(Determinants)

Recall:- For 2×2 matrix, $A=[a_{ij}]$, the determinant is the number $\det A = a_{11}a_{22} - a_{12}a_{21}$

Notation: For instance, if $A = \begin{bmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{bmatrix}$ the A_{32} is obtained

by crossing out row 3 and column 2

$$\begin{bmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{bmatrix} \Rightarrow A_{32} = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Remark:- We can now give a recursive definition of determinant. When $n=3$, determinant is defined using determinants of 2×2 submatrices A_{ij}. In general, so, an $n\times n$ determinant is defined by determinants of $(n-1)\times(n-1)$ submatrices.

Determinant:- For $n \geq 2$, the determinant of $n\times n$ matrix $A=[a_{ij}]$ is sum of terms of the form $\pm a_{ij} \det A_{ij}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, a_{13}, \dots, a_{1n}$ are from the first row of A. In symbols

$$\det A = a_{11}\det A_{11} - a_{12}\det A_{12} + \dots + (-1)^{1+n} a_{1n} \det A_{1n}$$

$$\det A = \sum_{j=1}^{n} a_{ij} (-1)^{i+j} A_{ij}$$

Example:-

Compute the determinant of $\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix}$$

$$= 1\begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} - 5\begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 0\begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix}$$

$$= 1(0-2) - 5(0-0) + 0(-4+0)$$

$$= 1(-2) - 5(0) + 0(-4)$$

$$= -2 + 0 + 0 = -2$$

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Notion:- Given $A = [a_{ij}]$ then (i,j)-cofactor of A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

Then, $\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$

The formula is called a $\boxed{\text{cofactor expansion}}$ across the first row of A.

Remark:-

(i) **Theorem:-** The determinant of $n \times n$ matrix A canbe computed by a cofactor expansion across any row or down any column. The expansion across the ith row using cofactors: $\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$ the expansion across the jth column using cofactors is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Example:- Compute determinant of $\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$

Using cofactor expansion:-

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= (-1)^{1+1} a_{11} \det A_{11} + (-1)^{1+2} a_{12} \det A_{12} + (-1)^{1+3} a_{13} \det A_{13}$$

$$= (-1)^2 (1) \begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} + (-1)^3 (5) \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + (-1)^4 (0) \begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix}$$

$$= +(0-2) + (-1)\, 5(0-0) + 0(-4+0)$$

$$= -2 - 0 + 0 = -2$$

Example:- Compute the determinant of

$$A = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$$

Expanding along first column.

$$\det A = 3 \begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix}$$

Again expanding along first column

$$\det A = (3)(2)\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix}$$ ⇒ Again expanding along first column

$$= 6(1)\begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} = 6(0-2) = 6(-2) = -12$$

Theorem:- If A is a triangular matrix, then $\det A$ is the product of entries on the main diagonal of A.

Example:- Compute the determinant of

$$A = \begin{bmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{bmatrix}$$

Expanding along third column

$$A = 2\begin{vmatrix} 0 & 3 & -4 \\ -5 & -8 & 3 \\ 0 & 5 & -6 \end{vmatrix}$$

Expanding along -5

$$A = (2)(-5) - \begin{vmatrix} 3 & -4 \\ 5 & -6 \end{vmatrix}$$

$$A = +10(-18-(-20)) = +10(-18+20) = +10(2)$$

$$A = +20$$

Example:- Compute determinants of the elementary matrices given in

$$E_1 = \begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det E_1 = k\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - 0 + 0 = k(1) = k$$

$$E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det E_2 = 0\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}$$

$$\det E_2 = 0 - 1(1) + 0 = -1$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{bmatrix}$$

$$\det E_3 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ k & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ k & 0 \end{vmatrix}$$

$$\det E_3 = 1(1) - 0 + 0 = 1$$

Theorem:- Let A be a square matrix

→ If a multiple of one row of A is added to another row to produce matrix B, then $\det B = \det A$

→ If two rows of A are interchanged to produce B, then $\det B = -\det A$.

→ If one row of A is multiplied by k to produce B, then $\det B = k \det A$.

Example:- Compute determinant $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$.

Reduce A to echelon form to form triangular matrix and then use fact determinant is product of diagonals

$$\det A = \begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix}$$

$R_2 + 2R_1$, $R_3 + R_1$

$$\det A = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix}$$

Swap R_2 and R_3 and sign is changed

$$\det A = -\begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix}$$

$$\det A = -(1)(3)(-5) = 15$$

Remark:- A common use of 3 axiom of Theorem in hand calculations is to factor out common multiple of one row matrix. For instance

$$\begin{vmatrix} * & * & * \\ * & * & * \\ 5k & -3k & 7k \end{vmatrix} = k\begin{vmatrix} * & * & * \\ * & * & * \\ 5 & -3 & 7 \end{vmatrix}$$ {where starred entries are unchanged

Example:- Compute the determinant of A, where

$$A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$$

$$\det A = 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix}$$

$R_2 - 3R_1$, $R_3 + 3R_1$, $R_4 - R_1$

$$\det A = 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{vmatrix}$$

$R_3 + 4R_2$

$$\det A = 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{vmatrix}$$

$R_4 - \frac{1}{2}R_3$

$$\det A = 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$\det A = 2(1)(3)(-6)(1) = -36$$

Remark:- Suppose a square matrix A has been reduced to an echelon form U by row replacements and row interchanges then if there are r interchanges

$$\det A = (-1)^r \det U.$$

and $\det U$ is just multiplication of diagonal elements of U. Thus we have

$$\det A = \begin{cases} (-1)^r (\text{Product of pivot in } U), & \text{when } r \text{ is invertible} \\ 0, & \text{when } r \text{ is not invertible} \end{cases}$$

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Theorem:- A square matrix A is invertible if and only if $\det A \neq 0$

Example:- Compute $\det A$, where $A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$

$$\det A = \begin{vmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{vmatrix}$$

$R_3 + 2R_1$

$$\det A = \begin{vmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 5 & -3 & -6 \\ -5 & -8 & 0 & 9 \end{vmatrix}$$

As R_2 and R_3 are same and dependent. So, we can say that $\det A = 0$ (don't exist) (not invertible)

Example:- Compute $\det A$, where $A = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{bmatrix}$

$$\det A = \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{vmatrix}$$

$R_4 + R_2$

$$\det A = \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{vmatrix}$$

Swap R_1 and R_2

$$\det A = -\begin{vmatrix} 2 & 5 & -7 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{vmatrix}$$

Expanding along R_1

$$\det A = -2\begin{vmatrix} 1 & 2 & -1 \\ 3 & 6 & 2 \\ 0 & -3 & 1 \end{vmatrix}$$

$R_2 - 3R_1$

$$\det A = -2\begin{vmatrix} 1 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & -3 & 1 \end{vmatrix}$$

$$\det A = -2\left[1\begin{vmatrix} 0 & 5 \\ -3 & 1 \end{vmatrix} - 2\begin{vmatrix} 0 & 5 \\ 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 0 & 0 \\ 0 & -3 \end{vmatrix}\right]$$

$$\det A = -2(0+15) - 0 - 0$$

$$\det A = -2(15)$$

$$\det A = -30$$

Theorem:- If A is an $n\times n$ matrix then
$$\det A^T = \det A$$

Theorem:- If A and B are $n\times n$ matrices then
$$\det(AB) = (\det A)(\det B)$$

Example:- Use determinant to decide if v_1, v_2, v_3 are linearly independent, when $v_1 = \begin{bmatrix} 5 \\ -7 \\ 9 \end{bmatrix}$, $v_2 = \begin{bmatrix} -3 \\ 3 \\ -5 \end{bmatrix}$, $v_3 = \begin{bmatrix} 2 \\ -7 \\ 5 \end{bmatrix}$

$$A = [v_1 \quad v_2 \quad v_3]$$
$$A = \begin{bmatrix} 5 & -3 & 2 \\ -7 & 3 & -7 \\ 9 & -5 & 5 \end{bmatrix}$$
$$\det A = \begin{vmatrix} 5 & -3 & 2 \\ -7 & 3 & -7 \\ 9 & -5 & 5 \end{vmatrix}$$
$$\det A = 5\begin{vmatrix} 3 & -7 \\ -5 & 5 \end{vmatrix} - (-3)\begin{vmatrix} -7 & -7 \\ 9 & 5 \end{vmatrix} + 2\begin{vmatrix} -7 & 3 \\ 9 & -5 \end{vmatrix}$$
$$\det A = 5(15-35) + 3(-35+63) + 2(35-27)$$
$$\det A = 5(-20) + 3(28) + 2(8)$$
$$\det A = -100 + 84 + 16 \qquad = -100+100$$
$$\det A = 0$$

So the matrix $[v_1 \quad v_2 \quad v_3]$ is not invertible. The columns are linearly dependent by Invertible matrix theorem

Practice Problems

Question:- Find determinant of matrices where $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5$

For multiple

$\boxed{\det B = k \det A}$

1) $\begin{vmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{vmatrix}$

$$= 5\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5(5) = 25$$

$$\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix}$$

$R_2 - R_1$

$$\begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2(5) = 10$$

$$\begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix}$$

$R_1 - R_3$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5$$

Question:- Use determinants to find out matrices are invertible or not

$$\begin{vmatrix} 2 & 3 & 0 \\ 1 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} - 3\begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}$$

$= 2(3-8) - 3(1-4) + 0$

$= 2(-5) - 3(-3)$

$= -10 + 9 = -1 \neq 0$

As $\det A \neq 0$, so matrix is invertible

$$\begin{vmatrix} 2 & 0 & 0 & 8 \\ 1 & -7 & -5 & 0 \\ 3 & 8 & 6 & 0 \\ 0 & 7 & 5 & 4 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 0 & 0 & 4 \\ 1 & -7 & -5 & 0 \\ 3 & 8 & 6 & 0 \\ 0 & 7 & 5 & 4 \end{vmatrix}$$

$R_2 - R_1$, $R_3 - 3R_1$

$$= 2\begin{vmatrix} 1 & 0 & 0 & 4 \\ 0 & -7 & -5 & -4 \\ 0 & 8 & 6 & -12 \\ 0 & 7 & 5 & 4 \end{vmatrix}$$

$$= 2(-1)\begin{vmatrix} 1 & 0 & 0 & 4 \\ 0 & 7 & 5 & 4 \\ 0 & 8 & 6 & -12 \\ 0 & 7 & 5 & 4 \end{vmatrix} = 2(-1)(0) = 0$$

As R_2, R_4 are linearly dependent. So, the matrix is not invertible

Question:- Use determinant to decide if v_1, v_2, v_3 are linearly independent, when

$$v_1 = \begin{bmatrix} 4 \\ 6 \\ -7 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -7 \\ 0 \\ 2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -3 \\ -5 \\ 6 \end{bmatrix}$$

$$A = [v_1 \quad v_2 \quad v_3]$$

$$A = \begin{bmatrix} 4 & -7 & -3 \\ 6 & 0 & -5 \\ -7 & 2 & 6 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 4 & -7 & -3 \\ 6 & 0 & -5 \\ -7 & 2 & 6 \end{vmatrix}$$

$$\det A = 4\begin{vmatrix} 0 & -5 \\ 2 & 6 \end{vmatrix} - (-7)\begin{vmatrix} 6 & -5 \\ -7 & 6 \end{vmatrix} + (-3)\begin{vmatrix} 6 & 0 \\ -7 & 2 \end{vmatrix}$$

$$= 4(0+10) + 7(36-35) - 3(12-0)$$

$$= 4(10) + 7(1) - 3(12) = 40 + 7 - 36 = 47 - 36$$

$$= 11 \neq 0$$

Hence the matrix $[v_1 \quad v_2 \quad v_3]$ is invertible and hence the columns are linearly independent by matrix invertible theorem.