## 3.2 Linear Combination

Given vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, ..., c_p$, the vector $\mathbf{y}$ defined

$$\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_p\mathbf{v}_p$$

is called a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p$ with weights $c_1, c_2, ..., c_p$.

**Example**: Selected linear combinations of $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

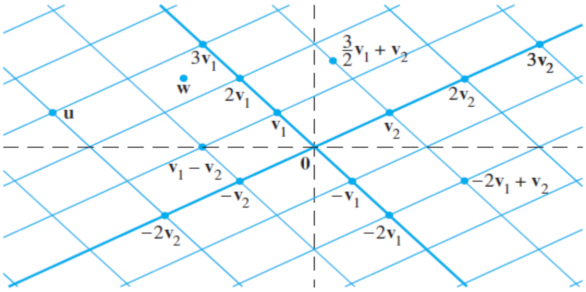


Figure 3.6: Some linear combinations of two vectors

**Example**: Write three different linear combinations of the the vectors

1. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$,

2. $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}$.

**Solution**: For $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, we can write the following three linear combinations, for example

$2\mathbf{u} + \mathbf{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$, $\quad \mathbf{u} + 2\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 10 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$ and $\mathbf{u} - \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$.

Indeed, you can write many linear combinations of these vectors. Try other linear combinations and do the other parts.

**Example**: Let $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$. Determine whether $\mathbf{b}$ can be written as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

**Solution**: We want to check that $\mathbf{b}$ is linear combination of the given vectors or not, i.e., we will check the solution of the system corresponding to equation $c_1\mathbf{u}+c_2\mathbf{v} = \mathbf{b}$.

The augmented matrix is $\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix}$ and the reduced echelon form is

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus the solution of the equation $c_1\mathbf{u}+c_2\mathbf{v}=\mathbf{b}$ is $c_1=3$ and $c_2=2$. Consequently, $\mathbf{b}$ is a liner combination of the vectors $\mathbf{u}$ and $\mathbf{v}$.

A vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_n\mathbf{v}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$[\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \ ... \ \ \ \mathbf{v}_n \ \ \mathbf{b}]. \tag{3.1}$$

In particular, $\mathbf{b}$ can be generated by a linear combination of $\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_n$ if and only if there exists a solution to the linear system corresponding to the matrix (3.1).

**Example**: Write the system of linear equations corresponding to each vector equation

1. $x_1 \begin{bmatrix} -2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

2. $x_1 \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 0 \end{bmatrix}$.

**Solution**:

1. $$-2x_1 + 7x_2 = 0$$
   $$3x_1 + 5x_2 = 1$$

   and

2. $$x_1 - x_2 = 7$$
   $$5x_1 + 6x_2 = 5$$
   $$-2x_1 + 3x_2 = 0$$

## 3.3 Spanning Set of Given Vectors

If $\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_p$ is denoted by $Span\{\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_p\}$ and is called the subset of $\mathbb{R}^n$ spanned (or generated) by $\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_p$.

That is, $Span\{\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_p\mathbf{v}_p$$

where $c_1, c_2, ..., c_p$ are scalars.

**Remark**: A given vector $\mathbf{b}$ is in $Span\{\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_p\}$ is the same as asking whether the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_p\mathbf{v}_p = \mathbf{b}$$

has a solution, or equivalently asking whether the linear system with augmented matrix $[\mathbf{v}_1 \;\; \mathbf{v}_2 ...\mathbf{v}_p \;\; \mathbf{b}]$ has a solution.

**A Geometric Description of $Span\{\mathbf{v}\}$ and $Span\{\mathbf{v}, \mathbf{u}\}$**

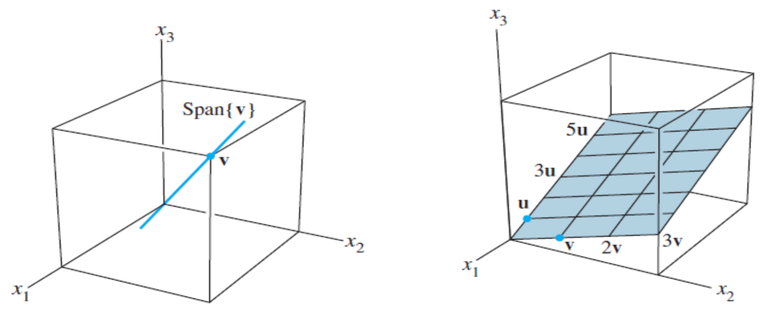


Figure 3.7: Geometry of spanning set of one and two vectors

**Example**: Let $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$. Then $Span\{\mathbf{u}, \mathbf{v}\}$ is a plane through the origin in $\mathbb{R}^3$. Is $\mathbf{b}$ in that plane?

**Solution**: We have to check whether the vector equation $c_1\mathbf{u}+c_2\mathbf{v} = \mathbf{b}$ is consistent or inconsistent.

The augmented matrix is

$$\begin{bmatrix} 1 & 5 & -3 \\ -2 & -13 & 8 \\ 3 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & -18 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

The third equation is $0 = -2$, which shows that that the system is inconsistent.

**Conclusion**: The vector equation $c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{b}$ has no solution, and $\mathbf{b}$ is not in the $Span\{\mathbf{u}, \mathbf{v}\}$.

**Example**: Let $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} h \\ -3 \\ -5 \end{bmatrix}$, Find the values of $h$ such that $\mathbf{b}$ is in the plane generated by $\mathbf{u}$ and $\mathbf{v}$.
**Solution**: The augmented matrix of the vector equation $c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{b}$ is

$$\begin{bmatrix} 1 & -2 & h \\ 0 & 1 & -3 \\ -2 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & h \\ 0 & 1 & -3 \\ 0 & 3 & -5+2h \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & h \\ 0 & 1 & -3 \\ 0 & 0 & 4+2h \end{bmatrix}$$

For the system to be consistent we must have $h + 2 = 0$, which gives $h = -2$. For $h = 2$, $\mathbf{b}$ is in the plane generated by $\mathbf{u}$ and $\mathbf{v}$.

## 3.4 Some Practice Problems

**Question**: Let $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} h \\ k \end{bmatrix}$, show that $\begin{bmatrix} h \\ k \end{bmatrix}$ is in the $Span\{\mathbf{u}, \mathbf{v}\}$ for all values of $h$ and $k$.

**Question**: Let $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} h \\ k \end{bmatrix}$, show that $\begin{bmatrix} h \\ k \end{bmatrix}$ is in the $Span\{\mathbf{u}, \mathbf{v}\}$ for all values of $h$ and $k$.

**Question**: Let $A = \begin{bmatrix} 2 & 0 & 6 \\ -1 & 8 & 5 \\ 1 & -2 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 10 \\ 3 \\ 7 \end{bmatrix}$. Is $\mathbf{b}$ in the span of columns of $A$? Show that the second column of $A$ is in the span of columns of $A$.

**Question**: Construct a $3 \times 3$ matrix $A$, with nonzero entries, and a vector $\mathbf{b}$ in $\mathbb{R}^3$ such that $\mathbf{b}$ is not in the set spanned by the columns of $A$.

**Question**: Write the vector equations corresponding to the linear systems

$$\begin{array}{rcrcrcr} 5x_1 & - & x_2 & + & 2x_3 & = & 7 \\ -2x_1 & + & 6x_2 & + & 9x_3 & = & 0 \\ -7x_1 & + & 5x_2 & - & 3x_3 & = & -7 \end{array}$$

and

$$\begin{array}{rcrcr} x_1 & - & 3x_2 & = & 0 \\ -x_1 & + & 6x_2 & = & 0 \\ x_1 & + & 4x_2 & = & 0 \end{array}.$$