CHAPTER 11

Lecture No. 11

11.1 Matrix Algebra

A matrix is an arrangement of numbers, for example a matrix A of order $m \times n$ often written as

$$A = [\mathbf{a}_1 \ \ \mathbf{a}_2 \ \ ... \ \ \mathbf{a}_n]$$

where each column $\mathbf{a}_i \in \mathbb{R}^m$.

The diagonal entries in an $m \times n$ matrix $A = a_{ij}$ are $a_{11}, a_{22}, a_{33}, ...$ and they form the main diagonal of A.

A diagonal matrix is a square $n \times n$ matrix whose nondiagonal entries are zero.

An example is the $n \times n$ identity matrix, I_n.

An $m \times n$ matrix whose entries are all zero is a zero matrix and is written as 0. The size of a zero matrix is usually clear from the context.

Sums and Scalar Multiples: The two matrices A and B could be added, i.e., $A + B$ is possible if and only if the have the same size (order).

Example: If $A = \begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ then

$$A + B = \begin{bmatrix} 2 & 1 & -4 & 8 \\ 2 & -2 & 2 & 1 \\ 5 & -8 & 8 & 1 \end{bmatrix}.$$

Can we add the matrices $C = \begin{bmatrix} 0 & 1 & -4 \\ 2 & -3 & 2 \\ 5 & -8 & 7 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 1 \\ 2 & -3 \\ 5 & -8 \end{bmatrix}$?

No both matrices C and D have different order.

Example: For A and B as in the above example then $3A = \begin{bmatrix} 0 & 3 & -12 & 24 \\ 6 & -9 & 6 & 3 \\ 15 & -24 & 21 & 3 \end{bmatrix}$

Theorem: Let A, B, and C be matrices of the same size, and let r and s be scalars.

1. $A + B = B + A$.
2. $(A + B) + C = A + (B + C)$
3. $A + 0 = 0 + A = A$

4. $r(A+B) = rA + rB$

5. $(r+s)A = rA + sA$

6. $r(sA) = (rs)A$

Matrix Multiplication: If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $\mathbf{b}_1, ..., \mathbf{b}_p$, then the product AB is the $m \times p$ matrix whose columns are $A\mathbf{b}_1, ..., A\mathbf{b}_p$. That is,

$$AB = A[\mathbf{b}_1 \ \ \mathbf{b}_2 \ \ ... \ \ \mathbf{b}_p] = [A\mathbf{b}_1 \ \ A\mathbf{b}_2 \ \ ... \ \ A\mathbf{b}_p].$$

Example: If $A = \begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ then

$$A\mathbf{b}_1 = \begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 10 \end{bmatrix}, \quad A\mathbf{b}_2 = \begin{bmatrix} 1 \\ -3 \\ -8 \end{bmatrix}.$$

$$A\mathbf{b}_3 = \begin{bmatrix} -4 \\ 2 \\ 7 \end{bmatrix}, \quad A\mathbf{b}_4 = \begin{bmatrix} 24 \\ 3 \\ 3 \end{bmatrix}, \quad AB = \begin{bmatrix} 0 & 1 & -4 & 24 \\ 4 & -3 & 2 & 3 \\ 10 & -8 & 7 & 3 \end{bmatrix}$$

Remark: Each column of AB is a linear combination of the columns of A using weights from the corresponding column of B.

Example: If A is a 3×5 matrix and B is a 5×2 matrix, what are the sizes of AB and BA, if they are defined?

Row-Column Rule for Computing AB: If the product AB is defined, then the entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B. If $(AB)_{ij}$ denotes the (i,j)-entry in AB, and if A is an $m \times n$ matrix, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{in}b_{nj}.$$

Example: Let $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$. Use the row-column rule to compute $(AB)_{13}$ and $(AB)_{22}$ entries in AB.

Remark:

$$row_i(AB) = row_i(A).B$$

Example: Let $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$. Find all the rows of AB by the above formula.

Properties of Matrix Multiplication:

Theorem: Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

1. $A(BC) = (AB)C$ (associative law of multiplication)
2. $A(B + C) = AB + AC$ (Left distributive law)
3. $(B + C)A = BA + CA$ (Right distributive law)
4. $r(AB) = (rA)B = A(rB)$ for any scalar r
5. $I_m A = A = AI_n$ (identity for matrix multiplication)

Example: Let $A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$. Check whether the commutative law holds or not.

Remark:

1. In general, $AB \neq BA$.
2. The cancelation laws do not hold for matrix multiplication. That is, if $AB = AC$, then it is not true in general that $B = C$.
3. If a product AB is the zero matrix, you cannot conclude in general that either $A = 0$ or $B = 0$.

Powers of a Matrix:

If A is an $n \times n$ matrix and if k is a positive integer, then A^k denotes the product of k copies of A

$$A^k = A...A \quad \text{(k times)}.$$

Transpose of a matrix: Given an $m \times n$ matrix A, the transpose of A is the $n \times m$ matrix, denoted by A^T , whose columns are formed from the corresponding rows of A.

Example: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} 5 & 2 \\ 3 & -2 \\ 0 & 5 \end{bmatrix}$, and $C = \begin{bmatrix} 5 & 1 & 1 & 1 \\ 3 & -2 & 2 & 3 \end{bmatrix}$ then

$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $B^T = \begin{bmatrix} 5 & 3 & 0 \\ 2 & -2 & 5 \end{bmatrix}$, and $C^T = \begin{bmatrix} 5 & 3 \\ 1 & -2 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$

Theorem: Let A and B denote matrices whose sizes are appropriate for the following sums and products.

1. $(A^T)^T = A$

2. $(A+B)^T = A^T + B^T$

3. For any scalar r, $(rA)^T = rA^T$

   $(AB)^T = B^T A^T$

Remark: The transpose of a product of matrices equals the product of their transposes in the reverse order.

Example: Let $A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 2 \\ 3 & -2 \end{bmatrix}$, then verify $(AB)^T = B^T A^T$ and $(A+B)^T = A^T + B^T$.

Invertible Matrix: An $n \times n$ matrix A is said to be invertible if there is an $n \times n$ matrix C such that

$$CA = I \qquad \text{and} \qquad AC = I$$

where $I = I_n$, the $n \times n$ identity matrix. Usually we denote $C = A^{-1}$.

Remark: The inverse of a matrix is unique (If it exists).

A matrix that is not invertible is sometimes called a **singular matrix**, and an invertible matrix is called a **nonsingular matrix**.

Example: If $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ and $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$ then $AC = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $CA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Thus $C = A^{-1}$.

Recall that :

Theorem: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is not invertible.

Example: Find the inverse of $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$.

Theorem: If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Example: Use inverse of the matrix of coefficients of the system

$$\begin{array}{rcrl} 3x_1 & - & 4x_2 = & 3 \\ 5x_1 & + & 6x_2 = & 7 \end{array}.$$

Theorem:

- If A is an invertible matrix, then A^{-1} is invertible and
$$(A^{-1})^{-1} = A.$$
- If A and B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is the product of the inverses of A and B in the reverse order. That is,
$$(AB)^{-1} = B^{-1}A^{-1}.$$
- If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. That is,
$$(A^T)^{-1} = (A^{-1})^T.$$

Elementary Matrices: An **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix.

Example: Let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 5 \end{bmatrix}$, $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

Compute E_1A, E_2A, and E_3A, and describe how these products can be obtained by elementary row operations on A.

$E_1A = \begin{bmatrix} a & b & c \\ d & e & f \\ g+4a & h+4b & i+4c \end{bmatrix}$, $E_2A = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}$ and $E_3A = \begin{bmatrix} d & e & f \\ a & b & c \\ 5g & 5h & 5i \end{bmatrix}$.

Remark: If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA, where the $m \times m$ matrix E is created by performing the same row operation on I_m.

Remark: Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I.

Example: Find the inverse of $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$

To transform E_1 into I , add $+4$ times row 1 to row 3. The elementary matrix that does this is

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

Theorem: An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

Example: Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$, if it exists.

Solution:

$$\begin{aligned}
[A \quad I] &= \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}.
\end{aligned}$$

Consequently, the inverse of the matrix is

$$A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}.$$

11.2 Some Practice Problems

Question: Let $A = \begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $\mathbf{b}_4 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

1. Find A^{-1}, and use it to solve the four equations

$$A\mathbf{x} = \mathbf{b}_1, \quad A\mathbf{x} = \mathbf{b}_2, \quad A\mathbf{x} = \mathbf{b}_3, \quad A\mathbf{x} = \mathbf{b}_4.$$

2. The four equations in part (1) can be solved by the same set of row operations, since the coefficient matrix is the same in each case. Solve the four equations in part (1) by row reducing the augmented matrix $[A \quad \mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \quad \mathbf{b}_4]$.

Question: Find the inverse the matrices $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$, if it exists.

Chapter 12

Lecture No. 12

Question: Let $A = \begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $\mathbf{b}_4 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

1. Find A^{-1}, and use it to solve the four equations

$$A\mathbf{x} = \mathbf{b}_1, \quad A\mathbf{x} = \mathbf{b}_2, \quad A\mathbf{x} = \mathbf{b}_3, \quad A\mathbf{x} = \mathbf{b}_4.$$

2. The four equations in part (1) can be solved by the same set of row operations, since the coefficient matrix is the same in each case. Solve the four equations in part (1) by row reducing the augmented matrix $[A \quad \mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \quad \mathbf{b}_4]$.

Solution:

12.1 Characterization of Invertible Matrices

Theorem: (The Invertible Matrix theorem) Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

1. A is an invertible matrix.
2. A is row equivalent to the $n \times n$ identity matrix.
3. A has n pivot positions.
4. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
5. The columns of A form a linearly independent set.
6. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
7. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in Rn.
8. The columns of A span $\mathbb{R}^n$.
9. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
10. There is an $n \times n$ matrix C such that $CA = I$.

11. There is an $n \times n$ matrix D such that $AD = I$.
12. A^T is an invertible matrix.

Example: Use the Invertible Matrix Theorem to decide if A is invertible:

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}.$$

Solution:

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}.$$

So A has three pivot positions and thus A is invertible by the Invertible Matrix Theorem.

Recall: One of the equivalent statement in the Invertible Matrix Theorem is that A has n pivot positions.
Invertible Linear Transformation: A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be invertible if there exists a function $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$S(T(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n$$

$$T(S(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n$$

We call S the inverse of T and write it as T^{-1}.

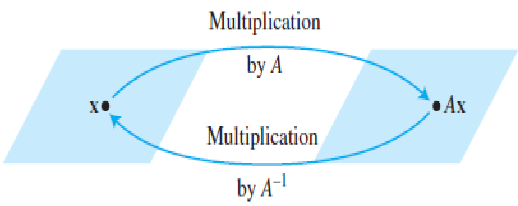

Figure 12.1: Invertible transformation

Theorem Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S : \mathbb{R}^n \to \mathbb{R}^n$ is the unique function satisfying the equations

$$S(T(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n$$

$$T(S(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n.$$

Example: What can you say about a one-to-one linear transformation T from $\mathbb{R}^n$ into $\mathbb{R}^n$?

The columns of the standard matrix A of T are linearly independent, because the system $A\mathbf{x} = \mathbf{0}$ has only trivial solution. So A is invertible, by the Invertible Matrix Theorem.

12.2 Partitioned Matrix

The matrix

$$A = \left[\begin{array}{ccc|cc|c} 3 & 0 & 1 & 2 & 2 & 2 \\ -5 & 3 & 4 & -1 & 5 & 1 \\ \hline -8 & -6 & 7 & 3 & 7 & 0 \end{array}\right]$$

can also be written into the 2×3 partitioned (or block) matrix

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

where

$$A_{11} = \begin{bmatrix} 3 & 0 & 1 \\ -5 & 3 & 4 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} -8 & -6 & 7 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 3 & 7 \end{bmatrix}, \quad A_{23} = \begin{bmatrix} 0 \end{bmatrix}$$

Addition of Partitioned Matrices:

If matrices A and B are the same size and are partitioned in exactly the same way, then it is natural to make the same partition of the ordinary matrix sum $A+B$. In this case, each block of $A+B$ is the (matrix) sum of the corresponding blocks of A and B. Multiplication of a partitioned matrix by a scalar is also computed block by block.

Example: The matrix $A = \left[\begin{array}{ccc|cc|c} 3 & 0 & 1 & 2 & 2 & 2 \\ -5 & 3 & 4 & -1 & 5 & 1 \\ \hline -8 & -6 & 7 & 3 & 7 & 0 \end{array}\right]$ can also be written into the 2×3 partitioned (or block) matrix

Multiplications of Partitioned Matrices: Partitioned matrices can be multiplied by the usual row-column rule as if the block entries were scalars, provided that for a product AB, the column partition of A matches the row partition of B.

Example: The matrix $A = \left[\begin{array}{ccc|cc} 2 & -3 & 1 & 0 & -4 \\ 1 & 5 & -2 & 3 & -1 \\ \hline 0 & -4 & -2 & 7 & -1 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $B =$

$$\left[\begin{array}{cc} 6 & 4 \\ -2 & 1 \\ -3 & 7 \\ \hline -1 & 3 \\ 5 & 2 \end{array}\right] = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \text{ then } AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} A_{11}B_1 + A_{12}B_2 \\ A_{21}B_1 + A_{22}B_2 \end{bmatrix} =$$

$$\left[\begin{array}{cc} -5 & 4 \\ -6 & 2 \\ \hline 2 & 1 \end{array}\right]$$

Column Row Expansion of AB:

Example: Let $A = \begin{bmatrix} -3 & 1 & 2 \\ 1 & -4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$. Verify that

$$AB = col_1(A)row_1(B) + col_2(A)row_2(B) + col_3(A)row_3(B).$$

Theorem: Column Row Expansion of AB

If A is $m \times n$ and B is $n \times p$, then

$$\begin{aligned} AB &= [col_1(A) \quad col_2(A) \quad \ldots \quad col_n(A)] \begin{bmatrix} row_1(B) \\ row_2(B) \\ . \\ . \\ . \\ row_n(B) \end{bmatrix} \\ &= col_1(A)row_1(B) + \ldots + col_n(A)row_n(B) \end{aligned}$$

Block Upper Triangular Matrix: The matrix $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ is said to be block upper triangular matrix.

Remark: A block diagonal matrix is a partitioned matrix with zero blocks off the main diagonal (of blocks). Such a matrix is invertible if and only if each block on the diagonal is invertible.

A factorization of a matrix A is an equation that expresses A as a product of two or more matrices.

12.3 The LU Factorizations

At first, assume that A is an $m \times n$ matrix that can be row reduced to echelon form, without row interchanges.

Then A can be written in the form $A = LU$, where L is an $m \times m$ lower triangular matrix with 1's on the diagonal and U is an $m \times n$ echelon form of A.

Such a factorization is called an LU factorization of A.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix} \begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

When $A = LU$, the equation $A\mathbf{x} = \mathbf{b}$ can be written as $L(U\mathbf{x}) = \mathbf{b}$. Writing $\mathbf{y}$ for $U\mathbf{x}$, we can find $\mathbf{x}$ by solving the pair of equations

$$L\mathbf{y} = \mathbf{b}, \qquad \text{and} \quad U\mathbf{x} = \mathbf{y}.$$

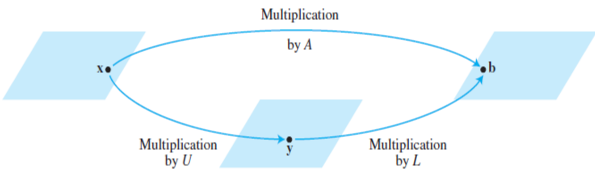

Figure 12.2: Solution of system by LU factorzation

Example: Verify that

$$A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Use this LU factorization of A to solve $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} -9 \\ 5 \\ 7 \\ 11 \end{bmatrix}$.

Solution: $[L \quad \mathbf{b}] = \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ -1 & 1 & 0 & 0 & 5 \\ 2 & -5 & 1 & 0 & 7 \\ -3 & 8 & 3 & 1 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$.

Remark: The solution of $L\mathbf{y} = \mathbf{b}$ needs only 6 multiplications and 6 additions, because the arithmetic takes place only in column 5.

$$[U \quad \mathbf{y}] = \begin{bmatrix} 3 & -7 & -2 & 2 & -9 \\ 0 & -2 & -1 & 2 & -4 \\ 0 & 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

Hence $\mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ -6 \\ -1 \end{bmatrix}$.

Remark: For $U\mathbf{x} = \mathbf{y}$, the "backward" phase of row reduction requires 4 divisions, 6 multiplications, and 6 additions.

Remark: To find $\mathbf{x}$ requires 28 arithmetic operations, or flops (floating point operations), excluding the cost of finding L and U. In contrast, row reduction of $[A \quad \mathbf{b}]$ to $[I \quad \mathbf{x}]$ takes 62 operations.

12.4 An LU Factorization Algorithm

1. Reduce A to an echelon form U by a sequence of row replacement operations, if possible.

2. Place entries in L such that the same sequence of row operations reduces L to I.

Example: Find an LU factorization of the matrix

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}.$$

Solution: L should be of order 4×4. The first column of L is the first column of A divided by the top pivot entry:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix}.$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & * & 1 & 0 \\ 3 & * & * & 1 \end{bmatrix}.$$

Then row reduce the matrix A into echelon form

$$\begin{aligned} A &= \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix}, \\ &\sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 7 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} = U \end{aligned}$$

and L take the form

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ 3 & 4 & 2 & 1 \end{bmatrix}.$$

12.5 Some Practice Problems

Question: Find the solution of the system $A\mathbf{x} = \mathbf{b}$ by using given LU factorization of A

$$A = \begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -7 \\ 5 \\ 2 \end{bmatrix}.$$

Question: Find LU factorization of the matrix $A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix}$.

Question: Can a square matrix with two identical rows be invertible? Justify your answer.

Question: Can a square matrix with two identical columns be invertible? Justify your answer.

Question If an $n \times n$ matrix G cannot be row reduced to I_n; what can you say about the columns of G? Justify your answer.

Question: Find LU factorization of the matrix $A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix}$.