CHAPTER 13

Lecture No. 13

13.1 Subspace of $\mathbb{R}^n$

A subspace of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has three properties:

1. The zero vector is in H.

2. For each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.

3. For each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

Example: If $\mathbf{v}_1$ and $\mathbf{v}_2$ are in $\mathbb{R}^n$ and $H = Span\{\mathbf{v}_1, \mathbf{v}_2\}$, then H is a subspace of $\mathbb{R}^n$.

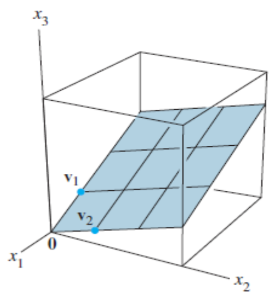


Figure 13.1: Subspace spanned by two vectors

Example: A line L not through the origin is not a subspace, because it does not contain the origin, as required. L is not closed under addition or scalar multiplication.

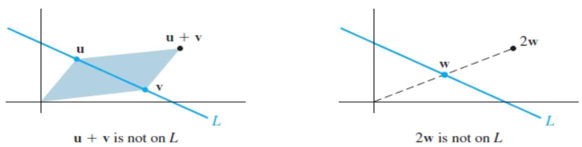


Figure 13.2: Line not passing through origin

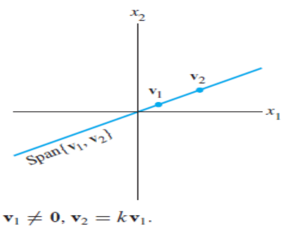


Figure 13.3: A line passing through origin is span of a nonzero vector

Example: For $\mathbf{v}_1, ..., \mathbf{v}_p$ in $\mathbb{R}^n$, the set of all linear combinations of $\mathbf{v}_1, ..., \mathbf{v}_p$ is a subspace of $\mathbb{R}^n$. The verification of this statement is similar to the argument given in previous Example.

Remark: We shall now refer to $Span\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ as the subspace spanned (or generated) by $\mathbf{v}_1, ..., \mathbf{v}_p$.

Remark: Note that $\mathbb{R}^n$ is a subspace of itself because it has the three properties required for a subspace. Another special subspace is the set consisting of only the zero vector in $\mathbb{R}^n$. This set, called the zero subspace, also satisfies the conditions for a subspace.

13.2 Column Space and Null Space of a Matrix

Column Space: The column space of a matrix A is the set $Col \;\; A$ of all linear combinations of the columns of A.

Remark: If $A = [\mathbf{v}_1 \quad ... \quad \mathbf{v}_n]$, with the columns in $\mathbb{R}^m$, then $Col \;\; A$ is the same as $Span\{\mathbf{v}_1, ..., \mathbf{v}_n\}$.

The column space of an $m \times n$ matrix is a subspace of $\mathbb{R}^m$.

Note that $Col \;\; A$ equals $\mathbb{R}^m$ only when the columns of A span $\mathbb{R}^m$. Otherwise, $Col \;\; A$ is only part of $\mathbb{R}^m$.

Example: Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$. Determine whether $\mathbf{b}$ is in the column space of A.

Solution: Let the columns of A be denoted by $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ then we have to check whether the system

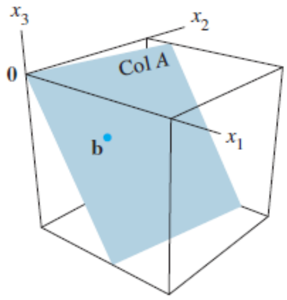
$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{b}$$

is consistent or inconsistent.

The augmented matrix of the above system and echelon form is

$$\begin{bmatrix} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & -2 & -6 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which shows that $\mathbf{b}$ is in the columns space of the given matrix.



Null Space: The null space of a matrix A is the set $Nul\ \ A$ of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Remark: When A has n columns, the solutions of $A\mathbf{x} = \mathbf{0}$ belong to $\mathbb{R}^n$, and the null space of A is a subset of $\mathbb{R}^n$.

In fact, $Nul\ \ A$ has the properties of a subspace of $\mathbb{R}^n$.

Theorem: The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions of a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

Example: Let $A = \begin{bmatrix} 1 & -1 & 5 \\ 2 & 0 & 7 \\ -3 & -5 & -3 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} -7 \\ 3 \\ 2 \end{bmatrix}$. Is $\mathbf{u}$ in Nul A? Is $\mathbf{u}$ in Col A? Justify your answer.

Solution: In order to check whether the given vector $\mathbf{u}$ is in the Null space of A or not we will check the matrix product

$$\begin{bmatrix} 1 & -1 & 5 \\ 2 & 0 & 7 \\ -3 & -5 & -3 \end{bmatrix} \begin{bmatrix} -7 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -7-3+10 \\ -14+0+14 \\ 21-15-6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which shows that $\mathbf{u}$ is in the Null space of the matrix A. For checking whether the vector is in column space of A or not, we will solve the system $A\mathbf{x} = \mathbf{u}$. If the system is consistent then $\mathbf{u}$ is in the column space of A otherwise not. The augmented matrix is

$$\begin{bmatrix} 1 & -1 & 5 & -7 \\ 2 & 0 & 7 & 3 \\ -3 & -5 & -3 & 2 \end{bmatrix}.$$

Check yourself or see the lecture.
Basis for a Subspace: A basis for a subspace H of $\mathbb{R}^n$ is a linearly independent set in H that spans H.

Example: The columns of an invertible $n \times n$ matrix form a basis for all of $\mathbb{R}^n$ because they are linearly independent and span $\mathbb{R}^n$, by the Invertible Matrix Theorem.

One such matrix is the $n \times n$ identity matrix. Its columns are denoted by $\mathbf{e}_1, ..., \mathbf{e}_n$.

The set $\{\mathbf{e}_1, ..., \mathbf{e}_n\}$ is called a standard basis for $\mathbb{R}^n$.
Example: Find a basis for the Null Space of the matrix

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

Solution:The echelon form of the augmented matrix of the system $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and the parametric vector form of the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

Which show that the basis for the Null space of the given matrix is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Basis for the column space of a matrix: Finding a basis for the column space of a matrix is actually less work than finding a basis for the null space.

Example: Find a basis for the column space of the matrix

$$\begin{bmatrix} 1 & 0 & 2 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution: Firs of all notice that $\mathbf{b}_3 = -3\mathbf{b}_1 + 2\mathbf{b}_2$ and $\mathbf{b}_4 = 5\mathbf{b}_1 - \mathbf{b}_2$.
This means that

$$Span\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4, \mathbf{b}_5\} = Span\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_5\}$$

and the vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_5$ are linearly independent and hence form a basis for the column space of the given matrix.
Theorem: The pivot columns of a matrix A form a basis for the column space of A.

Remark: Be careful to use pivot columns of A itself for the basis of $Col \quad A$. The columns of an echelon form B are often not in the column space of A.

Example: Suppose an $n \times n$ matrix A is invertible. What you can say about $Col \quad A$? About $Null \quad A$?

13.3 Some Practice Problems

Question Let

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 0 \\ 6 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 3 \\ 3 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ -5 \\ 5 \end{bmatrix} \quad \text{and} \quad \mathbf{p} = \begin{bmatrix} -6 \\ 1 \\ 17 \end{bmatrix}$$

Determine if $\mathbf{p}$ is in the columns space of $A[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]$? Is $\mathbf{p}$ is in the Null space of A?

Question: Give integers p and q such that $Nul \quad A$ is a subspace of $\mathbb{R}^p$ and $Col \quad A$ is a subspace of $\mathbb{R}^q$.

1. $A = \begin{bmatrix} 3 & 2 & 1 & -5 \\ -9 & -4 & 1 & 7 \\ 9 & 2 & -5 & 1 \end{bmatrix}$.

2. $A = \begin{bmatrix} 3 & 2 & 1 \\ -9 & -4 & 1 \\ 9 & 2 & -5 \end{bmatrix}$.