# Joint Probability distributions:

If 'X' & 'Y' are two discrete Random variables, the probability distribution for their simultaneous occurance can be represented by a function with values $f(x, y)$ for any pair of values $(x, y)$ within the range of the random variables 'X' & 'Y'. The function $f(x, y)$ is called joint probability distribution of 'X' & 'Y'.

Hence in discrete case $f(x, y) = P(X = x, Y = y)$.

| $Y \backslash X$ | $x_1$ | $x_2 \ldots$ | $x_i \ldots$ | $x_m$ | $h(y)$ |
|---|---|---|---|---|---|
| $y_1$ | $f(x_1, y_1)$ | $f(x_2, y_1) \ldots$ | $f(x_i, y_1) \ldots$ | $f(x_m, y_1)$ | $h(y_1)$ |
| $y_2$ | $f(x_1, y_2)$ | $f(x_2, y_2) \ldots$ | $f(x_i, y_2) \ldots$ | $f(x_m, y_2)$ | $h(y_2)$ |
| $\vdots$ | $\vdots$ | | | | $\vdots$ |
| $y_j$ | $f(x_1, y_j)$ | $f(x_2, y_j) \ldots$ | $f(x_i, y_j) \ldots$ | $f(x_m, y_j)$ | $h(y_j)$ |
| $\vdots$ | $\vdots$ | | | | $\vdots$ |
| $y_n$ | $f(x_1, y_n)$ | $f(x_2, y_n) \ldots$ | $f(x_i, y_n) \ldots$ | $f(x_m, y_n)$ | $h(y_n)$ |
| $g(x)$ | $g(x_1)$ | $g(x_2) \ldots$ | $g(x_i) \ldots$ | $g(x_m)$ | 1 |

# Exercise

3.39.

| O | | A | | B | | Total |
|---|---|---|---|---|---|---|
| 3 | + | 2 | + | 3 | = | 8 |

$X$ = No of oranges, $Y$ = No of apples.

$X = 0, 1, 2, 3$ , $Y = 0, 1, 2$

| $f(x,y)$ | | $x$ = 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| $y$ | 0 | $f(0,0)$ | $f(1,0)$ | $f(2,0)$ | $f(3,0)$ |
| | 1 | $f(0,1)$ | $f(1,1)$ | $f(2,1)$ | $f(3,1)$ |
| | 2 | $f(0,2)$ | $f(1,2)$ | $f(2,2)$ | $f(3,2)$ |

$$f(0,0) = P(X=0, Y=0) = \frac{\binom{3}{0}\binom{2}{0}\binom{3}{4}}{\binom{8}{4}} = 0$$ ✓

$$f(1,0) = P(X=1, Y=0) = \frac{\binom{3}{1}\binom{2}{0}\binom{3}{3}}{\binom{8}{4}} = \frac{3}{70}$$

$$f(2,0) = P(X=2, Y=0) = \binom{3}{2}\binom{2}{0}\binom{3}{2} \Big/ 70 = 9/70$$

$$f(3,0) = P(X=3, Y=0) = \binom{3}{3}\binom{2}{0}\binom{3}{1} \Big/ 70 = 3/70$$

$$f(0,1) = P(X=0, Y=1) = \binom{3}{0}\binom{2}{1}\binom{3}{3} \Big/ 70 = 2/70$$

(3)

$f(1,1) = P(X=1, Y=1) = \binom{3}{1}\binom{2}{1}\binom{3}{2}/70 = 18/70$

$f(2,1) = P(X=2, Y=1) = \binom{3}{2}\binom{2}{1}\binom{3}{1}/70 = 18/70$

$f(3,1) = P(X=3, Y=1) = \binom{3}{3}\binom{2}{1}\binom{3}{0}/70 = 2/70$

$f(0,2) = P(X=0, Y=2) = \dfrac{\binom{3}{0}\binom{2}{2}\binom{3}{2}}{70} = 3/70$

$f(1,2) = P(X=1, Y=2) = \binom{3}{1}\binom{2}{2}\binom{3}{1}/70 = 9/70$

$f(2,2) = P(X=2, Y=2) = \binom{3}{2}\binom{2}{2}\binom{3}{0}/70 = 3/70$

$f(3,2) = 0$

(a)

| $f(x,y)$ | | $x=0$ | 1 | 2 | 3 | $h(y)$ |
|---|---|---|---|---|---|---|
| $y$ | 0 | 0 | 3/70✓ | 9/70✓ | 3/70 | 15/70 |
| | 1 | 2/70✓ | 18/70✓ | 18/70 | 2/70 | 40/70 |
| | 2 | 3/70✓ | 9/70 | 3/70 | 0 | 15/70 |
| $g(x)$ | | 5/70 | 30/70 | 30/70 | 5/70 | 1 |

(b) $P(x+y \le 2) = f(0,0) + f(1,0) + f(2,0) + f(0,1) + f(1,1) + f(0,2)$

$= 0 + \frac{3}{70} + \frac{9}{70} + \frac{2}{70} + \frac{18}{70} + \frac{3}{70}$

$= 35/70 = 1/2 = 0.5$ Ans

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## 3.48 Def: Conditional distribution:-

Let 'X' & 'Y' be two random variables, discrete or continuous. The conditional distribution of Y given that $X = x$ is

$$f(y|x) = \frac{f(x,y)}{g(x)}, \quad g(x) > 0$$

Similarly,

$$f(x|y) = \frac{f(x,y)}{h(y)}$$

where

$$g(x) = \sum_y f(x,y) \quad \& \quad h(y) = \sum_x f(x,y)$$

or

$$g(x) = \int_y f(x,y)\,dy \quad \& \quad h(y) = \int_x f(x,y)\,dx$$

are marginal probability functions.

$$P(a<X<b \mid Y=y) = \sum_{a<x<b} f(x|y)$$

$$P(a<X<b \mid Y=y) = \int_a^b f(x|y)\,dx.$$

Note: The random variable X & Y are said to be independent iff $f(x,y) = g(x) * h(y)$ for all $(x,y)$ within range.

3.48 : Find (a) $f(y|2)\ \forall\ y$ (b) $P(Y=0|X=2)$

Sol:

(a) $$f(y|x) = \frac{f(x,y)}{g(x)}$$

$$f(y|2) = \frac{f(2,y)}{g(2)} = \frac{f(2,y)}{30/70}, \quad y=0,1,2$$

(b) $$P(Y=0|X=2) = \frac{f(2,0)}{g(2)} = \frac{9/70}{30/70} = \frac{\cancel{9}^{3}}{\cancel{30}_{10}} = 3/10$$

3.42:

$$f(x,y) = e^{-(x+y)}, \quad x>0,\ y>0$$

$$P(0<X<1|Y=2) = ?$$

$$f(x|y) = \frac{f(x,y)}{h(y)}$$

$$h(y) = \int_0^\infty e^{-(x+y)}\,dx = e^{-y}\int_0^\infty e^{-x}\,dx = e^{-y}\left|\frac{e^{-x}}{-1}\right|_0^\infty$$

$$h(y) = e^{-y}, \quad y>0$$

$$\Rightarrow f(x|y) = \frac{e^{-x}\times e^{-y}}{e^{-y}} = e^{-x}$$

$$P(0<X<1|Y=2) = \int_0^1 f(x|y)\,dx = \int_0^1 e^{-x}\,dx = \left|\frac{e^{-x}}{-1}\right|_0^1$$
$$= 1-e^{-1} = 0.63$$

3.40:-

$$f(x,y) = \frac{2}{3}(x+2y), \quad 0 \le x \le 1, \quad 0 \le y \le 1$$

(a) $g(x) = ?$

$$g(x) = \frac{2}{3}\int_0^1 (x+2y)\,dy$$

$$= \frac{2x}{3}\left|y\right|_0^1 + \frac{\cancel{4}}{3}\left|\frac{y^2}{\cancel{2}}\right|_0^1$$

$$g(x) = \frac{2x}{3} + \frac{2}{3} = \frac{2(x+1)}{3}, \quad 0 < x < 1$$

(b) $h(y) = ?$

$$h(y) = \frac{2}{3}\int_0^1 (x+2y)\,dx$$

$$= \frac{\cancel{2}}{3}\left|\frac{x^2}{\cancel{2}}\right|_0^1 + \frac{4}{3}y\left|x\right|_0^1$$

$$h(y) = \frac{1}{3} + \frac{4y}{3} = \frac{(1+4y)}{3}, \quad 0 < y < 1$$

(c) $$P(X < 1/2) = \int^{1/2} g(x)\,dx = \frac{2}{3}\int^{1/2} (x+1)\,dx = \frac{5}{12}$$

3.45: $f(x,y) = \frac{1}{y}, \quad 0 < x < y < 1$.

Find $P(X+Y > 1/2) = ?$

Sol:

$$P(X+Y > 1/2) = 1 - P(X+Y < 1/2) = 1 - \int_0^{1/4}\int_x^{1/2-x} (1/y)\,dy\,dx$$

$$= 1 - \int_0^{1/4} [\ln(1/2 - x) - \ln x]\,dx$$

$$= 1 + [(1/2 - x)\ln(1/2 - x) - x\ln x]\Big|_0^{1/4}$$

$$= 1 + 1/4 \ln(1/4) = 0.6534.$$

3.47:-

$$f(x,y) = 2, \quad 0 < x \leq y < 1.$$

(a) Determine if $X$ & $Y$ are independent.

(b) $P(1/4 < X < 1/2 \mid Y = 3/4)$.

Sol.

$$g(x) = 2\int_x^1 1\,dy = 2(1-x), \quad 0 < x < 1,$$

$$h(y) = 2\int_0^y dx = 2y, \quad 0 < y < 1,$$

Since $f(x,y) \neq g(x) * h(y)$, $X$ & $Y$ are not Independen

(b) $f(x|y) = \frac{f(x,y)}{h(y)} = \frac{1}{y}$ for $0 < x < y$.

$$P\left(\tfrac{1}{4} < X < \tfrac{1}{2} \,\middle|\, Y = \tfrac{3}{4}\right) = \frac{4}{3}\int_{1/4}^{1/2} dx = \tfrac{1}{3}.$$

**3.56:-**

$$f(x,y) = 6x, \quad 0<x<1, \quad 0<y<1-x$$

(a) Show that $X$ & $Y$ are not Independent?

(b) $P(X>0.3 \mid Y=0.5) = ?$

Sol:-

$$h(y) = 6\int_0^{1-y} x\,dx = 3(1-y)^2, \quad 0<y<1.$$

Since $f(x|y) = \dfrac{f(x,y)}{h(y)} = \dfrac{2x}{(1-y)^2}$ for $0<x<1-y$,

involves the variables $y$, $X$ & $Y$ are not independent.

(b) $P(X>0.3 \mid Y=0.5) = 8\int_{0.3}^{0.5} x\,dx = 0.64.$ Ans.