaller magnitude.

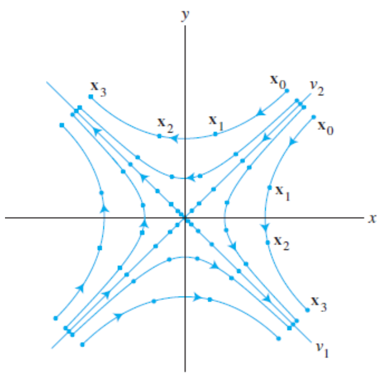
The direction of greatest repulsion is determined by an eigenvector for the eigenvalue of greater magnitude.

Example: Show that the origin is a saddle point for solutions of $\mathbf{x}_{k+1} = D\mathbf{x}_k$, when $A = \begin{bmatrix} 1.25 & -0.75 \\ -0.75 & 1.25 \end{bmatrix}$. Find the directions of greatest attraction and greatest repulsion.

Solution: The eigenvalues of the matrix A are $\lambda_1 = 2$ and $\lambda_2 = .5$. The corresponding eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Since $|2| > 1$ and $|.5| < 1$, the origin is a saddle point of the dynamical system.

An initial $\mathbf{x}_0$ can be written as $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ then

$$\mathbf{x}_k = c_1(2)^k \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2(.5)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$



On graph paper, draw axes through $\mathbf{0}$ and the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. The direction of greatest repulsion is the line through $\mathbf{0}$ and the eigenvector $\mathbf{v}_1$ whose eigenvalue is greater than 1 in magnitude.

If $\mathbf{x}_0$ is on this line, the c_2 in $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ is zero and $\mathbf{x}_k$ moves quickly away from $\mathbf{0}$.

The direction of greatest attraction is determined by the eigenvector $\mathbf{v}_2$ whose eigenvalue is less than 1 in magnitude.

31.2 Some Practice Problems

Question: The matrix A below has eigenvalues $1, 2/3$, and $1/3$, with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$:

$$A = \frac{1}{9}\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 5 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}.$$

Find the general solution of the equation $\mathbf{x}_{k+1} = D\mathbf{x}_k$ if $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 11 \\ -2 \end{bmatrix}$.

Question: What will happens to the sequence $\{\mathbf{x}_k\}$ in the above Problem as $k \to \infty$?

Question: Suppose a 3×3 matrix A has eigenvalues $3, 4/5$, and $3/5$, with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ given

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -3 \\ -3 \\ 7 \end{bmatrix}.$$

Find the general solution of the equation $\mathbf{x}_{k+1} = D\mathbf{x}_k$ if $\mathbf{x}_0 = \begin{bmatrix} -2 \\ -5 \\ 3 \end{bmatrix}$. Describe what will happens to the sequence $\{\mathbf{x}_k\}$ as $k \to \infty$?

Chapter 32

Lecture No. 32 (Revision)

In lecture number 32, I revised the important concepts in this lecture. You are advised to do the practice problems given in the handouts and from the exercises of your recommended text books.