CHAPTER 25

Lecture No. 25

Question: It can be shown that the algebraic multiplicity of an eigenvalue λ is always greater than or equal to the dimension of the eigenspace corresponding to λ. Find h in the matrix A below such that the eigenspace for $\lambda = 4$ is two-dimensional

$$\begin{bmatrix} 4 & 2 & 3 & 3 \\ 0 & 2 & h & 3 \\ 0 & 0 & 4 & 14 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Solution: For the basis of eigenspace we will solve the system $(A - 4I)\mathbf{x} = \mathbf{0}$, i.e.,

$$A - 4I = \begin{bmatrix} 0 & 2 & 3 & 3 \\ 0 & -2 & h & 3 \\ 0 & 0 & 0 & 14 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

the augmented matrix of the homogeneous system is

$$\begin{bmatrix} 0 & 2 & 3 & 3 & 0 \\ 0 & -2 & h & 3 & 0 \\ 0 & 0 & 0 & 14 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 2 & 3 & 3 & 0 \\ 0 & -2 & h & 3 & 0 \\ 0 & 0 & 0 & 14 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}.$$

We have $x_4 = 0$, $x_2 = \frac{hx_3}{2}$ also we have $x_2 = \frac{3x_3}{2}$, which gives the value of $h = 3$. For $h = 3$ the solution of the system $(A - 4I)\mathbf{x} = \mathbf{0}$ in parametric form is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 3/2 \\ 1 \\ 0 \end{bmatrix}$$

which shows that basis for the eigenspace corresponding to $\lambda = 4$ is 2.

25.1 Diagonalization

Example: If $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$ then calculate D^2, D^3 and D^k where $k \geq 1$.
Solution: The expressions are

$$D^2 = \begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix}, \qquad D^3 = \begin{bmatrix} 5^3 & 0 \\ 0 & 3^3 \end{bmatrix}$$

and

$$D^k = \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}.$$

Remark: If $A = PDP^{-1}$ for some invertible P and diagonal D, then A^k is also easy to compute.

Example: Let $\begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Find a formula for A^k, given that $A = PDP^{-1}$ where

$$P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}.$$

Solution: The standard formula for the inverse of a 2×2 matrix yields

$$P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}.$$

Then we can check that $A = PDP^{-1}$, also

$$A^2 = PD^2P^{-1}, \qquad A^3 = PD^3P^{-1}$$

and

$$A^k = PD^kP^{-1}.$$

Diagonalizable Matrix: A square matrix A is said to be diagonalizable if A is similar to a diagonal matrix, that is, if $A = PDP^{-1}$ for some invertible matrix P and some diagonal matrix D.

Theorem: The Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

Example: Diagonalize the following matrix, if possible

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

Solution: **Step I.** Find all the eigenvalues of the matrix A

$$\begin{aligned} \det(A - \lambda I) = 0 \Rightarrow -\lambda^3 - 3\lambda^2 + 4 &= 0 \\ -(\lambda - 1)(\lambda + 2)^2 &= 0. \end{aligned}$$

Step II. Find three linearly independent eigenvectors of A.

$$\text{Basis for } \lambda = 1: \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\text{Basis for} \lambda = -2: \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

Step III. Construct P from the vectors in step II.

The order of the vectors is unimportant.

$$P = [\mathbf{v}_1 \ \mathbf{v}_3 \ \mathbf{v}_3] = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Step IV. Construct D from the corresponding eigenvalues

It is essential that the order of the eigenvalues matches the order chosen for the columns of P.

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Verification: It is a good idea to check that P and D really work.

$$AP = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

$$PD = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

Example: Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

Solution: The characteristic equation of the given matrix is

$$\begin{aligned} \det(A - \lambda I) = 0 \Rightarrow -\lambda^3 - 3\lambda^2 + 4 &= 0 \\ -(\lambda - 1)(\lambda + 2)^2 &= 0. \end{aligned}$$

The eigenvalues are $\lambda = 1$ and $\lambda = -2$. The eigenspace for each eigenvalue is one dimensional

$$\text{Basis for}\lambda = 1: \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \text{Basis for}\lambda = -2: \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

There are no other eigenvalues, and every eigenvector of A is a multiple of either $\mathbf{v}_1$ or $\mathbf{v}_2$.

Hence A is not diagonalizable.

Theorem: An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Example: Determine if the following matrix is diagonalizable

$$A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}.$$

25.2 Matrices Whose Eigenvalues Are Not Distinct

Theorem: Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, ..., \lambda_p$.

1. For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.

2. The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n, and this happens if and only if

   (i) the characteristic polynomial factors completely into linear factors and

   (ii) the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.

3. If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $\mathcal{B}_1, ..., \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.

Example: Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}.$$

Solution: Since A is a triangular matrix, the eigenvalues are 5 and -3, each with algebraic multiplicity 2

$$\text{Basis for}\lambda = 5: \quad \mathbf{v}_1 = \begin{bmatrix} -8 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -16 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Basis for}\lambda = -3: \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent. So the matrix $P = [\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \mathbf{v}_3 \ \ \mathbf{v}_4]$ is invertible and $A = PDP^{-1}$, where

$$P = \begin{bmatrix} -8 & -16 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & -3 \end{bmatrix}.$$

25.3 Some Practice Problems

Question: Let $A = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Suppose you are told that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A. Use this information to diagonalize A.

Question: Let A be a 4×4 matrix with eigenvalues 5, 3, and -2, and suppose you know that the eigenspace for $\lambda = 3$ is two-dimensional. Do you have enough information to determine if A is diagonalizable?

Question: Diagonalize the given matrix, if possible

$$\begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & -2 & -2 \\ 3 & -3 & -2 \\ 2 & -2 & -2 \end{bmatrix}.$$

Question: A is a 7×7 matrix with three eigenvalues. One eigenspace is two-dimensional, and one of the other eigenspaces is three dimensional. Is it possible that A is not diagonalizable? Justify your answer.

Question: A is a 3×3 matrix with two eigenvalues. Each eigenspace is one-dimensional. Is A diagonalizable? Why?