CHAPTER 2

# Lecture No. 02

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**Question**: Find an equation involving $g, h$, and $k$ that makes the augmented matrix correspond to a consistent system $\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{bmatrix}$

**Solution**: After the elimination of the variable $x_1$ from the third equation and from the resulting matrix elimination of the $x_2$ variable from third row of the matrix gives us the following matrix

$$\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & 0 & 0 & h+k+2g \end{bmatrix}.$$

The system will be consistent if $h+k+2g=0$ otherwise we will have one equation of the form $0 = h+k+2g \neq 0$, which is absurd.

## 2.1 Elementary Row Operations

From the examples considered in the previous lecture we have seen that if a linear system is changed to another by one of these operations

1. An equation is **swapped with another**.

2. An equation has **both sides multiplied** by a nonzero constant.

3. An equation is **replaced** by the **sum of itself and a multiple of another**.

then the **two systems have the same set of solutions**.
**Elementary Row Operations**: The row operations corresponding to the above mentioned changes on a matrix are know as elementary row operations. These are

1. (**Interchange**) Interchange two rows.

2. (**Scaling**) Multiply all entries in a row by a nonzero constant.

3. (**Replacement**) Replace one row by the sum of itself and a multiple of another row.

## 2.2 Echelon and Reduced Echelon form of a Matrix

Let us define the **Leading entry** of a row refers to the leftmost nonzero entry (in a nonzero row).

**Echelon Form**:

A rectangular matrix is in echelon form (or row echelon form) if it has the following three properties:

- All nonzero rows are above any rows of all zeros.
- Each leading entry of a row is in a column to the right of the leading entry of the row above it.
- All entries in a column below a leading entry are zeros.

**Reduced Echelon Form**:

If a matrix in echelon form satisfies the following additional conditions, then it is in reduced echelon form (or reduced row echelon form):

- The leading entry in each nonzero row is 1.
- Each leading 1 is the only nonzero entry in its column.

**Example**: The matrices in the first two matrices are in echelon form where as the third and fourth matrices are in reduced echelon form.

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

**Remark**:

- Any nonzero matrix may be reduced to more than one echelon form, i.e., echelon form of a matrix is not unique.
- Each matrix is row equivalent to one and only one reduced echelon matrix, i.e., reduced echelon matrix of a matrix is unique.

**Pivot Positions**: A **pivot position** in a matrix $A$ is a location in $A$ that corresponds to a leading 1 in the **reduced echelon** form of $A$. A **pivot column** is a column of $A$ that contains a **pivot position**.

The matrix

$$\begin{bmatrix} 1 & 6 & 0 & 9 \\ 0 & -1 & -2 & -7 \\ 0 & 0 & 3 & 9 \end{bmatrix}$$

is in echelon form, with the pivot element in first, second and third columns.

The matrix

$$\begin{bmatrix} 1 & {\color{red}-2} & {\color{red}1} & 0 \\ 0 & 1 & {\color{red}-4} & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

is in echelon form and it has three pivot elements in first, second and third columns. These pivot elements are 1. The matrix is not in reduced echelon form as in pivot columns pivot elements are not the only nonzero elements. We can transform the above matrix into reduced echelon form by applying following row operations on the matrix $R_1 - R_3$, $R_2 + 4R_3$ gives

$$\begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

and then by the row operation $R_1 + 2R_2$ leads to the following reduced echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

**Example**: Row reduce the given matrix $A$ into echelon form and locate the pivot positions. $\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$

**Solution**:

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Pivot column

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Next pivot column

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns

## 2.3 The Row Reduction Algorithm

Transform the following matrix into echelon form and then in reduced echelon form by using elementary row operations.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}.$$

**Step I**: Choose the pivot column $\begin{bmatrix} \mathbf{0} & 3 & -6 & 6 & 4 & -5 \\ \mathbf{3} & -7 & 8 & -5 & 8 & 9 \\ \mathbf{3} & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$

**Step II**: Interchange rows 1 and 3 $\begin{bmatrix} \mathbf{3} & -9 & 12 & -9 & 6 & 15 \\ \mathbf{3} & -7 & 8 & -5 & 8 & 9 \\ \mathbf{0} & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$

**Step III**: add $-1$ times row 1 to row 2 $\begin{bmatrix} \mathbf{3} & -9 & 12 & -9 & 6 & 15 \\ \mathbf{0} & 2 & -4 & 4 & 2 & -6 \\ \mathbf{0} & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$

**Step IV**: Choose the pivot column $\begin{bmatrix} 3 & \mathbf{-9} & 12 & -9 & 6 & 15 \\ 0 & \mathbf{2} & -4 & 4 & 2 & -6 \\ 0 & \mathbf{3} & -6 & 6 & 4 & -5 \end{bmatrix}$

**Step V**: add $-3/2$ times row 2 to row 3 $\begin{bmatrix} 3 & -9 & 12 & -9 & \mathbf{6} & 15 \\ 0 & 2 & -4 & 4 & \mathbf{2} & -6 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 4 \end{bmatrix}$

**Step VI**: $\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

$Row1 + (-6)row3$ and $row2 + (-2)row3$ $\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

Row 2 scaled by 1/2 $\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

Row 1 + (9) row 2 $\begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

1/3 Row 1 $\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$ which is the reduced echelon form of the given matrix.

**Example**: The augmented matrix of a linear system is $\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ write down the solution of the system.
**Solution**: The given augmented matrix is in echelon form and we get the following solution of the system

$$x_2 = 4 - x_3, \qquad x_1 = 1 + 5x_3.$$

This solution shows that the values of the variables $x_1$ and $x_2$ can be obtained by taking some value of the variable $x_3$.
The variables $x_1, x_2$ are known as **Basic variables** and the variable $x_3$, is called **Free variable**.
**Example**: Find the solution of the linear system whose augmented matrix has been reduced to

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}.$$

**Solution**: The reduced Echelon form of the given matrix is

$$\begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}.$$

Then we have from third row $x_5 = 7$, from second row we have $x_3 = 5 + 4x_4$ and from first row we have $x_1 = -6x_2 - 3x_4$.

This shows that $x_2$ and $x_4$ are free variables where as $x_1, x_3, x_5$ are basic variables.
**Theorem**: A linear system is **consistent** if and only if the **rightmost column of the augmented matrix is not a pivot column** that is, if and only if an **echelon form of the augmented matrix** has no row of the form

$$[0 \ldots 0 \;\; b]$$

with $b$ **nonzero**.
If a linear system is **consistent**, then the solution set contains either

- a unique solution, when there are **no free variables**, or

- infinitely many solutions, when there is at **least one free variable**.

**Example**: Determine the existence and uniqueness of the solutions to the system

$$\begin{array}{rcrcrcrcrcr} & & 3x_2 & - & 6x_3 & + & 6x_4 & + & 4x_5 & = & -5 \\ 3x_1 & - & 7x_2 & + & 8x_3 & - & 5x_4 & + & 8x_5 & = & 9 \\ 3x_1 & - & 9x_2 & + & 12x_3 & - & 9x_4 & + & 6x_5 & = & 15 \end{array}$$

**Solution**: The augmented matrix of the system was row reduced to

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

The basic variables are $x_1, x_2$, and $x_5$; the free variables are $x_3$ and $x_4$. There is no equation such as $0 = 1$ that would indicate an inconsistent system, so we could use back substitution to find a solution.

Since there are free variables so the solution of the system is not unique, indeed we have infinite many solutions.

**Example**: Determine whether the system whose augmented matrix is row reduced to the following matrix is consistent or inconsistent

$$\begin{bmatrix} 1 & 0 & -9 & 0 & 4 \\ 0 & 1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

**Solution**: The augmented matrix is in echelon, indeed reduced echelon form and one of the row (4th row) is of the form $[0...0\ b]$ where $b \neq 0$ and the fourth row leads to 0=1 which is absurd. hence the system of linear equations is inconsistent.

**Example**: Determine the value(s) of $h$ such that the matrix is the augmented matrix of a consistent linear system

$(a) \begin{bmatrix} 1 & -1 & 4 \\ -2 & 3 & h \end{bmatrix}$,

$(b) \begin{bmatrix} 1 & -3 & 1 \\ h & 6 & -2 \end{bmatrix}$.

**Solution**: (a) The echelon form of the matrix is

$$\begin{bmatrix} 1 & -1 & 4 \\ 0 & 1 & h+8 \end{bmatrix}$$

We will never have a row of the type $[0...0\ b]$ whatever the value of $h$ we take. Hence the system is consistent for all values of $h$.

(b) Similarly, get the echelon form of the matrix

$$\begin{bmatrix} 1 & -3 & 1 \\ h & 6 & -2 \end{bmatrix}$$

and try to apply the theorem for deciding the value of $h$ to have consistent system.

## 2.4 Steps to Solve a Linear System

1. Write the augmented matrix of the system.

2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.

3. Continue row reduction to obtain the reduced echelon form.

4. Write the system of equations corresponding to the matrix obtained in step 3.

5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

## 2.5 Some Practice Problems

**Question**: choose $h$ and $k$ such that the system

$$\begin{array}{rcrcl} x_1 & - & 3x_2 & = & 1 \\ 2x_1 & + & hx_2 & = & k \end{array}$$

has (a) no solution, (b) a unique solution, and (c) many solutions.

**Question**: Find the general solutions of the systems whose augmented matrices are given by $\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & -5 & 0 & -8 & 3 \\ 0 & 1 & 4 & -1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

**Question**: Find the interpolating polynomial $P(t) = a_0 + a_1 t + a_2 t^2$ for the data $(1, 6), (2, 15), (3, 28)$.