The text book:

David C. Lay, Linear Algebra and Its Applications, Fourth Edition, Addison-Wesley, ISBN-13: 978-1408280560.

Reference books:

- **Gilbert Strang**, Introduction to Linear Algebra, Fourth Edition, Wellesley-Cambridge Press, ISBN: 9780980232714.

- **Lee W. Johnson, R. Dean Riess and Jimmy T. Arnold**, Introduction to Linear Algebra, Fifth Edition, Addison-Wesley, ISBN-13: 9780201658590.

Assessment Plan for the Course:

- Four Assignments 10%.
- Four Quiz 15%.
- First Sessional Exam 10%.
- Second Sessional Exam 15%.
- Final Exam 50%.

1.1 Motivation

Mass Balance: Although there are several examples from real world which can used as motivation for the study of system of linear equations. But here I choose a simple example of finding the unknown masses of two objects namely h and c. The three masses are placed on a rod and rod is balanced on a wedge as show in the figure.

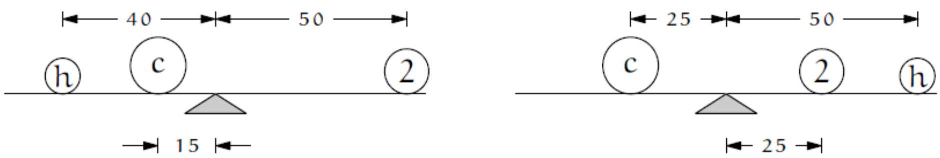

Figure 1.1: Mass balance

$$\text{The turning effect counterclockwise} = \text{Turning effect clockwise}$$

Then two equilibrium positions give rise to the following system of linear equations

$$\begin{aligned} 40h + 15c &= 100 \\ 25c &= 50 + 50h \end{aligned}$$

Some Other Applications:
Linear Programming: The airline industry, for instance, employs linear programs that **schedule flight crews**, **monitor the locations of aircraft**, or plan the varied schedules of support services such as maintenance and **terminal operations**.

Electrical Networks: Engineers use simulation software to design electrical circuits and microchips involving millions of transistors. Such software relies on linear algebra techniques and systems of linear equations.

1.2 System of Linear Equations

Example: The equations

$$3x_1 - 5x_2 = 4x_1 \quad \text{and} \quad x_1 - \sqrt{5}x_2 = 4x_2 + 5\sqrt{5}$$

are linear equations and can be simplified to

$$x_1 + 5x_2 = 0 \quad \text{and} \quad x_1 - (4 + \sqrt{5})x_2 = 5\sqrt{5}.$$

The equations

$$x_1 - x_2 + x_2x_1 = 0 \quad \text{and} \quad x_1 - \sqrt{5}x_2 = 4x_2 + \sqrt{x_2}$$

are not linear equations due to the terms x_2x_1 and $\sqrt{x_2}$, respectively.

A linear equation in the variables $x_1, \ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = d$$

where $a_1, \cdots, a_n$ are real or complex numbers (**usually known**) $d \in \mathbb{R}$ is the constant.
Examples:

$$\begin{aligned} 2x_1 - x_2 + 3x_3 &= 10 \\ -x_1 + 5x_2 + x_3 &= 5 \end{aligned}$$

is a system of linear equations with two equations and three unknowns.

The following system

$$-x_1 + 5x_2 + 3x_3 + x_4 = 10$$
$$2x_1 + 5x_2 + 2x_3 - 2x_4 = 5$$
$$9x_1 - 10x_2 + x_3 - 3x_4 = 5$$

has three equation and four variables (unknowns).

A system of linear equations with m equations and n variables

$$\begin{array}{ccccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \cdots & + & a_{1,n}x_n & = & d_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \cdots & + & a_{2,n}x_n & = & d_2 \\ & & & & & & & & \vdots \\ a_{m,1}x_1 & + & a_{m,2}x_2 & + & \cdots & + & a_{m,n}x_n & = & d_m \end{array}$$

1.3 Solution of system of Linear Equations

Example: Unique Solution

$$x_1 - 2x_2 = -1 \qquad l_1$$
$$-x_1 + 3x_2 = 3 \qquad l_2$$

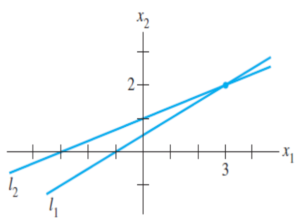

Figure 1.2: Unique solution

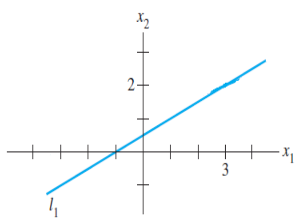

Figure 1.3: Infinite many solutions

Example: No solution and Infinite many solutions

$$(a) \qquad x_1 - 2x_2 = -1 \qquad l_1$$
$$-x_1 + 2x_2 = 3 \qquad l_2$$

$$(b) \qquad \begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 1 \end{aligned}$$

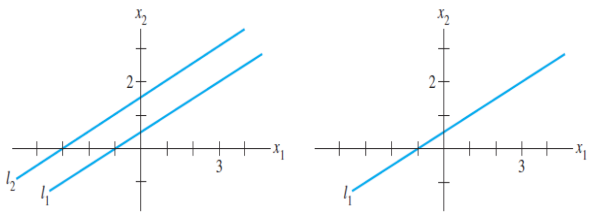

Figure 1.4: No solution and Infinite many solutions

Remark: For a system of linear equation with two variables and two unknown we have three possibilities, (i) system has unique solution, (ii) Infinite many solution, (iii) No solution.

Example: The ordered pair $(-1, 5)$ is a solution of this system. In contrast, $(5, -1)$ is not a solution.

$$\begin{array}{rcrcr} 3x_1 & + & 2x_2 & = & 7 \\ -x_1 & + & x_2 & = & 6 \end{array}$$

Solution: For $(-1, 5)$, we have $x_1 = -1$ and $x_2 = 5$ and the equations becomes

$$-3 + 10 = 7, \qquad 1 + 5 = 6$$

both equation are satisfied and hence the order pair $(-1, 5)$ is a solution of the system of linear equations. For the order pair $(5, -1)$, we have $x_1 = 5$ and $x_2 = -1$ then

$$15 - 2 = 7, \qquad -5 - 1 = 6$$

which shows that none of the equation is satisfied and hence order pair $(5, -1)$ is not a solution.

Example: Is $(3, 4, -2)$ a solution of the following system?

$$\begin{array}{rcrcrcr} 5x_1 & - & x_2 & + & 2x_3 & = & 7 \\ -2x_1 & + & 6x_2 & + & 9x_3 & = & 0 \\ -7x_1 & + & 5x_2 & - & 3x_3 & = & -7 \end{array}$$

Solution: We have $x_1 = 3, x_2 = 4$ and $x_3 = -2$ then the equation becomes

$$15 - 4 - 4 = 7, \quad -6 + 24 - 18 = 0, \quad -21 + 20 + 6 = -7$$

clearly the third equation is not satisfied. Consequently, $(3, 4, -2)$ is not a solution os the system of linear equations.

A system of linear equations with m equations and n variables

$$\begin{array}{rcrcccrcl} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \cdots & + & a_{1,n}x_n & = & d_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \cdots & + & a_{2,n}x_n & = & d_2 \\ & & & & & & & & \vdots \\ a_{m,1}x_1 & + & a_{m,2}x_2 & + & \cdots & + & a_{m,n}x_n & = & d_m \end{array}$$

has the solution $(s_1, s_2, \ldots, s_n)$ if that n-tuple is a solution of all of the equations in the system.
Recall: For a system of linear equation with two variables and two unknown we have three possibilities;

- System has a unique solution,
- Infinite many solution,
- No solution.

1.4 Consistent and Inconsistent System

A system of linear equations is said to be **consistent** if it has either one solution or infinitely many solutions; a system is inconsistent if it has no solution.

Question: Can a system of linear equations has only two solutions or only three solution or only 100 solutions?

How to find all solutions of a given system of linear equations?
Matrix: A matrix is a rectangular array of numbers. For example

$$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 6 & 1 & 0 & 2 \\ 3 & 1 & 0 & 0 \end{bmatrix}$$

is a matrix having three row and three columns.
The order of a matrix is defined as

$$\text{order} = \text{The number of rows} \times \text{the number of columns}.$$

The order of the matrix $\begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 6 & 1 & 0 & 2 \\ 3 & 1 & 0 & 0 \end{bmatrix}$ is 3×3.

Examples: $\begin{bmatrix} 1 \\ 1/3 \\ 1 \end{bmatrix}_{3\times 1}$ is called a columns matrix or vector. $\begin{bmatrix} -4 & 12 & 4 \\ 2 & -6 & -7 \end{bmatrix}_{2\times 3}$

For the system of linear equations

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & 3x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}$$

the matrix $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$ is known as **matrix of coefficients** of the system of linear equations.

The matrix $\begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$ is called **augmented matrix**.

Example:

$$\begin{array}{rcrcrcl} & & & & 3x_3 & = & 9 \\ x_1 & + & 5x_2 & - & 2x_3 & = & 2 \\ \frac{1}{3}x_1 & + & 2x_2 & & & = & 3 \end{array}$$

For the above system the **matrix of coefficients** is $\begin{bmatrix} 0 & 0 & 3 \\ 1 & 5 & -2 \\ \frac{1}{3} & 2 & 0 \end{bmatrix}$.

The matrix of **augmented matrix** is $\begin{bmatrix} 0 & 0 & 3 & 9 \\ 1 & 5 & -2 & 2 \\ \frac{1}{3} & 2 & 0 & 3 \end{bmatrix}$.

Example: If the matrix is the augmented matrix of a system of linear equations write down the system of linear equations. $\begin{bmatrix} 2 & 0 & -2 & 5 \\ 7 & 2 & 5 & 0 \\ 1 & 4 & 5 & 10 \end{bmatrix}$. The system of linear equations is

$$\begin{array}{rcrcrcl} 2x_1 & & & - & 2x_3 & = & 5 \\ 7x_1 & + & 2x_2 & + & 5x_3 & = & 0 \\ x_1 & + & 4x_2 & + & 5x_3 & = & 10 \end{array}$$

1.5 The Elimination Method

Solve the system of linear equations

$$\begin{array}{rcrcrcl} & & & & 3x_3 & = & 9 \\ x_1 & + & 5x_2 & - & 2x_3 & = & 2 \\ \frac{1}{3}x_1 & + & 2x_2 & & & = & 3 \end{array}$$

Solution: The augmented matrix of the system is $\begin{bmatrix} 0 & 0 & 3 & 9 \\ 1 & 5 & -2 & 2 \\ \frac{1}{3} & 2 & 0 & 3 \end{bmatrix}$.

The first transformation rewrites the system by **interchanging the first and third row**.

$$\text{swap row 1 with row 3} \quad \begin{array}{rcrcrcl} \frac{1}{3}x_1 & + & 2x_2 & & & = & 3 \\ x_1 & + & 5x_2 & - & 2x_3 & = & 2 \\ & & & & 3x_3 & = & 9 \end{array}$$

and augmented matrix becomes $\begin{bmatrix} \frac{1}{3} & 2 & 0 & 3 \\ 1 & 5 & -2 & 2 \\ 0 & 0 & 3 & 9 \end{bmatrix}$. The second transformation

rescales the first row by multiplying both sides of the equation by 3.

$$\text{multiply row 1 by 3}\qquad \begin{array}{rcrcrcr} x_1 & + & 6x_2 & & & = & 9 \\ x_1 & + & 5x_2 & - & 2x_3 & = & 2 \\ & & & & 3x_3 & = & 9 \end{array}$$

the corresponding change in the augmented matrix gives $\begin{bmatrix} 1 & 6 & 0 & 9 \\ 1 & 5 & -2 & 2 \\ 0 & 0 & 3 & 9 \end{bmatrix}$.

We **multiply both sides of the first row** by -1, and **add that to the second row**, and write the result in as the **new second row**.

$$\text{add } -1 \text{ times row 1 to row 2}\qquad \begin{array}{rcrcrcr} x_1 & + & 6x_2 & & & = & 9 \\ & & -x_2 & - & 2x_3 & = & -7 \\ & & & & 3x_3 & = & 9 \end{array}$$

$$\begin{bmatrix} 1 & 6 & 0 & 9 \\ 0 & -1 & -2 & -7 \\ 0 & 0 & 3 & 9 \end{bmatrix}.$$

The bottom equation shows that $x_3 = 3$. Substituting 3 for x_3 in the middle equation shows that $x_2 = 1$. Substituting those two into the top equation gives that $x_1 = 3$.

Thus the system has a **unique solution**; the solution set is $\{ (3, 1, 3) \}$.
Verification that the vector $\{ (3, 1, 3) \}$ is a solution set for the system of linear equations

$$\begin{array}{rcrcrcr} & & & & 3x_3 & = & 9 \\ x_1 & + & 5x_2 & - & 2x_3 & = & 2 \\ \frac{1}{3}x_1 & + & 2x_2 & & & = & 3 \end{array}$$

$$\begin{array}{rcrcrcr} & & & + & 3(3) & = & 9 \\ 3 & + & 5(1) & - & 2(3) & = & 2 \\ \frac{1}{3}(3) & + & 2(1) & & & = & 3 \end{array}$$

All equations of the system of linear equations are satisfied, hence the set $\{ (3, 1, 3) \}$ is a solution set of the system of linear equations.
Example: Solve the system of linear equations

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}$$

Solution: The augmented matrix of the given system is $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$.

Keep x_1 in the first equation and eliminate it from the other equations.

$$\begin{array}{lrcrcrcr} & x_1 & - & 2x_2 & + & x_3 & = & 0 \\ \text{4[Equation 1] +[Equation 3]} & & & 2x_2 & - & 8x_3 & = & 8 \\ & & - & 3x_2 & + & 13x_3 & = & -9 \end{array}$$

The corresponding change in the augmented matrix lead to the following matrix $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$.

Multiply equation 2 by 1/2 in order to obtain 1 as the coefficient for x_2.

$$\begin{array}{lrcrcrcr} & x_1 & - & 2x_2 & + & x_3 & = & 0 \\ \text{1/2[Equation 2]} & & & x_2 & - & 4x_3 & = & 4 \\ & & - & 3x_2 & + & 13x_3 & = & -9 \end{array}$$

The augmented matrix becomes $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$.

Use the x_2 in equation 2 to eliminate the $-3x_2$ in equation 3.

$$\begin{array}{lrcrcrcr} & x_1 & - & 2x_2 & + & x_3 & = & 0 \\ \text{3[Equation 2] +[Equation 3]} & & & x_2 & - & 4x_3 & = & 4 \\ & & & & & x_3 & = & 3 \end{array}$$

The corresponding augmented matrix takes the following form $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$.

$4[Equation3] + [Equation2]$ and $-1[Equation3] + [Equation1]$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & & & = & -3 \\ & & x_2 & & & = & 16 \\ & & & & x_3 & = & 3 \end{array}$$

$\begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$.

2[Equation 2]+[Equation 1] leads to

$$\begin{array}{rcrcrcr} x_1 & - & & & & = & 29 \\ & & x_2 & & & = & 16 \\ & & & & x_3 & = & 3 \end{array}$$

$\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$.

Solution of the system is $29, 16, 3$.

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}$$

$$\begin{array}{rcrcrcr} (29) & - & 2(16) & + & 3 & = & 0 \\ & & 2(16) & - & 8(3) & = & 8 \\ -4(29) & + & 5(16) & + & 9(3) & = & -9 \end{array}$$

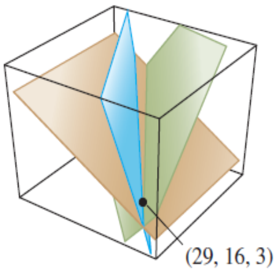

Figure 1.5: Solution of the system with three variables

Geometrically, unique solution for the system of linear equations with three variables is the point of intersection of the planes formed by these three equations.
Example: Determine if the following system is consistent.

$$\begin{array}{rcrcrcr} & & x_2 & - & 4x_3 & = & 8 \\ 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ 5x_1 & - & 8x_2 & + & 7x_3 & = & 1 \end{array}$$

Solution: The augmented matrix of the given system is $\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$.

To obtain an x_1 in the first equation, interchange rows 1 and 2:

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

To eliminate the $5x_1$ term in the third equation, add $-5/2$ times row 1 to row 3:

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -1/2 & 2 & -3/2 \end{bmatrix}$$

To eliminate the $-1/2x_2$ term from the third equation. Add $1/2$ times row 2 to row 3:

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}$$

The augmented matrix is in triangular form and we transform into equation notation

$$\begin{array}{rcrcrcr} 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ & & x_2 & - & 4x_3 & = & 8 \\ & & & & 0 & = & 5/2 \end{array}$$

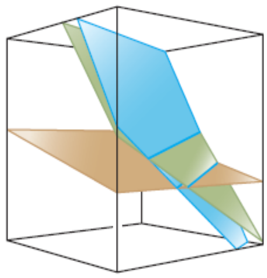

Figure 1.6: Inconsistent system with three variables

The planes formed by three equation don't have a common point of intersection as show in the figure and the system os inconsistent. **Example**: For what values of h and $k$is the following system consistent? $\begin{array}{rcrl} 2x_1 & - & x_2 = & h \\ -6x_1 & + & 3x_2 = & k \end{array}$

Solution: The augmented matrix of the system is $\begin{bmatrix} 2 & -1 & h \\ -6 & 3 & k \end{bmatrix}$.

3[Equation 1] + [Equation 2] or 3[Row 1] + [Row 2] $\begin{bmatrix} 2 & -1 & h \\ 0 & 0 & k+3h \end{bmatrix}$.

If $k + 3h \neq 0$ then we have $0 = k + 3h \neq 0$ implies the system is inconsistent.

So the system will be consistent if we have $k + 3h = 0$ or $k = -3h$.

For example: take $h = 2$ then $k = -9$ is one possibility. There are infinite many values of h and k satisfying $k + 3h = 0$.

1.6 Some Practice Problems

Question: Do the lines $2x_1 + 3x_2 = -1$, $6x_1 + 5x_2 = 0$, and $2x_1 - 5x_2 = 7$ have a common point of intersection?Justify your answer.

Question: Determine the value(s) of h such that the matrix is the augmented matrix of a consistent system

(a) $\begin{bmatrix} -4 & 12 & h \\ 2 & -6 & -3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 4 & -2 \\ 2 & h & -6 \end{bmatrix}$

Question: Determine whether the given system of linear equations are consistent or inconsistent.

$$\begin{array}{rcrcrcr} 2x_1 & & & - & 3x_3 & = & -8 \\ & & x_2 & - & 2x_3 & = & 3 \\ 3x_1 & + & 6x_2 & - & 2x_3 & = & -4 \end{array}$$

and

$$\begin{array}{rcrcrcr} x_1 & - & 5x_2 & + & 4x_3 & = & -3 \\ 2x_1 & - & 7x_2 & + & 3x_3 & = & -2 \\ -2x_1 & + & x_2 & + & 7x_3 & = & -1 \end{array}$$

Question: Find an equation involving g, h, and k that makes the augmented matrix correspond to a consistent system $\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{bmatrix}$