CS570 Spring 2025: Analysis of Algorithms Exam III

	Points		Points
Problem 1	20	Problem 4	20
Problem 2	9	Problem 5	20
Problem 3	16	Problem 6	15
			Total 100

First name	
Last Name	
Student ID	

Instructions:

1. This is a 2-hr exam. Closed book and notes. No electronic devices or internet access. 2. A single double sided 8.5in x 11in cheat sheet is allowed.
3. If a description to an algorithm or a proof is required, please limit your description or proof to within 150 words, preferably not exceeding the space allotted for that question. 4. No space other than the pages in the exam booklet will be scanned for grading. 5. If you require an additional page for a question, you can use the extra page provided within this booklet. However please indicate clearly that you are continuing the solution on the additional page.
6. Do not detach any sheets from the booklet.
7. If using a pencil to write the answers, make sure you apply enough pressure, so your answers are readable in the scanned copy of your exam.
8. Do not write your answers in cursive scripts.
9. This exam is printed double sided. Check and use the back of each page.

Student ID:

1) 20 pts (2 pts each)
Mark the following statements as **TRUE** or **FALSE** by circling the correct answer. No need to provide any justification.

[TRUE/FALSE]
In any graph, the size of a minimum Vertex Cover plus the size of a maximum Independent Set equals the number of vertices.

[TRUE/FALSE]
if A $\leq_p$ B and A∈NP, then B∈NP.

[TRUE/FALSE]
There exist problems X where X∈NP-hard and X∉NP

[TRUE/FALSE]
If a problem is NP-complete, it must have a polynomial-time certifier.

[TRUE/FALSE]
NP-Intermediate problems are those problems that neither have polynomial time solutions nor are they polynomial time reducible to NP-complete problems.

[TRUE/FALSE]
If we find a ½ - approximation to the independent set problem, we have proven that P=NP.

[TRUE/FALSE]
Assume P ≠ NP. Suppose problem X can be solved by an algorithm that first formulates an Integer Linear Program (ILP) in polynomial time and then solves this ILP formulation to find the solution to X. This means that X must be NP-hard.

[TRUE/FALSE]
Every LP has an optimal solution.

[TRUE/FALSE]
A ρ-approximation for the TSP with triangle inequalities, with $\rho < 1.5$ would imply that P = NP.

[TRUE/FALSE]
If X is NP-hard and X $\leq_p$ Y, then Y cannot belong to NP.

Student ID:

2) 9 pts - 3 pts each. Select all correct answers. No partial credit I. For

decision problems A and B, suppose A ≤□ B. Assuming P $\neq$ NP,

A) If B is in P, then A is in P.
B) If A is NP-hard, then B is NP-hard.
C) If B is NP-hard and belongs to NP, then A is NP-complete.
D) If B is NP-complete, then A is NP-hard.
Solution: A, B

II. In an undirected graph with n vertices, what is the maximum possible number of edges that two distinct Hamiltonian Cycles can share?

A) $\log_2 n$
B) $n/2$
C) n-2
D) n-1
Solution: C

III. Which of the following statements is true about Linear Programming

A) There exists a weakly polynomial time algorithm to solve Linear Programming.
B) If we can solve 3-SAT using LP in weakly polynomial time, we have proven that P=NP.
C) The Max flow problem with integer capacities can be solved using LP.
D) The Max flow problem with non-integer capacities can be solved using LP.
Solution: A,B,C,D

Student ID:

3) 16 pts

Suppose USC wants to send teams of students into a coding competition (again, yay!). Eligible students are a set of ***n*** freshmen $\{s_1 \ldots s_n\}$ and ***k*** sophomores $\{s_{n+1} \ldots s_{n+k}\}$ who need to be matched to form teams. Each team must have exactly one freshman and one sophomore and no student can be on multiple teams.

Each pair of freshman s_i ($i = 1, \ldots, n$), and sophomore s_j ($j = n+1, \ldots, n+k$) have a known conflict quotient of C_{ij}. To avoid negativity and chaos, we want to keep the combined conflict quotient of all the teams to at most C^*.

Formulate an Integer Linear Program (ILP) that will compute the maximum number of teams USC can send as per the constraints above.

a) Define the decision variables of the ILP (do NOT describe the known input parameters). (4 points)

Binary X_{ij} for whether 'freshman s_i and sophomore s_j are paired'

RUBRIC:

(1 point) Writing down the decision variable

(3 point) Explaining the decision variable as one that checks if a freshman and sophomore are paired

b) What is the objective of the ILP? (4 points)

max sum_{i,j} X_{ij}

RUBRIC:

(4 points) Correctly mention the maximization of the decision variable

(2 points) Correctly mention the maximization of an incorrect decision variable written in part a

c) What are the constraints of the ILP? (8 points)

1) sum_{i = 1 to n} X_{ij} <= 1 for all j = n+1, …, n+k
2) sum_{j = n+1 to n+k} X_{ij} <= 1 for all i = 1, …, n
3) sum_{i,j} X_{ij} C_{ij} <= C*
4) X_{ij} in {0,1} for all i,j

1,2 ensure each student in at most 1 team. 3 ensures combined conflict quotient at most C*.

RUBRIC:

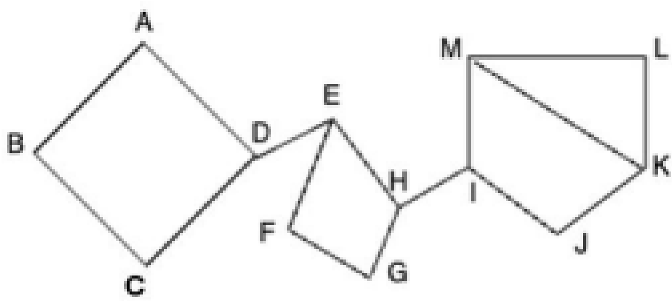
(2 points each) Correctly mention the contraints that limits each student to 1 team

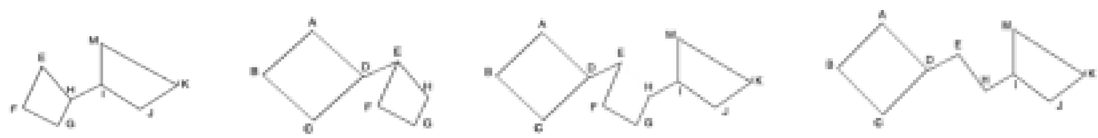
(3 points) Correctly mention the constraint that limits the combined conflict quotient to less than C*

(1 point) Correctly mention the constraint that limits the decision variable to be 0 or 1

4) 20 pts

The Dumbbell Containment Problem (DCP) asks: Given an undirected graph G = (V,E), and an integer k > 2, is there a *k-dumbbell* in G. A k-dumbbell consists of two disjoint simple cycles of size k connected by a simple path. A sample graph with its 4 distinct 4-dumbbells are shown below.





Show that DCP is

a) in NP. (5 points)

Certificate: 2 disjoint simple cycles of size k and a simple path connecting the two. Certifier:
- Check each cycle (as in Ham Cycle certifier)
- Check the cycles are disjoint
- Check the path has one end in one cycle and one end in the other -
Check path is simple

b) NP-hard. (15 points)

Solution: Reduction from Ham-Cycle. Given G with n nodes, create G' using two disjoint copies of G, (G1 and G2) and connect some node u from G1 to some node v from G2 with an edge. Set k = n.

Claim: G' has an n-dumbbell iff G has a ham-cycle.

Proof: …

Alternatively, you can create G' by connecting a copy of G to a simple cycle of length n by a simple path.

5) 20 pts

A Doctor and 99 Assistants have opened a laboratory startup to produce special medicine and bandages which require the collaboration of robots and humans. To produce an ounce of medicine, it takes 0.2 hours of human labor and 4 hours of robot labor. To produce an inch of bandage, it takes 0.5 hours of human labor and 2 hours of robot labor. An ounce of medicine sells for $30 and an inch of bandage sells for $20. Medicine and bandages can be sold in fractions of an ounce or inch respectively.

They want to maximize profit for the coming month so they can hire more staff. However, the Assistants are busy with other work so they can only devote 20 hours each to this project during the whole month. In addition, the doctor can only devote a total of 3800 robot hours for the month to this project since they are needed for other tasks. Formulate a linear program to help maximize the laboratory profit for the month, and to compute how much of each product to produce given the available human and robot hours. (No need to compute numerical answers to the LP)

A) Describe your LP variables (6 pts)

M: ounces of medicine produced

B: inches of bandage produced

B) Present your objective function (4 pts)

Maximize 30M + 20B

C) Constraints: (10 pts)

With only two variables M,B and just 3 constraints:

$0.2M+0.5B \leq 99*20$

$4M+2B \leq 3800$

$M,B \geq 0$

Rubric Principle: give full points to answers that have correct constraints. If the answer uses the first wrong solution(for mistaking AND for OR), but their solution is FULLY "correct" under that situation, students could have 6(for A, if correct) and 0(for B, if correct) and 10(for C, if it is correct) maximum.

Rubrics:

(A):

Full Points: Correctly and clearly defined variables that are used in the solution and related to the question.

-2: For each mistake or unclear description.

(B):

Full Points: Correct

-4: (for mistaking AND for OR but being correct under that situation)

-4: Incorrect, for example, $30M + 30B$ is incorrect.

(C):

Full Points: Fully Correct.

-4: For getting each of $0.2M+0.5B \leq 99*20$, $4M+2B \leq 3800$ wrong.(-8 if both incorrect)

-2: For getting $M,B \geq 0$ wrong.

6) 15 points

Recall the Maximum Independent Set (MIS) problem: For an undirected graph G = (V,E), S is an independent set of vertices if no two vertices in S share an edge in E. The MIS problem asks for the independent set with maximum size. We say S is a **Saturated** Independent Set (SIS) if no vertex v can be added to S so that S ∪ {v} is also an independent set.

For example, suppose graph G is simply a path ABCDE. Then, {B,D}, {B, E} and {A,C,E} are some examples of SIS, while {A,C}, {C,E}, and {B} are examples of independent sets that are not saturated.

a) (5 points) Describe an efficient algorithm that outputs an SIS for a given graph G. (Hint: *something very simple* should suffice)

Add an arbitrary node *v* to S and delete v and its neighbors. Repeat until the graph is empty.

Rubric:

+5pts Give an efficient algorithm for searching the graph and building Saturated IS.

+3pts Give an algorithm but miss the condition to add nodes and terminate.

Now, consider any such algorithm A that computes an arbitrary SIS (i.e., no additional property to its output other than the fact that it is an SIS). We want to analyze if this can serve as a good approximation to the MIS problem.

b) (2 points) Does there exist a constant $0 < \alpha < 1$, s.t., A is guaranteed to output an independent size of α *M(G) on every graph G, where M(G) denotes the size of a MIS in G. (Simply answer Yes or No)

No. (MIS doesn't have constant factor approximation)

+2pts State "No"

c) (8 points) If you answered Yes in part b), provide a value of α and prove the approximation guarantee. If you answered No, prove that no such α exists.

Consider a graph with one "central node" and n nodes connected to it. Choosing just the central node is a SIS of size 1. Choosing all the other nodes gives a MIS of size n. Ratio of 1/n, so can choose arbitrarily large n to disprove any α.

Rubric: (Precondition: state "No" in section b)

+8pts Proof with counter-example and ratio $lim_{n \to \infty} 1/n = 0$

+6pts Proof with counter-example but miss to state the ratio of approximation.

+2pts Proof with known facts or general results, e.g. P != NP without counter-example.