Total Sum Capacity Computation For Discrete Memoryless Multi-Access Channels

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Overview

Problem Statement

To understand, implement and numerically compare novel algorithms to compute inner and outer bounds for the MAC capacity region.

Solution Approach

1. Implementation of BA Algorithm for
   - Single-source systems (solved using the Lagrangian method).
   - Two-user input and output n-nary alphabet DMC-MAC (elementary case).
2. Generalisation to DMC-MAC.
3. Implementation of algorithms to compute inner and outer bounds for the MAC capacity region.
4. Implementation of a randomized algorithm for sum-capacity computation of MAC Channels.
5. Comparision of performance of existing vs new algorithm.

Applications

- This channel model has applications in the uplink of the cellular networks.
- Provide an estimation of max and min capacity of those real-world channels, which can modelled as DMC MACs.
- Theoretical advances in solving non-convex optimization problems having similar structure.
- Coming up with optimal encoding and decoding models.

- A **communication channel** is a medium to convey information.
- A **discrete** channel is a system consisting of
  1. an input alphabet X
  2. an output alphabet Y
  3. a probability transition matrix $p(y|x)$

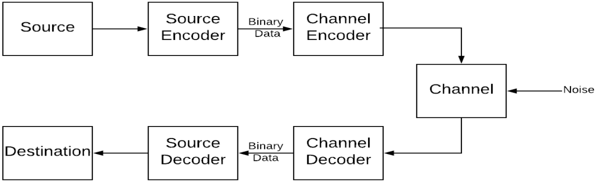


Figure: Shannon's Communication Model.

- The channel is said to be **memoryless** if:

$$P_{(Y^n|X^n)}(y^n|x^n) = \prod_{i=1}^{n}(P_{(Y|X)}(y[i]|x[i])) \tag{1}$$

- **Mutual information**: Information provided about the occurrence of an event by some other event.
- **Multi-terminal Channel**: A channel with multiple terminals.

Multi-terminal Channel

- **Multi-access Channel**: A type of multi-terminal channel, multiple senders transmit multiple messages over a shared medium to several destinations.

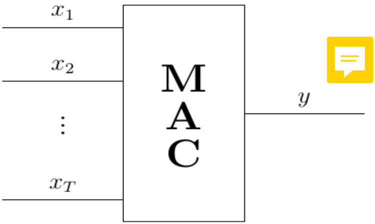


Figure: Schematic representation of a simple MAC Channel.

Shannon coding theorem

- **Noisy-channel coding theorem**: It establishes that for any given degree of noise in a channel, we can communicate discrete data nearly error-free up to a computable maximum rate.
- **Capacity**[1]: The highest rate in bits per channel use at which information can be sent, error-free.
- **Sum Capacity**: The sum capacity in a multi-user channel is the maximum aggregation of all the users data rates.

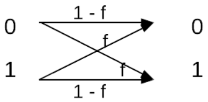


Figure: An elementary channel.

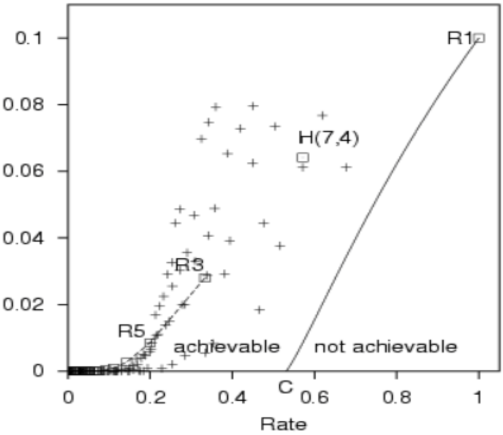


Figure: Shannon limit.

Optimisation Problem

- Capacity region determination for multi-terminal channels has attracted much attention in information theory.
  1. In many cases, single-letter representations are not known.
  2. In cases where such representations have been found, such as the discrete memoryless multiple-access channel (DMC-MAC), evaluation is difficult.
- Specifically, computation of the boundary of the capacity region for DMC-MACs is a non-convex constraint optimization problem.

Single User Case

- For single-user channel, capacity can be numerically approximated to arbitrary precision using this algorithm.
    1. Most common implementation: Expectation-Maximization(EM) paradigm
    2. Interior Point Methods
    3. Cutting plane Algorithms

Blahut-Arimoto Algorithm

For a DMC with transition matrix $p(y|x)$, input distribution $r(x)$ and reverse transition matrix $q(x|y)$, the capacity is given by:

$$C = \sup_{r>0} \max_{q} \sum_{x} \sum_{y} r(x)p(y|x) \log q(x|y)/r(x) \tag{2}$$

where maximisation is taken over all q such that
$q(x|y) = 0$ iff $p(y|x) = 0$
This(with recent results) gets converted to a convex optimization problem. Solved using the generalized Blahut-Arimoto Algorithm.

- E-step
  In the expectation step, we calculate $q(x|y)$ from the value of $r(x)$ computed in the last step; this is the inference we draw after observing the current input distribution and include in our prediction of what might have caused the current output.

$$q^k(x|y) = \frac{r^{(k-1)}(x)p(y|x)}{\sum_{x'} r^{(k-1)}(x')p(y|x')} \tag{3}$$

- M-step
  In the maximization step, $r(x)$ is calculated using the updated value of $q(x|y)$. This is the change we incorporate in our input distribution, after observing the behavior of the channel.

$$r^k(x) = \frac{\prod_y q^k(x|y)^{P(y|x)}}{\sum_{x'} \prod_y q^k(x'|y)^{p(y|x')}} \tag{4}$$

EM - continued...

After having adjusted the input distribution, C is calculated using (2).

- if increment is less than certain threshold we terminate the algorithm.
- else we continue with our E and M steps.

Generalized BA

- For a MAC, this convexity is missing
- Recent results [2] have shown that the Kuhn–Tucker condition is:
  - either sufficient for optimality in a MAC
  - or the channel can be decomposed into subchannels for which the Kuhn–Tucker condition is sufficient for optimality. In the this case, at least one subchannel has an optimal distribution that achieves the capacity of the original channel.
- The BA Algorithm is generalized for a MAC channel. We use ideas similar to that used for the single-user case. The optimization is taken over all input distributions (or users).

Total capacity (2) can be computed[3] as:

$$C_{total} = \max_{P_1(X_1)P_2(X_2)\ldots P_m(X_m)} I(X_{\mathcal{M}};Y) \tag{5}$$

where $X_{\mathcal{M}} = (X_1, X_2, ..., X_m)$

Generalized BA – continued...

Mutual Information is defined as:

$$I(x;Y) = D(P(Y|x)||P(Y)) \tag{6}$$

where $D(\cdot||\cdot)$ is the Kullback-Leibler distance

$$D(P(Y)||P'(Y)) = \sum P(Y) \log P(y)/P'(y). \tag{7}$$

For $S \subset \mathcal{M} \doteq 1, 2, ..., M$, let $x_S = x_i : i \in S$ and denote the marginalisation of $I(x_1, x_2, ..., x_M; Y)$ to S by

$$I_S(x_s;Y) = \sum_{x_{\bar{S}}} P_{\tilde{S}}(x_{\tilde{S}}) I(x_{\mathcal{M}};Y) \tag{8}$$

where $\tilde{S} \doteq \mathcal{M} \setminus S$

Generalized BA – continued...

The information function can be extended to a set of variables $X_i \in \mathcal{X}_\rangle$, namely

$$I(x_1, x_2, ..., x_M; Y) = D(P(Y|x1, x2, ..., x_M)||P(Y)). \tag{9}$$

The marginalized mutual information for each user becomes:

$$I_m(x_m; Y) = \sum_{x_{\tilde{m}}} P_{\tilde{m}}(x_{\tilde{m}}) I(x_{\mathcal{M}}; Y) \tag{10}$$

This marginalized mutual information is then used to adjust the input distributions of each user so that the capacity is maximized.

$$P_m^{r+1}(x_m) = P_m^r(x_m) \frac{f(I_m(x_m; Y))}{\sum_{x_m} P_m^r(x_m) f(I_m(x_m; Y))} \tag{11}$$

where f[4] is the exponential function.

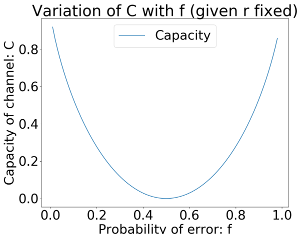
- E-step: In the expectation step, we compute the marginal mutual information for each user. This is done by using the current overall mutual information as input in the equation (10).
- M-step: In the maximization step, we compute the adjusted input distributions for each user using the results of the E-step as input to the the equation (11).
- After these two steps we compute the current capacity by using the mutual information equation (9).
    - if the increment is less than a certain threshold we terminate the algorithm
    - else we continue with our E and M steps.

Work Done till now

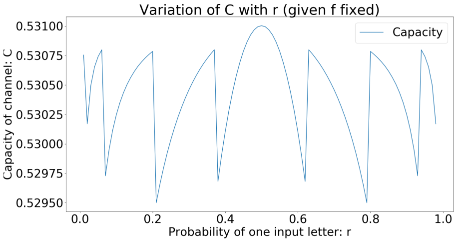
- Literature review.
- Analysis of implementation by Hitika Tiwari (SCEE, IIT Mandi)
- We have implemented BA Algorithm for single user n-ary alphabet.
- We are currently implementing generalised BA Algorithm with n inputs (binary alphabet).

Results

- These are the results for binary symmetric channel.



Figure



Figure

- We have also verified our implementation for Point to Point channel (n-ary alphabet).

Future Plan

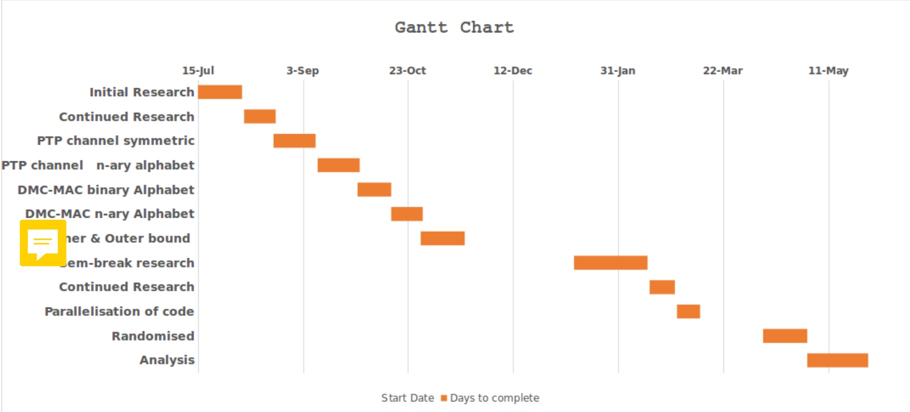


Figure: Gantt Chart

Conclusion

We have verified our code for Binary symmetric and Point to point channel. We are currently implementing DMC-MAC for binary alphabet and by the end–sem will be done with DMC-MAC for n-ary Alphabet. We will implement parallised version of the code. Next, randomised algorithm to calculate the capacity will be implemented.

References

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Thank You