Lagrange Multipliers

Why?

context : when you want to find the minimum of some function $f : M \to R$. and there is some constraint on the inputs M defined by $g : M \to R$.

M is from some arbitrary input space for the purpose of generality with assumptions of differentiability and smoothness (probably). For the purposes of this intro, let it be R^n.

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eg : **let** $f : R^2 \to R$ **s.t.** $f(x, y) = x^2 + y^2$**, and** $g : R^2 \to R$ **s.t.** $g(x, y) = 2x + y = k$**. Objective : Minimize** f

observations :

1. For f, A set of points in the domain that share an image define a circle. eg : $x^2 + y^2 = 5$
2. g describes a line with some fixed offset k
3. since we have to minimize f, it is like finding the circle with the smallest radius. The radius here happens to be the objective value. (look at Figure 1.)
4. The feasible set of inputs that satisfies the constraints will be the intersection points of circles that intersect the line.
5. the smallest circle, will have only 1 point intersecting with the line (constraint). the line will be tangent to the circle (objective function) at the intersection point. The radius of this circle will be the **optimal value** and the **solution** will be the coordinates of the intersection point.

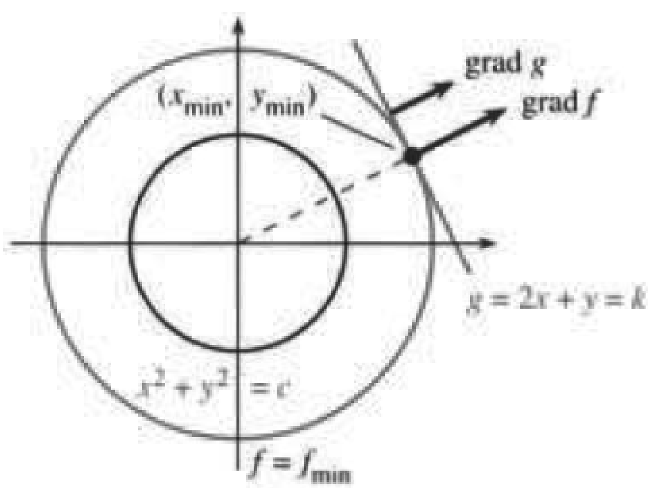

Figure 1: "calculus": Gibert Strang pg 592

Finding the optimal value, and the solutions :

when the optimal circle and the line intersect at point p_i, then from a Euclidean coordinate-based perspective: p_i will be where the vectors (f_x, f_y), (g_x, g_y) are parallel. the vectors are defined by the partial derivatives of the respective functions along each of the axes$(f_x, f_y), (g_x, g_y)$. Articulate the coordinate-free version of this.

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \; , \; \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$$

which gives us the following equations :

$$f_x = \lambda g_x \ , \ f_y = \lambda g_y \ , \ g = k$$

What function takes into account the constraint and let's us use known method like "setting derivatives to zero" ?

Let this ideal function be $L(x, y, \lambda)$. Since the above these equations : $f_x = \lambda g_x \ , \ f_y = \lambda g_y \ , \ g = k$ need to be satisfied for solutions, the derivatives of L can be engineered from these equations:

$$L_x = f_x - \lambda g_x = 0 \ , \ L_y = f_x - \lambda g_x = 0 \ , L_\lambda = g - k = 0$$

simply integrating the equations w.r.t x, y, z will give us the final function:

$$L(x, y, \lambda) = f(x, y) + \lambda(g(x, y) - k)$$

which is the Lagrange equation.

why not write constraint g(x,y) as $y = k - 2x$, and substitute that into f, giving f(x,k)