بسم الله الرحمن الرحيم

Design and Analysis of Algorithm

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Recap

Lecture No 10

Mathematical Analysis of non recursive Algorithm

Steps in mathematical analysis of non recursive algorithms

- Decide on parameter *n* indicating input size
- Identify algorithm's basic operation
- Determine worst, average, and best case for input of size *n*
- Set up summation for *C(n)* reflecting algorithm's loop structure
- Simplify summation using standard formulas

- Which algorithm is best?

ALGORITHM factorial(n)
// Input: A positive integer
// Output: The factorial of the positive integer n.
factorial ← 1 // initialize answer
i ← 1
while(i ≤ n) **do**
  factorial ← factorial * i
  i ← i + 1
return factorial

ALGORITHM $F(n)$
//Computes $n!$ recursively
//Input: A nonnegative integer n
//Output: The value of $n!$
if $n = 0$ **return** 1
else return $F(n-1) * n$

Lecture No 11

Analysis of Recursive Algorithm

(Analysis Framework.)

After completing this lecture you will be able

- To solve problem with recursive algorithm
- To convert an iterative algorithm into recursive algorithm.
- To compute Time complexity of recursive algorithm
- To compare iterative version with recursive version

What is Recursion?

- Dictionary definition:
- A problem-solving method of
  "decomposing bigger problems into smaller sub-problems that *are identical to itself.*"
- Recursion:
  - Process of solving a problem by reducing it to smaller versions of itself
- General Idea:
  - Solve simplest (smallest) cases DIRECTLY
    - usually these are very easy to solve
  - Solve bigger problems using smaller sub-problems
    - that are identical to itself (but smaller and simpler)
- Abstraction:
  - To solve a given problem, we first assume that we ALREADY know how to solve it for smaller instances!!

- Recursive algorithm:
  - Algorithm that finds the solution to a given problem by reducing the problem to smaller versions of itself.
  - Has one or more base cases.
  - Implemented using recursive methods.
- Recursive method:
  - Method that calls itself.
- Base case:
  - Case in recursive definition in which the solution is obtained directly.
  - Stops the recursion.
- General case:
  - Case in recursive definition in which a smaller version of itself is called.
  - Must eventually be reduced to a base case.

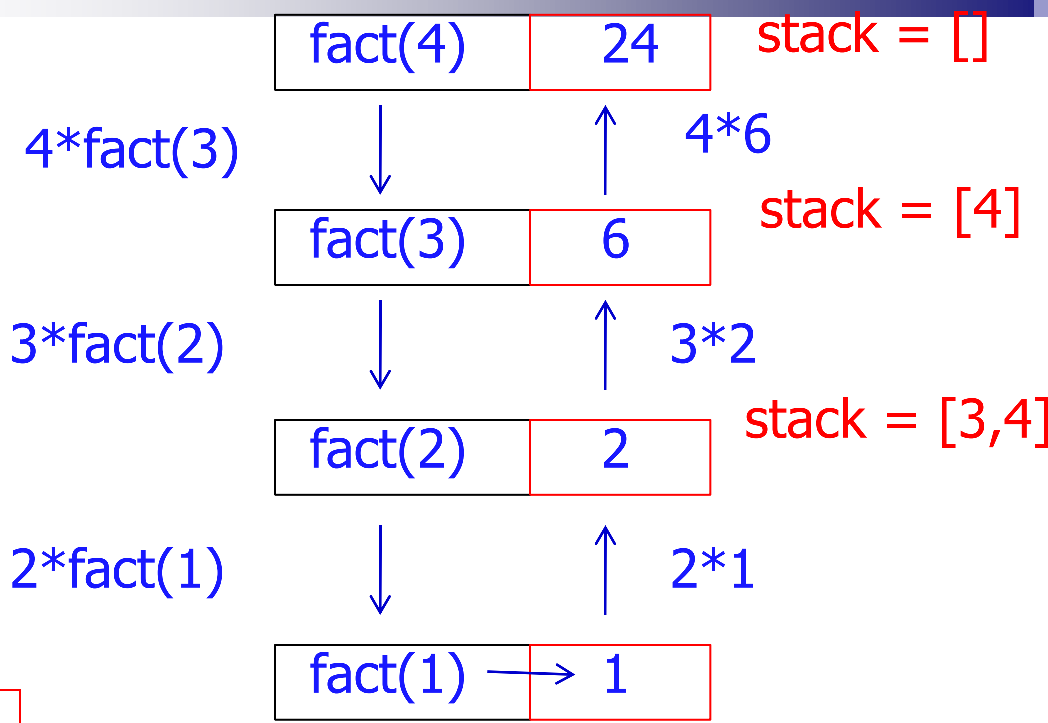
- Understand problem requirements.
- Determine limiting conditions.
- Identify base cases.
- Provide direct solution to each base case.
- Identify general cases.
- Provide solutions to general cases in terms of smaller versions of general cases.

fact(4): stack = [4]

fact(3): stack = [3,4]

fact(2): stack = [2,3,4]

fact(1): stack = [1,2,3,4]




```
fact(n)
   If n<=1 then return←1
            else return ←fact(n-1)*n
   endif
return return
```

Sequence of recursive calls

Back to the calling function

- Definition of Fibonacci numbers
  1. $F_1 = 1$,
  2. $F_2 = 1$,
  3. for $n>2$, $F_n = F_{n-1} + F_{n-2}$
- Problem: Compute F_n for any n.
- The above is a recursive definition.
  - F_n is computed in-terms of itself
  - actually, smaller copies of itself – F_{n-1} and F_{n-2}
- Actually, Not difficult:

$F_3 = 1 + 1 = 2$ $F_6 = 5 + 3 = 8$ $F_9 = 21 + 13 = 34$

$F_4 = 2 + 1 = 3$ $F_7 = 8 + 5 = 13$ $F_{10} = 34 + 21 = 55$

$F_5 = 3 + 2 = 5$ $F_8 = 13 + 8 = 21$ $F_{11} = 55 + 34 = 89$

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, …

ALGORITHM $Fib(n)$

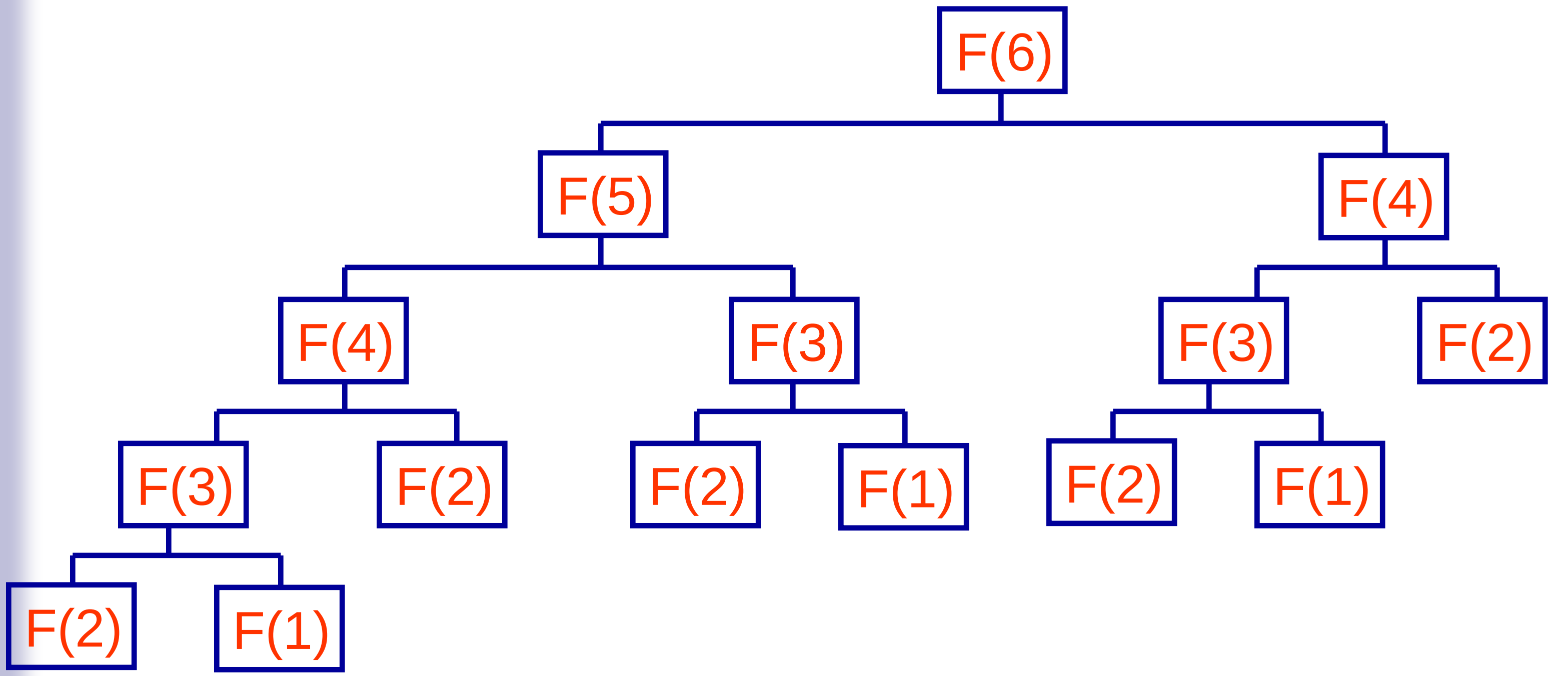
//Computes the nth Fibonacci number iteratively by using its definition
//Input: A nonnegative integer n
//Output: The nth Fibonacci number
$F[0] \leftarrow 0;\ F[1] \leftarrow 1$
for $i \leftarrow 2$ **to** n **do**
    $F[i] \leftarrow F[i-1] + F[i-2]$
return $F[n]$

- The below is a recursive algorithm
- It is simple to understand and elegant!
- But, very SLOW(**WHY**)

ALGORITHM $F(n)$

//Computes the nth Fibonacci number recursively by using its definition
//Input: A nonnegative integer n
//Output: The nth Fibonacci number
if $n \leq 1$ **return** n
else return $F(n-1) + F(n-2)$

- How slow is it?
  - E.g. To compute F(6)…



HW: Can we compute it faster?

Decide on parameter n indicating input size

Identify algorithm's basic operation

Determine worst, average, and best case for input of size n

Set up a recurrence relation and initial condition(s)

Solve the recurrence to obtain a closed form or estimate the order of magnitude of the solution

Determine Order of growth of C(n)

- A recurrence relation is an equation which is defined in terms of itself.
- It expresses the value of a function for an argument *n* in terms of the values of function for arguments less than *n*.
- Examples:

1. $$T(n)=\begin{cases}1 & if\ n=1\\ T(n-1)+1 & otherwise\end{cases}$$

2. $$T(n)=\begin{cases}1 & if\ n=1\\ 2T(n-1) & otherwise\end{cases}$$

How to derive a recurrence relation?

To derive a recurrence relation for the running time of an algorithm:

1) Figure out what "n", the ***problem size***, is.
2) See what value of n is used as the ***base of the recursion***. It will usually be a single value (e.g. $n = 1$), but may be multiple values. Suppose it is n_0.
3) Figure out what $T(n_0)$ is. You can usually use "**some constant c**", but sometimes a specific number will be needed.
4) The general $T(n)$ is usually a ***sum of various choices of T(m) (for the recursive calls), plus the sum of the other work done***. Usually the recursive calls will be solving a sub problems of the same size $f(n)$, giving a term "$a*T(f(n))$" in the recurrence relation.

ALGORITHM $BinRec(n)$

//Input: A positive decimal integer n

//Output: The number of binary digits in n's binary representation

if $n = 1$ **return** 1

else return $BinRec(\lfloor n/2 \rfloor) + 1$

T(1) = *d*

T(*n*) = T(*n*/2) + *c*

How to derive a recurrence relation?

To derive a recurrence relation for the running time of an algorithm:

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ALGORITHM $F(n)$

//Computes the nth Fibonacci number recursively by using its definition
//Input: A nonnegative integer n
//Output: The nth Fibonacci number
if $n \le 1$ **return** n
else return $F(n-1) + F(n-2)$

that can be defined by the simple recurrence

$$F(n) = F(n-1) + F(n-2) \quad \text{for } n > 1$$

and two initial conditions

$$F(0) = 0, \qquad F(1) = 1.$$

Deriving Recurrence Relations

To derive a recurrence relation for the running time of an algorithm:

1) Figure out what "n", the ***problem size***, is.
2) See what value of n is used as the ***base of the recursion***. It will usually be a single value (e.g. $n = 1$), but may be multiple values. Suppose it is n_0.
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```
procedure bugs(n)
  if n = 1 then do something
    else
      bugs(n − 1);
      bugs(n − 2);
      for i := 1 to n do
        something
```

$$T(n) = \left\{ \right.$$

Deriving Recurrence Relations

To derive a recurrence relation for the running time of an algorithm:

1) Figure out what "n", the ***problem size***, is.
2) See what value of n is used as the ***base of the recursion***. It will usually be a single value (e.g. $n = 1$), but may be multiple values. Suppose it is n_0.
3) Figure out what $T(n_0)$ is. You can usually use **"some constant c"**, but sometimes a specific number will be needed.
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```
procedure daffy(n)
  if n = 1 or n = 2 then do something
    else
      daffy(n − 1);
      for i := 1 to n do
        do something new
      daffy(n − 1);
```

$$T(n) = \begin{cases} \\ \\ \end{cases}$$

Deriving Recurrence Relations

To derive a recurrence relation for the running time of an algorithm:

1) Figure out what "n", the ***problem size***, is.
2) See what value of n is used as the ***base of the recursion***. It will usually be a single value (e.g. $n = 1$), but may be multiple values. Suppose it is n_0.
3) Figure out what $T(n_0)$ is. You can usually use **"some constant c"**, but sometimes a specific number will be needed.
4) The general $T(n)$ is usually a ***sum of various choices of T(m) (for the recursive calls), plus the sum of the other work done***. Usually the recursive calls will be solving a sub problems of the same size $f(n)$, giving a term "$a*T(f(n))$" in the recurrence relation.

```
procedure elmer(n)
  if n = 1 then do something
    else if n = 2 then do something else
    else
      for i := 1 to n do
        elmer(n − 1);
        do something different
```

$$T(n) = \begin{cases} \\ \\ \end{cases}$$

Deriving Recurrence Relations

To derive a recurrence relation for the running time of an algorithm:

1) Figure out what "n", the ***problem size***, is.
2) See what value of n is used as the ***base of the recursion***. It will usually be a single value (e.g. $n = 1$), but may be multiple values. Suppose it is n_0.
3) Figure out what $T(n_0)$ is. You can usually use **"some constant c"**, but sometimes a specific number will be needed.
4) The general $T(n)$ is usually a ***sum of various choices of T(m) (for the recursive calls), plus the sum of the other work done***. Usually the recursive calls will be solving a sub problems of the same size $f(n)$, giving a term "$a*T(f(n))$" in the recurrence relation.

```
procedure yosemite(n)
  if n = 1 then do something
    else
      for i := 1 to n − 1 do
        yosemite(i);
        do something completely different
```

$$T(n) = \begin{cases} \\ \\ \end{cases}$$

Deriving Recurrence Relations

To derive a recurrence relation for the running time of an algorithm:

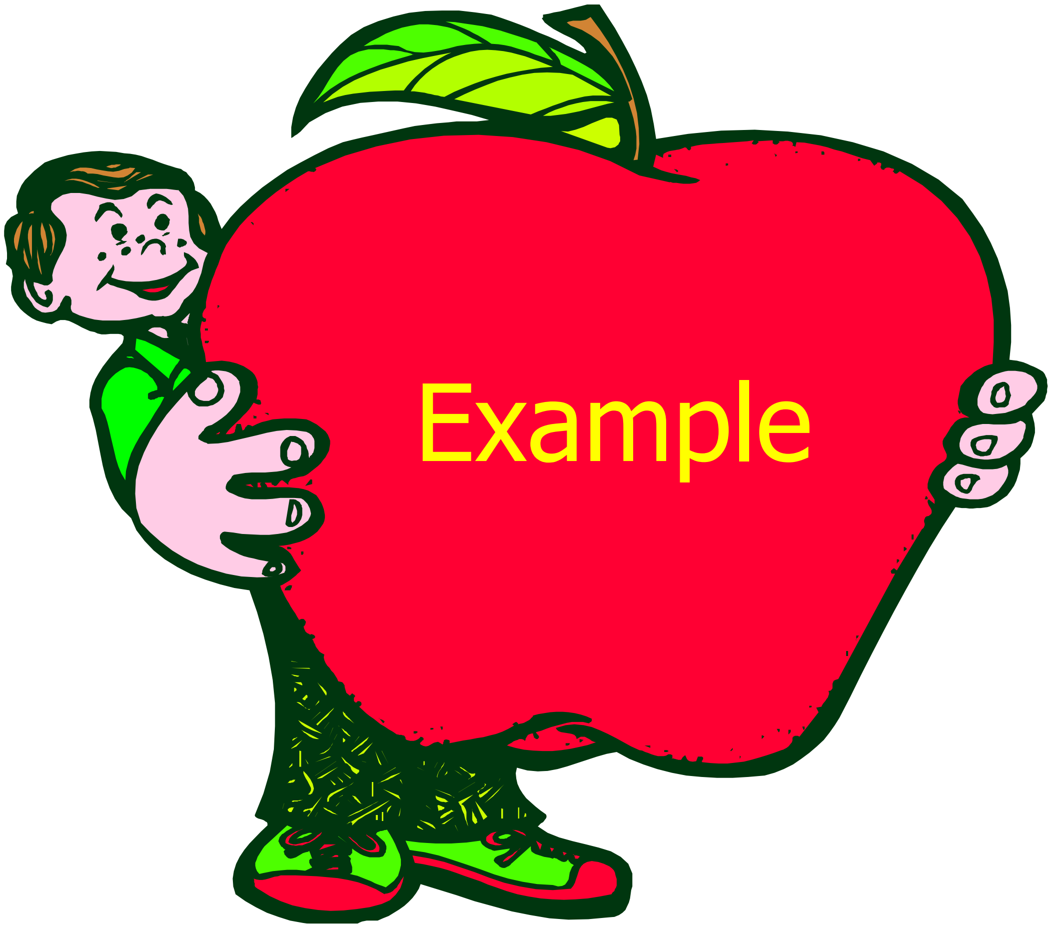
1) Figure out what "n", the ***problem size***, is.
2) See what value of n is used as the ***base of the recursion***. It will usually be a single value (e.g. $n = 1$), but may be multiple values. Suppose it is n_0.
3) Figure out what $T(n_0)$ is. You can usually use **"some constant c"**, but sometimes a specific number will be needed.
4) The general $T(n)$ is usually a ***sum of various choices of T(m) (for the recursive calls), plus the sum of the other work done***. Usually the recursive calls will be solving a sub problems of the same size $f(n)$, giving a term "$a*T(f(n))$" in the recurrence relation.

```
function multiply(y, z)
    comment return the product yz
1.  if z = 0 then return(0) else
2.  if z is odd
3.      then return(multiply(2y, ⌊z/2⌋)+y)
4.      else return(multiply(2y, ⌊z/2⌋))
```

D.Y.S = Do Your Self

Common Recurrence Types in Algorithm Analysis

- **Decrease-by-One** A decrease-by-one algorithm solves a problem by exploiting a relationship between a given instance of size *n* and a smaller instance of size *n* − 1. Specific examples include recursive evaluation of *n*! and insertion sort. The recurrence equation for investigating the time efficiency of such algorithms typically has the following form:
  - $T(n) = T(n-1) + f(n)$
- where function **f(n)** accounts for the time needed to reduce an instance to a smaller one and to extend the solution of the smaller instance to a solution of the larger instance.


Example

Common Recurrence Types in Algorithm Analysis

- **Decrease-by-One:** $T(n) = T(n-1) + f(n)$

```
ALGORITHM  factorial(n)
// Input: A positive integer
// Output: The factorial of the positive integer n.
factorial ← 1   // initialize answer
i ← 1
while( i ≤ n ) do
   factorial ← factorial * i
   i  ← i + 1
return  factorial
```

Convert iterative into recursive

n! = n*(n-1)!
0! = 1

ALGORITHM $F(n)$

//Computes $n!$ recursively
//Input: A nonnegative integer n
//Output: The value of $n!$
if $n = 0$ **return** 1
else return $F(n-1) * n$

Common Recurrence Types in Algorithm Analysis

- **Decrease-by-One: T (n) = T (n – 1) + f (n)**

ALGORITHM $F(n)$

//Computes $n!$ recursively
//Input: A nonnegative integer n
//Output: The value of $n!$
if $n = 0$ **return** 1
else return $F(n-1) * n$

- What is measure of an input's size?
  - The number of elements in the array, i.e., n.
- What is its basic operation/Primitive Operation?
  - Multiplication(*)

Common Recurrence Types in Algorithm Analysis

- **Decrease-by-One: T (n) = T (n – 1) + f (n)**

ALGORITHM $F(n)$

//Computes $n!$ recursively
//Input: A nonnegative integer n
//Output: The value of $n!$
if $n = 0$ **return** 1
else return $F(n-1) * n$

- The basic operation of the algorithm is multiplication, whose number of executions we denote M(n).

$$M(n) = \underset{\substack{\text{to compute}\\ F(n-1)}}{M(n-1)} + \underset{\substack{\text{to multiply}\\ F(n-1) \text{ by } n}}{1} \qquad \text{for } n > 0.$$

Common Recurrence Types in Algorithm Analysis

- **Decrease-by-One: T (n) = T (n – 1) + f (n)**

ALGORITHM $F(n)$

//Computes $n!$ recursively
//Input: A nonnegative integer n
//Output: The value of $n!$
if $n = 0$ **return** 1
else return $F(n-1) * n$

- Now we obtain initial condition by inspecting the condition that makes the algorithm stop its recursive calls:
  - **if** $n = 0$ **return** 1.

Common Recurrence Types in Algorithm Analysis

ALGORITHM $F(n)$

//Computes $n!$ recursively
//Input: A nonnegative integer n
//Output: The value of $n!$
if $n = 0$ **return** 1
else return $F(n-1) * n$

- **if** n = 0 **return** 1.

$M(0) = 0.$

the calls stop when $n = 0$ — no multiplications when $n = 0$

- This tells us two things.
  - First, since the calls stop when n = 0, the smallest value of n for which this algorithm is executed and hence $M(n)$ defined is 0.
  - Second, by inspecting the pseudocode's exiting line, we can see that when n = 0, the algorithm performs no multiplications.

Common Recurrence Types in Algorithm Analysis

ALGORITHM $F(n)$

```
//Computes n! recursively
//Input: A nonnegative integer n
//Output: The value of n!
if n = 0 return 1
else return F(n − 1) ∗ n
```

- Thus, we succeeded in setting up the recurrence relation and initial condition for the algorithm's number of multiplications *M(n)*:

$$M(n) = \begin{cases} 0, & \text{for } n = 0, \\ M(n-1) + 1 & \text{for } n > 0, \end{cases}$$

Common Recurrence Types in Algorithm Analysis

ALGORITHM $F(n)$

//Computes $n!$ recursively
//Input: A nonnegative integer n
//Output: The value of $n!$
if $n = 0$ **return** 1
else return $F(n-1) * n$

- We are dealing here with two recursively defined functions. The first is the factorial function F(n) itself; it is defined by the recurrence

n! = n*(n-1)!

0! = 1

Recurrence relation:

T(n) = T(n-1) + 1

T(1) = 1

- The second is the number of multiplications *M(n)* needed to compute *F(n)* by the recursive algorithm whose pseudocode is given

$$M(n) = \begin{cases} 0, & \text{for } n = 0, \\ M(n-1) + 1 & \text{for } n > 0, \end{cases}$$

YOUR TURN

Common Recurrence Types in Algorithm Analysis

- **Decrease-by-One: T (n) = T (n – 1) + f (n)**

ALGORITHM *BubbleSort*$(A[0..n-1])$

//Sorts a given array by bubble sort
//Input: An array $A[0..n-1]$ of orderable elements
//Output: Array $A[0..n-1]$ sorted in ascending order
for $i \leftarrow 0$ **to** $n-2$ **do**
  for $j \leftarrow 0$ **to** $n-2-i$ **do**
    if $A[j+1] < A[j]$ swap $A[j]$ and $A[j+1]$

Convert into recursive version

Recurrence relation?

T(n) = T(n-1) + n

ALGORITHM BubbleSort(A, *n*)
 for i ← 0 to n-2 **do**
  if A[i] > A[i+1] **then**
   swap A[i] and A[i+1]
 BubbleSort(A, *n-1*)

Common Recurrence Types in Algorithm Analysis

- **Decrease-by-a-Constant-Factor** A decrease-by-a-constant-factor algorithm solves a problem by reducing its instance of size *n* to an instance of size *n/b* (*b* = 2 for most but not all such algorithms), solving the smaller instance recursively, and then, if necessary, extending the solution of the smaller instance to a solution of the given instance. The most important example is binary search; other examples include exponentiation by squaring. The recurrence equation for investigating the time efficiency of such algorithms typically has the form

- **$T(n) = T(n/b) + f(n)$,**

Common Recurrence Types in Algorithm Analysis

- **$T(n) = T(n/b) + f(n)$,**

- Where $b > 1$ and function $f(n)$ accounts for the time needed to reduce an instance to a smaller one and to extend the solution of the smaller instance to a solution of the larger instance. Strictly speaking, equation is valid only for $n = b^k$, $k = 0, 1, \ldots$. For values of n that are not powers of b, there is typically some round off, usually involving the floor and/or ceiling functions.

Common Recurrence Types in Algorithm Analysis

- **Decrease-by-a-Constant-Factor** $T(n) = T(n/b) + f(n)$,

ALGORITHM *BinarySearch*$(A[0..n-1], K)$

//Implements nonrecursive binary search
//Input: An array $A[0..n-1]$ sorted in ascending order and
//   a search key K
//Output: An index of the array's element that is equal to K
//   or -1 if there is no such element
$l \leftarrow 0;\quad r \leftarrow n-1$
while $l \le r$ **do**
    $m \leftarrow \lfloor (l+r)/2 \rfloor$
    if $K = A[m]$ **return** m
    else if $K < A[m]$ $r \leftarrow m-1$
    else $l \leftarrow m+1$
return -1

$$\underbrace{A[0]\ldots A[m-1]}_{\text{search here if } K<A[m]}\ \overset{\overset{K}{\updownarrow}}{A[m]}\ \underbrace{A[m+1]\ldots A[n-1]}_{\text{search here if } K>A[m]}.$$

- Let us apply binary search to searching for $K = 70$ in the array

index	0	1	2	3	4	5	6	7	8	9	10	11	12
value	3	14	27	31	39	42	55	70	74	81	85	93	98
iteration 1	l						m						r
iteration 2								l		m			r
iteration 3								l,m	r				

Common Recurrence Types in Algorithm Analysis

- **Decrease-by-a-Constant-Factor** $T(n) = T(n/b) + f(n)$,

ALGORITHM *BinarySearch*$(A[0..n-1], K)$

```
//Implements nonrecursive binary search
//Input: An array A[0..n − 1] sorted in ascending order and
//       a search key K
//Output: An index of the array's element that is equal to K
//       or −1 if there is no such element
l ← 0;   r ← n − 1
while l ≤ r do
    m ← ⌊(l + r)/2⌋
    if K = A[m] return m
    else if K < A[m] r ← m − 1
    else l ← m + 1
return −1
```

Convert this iterative version into recursive version

←

```
ALGORITHM Binarysearch( A, l, r, key)
 if(l > r) then
   return -1
 m      (l + r)/2
   if ( key = A[m]) then
     return m
   if ( key < A[m]) then
        return  Binarysearch ( A , l, m-1 ,key)
   else
       return Binarysearch (A, m+1,r,key)
```

Common Recurrence Types in Algorithm Analysis

- **Decrease-by-a-Constant-Factor T (n) = T (n/b) + f (n),**

```
ALGORITHM Binarysearch( A, l, r, key)
 if(l > r) then
   return -1
  m ← (l + r)/2
   if ( key = A[m]) then
     return m
    if ( key < A[m]) then
        return  Binarysearch ( A , l, m-1 ,key)
   else
       return Binarysearch (A, m+1,r,key)
```

- The standard way to analyze the efficiency of binary search is to count the number of times the search key is compared with an element of the array.
- For the sake of simplicity, we will count the so-called three-way comparisons.
- This assumes that after one comparison of *key* with *A*[*m*], the algorithm can determine whether *key* is smaller, equal to, or larger than *A*[*m*].

Common Recurrence Types in Algorithm Analysis

- **Decrease-by-a-Constant-Factor** ~~T (n) = T (n/b) + f (n),~~

```
ALGORITHM Binarysearch( A, l, r, key)
 if(l > r) then
   return -1
  m      (l + r)/2
   if ( key = A[m]) then
     return m
    if ( key < A[m]) then
        return  Binarysearch ( A , l, m-1 ,key)
   else
       return Binarysearch (A, m+1,r,key)
```

- How many such comparisons does the algorithm make on an array of *n* elements?
- The answer obviously depends not only on *n* but also on the specifics of a particular instance of the problem.
- Let us find the number of key comparisons in the worst case $C_{worst}(n)$.

Common Recurrence Types in Algorithm Analysis

- **Decrease-by-a-Constant-Factor** **T (n) = T (n/b) + f (n),**

```
ALGORITHM Binarysearch( A, l, r, key)
 if(l > r) then
   return -1
  m      (l + r)/2
    if ( key = A[m]) then
       return m
     if ( key < A[m]) then
          return  Binarysearch ( A , l, m-1 ,key)
   else
        return Binarysearch (A, m+1,r,key)
```

- The worst-case inputs include all arrays that do not contain a given search key, as well as some successful searches.
- Since after one comparison the algorithm faces the same situation but for an array half the size, we get the following recurrence relation for $C_{worst}(n)$:

$$C_{worst}(n) = C_{worst}(\lfloor n/2 \rfloor) + 1 \quad \text{for } n > 1, \quad C_{worst}(1) = 1.$$

YOUR TURN

- Analyze the following algorithm

ALGORITHM *Bisection*($f(x), a, b, eps, N$)

//Implements the bisection method for finding a root of $f(x) = 0$
//Input: Two real numbers a and b, $a < b$,
// a continuous function $f(x)$ on $[a, b]$, $f(a)f(b) < 0$,
// an upper bound on the absolute error $eps > 0$,
// an upper bound on the number of iterations N
//Output: An approximate (or exact) value x of a root in (a, b)
//or an interval bracketing the root if the iteration number limit is reached
$n \leftarrow 1$ //iteration count
while $n \le N$ **do**
    $x \leftarrow (a + b)/2$
    if $x - a < eps$ **return** x
    $fval \leftarrow f(x)$
    if $fval = 0$ **return** x
    if $fval * f(a) < 0$
        $b \leftarrow x$
    else $a \leftarrow x$
    $n \leftarrow n + 1$
return "iteration limit", a, b

Common Recurrence Types in Algorithm Analysis

- **Divide-and-Conquer** A divide-and-conquer algorithm solves a problem by dividing its given instance into several smaller instances, solving each of them recursively, and then, if necessary, combining the solutions to the smaller instances into a solution to the given instance. Assuming that all smaller instances have the same size **n/b**, with *a* of them being actually solved, we get the following recurrence valid for $n = b^k$, $k = 1, 2, \ldots$:
  - **$T(n) = aT(n/b) + f(n)$,**
- where $a \geq 1$, $b \geq 2$, and $f(n)$ is a function that accounts for the time spent on dividing the problem into smaller ones and combining their solutions. Recurrence (B.14) is called the ***general divide-and-conquer recurrence.***

Common Recurrence Types in Algorithm Analysis

- **Divide-and-Conquer T (n) = aT (n/b) + f (n)**

ALGORITHM *MaxElement*$(A[0..n-1])$

//Determines the value of the largest element in a given array
//Input: An array $A[0..n-1]$ of real numbers
//Output: The value of the largest element in A
$maxval \leftarrow A[0]$
for $i \leftarrow 1$ **to** $n-1$ **do**
  if $A[i] > maxval$
    $maxval \leftarrow A[i]$
return $maxval$

Convert into recursive version

The recurrence for the number of element comparisons is

$$C(n) = \begin{cases} 0 & \text{for } n = 1 \\ C(\lceil n/2 \rceil) + C(\lfloor n/2 \rfloor) + 1 & \text{for } n > 1, \end{cases}$$

Recurrence relation?

Algorithm $MaxIndex(A[l..r])$
//Input: A portion of array $A[0..n-1]$ between indices l and r $(l \leq r)$
//Output: The index of the largest element in $A[l..r]$
if $l = r$ return l
else $temp1 \leftarrow MaxIndex(A[l..\lfloor (l+r)/2 \rfloor])$
  $temp2 \leftarrow MaxIndex(A[\lfloor (l+r)/2 \rfloor + 1..r])$
  if $A[temp1] \geq A[temp2]$
    return $temp1$
  else return $temp2$

SOLVING RECURRENCE RELATION

Solving Recurrence Relation

- There are four methods for solving a recurrence relation
  - **Iteration Method**
  - **Substitution method for recurrence relations**
  - **Recursion Tree**
  - **The Master Theorem**

Solving Recurrence Relations

Given a recurrence relation $T(n)$.

- Substitute a few times until you see a pattern
- Write a formula in terms of n and the number of substitutions i.
- Choose i so that all references to $T()$ become references to the base case.
- Solve the resulting summation

This will not always work, but works most of the time in practice.

T (n) = T (n – 1) + f (n)

$$\begin{aligned} T(n) &= T(n-1) + f(n) \\ &= T(n-2) + f(n-1) + f(n) \\ &= \cdots \\ &= T(0) + \sum_{j=1}^{n} f(j). \end{aligned}$$

For a specific function $f(x)$, the sum $\sum_{j=1}^{n} f(j)$ can usually be either computed exactly or its order of growth ascertained. For example, if $f(n) = 1$, $\sum_{j=1}^{n} f(j) = n$; if $f(n) = \log n$, $\sum_{j=1}^{n} f(j) \in \Theta(n \log n)$; if $f(n) = n^k$, $\sum_{j=1}^{n} f(j) \in \Theta(n^{k+1})$.

Solving Recurrence Relations

```
function multiply(y, z)
    comment return the product yz
    if z = 0 then return(0) else
    if z is odd
        then return(multiply(2y, ⌊z/2⌋)+y)
        else return(multiply(2y, ⌊z/2⌋))
```

Let $T(n)$ be the running time of multiply(y, z), where z is an n-bit natural number.

Then for some $c, d \in \mathbb{R}$,

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ T(n-1) + d & \text{otherwise} \end{cases}$$

Therefore, for large enough n,

$$\begin{aligned} T(n) &= T(n-1) + d \\ T(n-1) &= T(n-2) + d \\ T(n-2) &= T(n-3) + d \\ &\vdots \\ T(2) &= T(1) + d \\ T(1) &= c \end{aligned}$$

Repeated Substitution

$$\begin{aligned} T(n) &= T(n-1) + d \\ &= (T(n-2) + d) + d \\ &= T(n-2) + 2d \\ &= (T(n-3) + d) + 2d \\ &= T(n-3) + 3d \end{aligned}$$

There is a pattern developing. It looks like after i substitutions,

$$T(n) = T(n-i) + id.$$

Now choose $i = n - 1$. Then

$$\begin{aligned} T(n) &= T(1) + d(n-1) \\ &= dn + c - d. \end{aligned}$$

Warning

This is not a proof. There is a gap in the logic. Where did

$$\boxed{T(n) = T(n-i) + id}$$

come from? Hand-waving!

What would make it a proof? Either

- Prove that statement by induction on i, or
- Prove the result by induction on n.

- Convert the recurrence into a summation and try to bound it using known series
  - **Iterate** the recurrence **until the initial condition** is reached.
  - Use **back-substitution** to express the recurrence in terms of *n* and the initial (boundary) condition.

T (n) = T (n/b) + f (n)

$$\begin{aligned} T(b^k) &= T(b^{k-1}) + f(b^k) \\ &= T(b^{k-2}) + f(b^{k-1}) + f(b^k) \\ &= \cdots \\ &= T(1) + \sum_{j=1}^{k} f(b^j). \end{aligned}$$

For a specific function $f(x)$, the sum $\sum_{j=1}^{k} f(b^j)$ can usually be either computed exactly or its order of growth ascertained. For example, if $f(n) = 1$,

$$\sum_{j=1}^{k} f(b^j) = k = \log_b n.$$

If $f(n) = n$, to give another example,

$$\sum_{j=1}^{k} f(b^j) = \sum_{j=1}^{k} b^j = b\frac{b^k - 1}{b - 1} = b\frac{n - 1}{b - 1}.$$

$$\mathbf{T(n) = c + T(n/2)}$$

$$T(n/2) = c + T(n/4)$$
$$T(n/4) = c + T(n/8)$$

$$\begin{aligned} T(n) &= c + T(n/2) \\ &= c + c + T(n/4) \\ &= c + c + c + T(n/8) \end{aligned}$$

Assume $n = 2^k$

$$T(n) = \underbrace{c + c + \ldots + c}_{\text{k times}} + T(1)$$

$$= c\lg n + T(1)$$
$$= \Theta(\lg n)$$

T (n) = aT (n/b) + f (n)

$$\begin{aligned}
T(b^k) &= aT(b^{k-1}) + f(b^k) \\
&= a[aT(b^{k-2}) + f(b^{k-1})] + f(b^k) = a^2T(b^{k-2}) + af(b^{k-1}) + f(b^k) \\
&= a^2[aT(b^{k-3}) + f(b^{k-2})] + af(b^{k-1}) + f(b^k) \\
&= a^3T(b^{k-3}) + a^2f(b^{k-2}) + af(b^{k-1}) + f(b^k) \\
&= \cdots \\
&= a^kT(1) + a^{k-1}f(b^1) + a^{k-2}f(b^2) + \cdots + a^0f(b^k) \\
&= a^k[T(1) + \sum_{j=1}^{k} f(b^j)/a^j].
\end{aligned}$$

Since $a^k = a^{\log_b n} = n^{\log_b a}$, we get the following formula for the solution to recurrence (B.14) for $n = b^k$:

$$T(n) = n^{\log_b a}[T(1) + \sum_{j=1}^{\log_b n} f(b^j)/a^j].$$

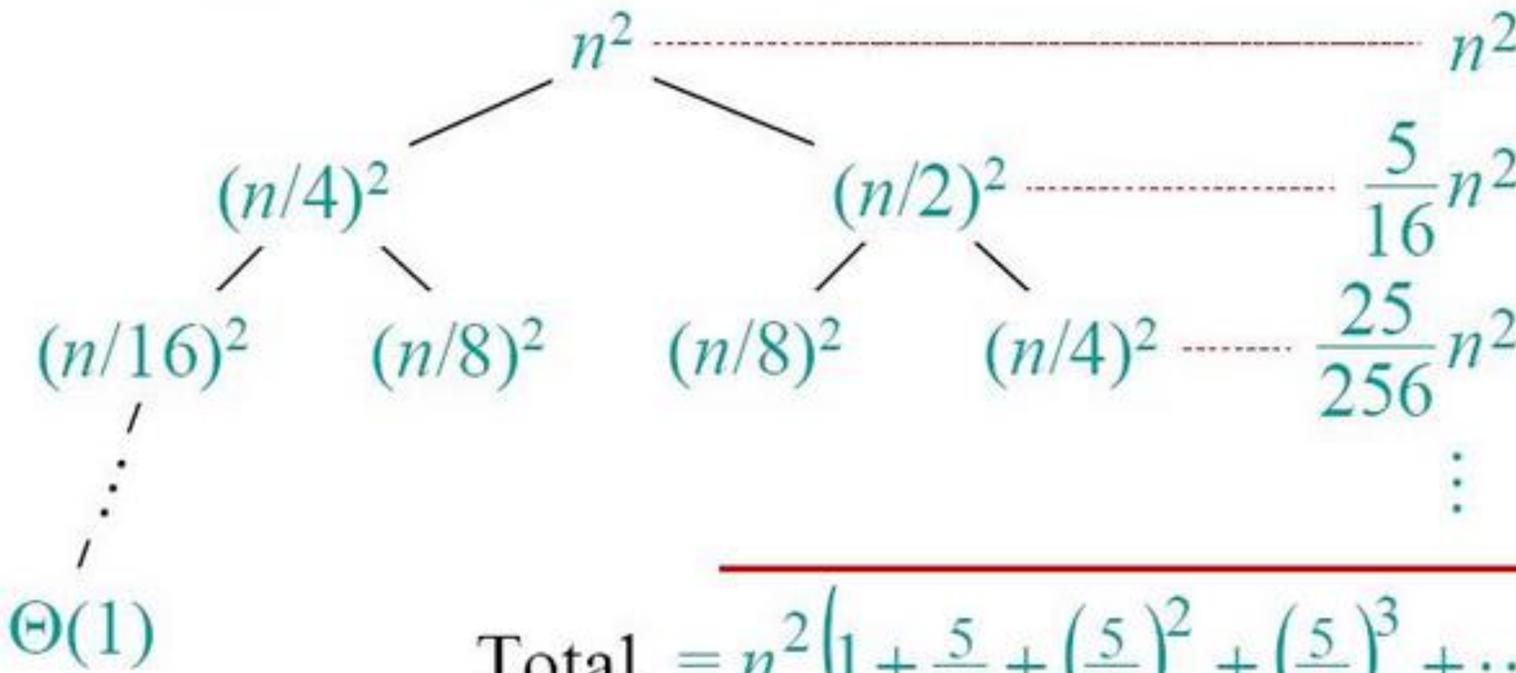
$$\mathbf{T(n) = n + 2T(n/2)} \qquad \text{Assume: } n = 2^k$$

$$T(n/2) = n/2 + 2T(n/4)$$

$$\begin{aligned}
T(n) &= n + 2T(n/2) \\
&= n + 2(n/2 + 2T(n/4)) \\
&= n + n + 4T(n/4) \\
&= n + n + 4(n/4 + 2T(n/8)) \\
&= n + n + n + 8T(n/8) \\
\dots \quad &= in + 2^iT(n/2^i) \\
&= kn + 2^kT(1) \\
&= n\lg n + nT(1) = \Theta(n\lg n)
\end{aligned}$$

- Expanding the recurrence into a tree
  - In a recursion tree, each node represents the cost of a single sub-problem somewhere in the set of recursive function invocations.
- Summing the cost at each level
  - We sum the costs within each level of the tree to obtain a set of per-level costs, and then we sum all the per-level costs to determine the total cost of all levels of the recursion.

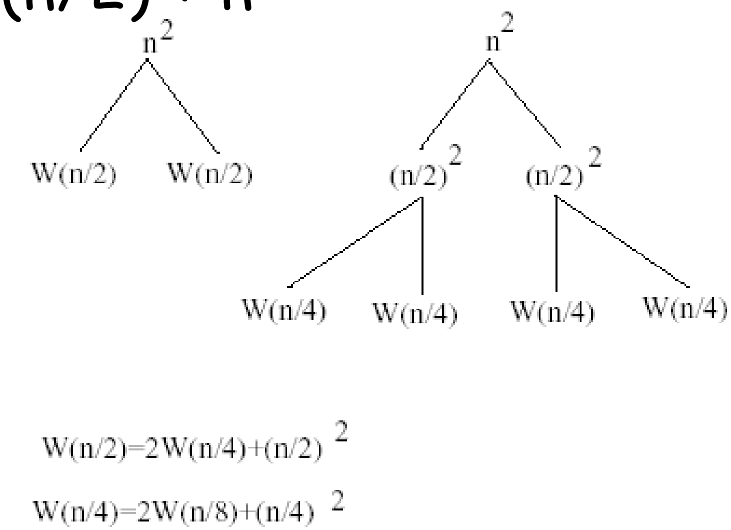
Solve $T(n) = T(n/4) + T(n/2) + n^2$:

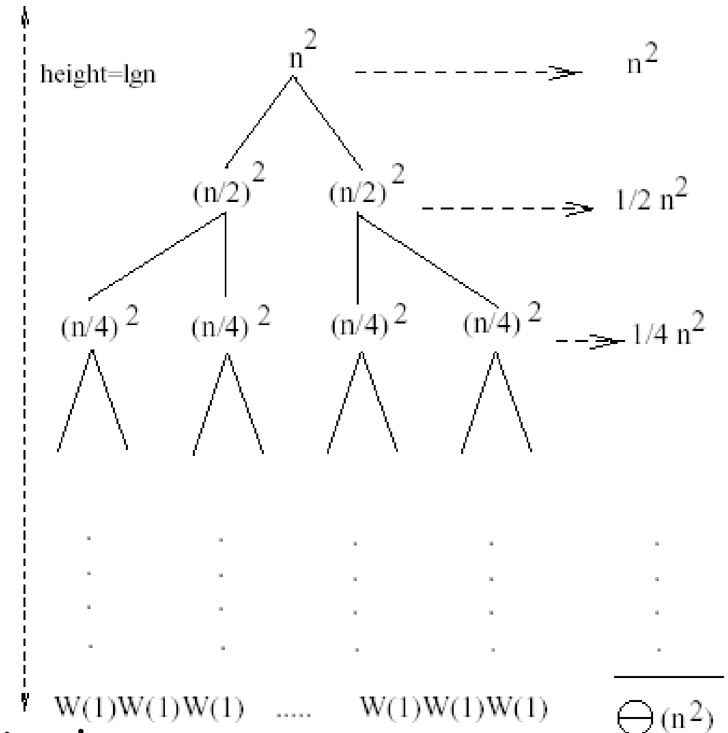


$$\text{Total} = n^2\left(1 + \frac{5}{16} + \left(\frac{5}{16}\right)^2 + \left(\frac{5}{16}\right)^3 + \cdots\right)$$

$$= \Theta(n^2) \quad \textit{geometric series}$$

$W(n) = 2W(n/2) + n^2$



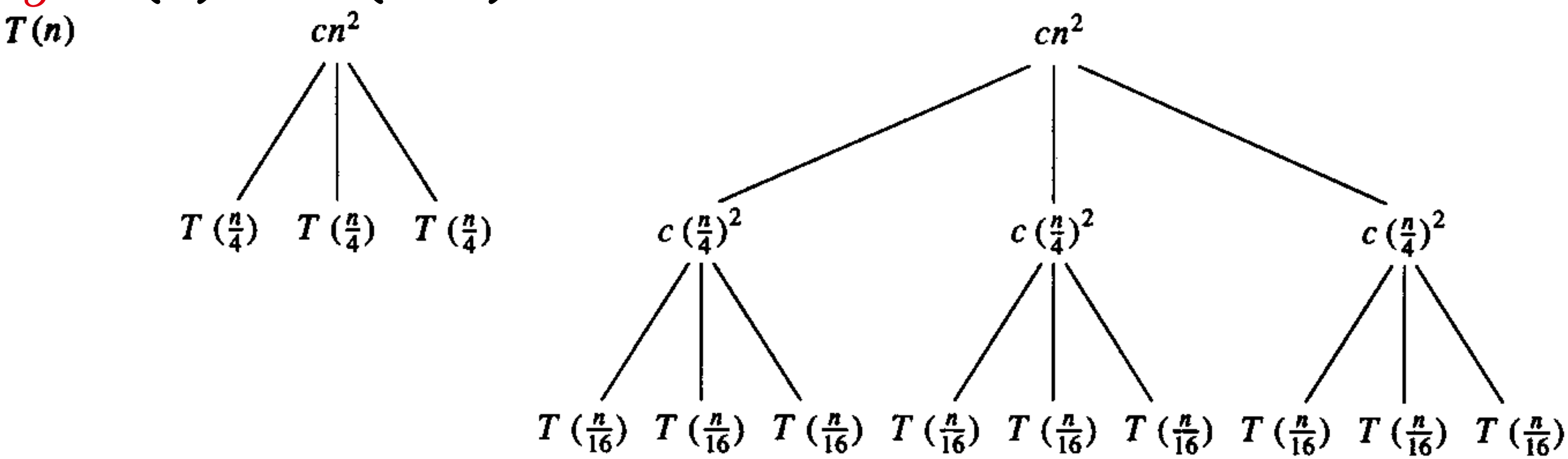


- Subproblem size at level i is: $n/2^i$
- Subproblem size hits 1 when $1 = n/2^i \Rightarrow i = \lg n$
- Cost of the problem at level i = $(n/2^i)^2$ No. of nodes at level i = 2^i
- Total cost:

$$W(n) = \sum_{i=0}^{\lg n-1} \frac{n^2}{2^i} + 2^{\lg n} W(1) = n^2 \sum_{i=0}^{\lg n-1} \left(\frac{1}{2}\right)^i + n \le n^2 \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i + O(n) = n^2 \frac{1}{1-\frac{1}{2}} + O(n) = 2n^2$$

$\Rightarrow W(n) = O(n^2)$

E.g.: $T(n) = 3T(n/4) + cn^2$



- Subproblem size at level i is: $n/4^i$
- Subproblem size hits 1 when $1 = n/4^i \Rightarrow i = \log_4 n$
- Cost of a node at level i = $c(n/4^i)^2$
- Number of nodes at level i = $3^i \Rightarrow$ last level has $3^{\log_4 n} = n^{\log_4 3}$ nodes
- Total cost:

$$T(n) = \sum_{i=0}^{\log_4 n-1} \left(\frac{3}{16}\right)^i cn^2 + \Theta\left(n^{\log_4 3}\right) \le \sum_{i=0}^{\infty} \left(\frac{3}{16}\right)^i cn^2 + \Theta\left(n^{\log_4 3}\right) = \frac{1}{1-\frac{3}{16}} cn^2 + \Theta\left(n^{\log_4 3}\right) = O(n^2)$$

$\Rightarrow T(n) = O(n^2)$

Convert the recurrence into a tree:

- Each node represents the cost incurred at various levels of recursion
- Sum up the costs of all levels

Used to "guess" a solution for the recurrence

The substitution method

1. Guess a solution

2. Use induction to prove that the solution works

Substitution method

- Guess a solution
  - $T(n) = O(g(n))$
  - Induction goal: apply the definition of the asymptotic notation
    - $T(n) \le d\, g(n)$, for some $d > 0$ and $n \ge n_0$
  - Induction hypothesis: $T(k) \le d\, g(k)$ for all $k < n$ (strong induction)
- Prove the induction goal
  - Use the **induction hypothesis** to find some values of the constants d and n_0 for which the **induction goal** holds

Example: Binary Search

$$T(n) = c + T(n/2)$$

- Guess: $T(n) = O(\lg n)$
  - Induction goal: $T(n) \le d \lg n$, for some d and $n \ge n_0$
  - Induction hypothesis: $T(n/2) \le d \lg(n/2)$
- Proof of induction goal:

$$T(n) = T(n/2) + c \le d \lg(n/2) + c$$

$$= d \lg n - d + c \le d \lg n$$

if: $-d + c \le 0$, $d \ge c$

- Base case?

Example 2

$$T(n) = T(n-1) + n$$

- Guess: $T(n) = O(n^2)$
  - Induction goal: $T(n) \le c\, n^2$, for some c and $n \ge n_0$
  - Induction hypothesis: $T(n-1) \le c(n-1)^2$ for all $k < n$
- Proof of induction goal:

$T(n) = T(n-1) + n \le c\,(n-1)^2 + n$

$= cn^2 - (2cn - c - n) \le cn^2$

if: $2cn - c - n \ge 0 \Leftrightarrow c \ge n/(2n-1) \Leftrightarrow c \ge 1/(2 - 1/n)$

  - For $n \ge 1 \Rightarrow 2 - 1/n \ge 1 \Rightarrow$ any $c \ge 1$ will work

Example 3

$$T(n) = 2T(n/2) + n$$

- Guess: $T(n) = O(n\lg n)$
  - Induction goal: $T(n) \le cn \lg n$, for some c and $n \ge n_0$
  - Induction hypothesis: $T(n/2) \le cn/2 \lg(n/2)$
- Proof of induction goal:

$$T(n) = 2T(n/2) + n \le 2c\,(n/2)\lg(n/2) + n$$

$$= cn \lg n - cn + n \le cn \lg n$$

$$\text{if: } -cn + n \le 0 \Rightarrow c \ge 1$$

- Base case?

Changing variables

$$T(n) = 2T(\sqrt{n}\,) + \lg n$$

- Rename: $m = \lg n \Rightarrow n = 2^m$

$T(2^m) = 2T(2^{m/2}) + m$

- Rename: $S(m) = T(2^m)$

$S(m) = 2S(m/2) + m \Rightarrow S(m) = O(m\lg m)$ (demonstrated before)

$T(n) = T(2^m) = S(m) = O(m\lg m)=O(\lg n\lg\lg n)$

Idea: transform the recurrence to one that you have seen before

Example 2 - Substitution

$$T(n) = 3T(n/4) + cn^2$$

- Guess: $T(n) = O(n^2)$
  - Induction goal: $T(n) \le dn^2$, for some d and $n \ge n_0$
  - Induction hypothesis: $T(n/4) \le d\,(n/4)^2$
- Proof of induction goal:

$T(n) = 3T(n/4) + cn^2$

$\le 3d\,(n/4)^2 + cn^2$

$= (3/16)\,d\,n^2 + cn^2$

$\le d\,n^2 \qquad$ if: $d \ge (16/13)c$

- Therefore: $T(n) = O(n^2)$

Example 3 (simpler proof)

$W(n) = W(n/3) + W(2n/3) + n$

- The longest path from the root to a leaf is:

  $n \rightarrow (2/3)n \rightarrow (2/3)^2\, n \rightarrow \ldots \rightarrow 1$

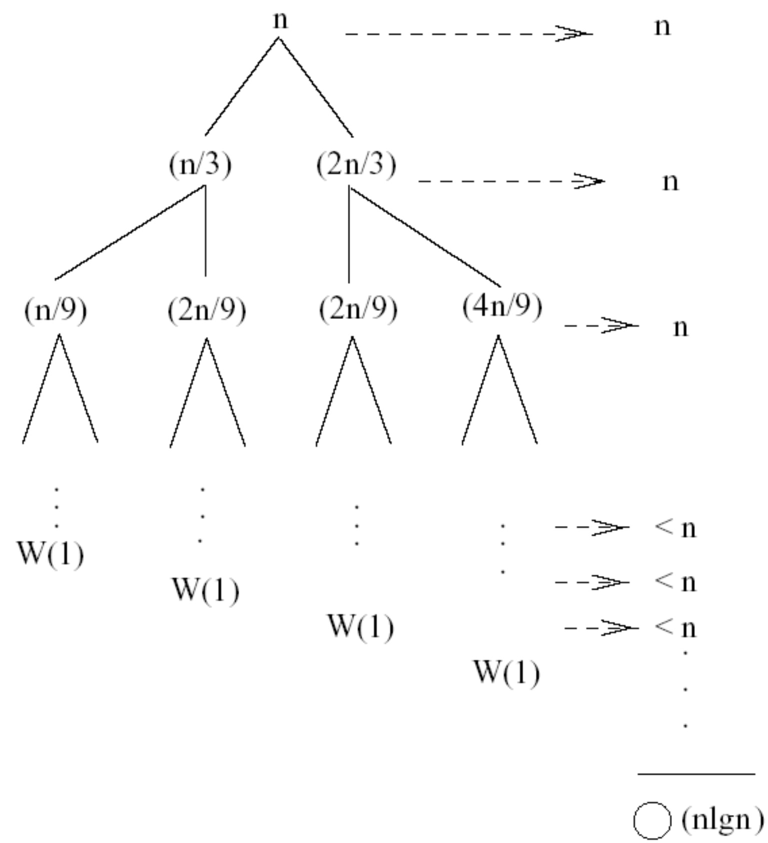
- Subproblem size hits 1 when

  $1 = (2/3)^i n \Leftrightarrow i = \log_{3/2} n$

- Cost of the problem at level i = n

- Total cost:

$$W(n) < n + n + \ldots = n(\log_{3/2} n) = n \frac{\lg n}{\lg \frac{3}{2}} = O(n \lg n)$$



Example 3

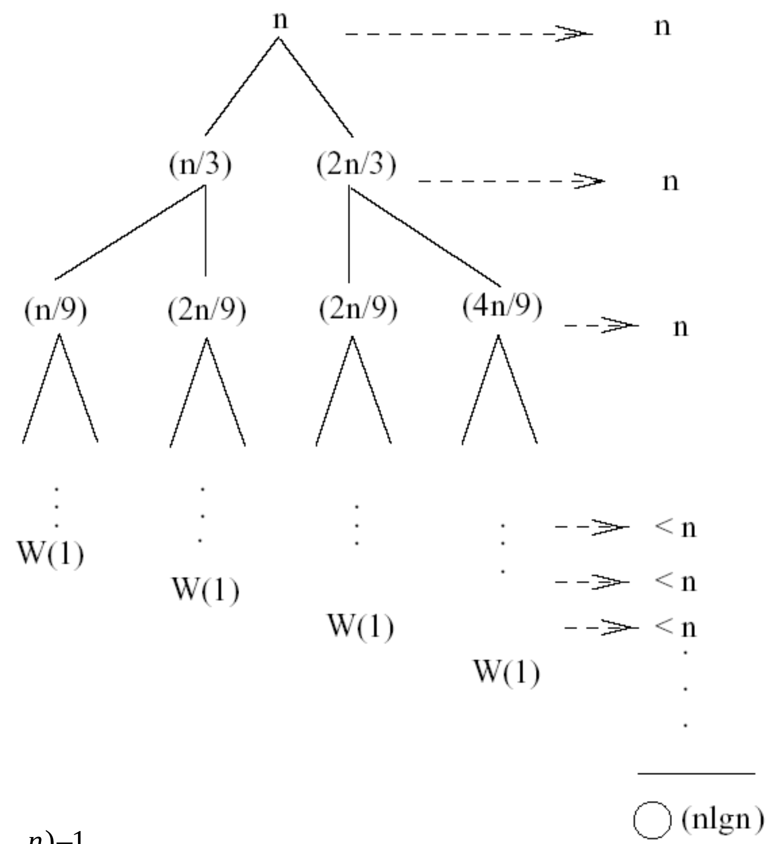
$W(n) = W(n/3) + W(2n/3) + n$

- The longest path from the root to a leaf is:

  $n \rightarrow (2/3)n \rightarrow (2/3)^2\, n \rightarrow \ldots \rightarrow 1$

- Subproblem size hits 1 when $1 = (2/3)^i n \Leftrightarrow i = \log_{3/2} n$
- Cost of the problem at level i = n
- Total cost: $W(n) < n + n + \ldots = \sum_{i=0}^{(\log_{3/2} n)-1} n + 2^{(\log_{3/2} n)} W(1) <$

$$< n \sum_{i=0}^{\log_{3/2} n} 1 + n^{\log_{3/2} 2} = n \log_{3/2} n + O(n) = n \frac{\lg n}{\lg 3/2} + O(n) = \frac{1}{\lg 3/2} n \lg n + O(n)$$



General Recurrence Relation of Divide and Conq.

Base of recursion

Running time for base

$$T(n) = \begin{cases} c & \text{if } n = n_0 \\ a.T(f(n)) + g(n) & \text{otherwise} \end{cases}$$

Number of times recursive call is made

All other processing not counting recursive calls

Size of problem solved by recursive call

Master Theorem: A general divide-and-conquer recurrence

$T(n) = aT(n/b) + g(n)$ where $g(n) \in \Theta(n^k)$

$a < b^k$ $T(n) \in \Theta(n^k)$
$a = b^k$ $T(n) \in \Theta(n^k \log n)$
$a > b^k$ $T(n) \in \Theta(n$ to the power of $(\log_b a))$

Note: the same results hold with O instead of Θ.

Div & Conq. (contd.)

- $T(n) = aT(n/b)+g(n)$, $a \geq 1$, $b > 1$
- Master Theorem:

  If $g(n) \in \Theta(n^d)$ where $d \geq 0$ then

What if a = 1?
Have we seen it?

$$T(n) \in \begin{cases} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \lg n) & \text{if } a = b^d \\ \Theta(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

So, $A(n) \in \Theta(n^{\log_2 2})$
Or, $A(n) \in \Theta(n)$

Without going through back-subs. we got it, but not quite…

For adding n numbers with divide and conquer technique, the number of additions A(n) is:
$A(n) = 2A(n/2)+1$

Here, a = ?, b = ?, d = ? **a = 2, b = 2, d = 0**

Which of the 3 cases holds ? **$a = 2 > b^d = 2^0$, case 3**

Div. & Conq. (contd.)

$T(n) = aT(n/b)+f(n)$, $a \geq 1$, $b > 1$

If $f(n) \in \Theta(n^d)$ where $d \geq 0$, then

$$T(n) \in \begin{cases} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \lg n) & \text{if } a = b^d \\ \Theta(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

$T(n) = 2T(n/2)+6n-1$?

$T(n) = 3\,T(n/2) + n$ $\quad$ $a = 3$, $b = 2$, $f(n) \in \Theta(n^1)$, so $d = 1$

$a=3 > b^d=2^1$ $\quad$ Case 3: $T(n) \in \Theta(n^{\log_2 3}) = \Theta(n^{1.5850})$

$T(n) = 3\,T(n/2) + n^2$ $\quad$ $a = 3$, $b = 2$, $f(n) \in \Theta(n^2)$, so $d = 2$

$a=3 < b^d=2^2$ $\quad$ Case 1: $T(n) \in \Theta(n^2)$

$T(n) = 4\,T(n/2) + n^2$ $\quad$ $a = 4$, $b = 2$, $f(n) \in \Theta(n^2)$, so $d = 2$

$a=4 = b^d=2^2$ $\quad$ Case 2: $T(n) \in \Theta(n^2 \lg n)$

$T(n) = 0.5\,T(n/2) + 1/n$ $\leftarrow$ Master th^m doesn't apply, $a<1$, $d<0$

$T(n) = 2\,T(n/2) + n/\lg n$ $\leftarrow$ Master th^m doesn't apply $f(n)$ not polynomial

$T(n) = 64\,T(n/8) - n^2 \lg n$ $f(n)$ is not positive, doesn't apply

$T(n) = 2^n\,T(n/8) + n$ a is not constant, doesn't apply

Important Recurrence Types

- One (constant) operation reduces problem size by one.
  $T(n) = T(n-1) + c$ $T(1) = d$
  Solution: $T(n) = (n-1)c + d$ *linear*

- A pass through input reduces problem size by one.
  $T(n) = T(n-1) + cn$ $T(1) = d$
  Solution: $T(n) = [n(n+1)/2 - 1]\, c + d$ *quadratic*

- One (constant) operation reduces problem size by half.
  $T(n) = T(n/2) + c$ $T(1) = d$
  Solution: $T(n) = c \log n + d$ *logarithmic*

- A pass through input reduces problem size by half.
  $T(n) = 2T(n/2) + cn$ $T(1) = d$
  Solution: $T(n) = cn \log n + d\, n$ *$n \log n$*

Important Recurrence Types

Recurrence	Algorithm	Big-Oh Solution
$T(n) = T(n/2) + O(1)$	Binary Search	$O(\log n)$
$T(n) = T(n-1) + O(1)$	Sequential Search	$O(n)$
$T(n) = 2\,T(n/2) + O(1)$	Tree Traversal	$O(n)$
$T(n) = T(n-1) + O(n)$	Selection Sort (other n^2 sorts)	$O(n^2)$
$T(n) = 2\,T(n/2) + O(n)$	Mergesort (average case Quicksort)	$O(n \log n)$