DIGITAL ASSIGNMENT - 1

Name : Kulvir Singh

Reg. No. : 19BCE2074

1) $$f(t) = \begin{cases} 5\sin t & 0 \le t \le \pi \\ 0 & , \pi \le t \le 2\pi \end{cases}$$

$$T = 2\pi$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t).$$

$$a_0 = \frac{2}{T}\int_0^{2\pi} f(t)\,dt$$

$$= \frac{2}{2\pi}\int_0^{\pi} 5\sin t\,dt + 0$$

$$= \frac{5}{\pi}\left(-\cos t\right)_0^{\pi}$$

$$= \frac{5}{\pi}\left(-\cos\pi - (-\cos 0)\right)$$

$$= \frac{5}{\pi}(2)$$

$$= \frac{10}{\pi}$$

$$a_n = \frac{2}{T}\int_0^{2\pi} f(t)\cos(nt)\,dt \qquad n=1,2,3\ldots \quad \omega = 1$$

$$= \frac{2}{2\pi}\int_0^{\pi} 5\sin t\cos(nt)\,dt + 0$$

$$= \frac{5}{\pi}\int_0^{\pi} \frac{1}{2}\left\{\sin(n+1)t + \sin(1-n)t\right\}dt.$$

$$= \frac{5}{2\pi}\left[-\frac{\cos(n+1)t}{n+1} - \frac{\cos(1-n)t}{(1-n)}\right]_0^{\pi}$$

$$= \frac{-5}{2\pi}\left[\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(1-n)\pi}{1-n} - \frac{1}{n+1} - \frac{1}{1-n}\right]$$

$$= \frac{-5}{2\pi}\left[\frac{\cos(n+1)\pi}{n+1} - \frac{\cos(n-1)\pi}{n-1} - \frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= \frac{5}{\pi}\left(\frac{-\cos(n\pi)-1}{n^2-1}\right)$$

$$= \frac{-5}{\pi}\left(\frac{(-1)^n+1}{n^2-1}\right)$$

$$b_n = \frac{2}{T}\int_0^{2\pi} f(t)\sin(nt)\,dt \qquad n=1,2,3\ldots \quad \omega = 1$$

$$= \frac{2}{2\pi}\int_0^{\pi} 5\sin t\sin(nt)\,dt + 0$$

$$= \frac{-5}{2\pi}\int_0^{\pi}\left[\cos(nt+t) + \cos(nt-t)\right]dt$$

$$= \frac{-5}{2\pi}\int_0^{\pi}\cos(n+1)t + \cos(n-1)t\,dt$$

$$= \frac{-5}{2\pi}\left[\frac{\sin(n+1)t}{(n+1)} + \frac{\sin(n-1)t}{(n-1)}\right]_0^{\pi}$$

$$= 0$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin tn$$

$$= \frac{10}{2\pi} + \sum_{n=1}^{\infty} \frac{-5}{\pi} \frac{\left((-1)^n + 1\right)}{n^2 - 1} \cos nt + 0 .$$

$$= \frac{5}{\pi} + \frac{5}{\pi} \sum_{n=1}^{\infty} \frac{\left((-1)^n + 1\right) \cos nt}{1 - n^2}$$

$$\boxed{f(t) = \frac{5}{\pi}\left(1 + \sum_{n=1}^{\infty} \frac{\left((-1)^n + 1\right) \cos nt}{1 - n2}\right)} \rightarrow \text{Ans.}$$

2) $$\int_{-\pi}^{\pi} \cos^4 x \, dx = \frac{3\pi}{4}$$

$$f(x) = \cos^2 x \qquad \pi < -x < x$$

using parseval's identity.

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} (\cos^2 x)^2 \, dx = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} [a_n^2 + b_n^2]$$

$$a_0 = \frac{2}{T} \int_{d}^{d+T} f(t) \, dt = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2 x \, dx$$

$$= \frac{\cancel{2}}{\pi} \times \frac{1}{\cancel{2}} \left[\int_{0}^{\pi} \cos 2x \, dx + \int_{0}^{\pi} dx \right]$$

$$= \frac{1}{\pi} \left(\left[\frac{\sin 2x}{2} \right]_0^{\pi} + [x]_0^{\pi} \right)$$

$$= \frac{1}{\pi} (\pi)$$

$$= 1$$

$$a_n = \frac{2}{T}\int_{d}^{d+T} f(t)\cos nt\, dt = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2 x \cos nx\, dx$$

$$= \frac{2}{\pi}\times\frac{1}{2}\int_{0}^{\pi} (\cos 2x + 1)\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{0}^{\pi} \cos 2x \cos nx\, dx + \int_{0}^{\pi} \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\int_{0}^{\pi} \frac{\cos(2+n)x}{2}\, dx + \int_{0}^{\pi} \frac{\cos(2-n)x}{2}\, dx + \int_{0}^{\pi} \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\left.\frac{\sin(2+n)x}{2(2+n)}\right|_{0}^{\pi} + \left.\frac{\sin(2-n)x}{2(2-n)}\right|_{0}^{\pi} + \left.\frac{\sin nx}{n}\right|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[0 + \left(\frac{\sin(2-n)\pi}{2(2-n)} - 0\right) + 0\right]$$

$$= \frac{1}{2\pi}\,\frac{\sin((2-n)\pi)}{(2-n)}$$

$n \in R \neq 2 \quad a_n = 0.$

for $n = 2$. we apply limits. (LH Rule)

$$\therefore \quad = \frac{1}{2\pi}\,\frac{(-\pi)\cos(2-n)\pi}{(-1)} = \frac{1}{2}\cos 0 = \frac{1}{2}.$$

$$\therefore a_n = \frac{1}{2}$$

$$b_n = \frac{2}{T}\int_{d}^{d+T} f(t)\sin(nt)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2 x \sin nx\,dx$$

$\because$ $\cos^2 x$ is even function
$\sin nx$ is odd function $\Big\}$ for interval $-\pi \le x \le \pi$

$$\therefore\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2 x \sin nx\,dx = 0.$$

$\therefore$ From given identity of Parseval

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^4 x\,dx = \frac{1}{4} + \frac{1}{2}\left(\frac{1}{2}\right)^2$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^4 x\,dx = \frac{1}{4} + \frac{1}{8}$$

$$\Rightarrow \int_{-\pi}^{\pi}\cos^4 x\,dx = \frac{3}{8}\times 2\pi$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi}\cos^4 x\,dx = \frac{3\pi}{4}}$$

$\hookrightarrow$ hence proved.

3)

t	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$
$f(t)$	3	4	5	3	-4	-12

$N = 6 \qquad h = \frac{2\pi}{N} = \frac{\pi}{3}$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{2} f(t_i)\cos(nt_i) + \sum_{n=1}^{2} f(t_i)\sin(nt_i)$$

t	$f(t)$	$\cos t$	$f(t)\cos t$	$\cos 2t$	$f(t)\cos 2t$	$\sin t$	$f(t)\sin t$	$\sin 2t$	$f(t)\sin 2t$
0	3	1	3	1	3	0	0	0	0
$\pi/3$	4	0.5	2	-0.5	-2	0.866	3.464	0.866	3.464
$2\pi/3$	5	-0.5	-2.5	-0.5	-2.5	0.866	4.330	-0.866	-4.330
π	3	-1	-3	1	3	0	0	0	0
$4\pi/3$	-4	-0.5	2	-0.5	2	-0.866	3.464	0.866	-3.464
$5\pi/3$	-12	0.5	-6	-0.5	6	-0.866	10.392	-0.866	10.392
	-1		-4.5		9.5		21.65		6.062

$$a_0 = \frac{2}{N}\sum_{i=0}^{N-1} f(t) = \frac{2}{6}(-1) = -\frac{1}{3} = -0.3334$$

$$a_1 = \frac{2}{N}\sum_{i=0}^{N-1} f(t)\cos(t) = \frac{2}{6}(-4.5) = -1.5$$

$$a_2 = \frac{2}{N}\sum_{i=0}^{N-1} f(t)\cos(2t) = \frac{2}{6}(9.5) = 3.166$$

$$b_1 = \frac{2}{N}\sum_{i=0}^{N-1} f(t)\sin(t) = \frac{2}{6}(21.65) = 7.216$$

$$b_2 = \frac{2}{N}\sum_{i=0}^{N-1} f(t)\sin(2t) = \frac{2}{6}(6.062) = 2.021$$

$$\therefore f(t) = \frac{-0.3334}{2} - 1.5\cos t + 3.166\cos 2t + 7.216\sin t + 2.021\sin 2t$$

$$\left[\therefore f(t) = -0.1667 - 1.5\cos t + 3.166\cos 2t + 7.216\sin t + 2.021\sin 2t\right] \rightarrow \text{Ans}$$

4) $x_1^2 - 6x_2^2 + 24x_1x_2 = 0$

$Q = X^T A X$ $\qquad x_0 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$Q = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 12 \\ 12 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Characteristic eq. of matrix

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 12 \\ 12 & -6-\lambda \end{vmatrix} = 0$

$\lambda^2 + 5\lambda - 6 - 144 = 0$

$\Rightarrow \lambda^2 + 5\lambda - 150 = 0$

$\lambda = 10, -15$

for $\lambda = 10$

$[A - \lambda I][X] = 0$

$\Rightarrow \begin{bmatrix} -9 & 12 \\ 12 & -16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\Rightarrow -9x_1 + 12x_2 = 0$

or

$12x_1 - 16x_2 = 0$

$\frac{x_1}{4} = \frac{x_2}{3} = \alpha$

$X_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \alpha \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

for $\underline{\underline{\lambda}} = 15$

$$[A - \lambda I][x] = 0$$

$$\Rightarrow \begin{bmatrix} 16 & 12 \\ 12 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \quad 16x_1 + 12x_2 = 0$$

$$\text{or} \quad 12x_1 + 9x_2 = 0$$

$$\frac{x_1}{-3} = \frac{x_2}{4} = \beta$$

$$X_2 = \beta \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$X_1^T X_2 = \begin{bmatrix} 4 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 4 \end{bmatrix} = 0 \qquad \therefore \text{ orthogonal}$$

Normalisation.

$$X_1 = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix} \qquad X_2 = \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$$

let $X = P\gamma$

$$Q = (P\gamma)^T A (P\gamma) = \gamma^T (P^T A P)\gamma$$

$$P = \frac{1}{5} \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} \quad \text{orthogonal modal matrix}$$

$$P^T A P = \frac{1}{25} \begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 12 \\ 12 & -6 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & -15 \end{bmatrix}$$

$$Q = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & -15 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 10y_1^2 - 15y_2^2 \rightarrow \text{Ans.}$$

5) (i) $A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

a) Characteristic Eqn.

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left((2-\lambda)(-1-\lambda) - 1\right) + 1\left((-1-\lambda) + 2\right) = 0$$

$$\Rightarrow (1-\lambda)\left(-(2-\lambda)(\lambda+1) - 1\right) + (1-\lambda) = 0$$

$$\Rightarrow (1-\lambda)\left(-(2-\lambda)(\lambda+1) \cancel{-1} \cancel{+1}\right) = 0$$

$$\Rightarrow (\lambda-1)(2-\lambda)(\lambda+1) = 0$$

$$\Rightarrow \lambda = -1, 1, 2. \rightarrow \text{Eigen Values.}$$

For $\lambda = -1$

$$[A - \lambda I][X] = 0$$

$$\Rightarrow \begin{bmatrix} 2 & 1 & -2 \\ -1 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$2x_1 + x_2 - 2x_3 = 0$$

$$-x_1 + 3x_2 + x_3 = 0$$

$$x_2 = 0.$$

$\therefore$

$$2x_1 = 2x_3 \Rightarrow x_1 = x_3$$

$$x_1 = x_3 = \alpha$$

$$x_2 = 0$$

$$\therefore X = \begin{bmatrix} \alpha \\ 0 \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

For $\lambda = 1$

$$[A - \lambda I]X = 0$$

$$\begin{bmatrix} 0 & 1 & -2 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$x_2 - 2x_3 = 0$$
$$-x_1 + x_2 + x_3 = 0$$
$$x_2 - 2x_3 = 0$$

$\therefore$
$$x_2 = 2x_3.$$
$$x_1 = 3x_3$$

$$\frac{x_1}{3} = \frac{x_2}{2} = \frac{x_3}{1} = \beta.$$

$$X = \begin{bmatrix} 3\beta \\ 2\beta \\ \beta \end{bmatrix} = \beta \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

For $\lambda = 2$

$$[A - \lambda I]X = 0$$

$$\begin{bmatrix} -1 & 1 & -2 \\ -1 & 0 & 1 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$-x_1 + x_2 - 2x_3 = 0$$
$$-x_1 + x_3 = 0.$$
$$x_2 - 3x_3 = 0$$

$$x_1 = x_3$$
$$x_2 = 3x_3$$

$$\therefore \quad \frac{x_1}{1} = \frac{x_2}{3} = \frac{x_3}{1} = \gamma$$

$$X = \begin{bmatrix} \gamma \\ 3\gamma \\ \gamma \end{bmatrix} = \gamma \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$\therefore$ Eigen Vectors are

$$X = \alpha \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \beta \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \gamma \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

(c) There $P = [x_1, x_2 \; x_3] = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$

$|P| = 1(2-3) + 1(9-2) = 6 \neq 0.$

$\therefore$ A is diagonalisable. (Similarity Transformation)

$$P^{-1} = \begin{bmatrix} -0.16 & -0.33 & 1.16 \\ 0.5 & 0 & -0.5 \\ -0.33 & 0.33 & 0.33 \end{bmatrix}$$ (from calculator)

Then. $P^{-1}AP = \begin{bmatrix} -0.16 & -0.33 & 1.16 \\ 0.5 & 0 & -0.5 \\ -0.33 & 0.33 & 0.33 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$

$$\boxed{D_\lambda = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}$$

(ii) $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

(a)

Characteristic Eqn.

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (2-\lambda)((3-\lambda)(2-\lambda)-2) - 1(2(2-\lambda)-2) + 1(2-(3-\lambda)) = 0$

$\Rightarrow (2-\lambda)((3-\lambda)(2-\lambda)-2) - (4-2\lambda-2) + (-1+\lambda) = 0$

$\Rightarrow (2-\lambda)(6-5\lambda+\lambda^2-2) - 2(1-\lambda) + (\lambda-1) = 0$

$\Rightarrow (2-\lambda)(\lambda^2-5\lambda+4) + 3(\lambda-1) = 0$

$\Rightarrow (2-\lambda)(\lambda^2-4\lambda-\lambda+4) + 3(\lambda-1) = 0$

$\Rightarrow (2-\lambda)(\lambda(\lambda-4) - 1(\lambda-4)) + 3(\lambda-1) = 0$

$\Rightarrow (2-\lambda)(\lambda-1)(\lambda-4) + 3(\lambda-1) = 0$

$\Rightarrow (\lambda-1)((2-\lambda)(\lambda-4)+3) = 0$

$\Rightarrow (\lambda-1)(2\lambda-8-\lambda^2+4\lambda+3) = 0$

$\Rightarrow (\lambda-1)(-\lambda^2+6\lambda-5) = 0$

$\Rightarrow (\lambda-1)(\lambda^2-5\lambda-\lambda+5) = 0$

$\Rightarrow (\lambda-1)(\lambda(\lambda-5) - 1(\lambda-5)) = 0$

$\Rightarrow (\lambda-1)(\lambda-1)(\lambda-5) = 0$

$$\lambda = 1, 1, 5.$$

For $\lambda = 1$

$$[A - \lambda I]\, X = 0$$

For $\lambda = 1$

$$[A - \lambda I]x = 0$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow x_1 + 2x_2 + x_3 = 0$

$$\Rightarrow \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix}$$

Let $x_2 = \alpha$ & $x_3 = 0$

$$X_\bullet = \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

Let $x_2 = 0$ & $x_3 = \beta$

$$X = \beta \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

For $\lambda = 5$.

$$[A - \lambda I]\, x = 0$$

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$-3x_1 + 2x_2 + x_3 = 0$$
$$x_1 - 2x_2 + x_3 = 0$$
$$x_1 + 2x_2 - 3x_3 = 0$$

$$\frac{x_1}{4} = \frac{x_2}{-(-4)} = \frac{x_3}{4} = \gamma$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \gamma \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Now $P = [x_1 \; x_2 \; x_3] = \begin{bmatrix} -2 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$|P| = -1(-2-(-1)) = -1(-2+1) = 1$

$\because \; |P| \neq 0$

$\therefore$ A is diagonalisable

$P^{-1} = \begin{bmatrix} -1 & 2 & -1 \\ 1 & -3 & 2 \\ 0 & 1 & 0 \end{bmatrix}$ from calculator

Then $P^{-1}AP = \begin{bmatrix} -1 & 2 & -1 \\ 1 & -3 & 2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} -2 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$$\boxed{D_\lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}}$$