CHAPTER 9

Lecture No. 09

9.1 A Homogeneous System in Economics

Suppose a nation's economy is divided into many sectors, such as various manufacturing, communication, entertainment, and service industries.

Suppose that for each sector we know its total output for one year and we know exactly how this output is divided or "exchanged" among the other sectors of the economy. Let the total dollar value of a sector's output be called the price of that output.

Leontief proved the following result:

There exist equilibrium prices that can be assigned to the total outputs of the various sectors in such a way that the income of each sector exactly balances its expenses.

Example: Suppose an economy consists of the Coal, Electric (power), and Steel sectors, and the output of each sector is distributed among the various sectors as shown in Table, where the entries in a column represent the fractional parts of a sector's total output.

TABLE 1 A Simple Economy

Distribution of Output from:			
Coal	**Electric**	**Steel**	**Purchased by:**
.0	.4	.6	Coal
.6	.1	.2	Electric
.4	.5	.2	Steel

The second column of Table 1, for instance, says that the total output of the Electric sector is divided as follows: 40% to Coal, 50% to Steel, and the remaining 10% to Electric. (Electric treats this 10% as an expense it incurs in order to operate its business.) Since all output must be taken into account, the decimal fractions in each column must sum to 1.

Denote the prices (i.e., dollar values) of the total annual outputs of the Coal, Electric, and Steel sectors by p_C, p_E, and p_S, respectively. If possible, find equilibrium prices that make each sector's income match its expenditures.

TABLE 1 A Simple Economy

Distribution of Output from:			
Coal	**Electric**	**Steel**	**Purchased by:**
.0	.4	.6	Coal
.6	.1	.2	Electric
.4	.5	.2	Steel

We have the following system of linear equations

$$\begin{array}{rcrcrcl} & & 0.6p_S & + & 0.4p_E & = & p_C \\ 0.6p_C & + & 0.1p_E & + & 0.2p_S & = & p_E \\ 0.4p_C & + & 0.5p_E & + & 0.2p_S & = & p_S \end{array}$$

the above system can be reduced to

$$\begin{array}{rcrcrcl} p_C & - & 0.4p_E & - & 0.6p_S & = & 0 \\ -0.6p_C & + & 0.9p_E & - & 0.2p_S & = & 0 \\ -0.4p_C & - & 0.5p_E & + & 0.8p_S & = & 0 \end{array}$$

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The augmented matrix is

$$\begin{bmatrix} 1 & -.4 & -.6 & 0 \\ -.6 & .9 & -.2 & 0 \\ -.4 & -.5 & .8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -.4 & -.6 & 0 \\ 0 & .66 & -.56 & 0 \\ 0 & -.66 & .56 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -.4 & -.6 & 0 \\ 0 & .66 & -.56 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -.4 & -.6 & 0 \\ 0 & 1 & -.85 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -.94 & 0 \\ 0 & 1 & -.86 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The equilibrium price vector is

$$\mathbf{p} = \begin{bmatrix} p_C \\ p_E \\ p_S \end{bmatrix} = \begin{bmatrix} .94p_S \\ .85p_S \\ p_S \end{bmatrix} = p_S \begin{bmatrix} .94 \\ .85 \\ 1 \end{bmatrix}.$$

9.2 Balancing Chemical Equations

Chemical equations describe the quantities of substances consumed and produced by chemical reactions. For instance, when propane gas burns, the propane (C_3H_8) combines with oxygen (O_2) to form carbon dioxide (CO_2) and water (H_2O), according to an equation of the form

$$(x_1)C_3H_8 + (x_2)O_2 \to (x_3)CO_2 + (x_4)H_2O.$$

To balance the above equation, we need to find the coefficients x_1, x_2 and x_3 such that

$$x_1 \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$

To solve we rewrite the system

$$x_1 \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

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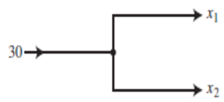
To solve we rewrite the system

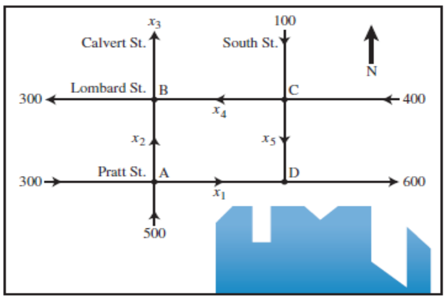
$$x_1 \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Network Flow: A network consists of a set of points called junctions, or nodes, with lines or arcs called branches connecting some or all of the junctions. The direction of flow in each branch is indicated, and the flow amount (or rate) is either shown or is denoted by a variable.

Assumption: The basic assumption of network flow is that the total flow into the network equals the total flow out of the network and that the total flow into a junction equals the total flow out of the junction.



Network Flow:The network shows the traffic flow (in vehicles per hour) over several one-way streets in downtown Baltimore during a typical early afternoon. Determine the general flow pattern for the network.



Intersection	Flow in		Flow out
A	$300 + 500$	=	$x_1 + x_2$
B	$x_2 + x_4$	=	$300 + x_3$
C	$100 + 400$	=	$x_4 + x_5$
D	$x_1 + x_5$	=	600

Network Flow:

Intersection	Flow in		Flow out
A	$300 + 500$	=	$x_1 + x_2$
B	$x_2 + x_4$	=	$300 + x_3$
C	$100 + 400$	=	$x_4 + x_5$
D	$x_1 + x_5$	=	600

$$\begin{array}{ccccccccccc}
x_1 & + & x_2 & & & & & & & = & 800 \\
 & & x_2 & - & x_3 & + & x_4 & & & = & 300 \\
 & & & & & & x_4 & + & x_5 & = & 500 \\
x_1 & & & & & & & + & x_5 & = & 600 \\
 & & & & x_3 & & & & & = & 400
\end{array}$$

Row reduction of the associated augmented matrix leads to

$$\begin{array}{ccccccccc}
x_1 & & & & & + & x_5 & = & 600 \\
 & x_2 & & & & - & x_5 & = & 200 \\
 & & x_3 & & & & & = & 400 \\
 & & & & x_4 & + & x_5 & = & 500
\end{array}$$

Network Flow: The general flow pattern for the network is described by

$$x_1 = 600 - x_5, \quad x_2 = 200 + x_5, \quad x_3 = 400, \quad x_4 = 500 - x_5, \quad x_5 \text{ is free}$$

Remark:

A negative flow in a network branch corresponds to flow in the direction opposite to that shown on the model. Since the streets in this problem are one way, none of the variables here can be negative. This fact leads to certain limitations on the possible values of the variables. For instance, $x_5 \leq 500$ because x_4 cannot be negative.

9.4 Some Practice Problems

Question: Consider an economy with three sectors: Fuels and Power, Manufacturing, and Services. Fuels and Power sells 80% of its output to Manufacturing, 10% to Services, and retains the rest. Manufacturing sells 10% of its output to Fuels and Power, 80% to Services, and retains the rest. Services sells 20% to Fuels and Power, 40% to Manufacturing, and retains the rest.

(i) Construct the exchange table for this economy.

(ii) Develop a system of equations that leads to prices at which each sector's income matches its expenses. Then write the augmented matrix that can be row reduced to find these prices.

Question: Balance the chemical equations. (a) Aluminum oxide and carbon react to create elemental aluminum and carbon dioxide: $Al_2O_3 + C \to Al + CO_2$.

(b) Alka-Seltzer contains sodium bicarbonate ($NaHCO_3$) and citric acid ($H_3C_6H_5O_7$). When a tablet is dissolved in water, the following reaction produces sodium citrate, water, and carbon dioxide (gas):

$$NaHCO_3 + H_3C_6H_5O_7 \to Na_3C_6H_5O_7 + H_2O + CO_2.$$

Balance the chemical equation.