Forward reachability

1 Components of a control loop

Given a system where $\mathcal{P}$ is the dynamics, π is the policy, x_t is the state at time t, u_t is the action at time t. the policy π is also called the "controller", and the dynamical system $\mathcal{P}$ is called the "plant", in controls.

$$x_{t+1} = \mathcal{P}(x_t, u_t) \qquad \text{open loop dynamics}$$

$$u_t = \pi(x_t) \qquad \text{NN / controller}$$

example of $\mathcal{P}$ would be $x_{t+1} = Ax_t + B\pi(x_t)$. Let's compile $\pi, \mathcal{P}$ into a single f s.t. $f : x_t \to x_{t+1}$

$$x_{t+1} = f(x_t; \pi) \qquad \text{closed loop dynamics}$$

Why does $\mathcal{P}$ define an open loop system, while f define a closed loop system?

$\mathcal{P}$ assumes the input action u_t can be some arbitrary valid action. Whereas, f defines a specific action using $\pi(x_t)$ No additional information is required by f given an initial state to recurse a loop that traces out a trajectory/ i.e: $x_{t+2} = f(f(x_t; \pi, \mathcal{P}); \pi, \mathcal{P}) = f^2(x_t; \pi, \mathcal{P})$

2 Reachable Sets

given a a state Set $\mathcal{X}_t$, the reachable set $\mathcal{R}_t$ is as follows :

$$\mathcal{R}_t(\mathcal{X}_t) = \{f(x_t; \pi, \mathcal{P}) | \forall x \in \mathcal{X}_t\} \qquad \text{single step reachable set}$$

$$\mathcal{R}_t(\mathcal{X}_t) = \{f^t(x_t; \pi, \mathcal{P}) | \forall x_t \in \mathcal{X}_t\} \qquad \text{t step reachability set}$$

3 Safety

if we have avoid-region A:

$$\mathcal{R}_t(\mathcal{X}_t) \cap A = \varnothing \mid \forall t \in [0, T]$$

If your reachable set at time $t = 1$ ($\mathcal{R}_1(\mathcal{X}_0)$)is a subset of your input state set $\mathcal{X}_0$, then $\mathcal{R}_t(\mathcal{X}_t)$ can never leave the initial input state set $\mathcal{X}_0$ $\forall t \in [0, \infty]$ (provable by induction)

4 Goal-reachability

If the G is the goal region

$$\exists\, t_{goal} \in [0, T]$$
$$s.t.\ \mathcal{R}_{t_{goal}}(\mathcal{X}_{t_{goal}}) \subseteq G$$

We may also wanna say that further states don't leave the goal region,

$$\mathcal{R}_{t_{goal+i}} \subseteq G \mid i \in [1, \infty]$$

5 Formulation

$\mathcal{R}_t(\mathcal{X}_t)$ can be found by finding the output bounds, given a set of input bounds using any of the verification methods previously covered like IBP, crown, MILP, etc.

To recall, the general formulation is as follows :

$$\underset{x_t \in \mathcal{X}_t}{Max}\ c^T x_{t+\tau}$$
$$s.t. : x_{t+\tau} = f^\tau(x_t; \pi, \mathcal{P})$$