Chapter 5

Lecture No. 05

Question: Suppose A is a 4×4 matrix and $\mathbf{b}$ is a vector in $\mathbb{R}^4$ with the property that $A\mathbf{x} = \mathbf{b}$ has a unique solution. Explain why the columns of A must span $\mathbb{R}^4$.

Solution: We are given that the system $A\mathbf{x} = \mathbf{b}$ has unique solution for $\mathbf{b} \in \mathbb{R}^4$, which means that the augmented matrix of the system in echelon form has a pivot element in every row and the columns of A are linearly independent and span $\mathbb{R}^3$.

5.1 Homogeneous Linear System

A system of linear equations is said to be homogeneous if it can be written in the form $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$ matrix and $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$.

Trivial and Nontrivial Solution: The zero solution is usually called the trivial solution. For a given equation $A\mathbf{x} = \mathbf{0}$; the important question is whether there exists a nontrivial solution, that is, a nonzero vector $\mathbf{x}$ that satisfies $A\mathbf{x} = \mathbf{0}$.

Remark: The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

Example: Determine if the following homogeneous system has a nontrivial solution.

$$\begin{array}{rcrcrcl} 3x_1 & + & 5x_2 & - & 4x_3 & = & 0 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ 6x_1 & + & x_2 & - & 8x_3 & = & 0 \end{array}$$

Solution: The augmented matrix is

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The reduced echelon form is

$$\begin{bmatrix} 3 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{cccl} x_1 & & -\frac{4}{3}x_3 & = 0 \\ & x_2 & & = 0 \\ & & 0 & = 0 \end{array}$$

which gives x_1, x_2 are basic variables and x_3 is free variable.

Example: A **single linear equation** can be treated as a very simple system of equations. Describe all solutions of the homogeneous system

$$10x_1 - 3x_2 - 2x_3 = 0.$$

Solution: From the given equation $10x_1 - 3x_2 - 2x_3 = 0$ we have

$$x_1 = \frac{1}{10}(3x_2 + 2x_3)$$

the solution of the system can be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 3/10 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1/5 \\ 0 \\ 1 \end{bmatrix}.$$

The above form of the solution is known as parametric vector form of the solution of homogeneous system.

Parametric Vector Form:

$$\mathbf{x} = s\mathbf{u} + t\mathbf{v}, \qquad (s, t \in \mathbb{R}).$$

Example: If possible write the nontrivial solution of the following system in parametric form.

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & 3x_3 & = & 0 \\ 2x_1 & + & x_2 & - & 3x_3 & = & 0 \\ -x_1 & + & x_2 & & & = & 0 \end{array}$$

Solution: The augmented matrix of the given homogeneous system of linear equations is

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & 1 & -3 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus we have $-3x_2 + 3x_3 = 0$ and $x_1 + 2x_2 - 3x_3 = 0$, if we let $x_3 = t$ where $t \in \mathbb{R}^3$ then the parametric vector from of the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{x} = t\mathbf{v}$$

where $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Example: Solve the nonhomogeneous system of linear equations

$$\begin{array}{rcrcrcr} 3x_1 & + & 5x_2 & - & 4x_3 & = & 7 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & -1 \\ 6x_1 & + & x_2 & - & 8x_3 & = & -4 \end{array}$$

Solution: The augmented matrix is

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{ccccc} x_1 & & -\frac{4}{3}x_3 & = & -1 \\ & x_2 & & = & 2 \\ & & 0 & = & 0 \end{array}$$

Thus $x_1 = -1 + \frac{4}{3}x_3, x_2 = 2$, and x_3 is free. In parametric form we can write the solution as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}.$$

or

$$\mathbf{x} = \mathbf{p} + x_3\mathbf{v}$$

where

$$\mathbf{p} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}.$$

Notice that $\mathbf{p}$ is a particular solution of the nonhomogeneous system and the second part is the solution of the associated homogenous system of the given system, i.e., $x_3\mathbf{v}$ is the solution of the homogenous system

$$\begin{array}{rcrcrcl} 3x_1 & + & 5x_2 & - & 4x_3 & = & 0 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ 6x_1 & + & x_2 & - & 8x_3 & = & 0 \end{array}$$

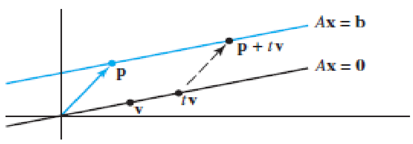

Figure 5.1: Relation between solution of nonhomogenous and homogenous systems

Example: Describe and compare the solution sets of $x_1 + 5x_2 - 3x_3 = 0$ and $x_1 + 5x_2 - 3x_3 = -2$.
Solution: The solution of the nonhomogenous equation i.e., $x_1 + 5x_2 - 3x_3 = -2$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.$$

Where as the solution of the homogenous equation $x_1 + 5x_2 - 3x_3 = 0$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.$$

Theorem: Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

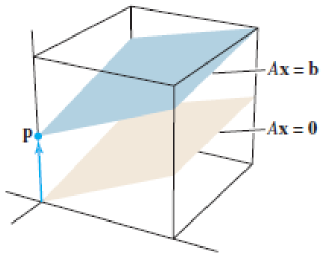

Figure 5.2: Solution of nonhomogenous system

Remark: The above result apply only to an equation $A\mathbf{x} = \mathbf{b}$ that has **at least one solution**. If the linear system is **inconsistent** the off course the solution set is empty.

Writing a solution set (of a consistent system) in parametric vector form:

1. Row reduce the augmented matrix to reduced echelon form.
2. Express each basic variable in terms of any free variables appearing in an equation.
3. Write a typical solution $\mathbf{x}$ as a vector whose entries depend on the free variables, if any.
4. Decompose $\mathbf{x}$ into a linear combination of vectors (with numeric entries) using the free variables as parameters.

Example: Describe the solutions of the following system in parametric vector form. Also, give a geometric description of the solution set and compare it to the homogeneous system of linear system

$$\begin{array}{rcrcrcr} 2x_1 & + & 2x_2 & + & 4x_3 & = & 8 \\ -4x_1 & - & 4x_2 & - & 8x_3 & = & -16 \\ & - & 3x_2 & - & 3x_3 & = & 12 \end{array}$$

Solution: The augmented matrix is

$$\begin{bmatrix} 2 & 2 & 4 & 8 \\ -4 & -4 & -8 & -16 \\ 0 & -3 & -3 & 12 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 4 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & -3 & 12 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 4 & 8 \\ 0 & -3 & -3 & 12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The solution of the system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

The associated homogeneous system has solution $x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ which is a line passing through origin and the the solution of nonhomogenous system is a line parallel to this line but passing through the point $\begin{bmatrix} 8 \\ -4 \\ 0 \end{bmatrix}$.

Example: Describe the solutions of the following system in parametric vector form. Also, give a geometric description of the solution set and compare it to the homogeneous system of linear system

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & 3x_3 & = & 5 \\ 2x_1 & + & x_2 & - & 3x_3 & = & 13 \\ -x_1 & + & x_2 & & & = & -8 \end{array}$$

Solution: The augmented matrix is

$$\begin{bmatrix} 1 & 2 & -3 & 5 \\ 2 & 1 & -3 & 13 \\ -1 & 1 & 0 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & -3 & 3 & 3 \\ 0 & 3 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & -3 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and repeat the same procedure as we did in he above example.

Example: Describe the solutions of the following system in parametric vector form (if possible). Also, give a geometric description of the solution set and compare it to the homogeneous system of linear system

$$\begin{array}{rcrcrcrcr} x_1 & & & + & x_3 & + & x_4 & = & -1 \\ 2x_1 & - & x_2 & & & + & x_4 & = & 3 \\ x_1 & + & x_2 & + & 3x_3 & + & 2x_4 & = & 1 \end{array}$$

Solution: $[A \quad \mathbf{b}] = \begin{bmatrix} 1 & 0 & 1 & 1 & -1 \\ 2 & -1 & 0 & 1 & 3 \\ 1 & 1 & 3 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 & -1 \\ 0 & -1 & -2 & -1 & 5 \\ 0 & 1 & 2 & 1 & 2 \end{bmatrix} \sim$ $\begin{bmatrix} 1 & 0 & 1 & 1 & -1 \\ 0 & -1 & -2 & -1 & 5 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}$ The third row gives us $0 = 7$ which is absurd and hence system is inconsistent.

Example: Describe the solutions of the following system in parametric vector form (if possible). Also, give a geometric description of the solution set and compare it to the homogeneous system of linear system

$$\begin{array}{rcrcrcrcr} x_1 & & & + & x_3 & + & x_4 & = & 0 \\ 2x_1 & - & x_2 & & & + & x_4 & = & 0 \\ x_1 & + & x_2 & + & 3x_3 & + & 2x_4 & = & 0 \end{array}$$

Solution: The augmented matrix is

$$[A \quad \mathbf{0}] = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 2 & -1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 \\ 0 & 1 & 2 & 1 & 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The solution set is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

5.2 Some Practice Problems

Question: If possible write the nontrivial solution of the following system in parametric form. $\begin{array}{rcrcrcl} 2x_1 & + & 2x_2 & + & 4x_3 & = & 0 \\ -4x_1 & - & 4x_2 & - & 8x_3 & = & 0 \\ & - & 3x_2 & - & 3x_3 & = & 0 \end{array}$

Question: Use the relationship between the solution of nonhomogeneous system of linear equation and homogeneous linear system to find the solution of the linear system $\begin{array}{rcrcrcr} 2x_1 & + & 2x_2 & + & 4x_3 & = & 8 \\ -4x_1 & - & 4x_2 & - & 8x_3 & = & -16 \\ & - & 3x_2 & - & 3x_3 & = & 12 \end{array}$.

Question: Describe all the solutions of homogeneous linear system $A\mathbf{x} = \mathbf{0}$ in parametric vector form where A is equivalent to the following matrix $\begin{bmatrix} 1 & -2 & 3 & -6 & 5 & 0 \\ 0 & 0 & 0 & 1 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Question: Does the equation $A\mathbf{x} = \mathbf{0}$ have a nontrivial solution and does the equation $A\mathbf{x} = \mathbf{b}$ have at least one solution for every possible $\mathbf{b}$. If A is a 3×3 matrix with three pivot positions.

CHAPTER 6

Lecture No. 06

Question: Describe all the solutions of homogeneous linear system $A\mathbf{x} = \mathbf{0}$ in parametric vector form where A is equivalent to the following matrix

$$\begin{bmatrix} 1 & -2 & 3 & -6 & 5 & 0 \\ 0 & 0 & 0 & 1 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution: The equations from the given matrix are

$$\begin{aligned} x_1 - 2x_2 + 3x_3 - 6x_4 + 5x_5 &= 0 \\ x_4 + 4x_5 - 6x_6 &= 0 \\ x_6 &= 0 \end{aligned}$$

and the parametric form of the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -29 \\ 0 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}.$$

Concept: Consider the vector equation $c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 7 \\ -1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.
What is the solution of the above vector equation? There are two possibilities

1. The vector equation has only trivial solution (the vectors are linearly independent)

2. The vector equation has nontrivial solution (the vectors are linearly dependent)

Solve the system and decide the vectors are linearly independent or dependent.

6.1 Linearly Independent set of vectors

An indexed set of vectors $\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_p\mathbf{v}_p = 0$$

has only the **trivial solution**.

The set $\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $\{c_1, ..., c_p\}$, **not all zero**, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_p\mathbf{v}_p = 0.$$

Example: Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

- Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is **linearly independent**.
- If possible, find a linear dependence relation among the vectors $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3\}$

Solution: The augmented matrix corresponding to the vector equation $c_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ is

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So we have the following equations

$$\begin{array}{rrrrl} x_1 & & - & 2x_2 = & 0 \\ & x_2 & + & x_3 = & 0 \\ & & & 0 = & 0 \end{array}$$

and say $x_3 = 2$, then we have the following linear combination

$$4\mathbf{v}_1 - 2\mathbf{v}_2 + 2\mathbf{v}_3 = \mathbf{0}.$$

Example: Find the value(s) of h for which the vectors are linearly dependent.

$$\begin{bmatrix} 3 \\ -6 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} 9 \\ h \\ 3 \end{bmatrix}.$$

Solution: We have to choose the the values of h such that the vector equation

$$c_1 \begin{bmatrix} 3 \\ -6 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix} + c_3 \begin{bmatrix} 9 \\ h \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has nontrivial solution. The augmented matrix of the system is

$$\begin{bmatrix} 3 & -6 & 9 & 0 \\ -6 & 4 & h & 0 \\ 1 & -3 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 & 0 \\ -6 & 4 & h & 0 \\ 3 & -6 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 & 0 \\ 0 & -14 & h+18 & 0 \\ 0 & 3 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -14 & h+18 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & h+32 & 0 \end{bmatrix}.$$

In order to system has nontrivial solution $h+32=0$, which gives the required value of the h.

Linear Independence of Matrix Columns: Suppose we have a matrix $A = [\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; ... \;\; \mathbf{a}_n]$. We can consider the columns of this matrix as a set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n\}$ and then we can talk about the **linearly independence** or **linearly dependence** of the vectors of this set, i.e.,

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + ... + c_n\mathbf{a}_n = \mathbf{0}.$$

Each **linear dependence** relation among the columns of A corresponds to a nontrivial solution of $A\mathbf{x} = \mathbf{0}$. Thus we have the following important fact.

The columns of a matrix A are **linearly independent** if and only if the **equation $A\mathrm{x} = 0$ has only the trivial solution**.

Example: Check whether the columns of the matrix $A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$ are linearly independent.

Solution: We should check whether the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only trivial solution? For this we take the augmented matrix

$$\begin{bmatrix} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & -2 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 13 & 0 \end{bmatrix}$$

It is clear that the homogeneous system has only trivial solution, hence the columns of the matrix are linearly independent.

Example: Given $\begin{bmatrix} 4 & 3 & -5 \\ -2 & -2 & 4 \\ -2 & -3 & 7 \end{bmatrix}$, observe that the first column minus three times the second column equals the third column. Find a nontrivial solution of $A\mathbf{x} = \mathbf{0}$.

Solution: The one nontrivial solution of the system is

$$\begin{bmatrix} 4 \\ -2 \\ -2 \end{bmatrix} - 3\begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix} - 1\begin{bmatrix} -5 \\ 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Hence $c_1 = 1, c_2 = -3$ and $c_3 = -1$.
Sets of one or two vectors: The set of one non zero vector is always linearly independent.

Example: Determine if the following sets of vectors are linearly independent

1. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -9 \\ -3 \end{bmatrix}$ Linearly dependent.

2. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -9 \\ 2 \end{bmatrix}$ Linearly independent.

Remark: A set of two vectors $\{\mathbf{v}_1; \mathbf{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other. The set is linearly independent if and only if neither of the vectors is a multiple of the other.
Theorem: **Characterization of Linearly Dependent Sets**
An indexed set $S = \{\mathbf{v}_1, ..., \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others.
In fact, if S is linearly dependent and $\mathbf{v}_1 \neq 0$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, ..., \mathbf{v}_{j-1}$.

Remark: The above result does not say that every vector in a linearly dependent set is a linear combination of the preceding vectors.
A vector in a linearly dependent set may fail to be a linear combination of the other vectors.

Example: Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -9 \\ 2 \\ 0 \end{bmatrix}$. Describe the set spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$, and explain why a vector $\mathbf{w}$ is in $Span\{\mathbf{v}_1, \mathbf{v}_2\}$ if and only if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}\}$ is linearly dependent. See the figure below.

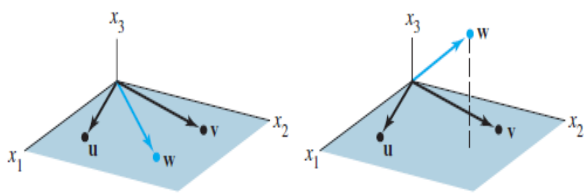

Figure 6.1: Span of two vectors

Theorem: If a set contains more vectors than there are entries in each vector, then the set is **linearly dependent**. That is, any set $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

Remark: The above result says nothing about the case in which the number of vectors in the set does not exceed the number of entries in each vector.

Theorem: If a set $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.
Example: Determine whether the given vectors are linearly dependent or linearly independent.

1. $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix}$ Linearly dependent.

2. $\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -9 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Linearly dependent.

3. $\begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -9 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ Linearly dependent.

4. $\begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -9 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ We can't say anything we have to check the vector equation.

6.2 Some Practice Problems

Question: Find a non trivial solution $A\mathbf{x} = \mathbf{0}$ where A is equivalent to the following matrix $\begin{bmatrix} 2 & 3 & 5 \\ -5 & 1 & -4 \\ -3 & -1 & -4 \\ 1 & 0 & 1 \end{bmatrix}$. Notice that the third column is the sum of the first two columns. (You don't need any computations for the answer of this question)

Question: Without any computations decide whether the given vectors are linearly dependent or linearly independent.

1. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$.

2. $\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -9 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

3. $\begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -9 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$