CHAPTER 20

Lecture No. 20

Question: Either show that the given set is a vector space or give counter example

1. $\left\{\begin{bmatrix} 3p-5q \\ 4q \\ p \\ q+1 \end{bmatrix}; p, q \in \mathbb{R}\right\}.$

2. $\left\{\begin{bmatrix} s-2t \\ 4s \\ s+t \\ -t \end{bmatrix}; s, t \in \mathbb{R}\right\}$

Solution: 1. The set is not a vector space as the set doesn't contain the null vector.
2. Notice that the generating element of the the set can be written as

$$\begin{bmatrix} s-2t \\ 4s \\ s+t \\ -t \end{bmatrix} = s\begin{bmatrix} 1 \\ 4 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

Thus the given set

$$\left\{\begin{bmatrix} s-2t \\ 4s \\ s+t \\ -t \end{bmatrix}; s, t \in \mathbb{R}\right\} = Span\left\{\begin{bmatrix} 1 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ -1 \end{bmatrix}\right\}.$$

Since the subspace of a vector space is itself vector space and we know that spanning set of a given vectors is a subspace. Consequently, the given set is a vector space.
Recall: An indexed set of vectors $\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ in V is said to be linearly independent if the vector equation

$$c_1\mathbf{v}_1 + ... + c_p\mathbf{v}_p = \mathbf{0}$$

has only trivial solution $c_1 = 0, ..., c_p = 0$.

The set $\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ is said to be linearly dependent if the above equation has a nontrivial solution, that is if there are some weights, $c_1, ..., c_p$, not all zero such that the equation

$$c_1\mathbf{v}_1 + ... + c_p\mathbf{v}_p = \mathbf{0}$$

and the relation is called linear dependence relation.

Theorem: An indexed set $\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ of two or more vectors, with $\mathbf{v}_1 \neq 0$, is dependent if and only if some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of thee preceding vectors, $\mathbf{v}_1, ..., \mathbf{v}_{j-1}$.
Example: Determine whether the polynomials $\mathbf{p}_1(t) = 1, \mathbf{p}_2(t) = t$ and $\mathbf{p}_3(t) = 5 - 2t$ are linearly independent or linearly dependent in $\mathbb{P}$. Justify your answer.
Solution: We have to check the solution os the equation

$$c_1\mathbf{p}_1(t) + c_2\mathbf{p}_2(t) + c_3\mathbf{p}_3(t) = \mathbf{0}$$

where $\mathbf{0}$ is the null polynomial. The above equation lead to

$$(c_1 + 5c_3) + (c_2 - 2c_3)t = 0 + 0t$$

which gives us the following system of linear equations

$$c_1 + 5c_3 = 0$$

$$c_2 - 2c_3 = 0$$

and the solution of this system is $c_1 = -5c_3$ and $c_2 = 2c_3$ where as c_3 is free variable. Thus by definition the set of vectors $\{\mathbf{p}_1(t), \mathbf{p}_2(t), \mathbf{p}_3(t)\}$ is linearly dependent.

You can see that we have $\mathbf{p}_3(t) = 5\mathbf{p}_1(t) - 2\mathbf{p}_2(t)$.

Example: Determine whether the set $\{\cos t, \sin t\}$ is linearly independent or linearly dependent in the vector space $C([0, 1])$.

The vectors are linearly independent. can you figure out why?

20.1 Basis of a Vector Space

Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, ..., \mathbf{b}_p\}$ in V is a basis for H if

1. $\mathcal{B}$ is a linearly independent set, and

2. the subspace spanned by $\mathcal{B}$ coincide with H; that is,

$$H = Span\{\mathbf{b}_1, ..., \mathbf{b}_p\}.$$

Remark: The definition of a basis applies to the case when $H = V$, because any vector space is a subspace of itself.

Remark: When $H \neq V$, the second condition also includes that the requirement that the vectors $\mathbf{b}_1, ..., \mathbf{b}_p$ must belong to H.
Example: Let A be an invertible $n \times n$ matrix say, $A = [\mathbf{a}_1 \ \ ... \ \mathbf{a}_n]$. Then the columns of A form a basis for $\mathbb{R}^n$ because they are linearly independent and they span $\mathbb{R}^n$, by the Invertible Matrix Theorem.

Example: Let $\mathbf{e}_1, ..., \mathbf{e}_n$ be the columns of the $n \times n$ identity matrix, I_n. That is

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ . \\ . \\ . \\ 0 \end{bmatrix}, \quad ..., \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 1 \end{bmatrix}$$

then the set $\{\mathbf{e}_1, ..., \mathbf{e}_n\}$ forms a basis for $\mathbb{R}^n$ and is called a standard basis of $\mathbb{R}^n$.
Example: Let

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, \text{ and } \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}.$$

Determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$.
Solution: Consider the matrix $A = [\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \mathbf{v}_3]$ and check whether the matrix is invertible and use Invertible Matrix Theorem to decide the given vectors form basis for $\mathbb{R}^3$ or not.
Example: Let $S = \{1, t, t^2, ..., t^n\}$. Verify that S is basis for $\mathbb{P}_n$. This basis is called the standard basis for $\mathbb{P}_n$.
Solution: Any nth order polynomial can be generated from the polynomials $\{1, t, t^2, ..., t^n\}$ and the set forms linearly independent set of vectors. Hence a basis for the vector space $\mathbb{P}_n$.
Example: Let

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \text{ and } \mathbf{v}_3 = \begin{bmatrix} 6 \\ 16 \\ -5 \end{bmatrix}$$

and $H = Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Note that $\mathbf{v}_3 = 5\mathbf{v}_1 + 3\mathbf{v}_2$, and show that $Span\{\mathbf{v}_1, \mathbf{v}_2\} = Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Find a basis for the subspace H.
Solution: It is clear that $Span\{\mathbf{v}_1, \mathbf{v}_2\} \subseteq Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Now any vector $\mathbf{u} \in Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ will have the form

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3.$$

We are also given $\mathbf{v}_3 = 5\mathbf{v}_1 + 3\mathbf{v}_2$ put the expression of $\mathbf{v}_3$ in the above expression we have

$$\mathbf{u} = (c_1 + 5c_3)\mathbf{v}_1 + (c_2 + 3c_3)\mathbf{v}_2.$$

which shows that $\mathbf{u} \in Span\{\mathbf{v}_1, \mathbf{v}_2\}$. Hence $Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subseteq Span\{\mathbf{v}_1, \mathbf{v}_2\}$

Consequently, we have

$$Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = Span\{\mathbf{v}_1, \mathbf{v}_2\}.$$

As the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are not multiple of each other, so the vectors are linearly independent and the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ forms a basis for the space spanned by these two vectors.

Theorem: Let $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\}$ be a set in V, and let $H = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\}$.

1. If one of the vectors in S say $\mathbf{v}_k$ is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.

2. If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.

Example: Find a basis for the Col A, where

$$A = [\mathbf{a}_1 \ \ \mathbf{a}_2 \ \ ... \ \ \mathbf{a}_5] = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution: We know that the pivot columns are the linearly independent columns and $Span\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5\} = Span\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\}$ Thus basis for the column space of the given matrix is the set

$$S = \{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5\}.$$

Remark: What about a matrix A that is not in reduced echelon form?

Remark: Recall that any linear dependence relationship among the columns of A can be expressed in the form $A\mathbf{x} = 0$, where $\mathbf{x}$ is a column of weights. (If some columns are not involved in a particular dependence relation, then their weights are zero.)

When A is row reduced to a matrix B, the columns of B are often totally different from the columns of A.

However, the equations $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have exactly the same set of solutions. If $A = [\mathbf{a}_1 \ \ \mathbf{a}_2 \ \ ... \ \ \mathbf{a}_n]$ and $B = [\mathbf{b}_1 \ \ \mathbf{b}_2 \ ; ... \ ; \mathbf{b}_n]$, then the vector equations

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + ... + x_n\mathbf{a}_n = \mathbf{0} \quad \text{and} \quad x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + ... + x_n\mathbf{b}_n = \mathbf{0}$$

also have the same set of solutions.

That is, the columns of A have exactly the same linear dependence relationships as the columns of B.

Example: The matrix B

$$B = [\mathbf{b}_1 \ \ \mathbf{b}_2 \ \ ... \ \ \mathbf{b}_5] = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$$

is row equivalent to the matrix A

$$A = [\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \ldots \;\; \mathbf{a}_5] = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a basis for the Col B.

Solution: Basis for the column space of B is the set

$$S = \{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5\}.$$

The columns corresponding to the pivot columns in the echelon form.
Theorem: The pivot columns of a matrix A form a basis for Col A.
Remark: The pivot columns of a matrix A are evident when A has been reduced only to echelon form.

But, be careful to use the pivot columns of A itself for the basis of Col A. Row operations can change the column space of a matrix. The columns of an echelon form B of A are often not in the column space of A.

Two views of a basis: Basis of a vector space is the smallest spanning set and largest linearly independent set.
Example: The following three sets in $\mathbb{R}^3$ show how a linearly independent set can be enlarged to a basis and how further enlargement destroys the linear independence of the set.

Also, a spanning set can be shrunk to a basis, but further shrinking destroys the spanning property.

Linearly independent set but does not span $\mathbb{R}^3$: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \right\}.$

A basis for $\mathbb{R}^3$: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}.$

Spans $\mathbb{R}^3$ **but is linearly dependent**: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}.$

Example: Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 7 \\ -9 \end{bmatrix}$. Determine if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is basis for $\mathbb{R}^3$. Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ a basis for $\mathbb{R}^2$?

Example: Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $H = \left\{ \begin{bmatrix} s \\ s \\ 0 \end{bmatrix} : s \in \mathbb{R} \right\}$ Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ a basis for H.

Solution: The given vectors didn't form a basis for the space $\mathbb{R}^3$ as the vectors didn't span the space $\mathbb{R}^3$.

The vectors can't be basis for $\mathbb{R}^2$ as vectors are not vectors of $\mathbb{R}^2$.

Solution: Yes, the given vectors form a basis for the space H, because the vectors are linearly independent and

$$H = \left\{ \begin{bmatrix} s \\ s \\ 0 \end{bmatrix} : s \in \mathbb{R} \right\} = Span\{\mathbf{v}_1, \mathbf{v}_2\}.$$

Example: Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix} -4 \\ -8 \\ 9 \end{bmatrix}$.
Find a basis for the subspace W spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

Solution: Check whether the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ form linearly independent set or not. If the set is linearly independent then the set forms basis for W, otherwise one of the vector can be written as linear combination of other vectors. Delete that vector from the set and check the remaining vectors form a linearly independent set or not keep on doing this process unless you reach a linearly independent set of vector which will form a basis for the space W.

20.2 Some Practice Problems

Question:

1. Find a basis for the set of vectors in $\mathbb{R}^3$ in the plane $x - 3y + 2z = 0$.
2. Find a basis for the set of vectors in $\mathbb{R}^2$ on the line $y = -3x$.

Question: Determine whether the sets

1. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$
2. $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ 6 \end{bmatrix} \right\}.$

are basis for $\mathbb{R}^3$. Of the sets that are not basis, determine which ones are linearly independent and which ones span $\mathbb{R}^3$. Justify your answers.

Question: In the vector space of all real-valued functions, find a basis for the subspace spanned by $\{\sin t, \sin 2t, \sin t \cos t\}$.

CHAPTER 21

Lecture No. 21

Question:

1. Find a basis for the set of vectors in $\mathbb{R}^3$ in the plane $x - 3y + 2z = 0$.
2. Find a basis for the set of vectors in $\mathbb{R}^2$ on the line $y = -3x$.

Solution: 1. The parametric vector form of the solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, the given plane in $\mathbb{R}^3$ can be written as

$$Span\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \}.$$

Consequently, the basis for the given plane is

$$\mathcal{B} = \{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \}$$

and the dimension of the plane is 2.
2. Do yourself in the same way as we did in part 1.
Question: In the vector space of all real-valued functions, find a basis for the subspace spanned by $\{\sin t, \sin 2t, \sin t \cos t\}$.
Solution: Notice that $\sin 2t = 2 \sin t \cos t$. Now it is evident that the set of vectors $\{\sin t, \sin 2t, \sin t \cos t\}$ is linearly dependent. We will delete the vector $\sin 2t$ from the given set. Then the set

$$\{\sin t, \sin t \cos t\}.$$

Which is linearly independent set and forma basis for the space spanned by the given set.
Recall: The coordinates of each $\mathbf{x} \in \mathbb{R}^n$ with respect to a given basis $\mathcal{B}$ of $\mathbb{R}^n$.
Remark: An important reason for specifying a basis $\mathcal{B}$ for a vector space V is to impose a "coordinate system" on V.
Theorem: The Unique Representation Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, ..., \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, ..., c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1, ..., c_n\mathbf{b}_n.$$

21.1 Coordinate of x relative to the basis $\mathcal{B}$

Suppose $\mathcal{B} = \{\mathbf{b}_1, ..., \mathbf{b}_n\}$ is a basis for V and $\mathbf{x}$ is in V. The coordinates of $\mathbf{x}$ relative to the basis $\mathcal{B}$ (or the $\mathcal{B}$-coordinates of $\mathbf{x}$) are the weights $c_1, ..., c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1, ..., c_n\mathbf{b}_n.$$

The mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is the coordinate mapping (determined by $\mathcal{B}$)
Recall: Consider a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ for $\mathbb{R}^2$, where $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
Suppose an $\mathbf{x} \in \mathbb{R}^2$ has the coordinates $\mathbf{x}_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Find $\mathbf{x}$.

Example: The entries in the vector $\mathbf{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ are the coordinates of $\mathbf{x}$ relative to the standard basis of $\mathbb{R}^2$. Why?
Example: In crystallography, the description of a crystal lattice is aided by choosing a basis $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ for $\mathbb{R}^3$ that corresponds to three adjacent edges of one "unit cell" of the crystal.

An entire lattice is constructed by stacking together many copies of one cell. There are fourteen basic types of unit cells; three are displayed in Fig

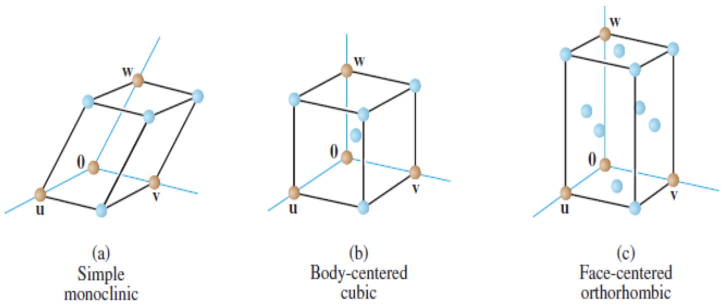

Figure 21.1: Coordinate map in crystallography

Remark: When a basis $\mathcal{B}$ for $\mathbb{R}^n$ is fixed, the $\mathcal{B}$-coordinate vector of a specified $\mathbf{x}$ is easily found.
Example: Let $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ of $\mathbf{x}$ relative to $\mathcal{B}$.
Remark: An analogous change of coordinates can be carried out in $\mathbb{R}^n$ for a basis $\mathcal{B} = \{\mathbf{b}_1, ..., \mathbf{b}_n\}$. Let

$$P_{\mathcal{B}} = [\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n].$$

Then the vector equation $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + ... + c_n\mathbf{b}_n$ is equivalent to

$$\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}.$$

We call $P_{\mathcal{B}}$ the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$.

Since the columns of $P_{\mathcal{B}}$ form a basis for $\mathbb{R}^n$, $P_{\mathcal{B}}$ is invertible (by the Invertible Matrix Theorem).

Left-multiplication by $P_{\mathcal{B}}^{-1}$ converts $\mathbf{x}$ into its $\mathcal{B}$-coordinate vector, $P_{\mathcal{B}}^{-1}\mathbf{x} = [\mathbf{x}]_{\mathcal{B}}$.

The correspondence $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$, produced here by $P_{\mathcal{B}}^{-1}$, is the coordinate mapping mentioned earlier.

Since $P_{\mathcal{B}}^{-1}$ is an invertible matrix, the coordinate mapping is a one-to-one linear transformation from $\mathbb{R}^n$ onto $\mathbb{R}^n$, by the Invertible Matrix Theorem.

21.2 The Coordinate Mapping

Theorem: Let $\mathcal{B} = \{\mathbf{b}_1, ..., \mathbf{b}_n\}$. be a basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a one to one linear transformation from V onto $\mathbb{R}^n$.

Remark: In general, a one-to-one linear transformation from a vector space V onto a vector space W is called an **isomorphism** from V onto W (iso from the Greek for "the same," and morph from the Greek for "form" or "structure").

The notation and terminology for V and W may differ, but the two spaces are indistinguishable as vector spaces.

Every vector space calculation in V is accurately reproduced in W, and vice versa.

In particular, any real vector space with a basis of n vectors is indistinguishable from $\mathbb{R}^n$, i.e., isomorphic to $\mathbb{R}^n$.

Example: Let $\mathcal{B}$ be the standard basis of the space $\mathbb{P}_3$ of polynomials; that is, let $\mathcal{B} = \{1, t, t^2, t^3\}$. Find the coordinate of arbitrary polynomial in the space $\mathbb{P}_3$ and define the coordinate mapping and decide the vector space $\mathbb{P}_3$ is isomorphic to which Euclidean space.

Solution: The arbitrary vector of $\mathbb{P}_3$ is $\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$, where $a_0, a_1, a_2, a_3 \in \mathbb{R}$ then the coordinate vector of that polynomial is

$$[\mathbf{p}(t)]_{\mathcal{B}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Since the dimension of $\mathbb{P}_3$ is 4 and its is isomorphic to $\mathbb{R}^4$.

Example: Use coordinate vectors to verify that the polynomials $1+2t^2$, $4+t+5t^2$, and $3+2t$ are linearly dependent in $\mathbb{P}_2$.

Solution: The standard basis for $\mathbb{P}_2$ is $\mathcal{B} = \{1, t, t^2\}$ and the coordinates of the given vectors with respect to basis $\mathcal{B}$ is

$$[1+2t^2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \qquad [4+t+5t^2]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}, \qquad [3+2t]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}.$$

The given vectors are linearly independent if and only if the coordinate vectors of these polynomials are linearly independent. So the augmented matrix is

$$\begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & 5 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which shows that the given polynomials are linearly dependent.

Example: Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$ and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$. Then $\mathcal{B}$ is basis for $H = Span\{\mathbf{v}_1, \mathbf{v}_2\}$. Determine if $\mathbf{x}$ is in H, and if it is, find the coordinate vector of $\mathbf{x}$ relative to $\mathcal{B}$.

Solution: We will solve the following vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{x}$. The augmented matrix is

$$\begin{bmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $c_1 = 3$ and $c_2 = 2$ and the coordinate of $\mathbf{x}$ is

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Example: Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}$,and $\mathbf{b}_3 = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} -8 \\ 2 \\ 3 \end{bmatrix}$.

1. Show that the set $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$. is basis for $\mathbb{R}^3$.
2. Find the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis.
3. Write the equation that relates $\mathbf{x} \in \mathbb{R}^3$ to $[\mathbf{x}]_{\mathcal{B}}$.
4. Find $[\mathbf{x}]_{\mathcal{B}}$, for the $\mathbf{x}$ given above.

Solution: 1. The matrix $A = [\mathbf{b}_1 \;\; \mathbf{b}_2 \;\; \mathbf{b}_3]$ has pivot in every row hence the set $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is linearly independent and forms a basis for the space $\mathbb{R}^3$.
2. The change of coordinate matrix from $\mathbb{B}$ to the standard basis of $\mathbb{R}^3$ is

$$\begin{bmatrix} 1 & -3 & 3 \\ 0 & 4 & -6 \\ 0 & 0 & 3 \end{bmatrix}.$$

3. Can you write the equation?
4. For $[\mathbf{x}]_{\mathcal{B}}$, we will solve the system

$$\begin{bmatrix} 1 & -3 & 3 & -8 \\ 0 & 4 & -6 & 2 \\ 0 & 0 & 3 & 3 \end{bmatrix}.$$

Example: The set $\mathcal{B} = \{1+t, 1+t^2, t+t^2\}$. is a basis for $\mathbb{P}_2$. Find the coordinate vector of $\mathbf{p}(t) = 6+3t-t^2$ relative to $\mathcal{B}$.
Solution: The one way of getting the answer is to write down the coordinate of the polynomials with respect to standard basis and then solve the system.

$$[1{+}t]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad [1{+}t^2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad [t{+}t^2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad [6{+}3t{-}t^2]_{\mathcal{B}} = \begin{bmatrix} 6 \\ 3 \\ -1 \end{bmatrix}.$$

For coordinate vector of $\mathbf{p}(t)$ we solve the system whose augmented matrix is

$$\begin{bmatrix} 1 & 1 & 0 & 6 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 6 \\ 0 & -1 & 1 & -3 \\ 0 & 1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 6 \\ 0 & -1 & 1 & -3 \\ 0 & 0 & 2 & -4 \end{bmatrix}.$$

21.3 Some Practice Problems

Question: The set $\mathcal{B} = \{1+t^2, t+t^2, 1+2t+t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $\mathbf{p}(t) = 1+4t+7t^2$.

Question: Determine whether the sets of polynomials form a basis for $\mathbb{P}_3$. justify your answer

1. $3+7t,\ 5+t-2t^3,\ t_2t^2,\ 1+16t-6t^2+2t^3$.
2. $5-3t+4t^2+2t^3,\ 9+t+8t^2-6t^3,\ 6-2t+5t^2,\ t^3$.

Question: Find the vector $\mathbf{x}$ determined by the given coordinate $[\mathbf{x}]_{\mathcal{B}}$ and the given basis $\mathcal{B}$.

1. $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} \right\}$, $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$.
2. $\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} \right\}$, $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}$.

Chapter 22

Lecture No. 22

Theorem: If a vector space V has a basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$, then any set in V containing more than n vectors must be linearly dependent.

Theorem: If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

22.1 Infinite Dimensional and Finite Dimensional Vector Spaces

If V is spanned by a finite set, then V is said to be finite dimensional, and the dimension of V, written as dim V, is the number of vectors in a basis for V.

The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero.

If V is not spanned by a finite set, then V is said to be infinite dimensional.
Example: The standard basis for $\mathbb{R}^n$ contains n vectors, so dim $\mathbb{R}^n = n$.

The standard polynomial basis $\{1, t, t^2\}$ show that the dim $\mathbb{P}_2 = 3$. In general dim $\mathbb{P}_n = n + 1$. The space of all polynomials $\mathbb{P}$ is infinite dimensional space.

Example: Let $H = Span\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. Find the dimension of the space H.
Solution: Since the two vectors are not multiple of each other hence the se $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent and form a basis for the space H. Hence the dimension of the space H is 2.

Example: Find the dimension of the subspace

$$H = \left\{ \begin{bmatrix} a - 3b + 6c \\ 5a + 4d \\ b - 2c - d \\ 5d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}.$$

Solution: The defining vector of the subspace H can be written as

$$\begin{bmatrix} a-3b+6c \\ 5a+4d \\ b-2c-d \\ 5d \end{bmatrix} = a\begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 6 \\ 0 \\ -1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix}.$$

The set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix} \right\}$$

is linearly independent and hence the dimension of H is 4.
Example: The subspaces of $\mathbb{R}^3$ can be classified by dimension

1. 0 dimensional subspace, i.e., subspace containing origin only.
2. 1 dimensional subspaces, i.e., subspace spanned a nonzero vector.
3. 2 dimensional subspaces, i.e., subspace spanned by two non parallel vectors.

22.2 Subspaces of a Finite Dimensional Space

Theorem: Let H be a subspace of a finite dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite dimensional and

$$\dim\ H \leq \dim\ V.$$

The Basis Theorem: Let V be a p dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V.

Any set of exactly p elements that spans V is automatically a basis for V.

The Dimensions of Nul A and Col A: The dimension of Nul A is the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$, and the dimension of Col A is the number of pivot columns in A.
Example: Find the dimensions of the null space and the column space of

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

Solution: Row reduce the augmented matrix $[A \quad \mathbf{0}]$ to echelon form

$$A \sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The pivot columns in the echelon form of the given matrix form the basis for the column space of the given matrix. Since there are two pivot columns so the dimension of the column space of the matrix is 2.

According to the rank theorem the dimension of the null space of the matrix is 3.

You can get the dimension of the Null space by writing the parametric vector form of the above matrix.

Example: Find the dimension of the subspace spanned by the given vectors

1. $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}$

2. $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 5 \end{bmatrix}$

Solution: In both parts the given vectors are linearly dependent, find a vector which is linear combination of the other vectors, delete that matrix from the set and check the remaining vectors are linearly independent or not.

Example: The first four Laguerre polynomials are $1, 1-t, 2-4t+t^2$, and $6-18t+t^2-t^3$. Show that these polynomials form a basis of $\mathbb{P}_3$.

Solution: The given vectors, i.e., polynomials are 4 and the dimension of the vector space $\mathbb{P}_3$ is also 4. Hence we have to show that the given polynomials are linearly independent, then these polynomials automatically span the vector space $\mathbb{P}_3$. For checking the linearly independence we write the coordinate vectors of the given matrices with respect to standard basis $\mathcal{B} = \{1, t, t^2, t^3\}$ of vector space $\mathbb{P}_3$.

$$[1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [1-t]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad [2-4t+t^2]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -4 \\ 1 \\ 0 \end{bmatrix},$$

$$[6-18t+t^2-t^3]_{\mathcal{B}} = \begin{bmatrix} 6 \\ -18 \\ 1 \\ -1 \end{bmatrix}.$$

Then the augmented matrix is

$$\begin{bmatrix} 1 & 1 & 2 & 6 & 0 \\ 0 & -1 & -4 & -18 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

Which shows that the vectors are linearly independent and hence the Lagurre polynomials are Linearly independent hence forma basis for vector space $\mathbb{P}_3$.

22.3 The Row Space

If A is an $m \times n$ matrix, each row of A has n entries and thus can be identified with a vector in $\mathbb{R}^n$. The set of all linear combinations of the row vectors is called the row space of A and is denoted by Row A.

Notice that Row $A =$ Col A^T.

Example: Let $A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$. Then Row A will be space of which Euclidean space? Find the basis for the Row A.

Solution: The matrix A is row equivalent to the following matrix

$$A \sim B = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The nonzero rows form basis for the row space. hence dimension of the row space is 3 and row space is subspace of $\mathbb{R}^5$.

Theorem: If two matrices A and B are row equivalent, then their row spaces are the same.

If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.

Theorem: The rank of A is the dimension of the column space of A.

Theorem The Rank Theorem

The dimensions of the column space and the row space of an $m \times n$ matrix A are equal. This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation

$$\text{rank}A + \dim \text{Nul}A = n.$$

number of pivot columns +number of non pivot columns = number of columns

Example:

1. If A is a 7×9 matrix with a two dimensional null space, what is the rank of A?

2. Could a 6×9 matrix have a two-dimensional null space?

Solution: Use the above Theorem to answer the questions.

Example: Let $A = \begin{bmatrix} 3 & 0 & -1 \\ 3 & 0 & -1 \\ 4 & 0 & 5 \end{bmatrix}$

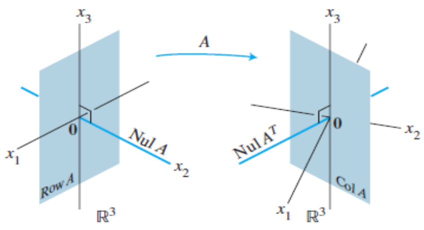

Figure 22.1: Row, column and null space of a matrix

Theorem The Invertible Matrix Theorem (Contd) Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

1. The columns of A form a basis of $\mathbb{R}^n$.

2. Col $A = \mathbb{R}^n$.

3. dim Col $A = n$.

4. rank $A = n$.

5. Nul $A = \{\mathbf{0}\}$.

6. dim Nul $A = 0$.

Example: The matrices below are row equivalent

$$A = \begin{bmatrix} 2 & -1 & 1 & -6 & 8 \\ 1 & -2 & -4 & 3 & -2 \\ -7 & 8 & 10 & 3 & -10 \\ 4 & -5 & -7 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 & -4 & 3 & -2 \\ 0 & 3 & 9 & -12 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1. Find rank A and dim Nul A.

2. Find basis for Col A and Row A.

3. What is the next step to perform to find a basis for Nul A?

4. How many pivot columns are in a row echelon form of A^T?

Solution: 1. We know that "rank + dim Null space =n", where n is number of columns of the matrix. Also, we know that the number of nonzero rows form basis for the row space of the matrix. So the given matrix has dimension of the row space as 2.

Null space of the dimension 3.

2. The columns in A corresponding to the pivot columns in B form the basis for the column space of A. The non zero rows in B form the basis of row space of A and B as well. Remember that row spaces of a matrix and its echelon form are same.
3. The next step is to write down the parametric vector form of the solution of the system $A\mathbf{x} = \mathbf{0}$ or reduce the matrix to the reduced echelon form and decide about the free and basic variables.
4. Two columns.

22.4 Some Practice Problems

Question: Find the dimension of the subspace of all vectors in $\mathbb{R}^3$ whose first and third entries are equal.

Question: Explain why the space $\mathbb{P}$ of all polynomials is an infinite dimensional space.

Question: Let H be an n-dimensional subspace of an n-dimensional vector space V. Show that $H = V$.

Question: If the null space of an 8×7 matrix A is 5-dimensional, what is the dimension of the column space of A? If the null space of an 8×7 matrix A is 3-dimensional, what is the dimension of the row space of A?
Question: Find the Null, Row and Column spaces of the matrix $A = \begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 2 & 6 & 6 & 0 & -3 \\ 3 & 9 & 3 & 6 & -3 \\ 3 & 9 & 0 & 9 & 0 \end{bmatrix}$.

Chapter 23

Lecture No. 23

Question: Find the dimension of the subspace of all vectors in $\mathbb{R}^3$ whose first and third entries are equal.
Solution: The given set can be written as

$$H = \left\{ \begin{bmatrix} s \\ t \\ s \end{bmatrix} : s, t \in \mathbb{R} \right\}.$$

The generating element of the above set can be written as

$$\begin{bmatrix} s \\ t \\ s \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Which gives

$$H = \left\{ \begin{bmatrix} s \\ t \\ s \end{bmatrix} : s, t \in \mathbb{R} \right\} = Span\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \}.$$

The dimension of the space is 2.

Question: Explain why the space $\mathbb{P}$ of all polynomials is an infinite dimensional space.

Solution: The set $\{1, t, t^2, ..., t^n, ...\}$ forms basis for the space $\mathbb{P}$ and hence dimension of ı $= \infty$.

Remark: When a basis $\mathcal{B}$ is chosen for an n-dimensional vector space V, the associated coordinate mapping onto $\mathbb{R}^n$ provides a coordinate system for V.

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad [\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

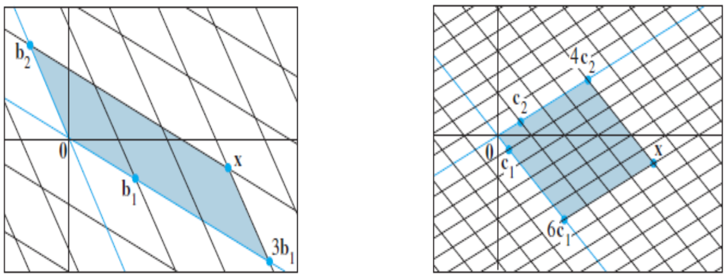

Figure 23.1: Coordinate system with respect to two different basis of the same space

Remark: Our problem is to find the connection between the two coordinate vectors.

23.1 Change of Coordinate Matirx

Example: Consider two bases $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ for a vector space V, such that

$$\mathbf{b}_1 = 4\mathbf{c}_1 + \mathbf{c}_2, \quad \text{and} \quad \mathbf{b}_2 = -6\mathbf{c}_1 + \mathbf{c}_2.$$

Suppose $\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2$, that is, suppose $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Find $[\mathbf{x}]_{\mathcal{C}}$.

Solution: The coordinate mapping is a linear transformation. So we can have

$$[\mathbf{x}]_{\mathcal{C}} = 3[\mathbf{b}_1]_{\mathcal{C}} + [\mathbf{b}_2]_{\mathcal{C}}.$$

We are also given

$$\mathbf{b}_1 = 4\mathbf{c}_1 + \mathbf{c}_2, \quad \text{and} \quad \mathbf{b}_2 = -6\mathbf{c}_1 + \mathbf{c}_2,$$

that is

$$[\mathbf{b}_2]_{\mathcal{C}} \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \qquad [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}.$$

Consequently, we have

$$[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$

The above coordinate vector $[\mathbf{x}]_{\mathcal{C}}$ can be obtained as

$$[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} [\mathbf{x}]_{\mathcal{B}}.$$

The matrix

$$\begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix}$$

is know as change of coordinate matrix from basis $\mathcal{B}$ to the basis $\mathcal{C}$.

Theorem: Let $\mathcal{B} = \{\mathbf{b}_1, ..., \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, ..., \mathbf{c}_n\}$ be basis of a vector space V. The there is a unique $n \times n$ matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$such that

$$[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}.$$

The columns of $P_{\mathcal{C}\leftarrow\mathcal{B}}$ are the $\mathcal{C}$-coordinate vectors of the vectors in the basis $\mathcal{B}$. That is,

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = [[\mathbf{b}_1]_{\mathcal{C}} \ \ [\mathbf{b}_2]_{\mathcal{C}} \ \ ... \ \ [\mathbf{b}_n]_{\mathcal{C}}].$$

Remark: The matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is called the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

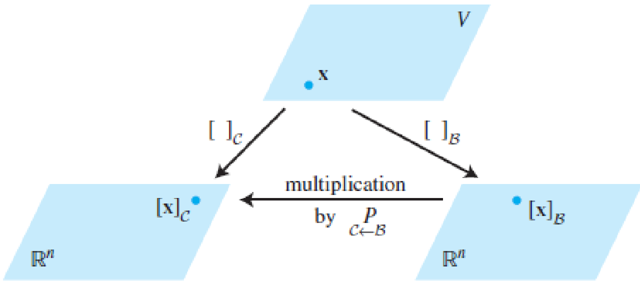

Figure 23.2: Change of coordinate matrix

Remark: The columns of $P_{\mathcal{C}\leftarrow\mathcal{B}}$ are linearly independent because they are the coordinate vectors of the linearly independent set $\mathcal{B}$.

Remark: The matrix of $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is invertible (by invertible matrix theorem) and we have

$$(P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1}[\mathbf{x}_{\mathcal{C}}] = [\mathbf{x}_{\mathcal{B}}].$$

Thus $(P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1}$ is the matrix that converts $\mathcal{C}$-coordinates into $\mathcal{B}$-coordinates. That is,

$$(P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1} = P_{\mathcal{B}\leftarrow\mathcal{C}}.$$

Change of Basis in $\mathbb{R}^n$: If $\mathcal{B} = \{\mathbf{b}_1, ..., \mathbf{b}_n\}$ and $\mathcal{E} = \{\mathbf{e}_1, ..., \mathbf{e}_n\}$ be the standard basis of $\mathbb{R}^n$ then change of coordinate matrix is

$$P_{\mathcal{B}} = [\mathbf{b}_1 \ \ ... \ \ \mathbf{b}_n].$$

Example: Let $\mathbf{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, $\mathbf{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$, and consider the basis for $\mathbb{R}^2$ given by $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$. Find the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

Solution: The matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ involves the $\mathcal{C}$-Coordinate of the vectors $\mathbf{b}_1$ and $\mathbf{b}_2$. Let $[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $[\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. By definition, we have

$$[\mathbf{c}_1 \quad \mathbf{c}_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{b}_1 \quad \text{and} \quad [\mathbf{c}_1 \quad \mathbf{c}_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{b}_2.$$

$$\left[\begin{array}{cc|cc} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{array}\right].$$

Thus we have

$$[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} 6 \\ -5 \end{bmatrix} \qquad [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}.$$

Consequently, the change of coordinate matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = [[\mathbf{b}_1]_{\mathcal{C}} \quad [\mathbf{b}_2]_{\mathcal{C}}] = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}.$$

Example: Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, $\mathbf{c}_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$, and consider the basis for $\mathbb{R}^2$ given by $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$.

1. Find the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$.

2. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

Solution: 1. Notice that we have to find out the matrix $P_{\mathcal{B}\leftarrow\mathcal{C}}$ involves the $\mathcal{B}$-Coordinate of the vectors $\mathbf{c}_1$ and $\mathbf{c}_2$. By definition, we have

$$[\mathbf{b}_1 \quad \mathbf{b}_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{c}_1 \quad \text{and} \quad [\mathbf{b}_1 \quad \mathbf{b}_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{c}_2.$$

which gives

$$\left[\begin{array}{cc|cc} 1 & -2 & -7 & -5 \\ -3 & 4 & 9 & 7 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 5 & 3 \\ 0 & 1 & 6 & 4 \end{array}\right].$$

Thus we have

$$[\mathbf{c}_1]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \qquad [\mathbf{c}_2]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Consequently, the change of coordinate matrix $P_{\mathcal{B}\leftarrow\mathcal{C}}$ is

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = [[\mathbf{c}_1]_{\mathcal{B}} \quad [\mathbf{c}_2]_{\mathcal{B}}] = \begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}.$$

2. $P_{\mathcal{C}\leftarrow\mathcal{B}} = (P_{\mathcal{B}\leftarrow\mathcal{C}})^{-1}$ and is given by

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 2 & -3/2 \\ -3 & 5/2 \end{bmatrix}.$$

Example: Let $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ and $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ be basis for a vector space V, and suppose $\mathbf{f}_1 = 2\mathbf{d}_1 - \mathbf{d}_2 + \mathbf{d}_3$, $\mathbf{f}_2 = 3\mathbf{d}_2 + \mathbf{d}_3$, $\mathbf{f}_3 = -3\mathbf{d}_1 + 2\mathbf{d}_3 + \mathbf{d}_3$.

1. Find the change-of-coordinates matrix from $\mathcal{F}$ to $\mathcal{D}$.

2. Find $[\mathbf{x}]_{\mathcal{D}}$ for $\mathbf{x} = \mathbf{f}_1 - 2\mathbf{f}_2 + 2\mathbf{f}_3$.

Solution: 1. The change of coordinate matrix $P_{\mathcal{D}\leftarrow\mathcal{F}}$ is

$$P_{\mathcal{D}\leftarrow\mathcal{F}} = [[\mathbf{f}_1]_{\mathcal{D}} \quad [\mathbf{f}_2]_{\mathcal{D}} \quad [\mathbf{f}_3]_{\mathcal{D}}] = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

2.

$$[\mathbf{x}]_{\mathcal{D}} = P_{\mathcal{D}\leftarrow\mathcal{F}}[\mathbf{x}]_{\mathcal{F}} = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \\ 1 \end{bmatrix}.$$

Example: In $\mathbb{P}_2$, find the change of coordinates matrix from the basis $\mathcal{B} = \{1 - 2t + t^2, 3 - 5t + 4t^2, 2t + 3t^2\}$ to the standard basis $\mathcal{C} = \{1, t, t^2\}$. Then find $\mathcal{B}$-coordinate vector for $-1 + 2t$.
Solution: The change of coordinate matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = [[1 - 2t + t^2]_{\mathcal{C}} \quad [3 - 5t + 4t^2]_{\mathcal{C}} \quad [2t + 3t^2]_{\mathcal{C}}] = \begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix}.$$

Now we have

$$[-1 + 2t]_{\mathcal{B}} = (P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1}[\mathbf{x}]_{\mathcal{C}}.$$

The inverse of the matrix $(P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1}$ is

$$\begin{bmatrix} -23 & -9 & 6 \\ 8 & 3 & -2 \\ -3 & -1 & 1 \end{bmatrix}.$$

Consequently, we have

$$[-1 + 2t]_{\mathcal{B}} = (P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1}[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} -23 & -9 & 6 \\ 8 & 3 & -2 \\ -3 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}.$$

Can you verify the above calculations?

Example: Let

$$P = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -5 & 0 \\ 4 & 6 & 1 \end{bmatrix}, \qquad \mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -8 \\ 5 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -7 \\ 2 \\ 6 \end{bmatrix}.$$

Find a basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ for $\mathbb{R}^3$ such that P is the change of coordinate matrix from $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ to the basis $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Solution: If P is the change of coordinate matrix from $\mathcal{U}$ to $\mathcal{V}$ then the columns of P must satisfy

$$[\mathbf{u}_1]_{\mathcal{V}} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}, \quad [\mathbf{u}_2]_{\mathcal{V}} = \begin{bmatrix} 2 \\ -5 \\ 6 \end{bmatrix}, \quad [\mathbf{u}_3]_{\mathcal{V}} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Consequently, the required vectors are

$$\mathbf{u}_1 = 1\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} - 3\begin{bmatrix} -8 \\ 5 \\ 2 \end{bmatrix} + 4\begin{bmatrix} -7 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \\ 21 \end{bmatrix}$$

$$\mathbf{u}_2 = 2\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} - 5\begin{bmatrix} -8 \\ 5 \\ 2 \end{bmatrix} + 6\begin{bmatrix} -7 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ -9 \\ 32 \end{bmatrix}$$

$$\mathbf{u}_3 = -1\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} + 0\begin{bmatrix} -8 \\ 5 \\ 2 \end{bmatrix} + 1\begin{bmatrix} -7 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} -8 \\ 0 \\ 3 \end{bmatrix}.$$

23.2 Some Practice Problems

Question: In $\mathbb{P}_2$, find the change of coordinates matrix from the basis $\mathcal{B} = \{1 - 3t^2, 2 + t - 5t^2, 1 + 2t\}$ to the standard basis. Then write t^2 as linear combination of the polynomials in $\mathcal{B}$.

Question: Mark each statement as true or false

1. The columns of the change of coordinates matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ are $\mathcal{B}$-coordinate vector of the vectors in $\mathcal{C}$.
2. The columns of $P_{\mathcal{C}\leftarrow\mathcal{B}}$ are linearly independent.
3. If $V = \mathbb{R}^2$, $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$, then row reduction of $[\mathbf{c}_1 \ \ \mathbf{c}_2 \ \ \mathbf{b}_1 \ \ \mathbf{b}_2 \ \ ]$ to $[I \ \ P]$ produces a matrix P that satisfies $[\mathbf{x}]_{\mathcal{B}} = P[\mathbf{x}]_{\mathcal{C}}$ for all $\mathbf{x}$ in V.

Question: Let $P = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -5 & 0 \\ 4 & 6 & 1 \end{bmatrix}$ $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. Find a basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ for $\mathbb{R}^3$ such that P is the change of coordinate matrix from $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.