CHAPTER 7

Lecture No. 07

Concept: Consider the following matrix equations $\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$ and $\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

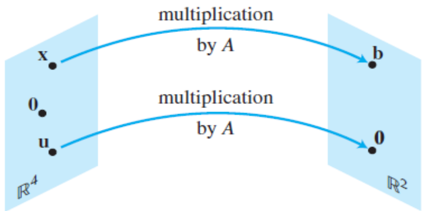


Figure 7.1: Matrix Transformation

7.1 Transformation

A transformation (or function or mapping) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$. The set $\mathbb{R}^n$ is called the **domain** of T, and $\mathbb{R}^m$ is called the **codomain** of T. The notation $T : \mathbb{R}^n \to \mathbb{R}^m$ indicates that the domain of T is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$. For $\mathbf{x}$ in $\mathbb{R}^n$, the vector $T(\mathbf{x})$ in $\mathbb{R}^m$ is called the image of $\mathbf{x}$ (under the action of T). The set of all images $T(\mathbf{x})$ is called the range of T.

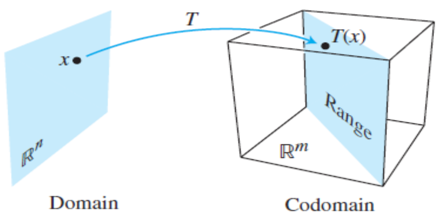


Figure 7.2: Domain and codomain

Matrix Transformation: For each $\mathbf{x}$ in $\mathbb{R}^n$, $T(\mathbf{x})$ is computed as $A\mathbf{x}$, where A is an $m \times n$ matrix.

For simplicity, we sometimes denote such a matrix transformation by $\mathbf{x} \mapsto A\mathbf{x}$. Observe that the domain of T is $\mathbb{R}^n$ when A has n columns and the codomain of T is $\mathbb{R}^m$ when each column of A has m entries. The range of T is the set of all linear combinations of the columns of A, because each image $T(\mathbf{x})$ is of the form $A\mathbf{x}$.

Example: $A = \begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}$ then find the transformation defined by the given matrix.
Solution: The order of the matric is 2×4, thus matrix gives a transformation from $\mathbb{R}^4$ to $\mathbb{R}^2$.
Example: Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ and define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$, so that

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$$

1. Find $T(\mathbf{u})$, the image of $\mathbf{u}$ under the transformation.
2. Find an $\mathbf{x}$ in $\mathbb{R}^2$ whose image under T is $\mathbf{b}$.
3. Is there more than one $\mathbf{x}$ whose image under T is $\mathbf{b}$?
4. Determine if $\mathbf{c}$ is in the range of the transformation T.

Solution: 1. $T(\mathbf{u}) = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$.

2. Solve $T(\mathbf{x}) = \mathbf{b}$ for $\mathbf{x}$, i.e., solve $A\mathbf{x} = \mathbf{b}$, the augmented matrix of the system is

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{bmatrix}$$

Which shows that the system has a unique solution and there is one unique vector in $\mathbb{R}^2$, i.e., $x_1 = 1.5$ and $x_2 = -0.5$ such that $\mathbf{b}$ is the under the given transformation.

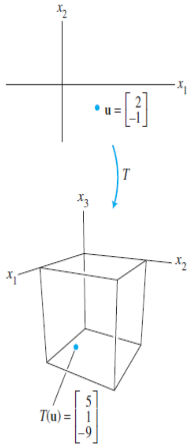


Figure 7.3: Image of $\mathbf{b}$ under matrix transformation

3. No there is one and only one $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{b}$ holds.

4. Solve $T(\mathbf{x}) = \mathbf{c}$ for $\mathbf{x}$, i.e., solve $A\mathbf{x} = \mathbf{c}$, the augmented matrix of the system is $\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 14 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -35 \end{bmatrix}$ which shows that the system is inconsistent, hence there doesn't exist any $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{c}$ holds.

Example: Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -3 \\ 2 & -5 & 6 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -6 \\ -4 \\ -5 \end{bmatrix}$. Find a vector $\mathbf{x}$ such that $T(\mathbf{x}) = A\mathbf{x} = \mathbf{b}$ satisfy. Is such an $\mathbf{x}$ is unique?
Solution: The augmented matrix is

$$\begin{bmatrix} 1 & -2 & 3 & -6 \\ 0 & 1 & -3 & -4 \\ 2 & -5 & 6 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & -6 \\ 0 & 1 & -3 & -4 \\ 0 & -9 & 12 & -17 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & -6 \\ 0 & 1 & -3 & -4 \\ 0 & 0 & -15 & -53 \end{bmatrix}.$$

The above augmented matrix in echelon form shows that the system has a unique solution.

Example: If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ projects points in $\mathbb{R}^3$ onto x_1x_2-plane because

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}.$$

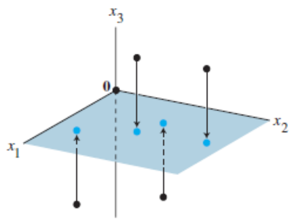


Figure 7.4: Projection transformation

Example: The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$.

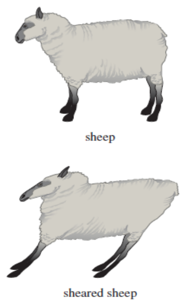


Figure 7.5: Shear transformation

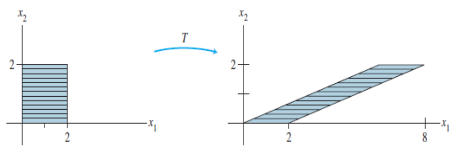


Figure 7.6: Another example of shear transformation

Shear transformations appear in physics, geology, and crystallography.

7.2 Linear Transformation

A transformation (or mapping) T is linear if

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}$, $\mathbf{v}$ in the domain of T;

2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all $\mathbf{u}$ in the domain of T.

Remark: We know that a given matrix A of order $m \times n$ satisfies the following properties

1. $A(\mathbf{u}+\mathbf{v}) = A(\mathbf{u}) + A(\mathbf{v})$ and
2. $T(c\mathbf{u}) = cT(\mathbf{u})$.

Every matrix transformation is a linear transformation.
Remark: If T is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0}, \quad \text{and,} \quad T(c\mathbf{u}+d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

for all vectors $\mathbf{u}, \mathbf{v}$ in the domain of T and scalars c, d.

Example: Let T be a linear transformation such that $T(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix}$ and $T(\begin{bmatrix} 0 \\ 2 \end{bmatrix}) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. Find the image of $\begin{bmatrix} 5 \\ 9 \end{bmatrix}$ under the linear transformation T.

Example: Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{x}$ into $x_1\mathbf{u} + x_2\mathbf{v}$ where $\mathbf{u} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$. Find a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for each $\mathbf{x}$.

Contraction and Dilation: Given a scalar r, the mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = r\mathbf{x}$. The mapping T is called a contraction when $0 \leq r \leq 1$ and a dilation if $r > 1$.

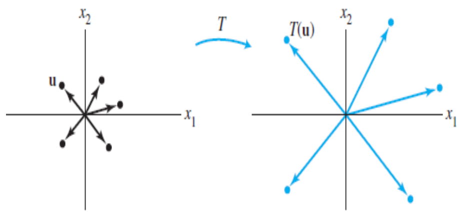


Figure 7.7: Contraction and dilation

Example: Define a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(\mathbf{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}.$$

Find the images under T of $\mathbf{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{u}+\mathbf{v} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$.

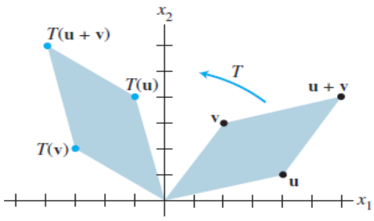


Figure 7.8: Image under linear transformation

7.3 Some Practice Problems

Question: Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -3 \\ 2 & -5 & 6 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -6 \\ -4 \\ -5 \end{bmatrix}$. Find a vector $\mathbf{x}$ such that $T(\mathbf{x}) = A\mathbf{x}$ satisfy. Is such an $\mathbf{x}$ is unique?

Question: Define $T : \mathbb{R} \to \mathbb{R}$ by $T(x) = mx + b$.

1. Show that T is a linear transformation when $b = 0$.
2. Find a property of a linear transformation that is violated when $b \neq 0$.

Question: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$(i) \quad T(\mathbf{x}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad (ii) \quad T(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Describe geometrically what T does to each vector in $\mathbb{R}^2$.

Chapter 8

Lecture No. 08

Question: Define $T : \mathbb{R} \to \mathbb{R}$ by $T(x) = mx + b$.

1. Show that T is a linear transformation when $b = 0$.
2. Find a property of a linear transformation that is violated when $b \neq 0$.

Solution: 1. Check the two conditions of linear transformation.
2. The first property of the linear transformation is not satisfied. Verify

Identity Matrix: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times 4}$.

Example: The columns of $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ such that

$$T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} \qquad \text{and} \qquad T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}.$$

With no additional information, find a formula for the image of an arbitrary $\mathbf{x}$ in $\mathbb{R}^2$.

Solution: Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ then we can write

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Since T is a linear transformation then we have

$$T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = x_T(1\begin{bmatrix} 1 \\ 0 \end{bmatrix}) + x_2 T(\begin{bmatrix} 0 \\ 1 \end{bmatrix})$$

and thus

$$T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = x_1 \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ -7 & 8 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Consequently, we can say that $A = [T(\mathbf{e}_1) \;\; T(\mathbf{e}_2)]$.

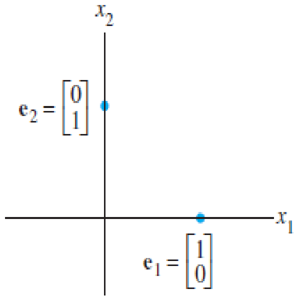


Figure 8.1: The I_2 columns

Theorem: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x}, \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n.$$

In fact, A is the $m \times n$ matrix whose jth column is he vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the jth column of the identity matrix in $\mathbb{R}^n$:

$$A = [T(\mathbf{e}_1) \;\; T(\mathbf{e}_2) \quad ... \quad T(\mathbf{e}_n)].$$

Remark: The matrix A such that $T(\mathbf{x}) = A\mathbf{x}$, is called a matrix of transformation for T.

Example: Assume that T is a linear transformation. Find the standard matrix of T

1. $T : \mathbb{R}^2 \to \mathbb{R}^4$, $T(\mathbf{e}_1) = (3, 1, 3, 1)$, and $T(\mathbf{e}_2) = (-5, 2, 0, 0)$, where $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$.
2. $T : \mathbb{R}^3 \to \mathbb{R}^2$, $T(\mathbf{e}_1) = (1, 4)$, and $T(\mathbf{e}_2) = (-2, 9)$, and $T(\mathbf{e}_3) = (3, -8)$ where $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are the columns of the 3×3 identity matrix.

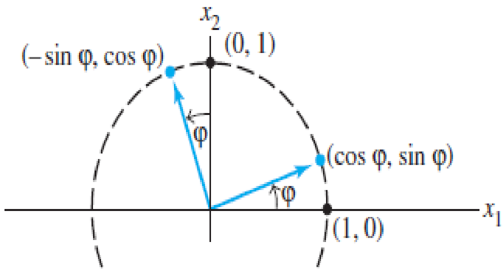
Solution: 1. The standard matrix of linear transformation is

$$A = [T(\mathbf{e}_1) \;\; T(\mathbf{e}_2)] = \begin{bmatrix} 3 & -5 \\ 1 & 2 \\ 3 & 0 \\ 1 & 0 \end{bmatrix}$$

2. The standard matrix of linear transformation is

$$A = [T(\mathbf{e}_1) \;\; T(\mathbf{e}_2) \;\; T(\mathbf{e}_3)] = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 9 & -8 \end{bmatrix}.$$

Example: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin through an angle φ, with counterclockwise rotation for a positive angle. We could show geometrically that such a transformation is linear. Find the standard matrix A of this transformation.



8.1 Finding the matrix of transformation of a linear transformation defined geometrically

:

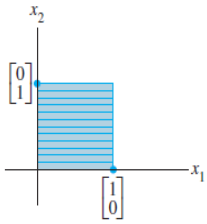


Figure 8.2: The I_2 columns and rectangle

Reflection about a line: You have to understand reflection about a line see the lecture in which I have explained this concept.
Reflection through the x_1-axis:

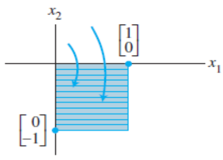


Figure 8.3: Reflection about x_1-axis

The matrix of Transformation: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection through the x_2-axis:

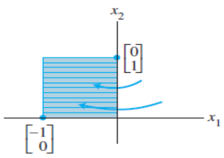


Figure 8.4: Reflection about x_2-axis

The matrix of Transformation: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Reflection through the the line $x_2 = x_1$:

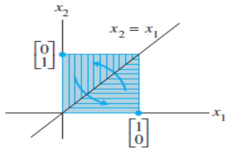


Figure 8.5: Reflection about the line $x_2 = x_1$

The matrix of Transformation: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Reflection through the line $x_2 = -x_1$:

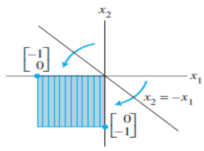
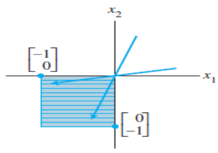


Figure 8.6: Reflection about the line $x_2 = -x_1$

The matrix of Transformation: $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

Reflection through the origin:



The matrix of Transformation: $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Horizontal / Vertical contraction and expansion:

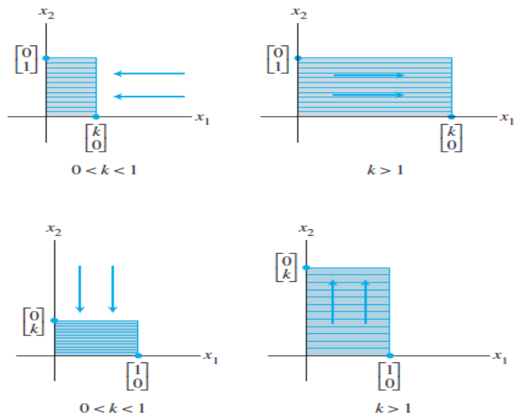


Figure 8.7: Horizontal and vertical contraction and expansion

The matrix of Transformations: $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

Horizontal / Vertical Shears:

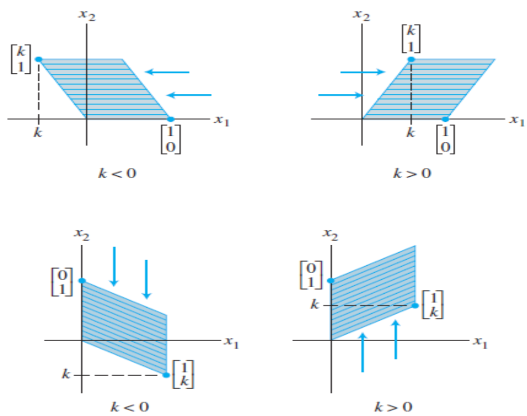


Figure 8.8: Horizontal and vertical shears

The matrix of Transformations: $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

Projections:

Projection onto x_1-axis:

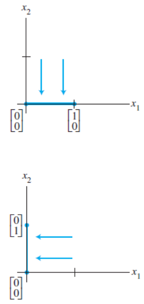


Figure 8.9: Projection on x_1-axis and x_2-axis

Projection onto x_2-axis:

The matrix of Transformations: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Example: Assume that T is a linear transformation. Find the standard matrix of T when

1. T is a vertical shear transformation that maps $\mathbf{e}_1$ into $\mathbf{e}_1 - 3\mathbf{e}_2$ and leaves $\mathbf{e}_2$ unchanged.

2. T is a horizontal shear transformation that maps $\mathbf{e}_2$ into $\mathbf{e}_2 + 2\mathbf{e}_1$ and leaves $\mathbf{e}_1$ unchanged.

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at least one $\mathbf{x}$ in $\mathbb{R}^n$.

Remark: T is onto if codomain of $T = \mathbb{R}^m$.

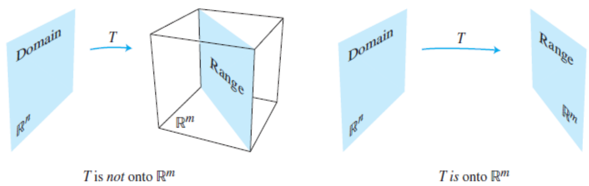


Figure 8.10: Onto mapping

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one to one** if each $\mathbf{b}$ is the image of at most one $\mathbf{x}$ in $\mathbb{R}^n$.

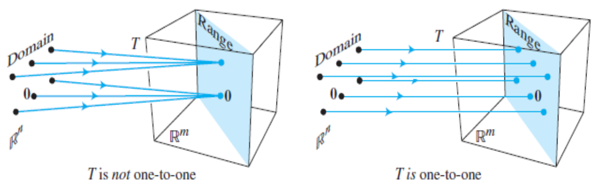


Figure 8.11: One to one transformation

Example: Let T be the linear transformation whose standard matrix is

$$A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T a one to one mapping?
Solution: Let $\mathbf{b} \in \mathbb{R}^3$ be arbitrary element then we have to check the system of linear equations $A\mathbf{x} = \mathbf{b}$. For given A the augmented matrix is

$$\begin{bmatrix} 1 & -4 & 8 & 1 & b_1 \\ 0 & 2 & -1 & 3 & b_2 \\ 0 & 0 & 0 & 5 & b_3 \end{bmatrix}.$$

The augmented matrix is in echelon form and for every $\mathbf{b}$ the system $A\mathbf{x} = \mathbf{b}$ has solution (indeed not unique, verify).

T is not one to one map. Can you justify.
Theorem: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

Theorem: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T, then

1. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$;

2. T is one to one if and only if the columns of A are linearly independent.

Example: Let $T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$. Show that T is a one-to-one linear transformation. Does T map $\mathbb{R}^2$ onto $\mathbb{R}^3$?
Solution: The matrix of transformation is

$$[T(\mathbf{e}_1 \;\; T(\mathbf{e}_2))] = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix}$$

clearly the columns of the matrix are linearly independent and hence the transformation is one to one.

The transformation T is not onto. Can you justify?

8.2 Some Practice Problems

Question: Assume that T is a linear transformation. Find the standard matrix of T

1. $T : \mathbb{R}^2 \to \mathbb{R}^3$, $T(\mathbf{e}_1) = (3, 0, 3)$, and $T(\mathbf{e}_2) = (2, 1, 5)$, where $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$.

2. $T : \mathbb{R}^3 \to \mathbb{R}^3$, $T(\mathbf{e}_1) = (1, 4, 0)$, and $T(\mathbf{e}_2) = (-2, 0, 2)$, and $T(\mathbf{e}_3) = (0, 3, 5)$ where $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are the columns of the 3×3 identity matrix.

3. $T : \mathbb{R}^2 \to \mathbb{R}^2$ first performs a horizontal shear that transforms $\mathbf{e}_2$ into $\mathbf{e}_2 + 2\mathbf{e}_1$ and leaves $\mathbf{e}_1$ unchanged and then reflects points through the line $x_2 = -x_1$

4. $T : \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points through the horizontal x_1-axis and then reflects points through the line $x_2 = x_1$.

Question: Show that T is a linear transformation by finding a matrix that implements the mapping

1. $T(x_1, x_2) = (x_1 + 4x_2, 0, x_1 - 3x_2, x_1)$.

2. $T(x_1, x_2, x_3, x_4) = 3x_1 + 4x_3 - 2x_4$.