Thus the columns of the matrix are linearly independent and dimension of the space H is 3.
Rank of a Matrix: The rank of a matrix A, denoted by rank A, is the dimension of the column space of A.

Example: Determine the rank of the matrix

$$A = \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix}.$$

Solution: Reduce A to echelon form

$$A \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & -5 & -7 \\ 0 & -6 & 4 & 14 & -20 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix} \sim ... \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & -5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The basis for the column space is the set

$$\left\{ \begin{bmatrix} 2 \\ 4 \\ 6 \\ 0 \end{bmatrix}, \left\{ \begin{bmatrix} 5 \\ 7 \\ 9 \\ -9 \end{bmatrix}, \left\{ \begin{bmatrix} -4 \\ -3 \\ 2 \\ 5 \end{bmatrix} \right\}.$$

The above set contains the columns corresponding to the pivot columns and hence dimension of the column space is 3, which is also rank of the matrix.
The Rank Theorem: If a matrix A has n columns, then

$$\text{rank } A + \text{dim Nul } A = n.$$

The Basis Theorem: Let H be a p-dimensional subspace of $\mathbb{R}^n$. Any linearly independent set of exactly p elements in H is automatically a basis for H. Also, any set of p elements of H that spans H is automatically a basis for H.

The Invertible Matrix Theorem: Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

1. The columns of A form a basis of $\mathbb{R}^n$.

2. Col $A = \mathbb{R}^n$.

3. dim Col $A = n$.

4. rank $A = n$.

5. Nul $A = \{\mathbf{0}\}$.

6. dim Nul $A = 0$.

Example:

1. Suppose a 4×7 matrix A has three pivot columns. Is Col $A = \mathbb{R}^3$. What is the dimension of Nul A? Explain your answers

2. Suppose a 4×6 matrix A has four pivot columns. Is Col $A = \mathbb{R}^4$. Is Nul $A = \mathbb{R}^2$? Explain your answers

Solution: 1. Yes, Col $A = \mathbb{R}^3$. The dimension of Null space will be 4 because we know that

$$rank + Nul \;\; A = 7.$$

Remember that Rank = dimension of the column space of the matrix.
2. Yes, Col $A = \mathbb{R}^4$ as dimension of the column space of the matrix will be 4. Null space of the matrix will be two dimensional space and Nul $A = \mathbb{R}^2$.

14.2 Some Practice Problems

Question: Consider the basis $\mathcal{B} = \left\{ \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}\right.$ for $\mathbb{R}^2$. If $[\mathbf{x}]_{\mathcal{B}} = \left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right.$, what is $\mathbf{x}$?
Question: Determine the dimension of the subspace H of $\mathbb{R}^4$ spanned by the vectors $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$, (First, find a basis for H.)

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \\ -4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 9 \\ -6 \\ 12 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} -4 \\ 5 \\ -3 \\ 7 \end{bmatrix}.$$

Question: Find bases for Col A and Nul A, and then state the dimensions of these subspaces for the matrix

$$A = \begin{bmatrix} 1 & -2 & -1 & 5 & 4 \\ 2 & -1 & 1 & 5 & 6 \\ -2 & 0 & -2 & 1 & -6 \\ 3 & 1 & 4 & 1 & 5 \end{bmatrix}.$$

CHAPTER 15

Lecture No. 15

Recall: For a 2×2, $A = [a_{ij}]$, then determinant is the number det $A = a_{11}a_{22} - a_{12}a_{21}$.

Notion: For instance, if $A = \begin{bmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{bmatrix}$ then A_{32} is obtained by crossing out row 3 and column 2,

$$\begin{bmatrix} 1 & \mathbf{-2} & 5 & 0 \\ 2 & \mathbf{0} & 4 & -1 \\ \mathbf{3} & \mathbf{1} & \mathbf{0} & \mathbf{7} \\ 0 & \mathbf{4} & -2 & 0 \end{bmatrix} \quad \text{so that} \quad A_{32} = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}.$$

Remark: We can now give a recursive definition of a determinant. When $n = 3$, det A is defined using determinants of the 2×2 submatrices A_{1j}.

When $n = 4$, det A uses determinants of the 3×3 submatrices A_{1j}. In general, an $n \times n$ determinant is defined by determinants of $(n-1) \times (n-1)$ submatrices.

15.1 Determinant

For $n \geq 2$, the determinant of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j}$ detA_{1j}, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, ..., a_{1n}$ are from the first row of A. In symbols

$$\begin{aligned} \det A &= a_{11}\det A_{11} - a_{12}\det A_{12} + ... + (-1)^{1+n}a_{1n}\det A_{1n} \\ &= \sum_{j=1}^{n} a_{1j}(-1)1 + j\ A_{1j} \end{aligned}$$

Example: Compute the determinant of $\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$.

Solution: The determinant of the matrix is -2.

Notion: Given $A = [a_{ij}]$ then (i, j)-Cofactor of A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j}\det A_{ij}.$$

Then

$$\det A = a_{11}C_{11} + a_{12}C_{12} + ... + a_{1n}C_{1n}.$$

This formula is called a cofactor expansion across the first row of A.
Theorem: The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the ith row using the cofactors

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + ... + a_{in}C_{in}.$$

The cofactor expansion down the jth column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + ... + a_{nj}C_{nj}.$$

Example: Compute the determinant of $\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$.
Solution: Use cofactor notion to show that the determinant of the matrix is -2.
Example: Compute the determinant of

$$A = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}.$$

Solution: Expanding along the first column gives us

$$\begin{aligned} det\ A &= 3\begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix} = (3)(2)\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} \\ &= = (3)(2)(1)\begin{vmatrix} 1 & 5 \\ 0 & -2 \end{vmatrix} = (3)(2)(1)(-2) = -12. \end{aligned}$$

Theorem: If A is a triangular matrix, then det A is the product of the entries on the main diagonal of A.
Example: Compute the determinant of

$$A = \begin{bmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{bmatrix}.$$

Example: Compute the determinants of the elementary matrices given in

$$E_1 = \begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{bmatrix}.$$

Solution: These are very important examples and the determinants are

$$det\ E_1 = k, \qquad det\ E_2 = -1 \qquad det\ E_3 = 1.$$

Theorem: Let A be a square matrix.

1. If a multiple of one row of A is added to another row to produce a matrix B, then det B =det A.

2. If two rows of A are interchanged to produce B, then det$B = -$det A.

3. If one row of A is multiplied by k to produce B, then det$B = k$ det$A$.

Example: Compute the determinant $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$.

Solution: The strategy is to reduce A to echelon form and then to use the fact that the determinant of a triangular matrix is the product of the diagonal entries.

$$\det A = \begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix}.$$

An interchange of rows 2 and 3 reverses the sign of the determinant

$$\det A = - \begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} = -(1)(3)(-5) = 15.$$

Remark: A common use of 3 axiom of Theorem in hand calculations is to factor out a common multiple of one row of a matrix. For instance

$$\begin{vmatrix} * & * & * \\ * & * & * \\ 5k & -3k & 7k \end{vmatrix} = k \begin{vmatrix} * & * & * \\ * & * & * \\ 5 & -3 & 7 \end{vmatrix}$$

where the starred entries are unchanged.

Example: Compute the determinant of A, where $A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$.

Solution:

$$\begin{aligned}
\det A &= 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{vmatrix} \\
&= 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} \\
&= 2(1)(3)(-6)(1) = -36.
\end{aligned}$$

Remark: Suppose a square matrix A has been reduced to an echelon form U by row replacements and row interchanges then If there are r interchanges

$$\det A = (-1)^r \det U$$

and det U is just the multiplication of the diagonal elements of U. Thus we have

$$\det A = \begin{cases} (-1)^r(\text{product of pivots in} U), & \text{when } A \text{ is invertible,} \\ 0 & \text{when } A \text{ is not invertible.} \end{cases}$$

Theorem: A square matrix A is invertible if and only if det $A \neq 0$.

Example: Compute det A, where $A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$.

Solution: Add 2 times row 1 to row 3 to obtain $A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 5 & -3 & -6 \\ -5 & -8 & 0 & 9 \end{bmatrix}$.
because the second and third rows of the second matrix are equal.

Example: Compute det A, where $A = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{bmatrix}$.

Solution:

$$\begin{aligned}
\det A &= \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{vmatrix} = -2\begin{vmatrix} 1 & 2 & -1 \\ 3 & 6 & 2 \\ 0 & -3 & 1 \end{vmatrix} \\
&= -2\begin{vmatrix} 1 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & -3 & 1 \end{vmatrix} = -2\begin{vmatrix} 1 & 2 & -1 \\ 0 & -3 & 1 \\ 0 & 0 & 5 \end{vmatrix} \\
&= -2(1)(-3)(5) = -30
\end{aligned}$$

Theorem: If A is an $n \times n$ matrix, then det $A^T =$ det A.

Theorem: If A and B are $n \times n$ matrices, then det $(AB) =$ (det A)(det B).

Example: Use a determinant to decide if $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly independent, when $\mathbf{v}_1 = \begin{bmatrix} 5 \\ -7 \\ 9 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -3 \\ 3 \\ -5 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 2 \\ -7 \\ 5 \end{bmatrix}$.

Solution: Consider the matrix $A = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]$, the vectors will be linearly independent if det $A \neq 0$. Otherwise vectors will be linearly dependent.

15.2 Some Practice Problems

Question: Find the determinant of the matrices, where $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5$.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{vmatrix}, \quad \begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix}, \quad \begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix}.$$

Question: Use determinants to find out the matrices are invertible or not

$$\begin{vmatrix} 2 & 3 & 0 \\ 1 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix}, \quad \begin{vmatrix} 2 & 0 & 0 & 8 \\ 1 & -7 & -5 & 0 \\ 3 & 8 & 6 & 0 \\ 0 & 7 & 5 & 4 \end{vmatrix}.$$

Question: Use a determinant to decide if $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly independent, when $\mathbf{v}_1 = \begin{bmatrix} 4 \\ 6 \\ -7 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -7 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -3 \\ -5 \\ 6 \end{bmatrix}$.