DIGITAL ASSIGNMENT 2

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(1) $E = \delta(t-2)$

$$E(t) = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C}$$

$$\Rightarrow L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = \delta(t-2)$$

$$\Rightarrow 0.1\frac{d^2q}{dt^2} + 4\frac{dq}{dt} + \frac{q}{25} \times 1000 = \delta(t-2)$$

$$\Rightarrow \frac{d^2q}{dt^2} + 40\frac{dq}{dt} + 400q = 10\,\delta(t-2).$$

Laplace Transform both sides.

$$\Rightarrow L[q''(t)] + 40L[q'(t)] + 400L[q(t)] = 10L[\delta(t-2)]$$

$$\Rightarrow [s^2\tilde{q}(s) - s\cancel{q(0)}^{0} - \cancel{q'(0)}^{0}] + 40[s\tilde{q}(s) - \cancel{q(0)}^{0}] + 400\tilde{q}(s) = 10e^{-2s}$$

$$\Rightarrow s^2\tilde{q}(s) + 40s\tilde{q}(s) + 400\tilde{q}(s) = 10e^{-2s}$$

$$\Rightarrow \tilde{q}(s)[s^2 + 40s + 400] = 10e^{-2s}$$

$$\Rightarrow \tilde{q}(s) = \frac{10e^{-2s}}{s^2+40s+400}$$

$$\Rightarrow \tilde{q}(s) = \frac{10e^{-2s}}{(s+20)^2}$$

Inverse Laplace on both sides.

$$\Rightarrow L^{-1}[q(s)] = 10 \quad L^{-1}\left[\frac{e^{-2s}}{(s+20)^2}\right]$$

$$\Rightarrow \quad q(t) = 10 \; H(t-2) \; f(t-2)$$

$$\Rightarrow \boxed{q(t) = 10 \; H(t-2) e^{-20(t-2)} \; (t-2)} \rightarrow \text{Ans.}$$

(2)

$m = 2\,kg \qquad k = 24\,Nm^{-1} \qquad \beta = x \qquad y(0) = 0.4 \qquad y'(0) =$

$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = 0$$

$$\Rightarrow 2\frac{d^2y}{dt^2} + x\frac{dy}{dt} + 24y = 0$$

a) Under Damping

$$x^2 < 4mk$$

$$x^2 < 4 \times 2 \times 24$$

$$x < 14$$

Let $x = 12$

$$2\frac{d^2y}{dt^2} + \frac{12\,dy}{dt} + 24y = 0$$

$$\Rightarrow \frac{d^2y}{dt^2} + \frac{6\,dy}{dt} + 12y = 0$$

$$\Rightarrow m^2 + 6m + 12 = 0$$

$$\Rightarrow m = \frac{-6 \pm \sqrt{36 - 48}}{2} = -3 \pm \sqrt{-3}.$$

$\therefore \quad m = -3 \pm \sqrt{3}\, i$

$y(t) = e^{-3t}\left[c_1 \cos\sqrt{2}t + c_2 \sin\sqrt{3}t\right]$

$y(0) = c_1 \qquad \therefore \quad c_1 = 0.4$

$y'(t) = e^{-3t}\left[-\sqrt{3}\, c_1 \sin\sqrt{3}t + \sqrt{2}\, c_2 \cos\sqrt{3}t\right] - 3e^{-3t}\left[c_1 \cos\sqrt{3}t + c_2 \sin\sqrt{3}t\right]$

$\Rightarrow \quad 0 = \sqrt{3}\, c_2 - 3c_1$

$\Rightarrow \quad c_2 = \sqrt{3} \times 0.4 =$

$\Rightarrow \quad c_2 = 0.69 \approx 0.7$

$\therefore \quad \boxed{y(t) = e^{-3t}\left[0.4 \cos\sqrt{3}t + 0.7 \sin\sqrt{3}t\right]}$ ← Ans

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$x_1' = 5x_1 + 3x_2$ $\quad x_1(0) = 0$

$x_2' = x_1 + 3x_2$ $\quad x_2(0) = 4$

$x' = Ax + f$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{vmatrix} 5-\lambda & 3 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (5-\lambda)(3-\lambda) - 3 = 0$

$\Rightarrow 15 - 5\lambda - 3\lambda + \lambda^2 - 3 = 0$

$\Rightarrow 12 - 8\lambda + \lambda^2 = 0$

$\Rightarrow \lambda = 2, 6$

$\lambda_1 = 2$

$$A - \lambda I = \begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$\therefore x_1 + x_2 = 0 \Rightarrow x_1 = -x_2$

$$X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\lambda_2 = 6$

$A - \lambda I = 0$

$$\begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$x_1 - 3x_2 = 0$

$\Rightarrow x_1 = 3x_2$

$$X_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$X = c_1 e^{2t} X_1 + c_2 e^{6t} X_2$$

$$\Rightarrow X = c_1 e^{2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \cdot e^{6t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$x_1 = -c_1 e^{2t} + 3c_2 e^{6t}$$
$$x_2 = c_1 e^{2t} + c_2 e^{6t}$$

$$\Rightarrow 0 = -c_1 + 3c_2$$
$$4 = c_1 + c_2$$

$c_2 = 1$

$\therefore c_1 = 3.$

$$\boxed{\begin{aligned} x_1 &= -3e^{2t} + 3e^{6t} \\ x_2 &= 3e^{2t} + e^{6t} \end{aligned}} \rightarrow \text{Ans}$$

(ii) <u>Diagonalisation</u>

$$A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix} \qquad \lambda = 3, 6$$

$$x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad x_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix} \qquad \therefore \; P^{-1} = \frac{1}{4}\begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$$

$$D = P^{-1}AP$$

$$= \frac{1}{4}\begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$$

$$x = PY \Rightarrow x' = PY'$$

$$x' = Ax + F$$

$$PY' = APY + F$$

$$P^{-1}PY' = P^{-1}APY + P^{-1}F$$

$$Y' = DY \qquad \text{as } F = 0$$

$$\Rightarrow Y' = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$y_1' = 2y_1 \quad \& \quad y_2' = 6y_2$$

$\Rightarrow \frac{dy_1}{dt} = 2y_1$

$\Rightarrow \int \frac{dy_1}{y_1} = \int 2dt$

$\Rightarrow \ln(y_1) = 2t + c_1$

$\therefore \boxed{y_1 = c_1 e^{2t}}$

$\Rightarrow \frac{dy_2}{dt} = 6y_2$

$\Rightarrow \int \frac{dy_2}{y_2} = \int 6\,dt$

$\Rightarrow \ln(y_2) = 6t + c_2$

$\therefore \boxed{y_2 = c_2 e^{6t}}$

$X = PY$

$\Rightarrow X = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{2t} \\ c_2 e^{6t} \end{bmatrix}$

$x_1 = -c_1 e^{2t} + 3c_2 e^{6t}$

$x_2 = c_1 e^{2t} + c_2 e^{6t}$

$\because c_2 = 1$

$c_1 = 3$

$\therefore \boxed{\begin{aligned} x_1 &= -3e^{2t} + 3e^{6t} \\ x_2 &= 3e^{2t} + e^{6t} \end{aligned}} \rightarrow \underline{\underline{\text{Ans}}}$

4. (i) Matrix Method.

$$x'' = Ax + F \qquad \because F = 0.$$

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\lambda = 3, 1$ For A

$\lambda_1 = 3$

$[A - \lambda I]x_1 = 0$

$$\Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 0$$

$$z_1 - z_2 = 0$$

$$\Rightarrow z_1 = z_2.$$

$$X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda_2 = 1$

$[A - \lambda I]X_2 = 0$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 0$$

$$z_1 + z_2 = 0$$

$$X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$X = x_1 c_1 e^{\sqrt{\lambda_1} t} + x_1 c_2 e^{-\sqrt{\lambda_1} t} + x_2 c_3 e^{\sqrt{\lambda_2} t} + x_2 c_4 e^{-\sqrt{\lambda_2} t}$$

$$x_1 = -c_1 e^{t} - c_2 e^{-t} + c_3 e^{\sqrt{3}t} + c_4 e^{-\sqrt{3}t}$$
$$x_2 = c_1 e^{t} + c_2 e^{-t} + c_3 e^{\sqrt{3}t} + c_4 e^{-\sqrt{3}t}$$

(ii) <u>Diagonalisation Method</u>.

from (i)

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$\lambda = 1, 3$

$$X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$X'' = AX$$

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Let $X = PY$

$X'' = PY''$ where $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

since P is similar.

$$P^{-1} = P^{T}$$

$\Rightarrow PY'' = APY$

$\Rightarrow P^{-1}PY'' = P^{-1}APY$

$\Rightarrow Y'' = DY$

$\Rightarrow Y'' = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$$Y'' = \begin{bmatrix} y_1 \\ 3y_2 \end{bmatrix}$$

$y_1'' = y_1$ & $y_2'' = 3y_2$

$\Rightarrow y_1'' - y_1 = 0$ & $y_2'' - 3y_2 = 0$

$\Rightarrow m^2 - 1 = 0$ $\Rightarrow m^2 - 3 = 0$

$\Rightarrow m = \pm 1$ $\Rightarrow m = \pm\sqrt{3}$

$y_1 = c_1 e^{t} + c_2 e^{-t}$ $y_2 = c_3 e^{\sqrt{3}t} + c_4 e^{-\sqrt{3}t}$

$$X = PY$$

$$\Rightarrow X = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{t} + c_2 e^{-t} \\ c_3 e^{\sqrt{3}t} + c_4 e^{-\sqrt{3}t} \end{bmatrix}$$

$$\Rightarrow X = \begin{bmatrix} -c_1 e^{t} - c_2 e^{-t} + c_3 e^{\sqrt{3}t} + c_4 e^{-\sqrt{3}t} \\ c_1 e^{t} + c_2 e^{-t} + c_3 e^{\sqrt{3}t} + c_4 e^{-\sqrt{3}t} \end{bmatrix}$$

$$\boxed{\begin{aligned} x_1 &= -c_1 e^{t} - c_2 e^{-t} + c_3 e^{\sqrt{3}t} + c_4 e^{-\sqrt{3}t} \\ x_2 &= c_1 e^{t} + c_2 e^{-t} + c_3 e^{\sqrt{3}t} + c_4 e^{-\sqrt{3}t} \end{aligned}} \rightarrow \text{Ans}$$

(5) $\frac{d}{dx}\left(1\frac{dy}{dx}\right) + \lambda.1 = 0$

Case I → $\lambda > 0$

Let $y = e^{mx}$ be the solⁿ.

$m^2 + \lambda = 0$

$\Rightarrow m = \pm i\sqrt{\lambda}$

$y(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$

$\Rightarrow y'(x) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$

$\Rightarrow y(0) + y'(0) = 0$

$\Rightarrow c_1 + c_2\sqrt{\lambda} = 0$

$\Rightarrow c_1 = -c_2\sqrt{\lambda}$

$y(1) + y'(1) = 0$

$\Rightarrow c_1 \cos\sqrt{\lambda} + c_2 \sin\sqrt{\lambda} - c_1\sqrt{\lambda}\sin\sqrt{\lambda} + c_2\sqrt{\lambda}\cos\sqrt{\lambda}$

$- c_2\sqrt{\lambda}\cos\sqrt{\lambda} + c_2 \sin\sqrt{\lambda} + c_2\sqrt{\lambda}\sin\sqrt{\lambda} + c_2\sqrt{\lambda}\sin\sqrt{\lambda} = 0$

$\Rightarrow c_2(\sin\sqrt{\lambda} + \lambda\sin\sqrt{\lambda}) = 0$

$\Rightarrow \sin\sqrt{\lambda}(1+\lambda) = 0$

$\Rightarrow \sin\sqrt{\lambda} = 0$

$\lambda = n^2\pi^2$

$n = 0, 1, 2 \ldots$

$$\boxed{y(x) = c_1\cos(n\pi x) + c_2\sin(n\pi x)}$$

eigen fⁿ.

Case II $\rightarrow$ $\lambda = 0$.

$$y'' = 0$$

$$m^2 = 0 \quad , \quad m = 0$$

$$y(x) = c_1 + c_2 x$$

$$y'(x) = c_2$$

$$y(1) = c_1 + c_2 = 0$$

$$y(0) = c_1 = -c_2$$

$$y'(1) = c_2$$

$$y'(0) = c_2$$

$$c_1 + 2c_2 = 0$$

$$c_1 = c_2 = 0.$$

$$\therefore \boxed{y(x) = 0}.$$

Not an eigen function.

Case III $\rightarrow$ $\lambda = 0$

$$m^2 - \lambda = 0$$

$$m = \pm\sqrt{\lambda}$$

$$y(x) = c_1 e^{\sqrt{\lambda} x} + c_2 e^{-\sqrt{\lambda} x}$$

$$y'(x) = \sqrt{\lambda}\, c_1 e^{\sqrt{\lambda} x} - \sqrt{\lambda}\, c_2 e^{-\sqrt{\lambda} x}$$

$y(0) + y'(0) = 0$

$\Rightarrow c_1 + c_2 + c_1\sqrt{\lambda} - \sqrt{\lambda}\, c_2 = 0$

$\Rightarrow c_1(1+\sqrt{\lambda}) + c_2(1-\sqrt{\lambda}) = 0$

$\cancel{y(1) + y'(0)}$

$y(1) + y'(1) = 0$

$\Rightarrow c_1 e^{\sqrt{\lambda}} + c_2 e^{-\sqrt{\lambda}} + c_1\sqrt{\lambda}\, e^{\sqrt{\lambda}} - c_2\sqrt{\lambda}\, e^{-\sqrt{\lambda}} = 0$

$\Rightarrow c_1 e^{\sqrt{\lambda}}(1-\sqrt{\lambda}) + e^{-\sqrt{\lambda}}(1-\sqrt{\lambda}) = 0$

$\Rightarrow \left(c_1 e^{\sqrt{\lambda}} + e^{-\sqrt{\lambda}}\right)(1-\sqrt{\lambda}) = 0$

$\therefore \boxed{\lambda = 1}$

$e^{2\sqrt{\lambda}} = 1$

$\Rightarrow 2\sqrt{\lambda} = 0$

$\Rightarrow \lambda = 0$

$\therefore \boxed{y = c_1 + c_2}$

Eigen function

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(6) $xy'' + y' + 2y = 0$
$y(1) = 1 \qquad y'(1) = 2$

$\lim_{x \to 1} \frac{1}{x}$ and $\lim_{x \to 1} \frac{2}{x}$ exist.

$\therefore$ $\underline{x = 1}$ is an ordinary point.

let $x - 1 = x$
$x = x + 1$

$xy'' + y' + 2y = 0$
$\Rightarrow (x+1)y'' + y' + 2y = 0$
$\Rightarrow xy'' + y'' + y' + 2y = 0$ — (1)

let
$y = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$
$y' = a_1 + 2a_2 x + 3a_3 x^2 + \dots\, n a_n x^{n-1}$
$y'' = 2a_2 + 6a_3 x + \dots + n(n-1)a_n x^{n-2}$

Substitute this in (1).

$2a_2 x + 6a_3 x^2 + \dots + n(n-1) a_n x^{n-1}$
$n(n+1) a_n x^n + 2a_2 + 6a_3 x + \dots + (2+n)(n+1) a_{n+2} x^n$
$+ a_1 + 2a_2 x + \dots + (n+1) a_{n+1} x^n +$
$2a_0 + 2a_1 x + 2a_2 x^2 + \dots + 2a_n x^n = 0$

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co-eff of x^0

$$2a_2 + a_1 + 2a_0 = 0$$

$$\Rightarrow 2a_2 = -a_1 - 2a_0$$

$$\Rightarrow a_2 = \frac{-a_1 - 2a_0}{2}$$

co-eff of x

$$2a_2 + 6a_3 + 2a_2 + 2a_1 = 0$$

$$\Rightarrow 3a_3 = -2a_2 - a_1$$

$$\Rightarrow a_3 = \frac{-2a_2 - a_1}{3}$$

co-eff of x^n

$$n(n+1)a_{n+1} + (n+1)(n+2)a_{n+2} + (n+1)a_{n+1} + 2a_n = 0$$

$$\Rightarrow (n+1)(n+2)a_{n+2} + (n^2+2n+1)a_{n+1} + 2a_n = 0$$

$$\Rightarrow a_{n+2} = \frac{-2a_n - (n^2+2n+1)a_{n+1}}{(n+1)(n+2)}$$

Put $n = 0$

$$a_2 = \frac{-a_1 - 2a_0}{2}$$

$$y(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

$$y(x) = y(1) = 1$$

$$y(x-1) = y(x)$$

$$y(x) = a_0 + a_1(x-1) + a_2(x-1)^2 + \dots + a_n(x-1)^n$$

$$\Rightarrow y(1) = a_0 = 1$$

$$y'(x) = a_1 + 2a_2(x-1)^2 + \cdots n a_n (x-1)^n$$

$$y(1) = a_1$$

$$1 = a_1$$

$$a_0 = 1 \quad , \quad a_1 = 2$$

$$a_2 = \frac{-a_1 - 2a_0}{2} = -2$$

$$a_3 = \frac{-2a_2 - a_1}{3} = \frac{2}{3}$$

$$\boxed{\therefore y(x) = 1 + 2(x-1) - 2(x-1)^2 + \frac{2}{3}(x-1)^3 + \cdots}$$

Ans

7) a) $xy'' + y\sin x = 0$

$$y'' + y\frac{\sin x}{x} = 0$$

$P(x) = 0$. $Q(x) = \lim_{x\to 0} \frac{\sin x}{x}$

$= 1 \neq \infty$

for $x = 1$

$\lim_{x\to 1} (x-1)\frac{\sin x}{x} = 0$

$\boxed{x = 0 \text{ is an Irregular Singular Point}}$

b) $x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$

$\Rightarrow y'' + \frac{x}{x^2}y' + \frac{\left(x^2 - \frac{1}{4}\right)}{x^2}y = 0$

$P(x) = \frac{1}{x}$ $Q(x) = \frac{\left(x^2 - \frac{1}{4}\right)}{x^2}$

at $x = 0$

$\lim_{x\to 0} \frac{1}{x}$ = undefined | $\lim_{x\to 0} \frac{\left(x^2 - \frac{1}{4}\right)}{x^2}$ = undefined

$\therefore$ $\boxed{x = 0 \text{ is a Singular Point}}$

c) $2x^2 y'' + xy' - (x+1)y = 0$

$$\Rightarrow y'' + \frac{x}{2x^2} y' - \frac{(x+1)}{2x^2} y = 0$$

$$P(x) = \frac{1}{2x}$$

$$Q(x) = \frac{-(x+1)}{2x^2}$$

at $x = 0$

$$\lim_{x \to 0} P(x)$$

$$= \lim_{x \to 0} \frac{1}{2x} = \cancel{\text{undefined}}$$

$$= \infty$$

$$\lim_{x \to 0} Q(x)$$

$$= \lim_{x \to 0} \frac{-(x+1)}{2x^2} = \infty$$

$$\boxed{x = 0 \text{ is a Singular Point}}$$