SKDAV GOVT. POLYTECHNIC ROURKELA

DEPARTMENT OF MATHEMATICS & SCIENCE

LECTURE NOTES

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Subject Code/Name: TH-1, ENGG. MATHEMATICS III

Prepared by

Smt. Subhashree Jena, Lect. (MATH)

COMPLEX NUMBER SYSTEM

Set of intezer $Z = \{0, \pm 1 \pm 2 \cdots\cdots\}$ +ve and -ve whole Number

Natural Number, $N = \{1, 2, 3 \cdots\cdots\}$ +ve Number

Set of rational, $Q = \{P/q, q \neq 0\}$ Fractions

The set of real Number - $R = Q \cup Q'$

$N \subset Z \subset Q \subset R$

* $x^2 + 4 = 0$

$x^2 = -4$

$x = \sqrt{-4} \notin R$

→ To solve this problem Complex Number System is introduced.

→ Here, let $\sqrt{-1} = i$

then, $x = \sqrt{-1 \times 4}$

$= \sqrt{-1} \times \sqrt{4}$

$= i \times 2$

$x = 2i \in \mathbb{C}$

The solution of $x^2 + 4 = 0$ is $2i$

→ A Complex Number is denoted by

$Z = a + ib$,

Where, $i = \sqrt{-1}$

$a, b =$ Real Number

$a, b \in R$

Here,

a = Real (z)

b = imaginary (z)

* $z = 5+3i$ This is a Complex Number

$z_1 = 3-2i$ This is a Complex Number

$z_2 = -1+2i \in \mathbb{C}$

$z_3 = 9 \in R$

$= 9+0 \times i \in \mathbb{C}$

* Every Real Number Can be a Complex Number

ie $R \in \mathbb{C}$

Addition :-

Let $z_1 = a+ib$ & $z_2 = p+iq$

$z_1+z_2 = (a+ib)+(p+iq)$

$= a+p+ib+iq$

$= (a+p)+i(b+q)$

Substraction :-

$z_1-z_2 = (a+ib)-(p-iq)$

$= a+ib - p - iq$

$= a-p+ib-iq$

$= (a-p)+i(b-q)$

multiplication:-

$$Z_1 \times Z_2 = (a+ib) \times (p+iq)$$

$$= a(p+iq) + ib(p+iq)$$

$$= ap + iaq + ibp + i^2bq$$

$$= ap + i(aq+bp) + (\sqrt{-1})^2 bq$$

$$= ap + i(aq+bp) - bq$$

$$Z_1 \times Z_2 = (ap-bq) + i(aq+bp)$$

* $i = \sqrt{-1}$

$i^2 = (\sqrt{-1})^2 = 1$

$i^3 = (\sqrt{-1})^3 = (\sqrt{-1})^2(\sqrt{-1}) = (-1)i = -i$

$i^4 = (\sqrt{-1})^4 = (\sqrt{-1})^2(\sqrt{-1})^2 = (-1)(-1) = +1$

$i^5 = (\sqrt{-1})^5 = (\sqrt{-1})^2(\sqrt{-1})^2(\sqrt{-1}) = (-1)(-1)i = i$

Q i^{101}

$= i^{4\times 25+1}$

$= (i^4)^{25} \times i^1$

$= 1 \times i$

$= i$

Q i^{88}

$= i^{4\times 22}$

$= (i^4)^{22}$

$= (i)^{22}$

$= 1$

Q i^{87}

$= i^{4\times 21+3}$

$= (i^4)^{21} \times i^3$

$= 1 \times (-i) = -i$

Q write $\frac{1+i}{2-i}$ is $a+ib$ form.

Division

Let $z_1 = a+ib$ and $z_2 = c+id$

$$\frac{z_1}{z_2} = \frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)}$$

$$= \frac{a(c-id) + ib(c-id)}{c^2 - (id)^2}$$

$$= \frac{ac - aid + ibc - i^2 bd}{c^2 - i^2 d^2}$$

$$= \frac{ac - iad + ibc - (-1) bd}{c^2 - (-1) d^2}$$

$$= \frac{(ac+bd) + i(bc-ad)}{c^2 + d^2}$$

$$= \left(\frac{ac+bd}{c^2+d^2}\right) + i\left(\frac{bc-ad}{c^2+d^2}\right)$$

S $\frac{1+i}{2-i} = \frac{(1+i)(2+i)}{(2-i)(2+i)}$

$$= \frac{1(2+i) + i(2+i)}{2^2 - i^2}$$

$$= \frac{2 + i + 2i + i^2}{4 - (-1)}$$

$$= \frac{2 - 1 + 3i}{4+1}$$

$$= \frac{1}{5} + \frac{3i}{5}$$

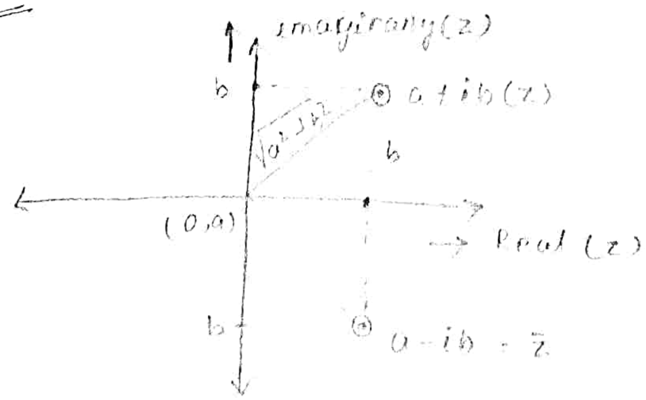
Q write $\frac{3}{5i}$ in $a+ib$

$$\frac{3}{5i} = \frac{3 \times i}{5i \times i} = \frac{3i}{5(-1)}$$

$$= \frac{-3i}{5} = 0 - \frac{3}{5}i$$

$\Rightarrow a = 0, \; b = -3/5$

Geometrical Representation:-



$|z|$ = modulus of z

= $\sqrt{a^2+b^2}$ = distance from origine

$\bar{z}$ = Conjugate of z

= a - ib = image of z through real (z)

$\rightarrow z = a+ib$, $\bar{z} = a - ib$, $|z| = \sqrt{a^2+b^2}$

* $|\bar{z}| = \sqrt{a^2+(-b)^2}$

$= \sqrt{a^2+b^2}$

$\Rightarrow |z| = |\bar{z}|$

* $z_1 = a+ib$, $z_2 = c+id$

$\bar{z}_1 = a - ib$, $\bar{z}_2 = c - id$

① $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$

② $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

③ $\overline{z_1 z_2} = \bar{z}_1 \times \bar{z}_2$

④ $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

Q write $\bar{z}$ if

① $z_1 = 2 - 3i \Rightarrow \overline{z_1} = 2 + 3i$

② $z_2 = -2 + 5i \Rightarrow \overline{z_2} = -2 - 5i$

③ $z_3 = -1 - 2i \Rightarrow \overline{z_3} = -1 + 2i$

Cube roots of unity :-

$\sqrt[3]{x}$ $\quad \sqrt[3]{1} = 1$

we are to find $\sqrt[3]{1} = ?$

Let $x = \sqrt[3]{1}$

$x^3 = 1$

$\Rightarrow x^3 - 1 = 0$

$\Rightarrow x^3 - 1^2 = 0$

But. $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$\Rightarrow (x-1)(x^2 + x + 1) = 0$

$\Rightarrow x - 1 = 0$ or $x^2 + x + 1 = 0$

$\Rightarrow x = 1$ $\qquad \Rightarrow x = \dfrac{-1 \pm \sqrt{1^2 - 4x \times 1}}{2 \times 1}$

$\Rightarrow x = \dfrac{-1 \pm \sqrt{-3}}{2}$

$\Rightarrow x = 1, \dfrac{-1 \pm \sqrt{3}i}{2}$

$\Rightarrow 1, \dfrac{-1 + \sqrt{3}i}{2}, \dfrac{-1 - \sqrt{3}i}{2}$

$*\ x^3 - 1^3$

$= (x-1)(x^2 + x \times 1 + 1^2)$

$= (x-1)(x^2 + x + 1)$

$*\ ax^2 + bx + c = 0$

$\Rightarrow x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Properties :-

① Let $\omega = \frac{-1+\sqrt{3}i}{2}$

$\Rightarrow \omega^2 = \left(\frac{-1+\sqrt{3}i}{2}\right)^2$

$= \frac{1}{4}(-1+\sqrt{3}i)^2$

$= \frac{1}{4}\left[(-1)^2 + (\sqrt{3}i)^2 - 2\times(-1)\times\sqrt{3}i\right]$

$= \frac{1}{4}\left[1+3i^2 - 2\sqrt{3}i\right] = \frac{1}{4}(1-3-2\sqrt{3}i)$

$= \frac{1}{4}(-2-2\sqrt{3}i) = \frac{-1-\sqrt{3}i}{2} = (\omega^2)$

1- The Complex roots are square of each other.

ie $(\omega^2)^2 = \left(\frac{-1-\sqrt{3}i}{2}\right)^2 = \left(\frac{-1}{2}\right)^2\left(1+\sqrt{3}i\right)^2$

$= \frac{1}{4}(1+(\sqrt{3}i)^2 + \sqrt{3}i)$

$= \frac{1}{4}(1+3i^2+2\sqrt{3}i)$

$= \frac{1}{4}(1-3+2\sqrt{3}i)$

$= \frac{1}{4}(-2+2\sqrt{3}i)$

$\Rightarrow \omega^4 = \frac{-1+\sqrt{3}i}{2} = \omega$

2 - $\omega^4 = \omega$

$\Rightarrow \omega^3 \times \cancel{\omega} = \cancel{\omega}$

$\Rightarrow \omega^3 = 1$

ie $\boxed{1\times\omega\times\omega^2 = 1}$

3 - $1+\omega+\omega^2=0$

ie $1+\omega+\omega^2$

$= 1+\frac{-1+\sqrt{3}i}{2} + \frac{-1-\sqrt{3}i}{2}$

$\Rightarrow 1+\frac{-1+\cancel{\sqrt{3}i}-1-\cancel{\sqrt{3}i}}{2}$

$\Rightarrow 1+ -2/2$

$\Rightarrow 1-1=0$

The sum of the cube root is zero.

Q. Evaluate $(1+\omega-\omega^2)^5+(1-\omega+\omega^2)^5$

$\rightarrow (1+\omega-\omega^2)^5+(1-\omega+\omega^2)^5$

$= (-\omega^2-\omega^2)^5+(1+\omega^2-\omega)^5$

$= (-2\omega^2)^5+(-\omega-\omega)^5$

$1+\omega+\omega^2=0$

$= (-32\omega^{10})+(-2\omega)^5$

$\Rightarrow 1+\omega=-\omega^2$

$= (-32\omega^{10})+(-32\omega^5)$

& $1+\omega^2=-\omega$

$= (-32)(\omega^{10}+\omega^5)$

Also $= \omega+\omega^2=-1$

$= (-32)\times(\omega^{9+1}+\omega^{3+2})$

$= (-32)\times(\omega+\omega^2)$

$\omega^3=1$

$\omega^{15}=(\omega^3)^5$

$= (-32)\times(-1)$

$=1^5=1$

$= 32$

Q prove that

$$(1+\omega^2-\omega)^3 + (\omega-\omega^2+1)^3 = -16$$

LHS

$\longrightarrow (1+\omega^2-\omega)^3 + (\omega-\omega^2+1)^3 = -16$

$= (1+\omega^2-\omega)^3 + (1+\omega-\omega^2)^3$

$= (-\omega-\omega)^3 + (-\omega^2-\omega^2)^3$

$= (-2\omega)^3 + (-2\omega^2)^3$

$= (-8\omega^3) + (-8\omega^6)$

$= -8(\omega^3+\omega^6)$

$= -8(1+1)$

$= -16$

Square Root of a Complex Number :-

Q evaluate square root of $-5+12i$

$\longrightarrow$ Let $\sqrt{-5+12i} = a+ib$

squaring both sides

$$(\sqrt{-5+12i})^2 = (a+ib)^2$$

$$\Rightarrow -5+12i = a^2+(ib)^2+2aib$$

$$= a^2+i^2b^2+i2ab$$

$$= (a^2-b^2)+i2ab$$

$$\left.\begin{aligned} &\Rightarrow a^2-b^2 = -5 \\ &\Rightarrow 2ab = 12 \end{aligned}\right\} \longrightarrow (1)$$

We know that $(a+b)^2 = (a-b)^2 + 4ab$

Let $a = x^2$, $b = y^2$

$$\boxed{(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2}$$

We know,

$$(a^2+b^2)^2 = (a^2-b^2)^2 + 4a^2b^2$$
$$= (-5)^2 + (2ab)^2$$
$$= (-5)^2 + (12)^2$$
$$= 25 + 144$$
$$= 169$$
$$a^2+b^2 = \sqrt{169} = 13 \quad \text{—— (ii)}$$

Solve eqn (i) and (ii)

$$a^2 - \cancel{b^2} = -5$$
$$a^2 + \cancel{b^2} = 13$$
$$2a^2 = 8$$

$\Rightarrow a^2 = \frac{8}{2} = 4$

$\Rightarrow a = \sqrt{4} = 2$

$\Rightarrow a = \pm 2$

put a^2 is eqn (ii)

$a^2 + b^2 = 13$

$\Rightarrow 4 + b^2 = 13$

$\Rightarrow b^2 = 13 - 4 = 9$

$\Rightarrow b = \sqrt{9} = 3$

$\Rightarrow b = \pm 3$

$\therefore \sqrt{-5+12i} = a+ib$

$= \pm 2 + i \times 3$

$= \pm (2+3i)$ Ans

Q Evaluate the square root of $3-4\sqrt{-1}$

$\rightarrow$ Let $\sqrt{3-4\sqrt{-1}} = \sqrt{3-4i} = x+iy$

squaring.

$(\sqrt{3-4i})^2 = (x+iy)^2$

$\Rightarrow 3-4i = (x+y)^2$

$= (x^2+i^2y^2+2ixy)$

$= (x^2-y^2+i \cdot 2xy)$

$\left.\begin{array}{l}\Rightarrow x^2-y^2 = 3 \\ \Rightarrow 2xy = -4\end{array}\right\}$ —— (I)

put the value

$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$

$= (x^2-y^2)^2 + (2xy)^2$

$= 3^2 + (-4)^2$

$= 9+16 = 25$

$x^2+y^2 = \sqrt{25} = 5$ —— (II)

Solve eqn (i) and (ii)

$$x^2 + y^2 = 5$$

$$x^2 - y^2 = 3$$

$$2x^2 = 8$$

$$\Rightarrow x^2 = \frac{8}{2} = 4$$

$$\Rightarrow x = \sqrt{4} = 2$$

$$\Rightarrow x = \pm 2$$

put x^2 is eqn (ii)

$$x^2 + y^2 = 5$$

$$\Rightarrow 4 + y^2 = 5$$

$$\Rightarrow y^2 = 5 - 4 = 1$$

$$\Rightarrow y = \pm 1$$

$$\therefore \sqrt{3-4i} = x + iy$$

$$= \pm (2 + i)$$ Ans

DE-MOIVER'S THEOREM

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

polar form of $z = a + ib$

Let $a = r\cos\theta$ &

$b = r\sin\theta$

then,

$r = ?$

$\theta = ?$

$a^2 + b^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$= r^2(\cos^2\theta + \sin^2\theta)$

$\Rightarrow r^2 = a^2 + b^2$

$\Rightarrow r = \sqrt{a^2+b^2}$

Again $b/a = \dfrac{r\sin\theta}{r\cos\theta} = \tan\theta$

$\Rightarrow \theta = \tan^{-1}(b/a)$

Now, $z = a + ib$

$= r\cos\theta + ir\sin\theta$

$\Rightarrow \boxed{z = r\cos\theta + i\sin\theta}$

Application theorem:-

We can find out / evaluate z^n

Q Evaluate $(1+i)^5$

→ We are to convert z in to polar form

Here $a = 1 = r\cos\theta$

$b = 1 = r\sin\theta$

$\Rightarrow r = \sqrt{a^2+b^2}$

$\Rightarrow r = \sqrt{1+1} = \sqrt{2}$

$\theta = \tan^{-1}(b/a)$

$= \tan^{-1}(1/1) = \pi/4$

$\therefore z = r(\cos\theta + i\sin\theta)$

$\Rightarrow 1+i = \sqrt{2}(\cos\pi/4 + i\sin\pi/4)$

$\Rightarrow (1+i)^5 = [\sqrt{2}(\cos\pi/4 + i\sin\pi/4)]^5$

$= 2^{5/2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

$= 4\sqrt{2}[-\cos\pi/4 - i\sin\pi/4]$

$\Rightarrow \sqrt{32}[-\cos\pi/4 - i\sin\pi/4]$

$\Rightarrow (1+i)^5 = 4\sqrt{2}(-1/\sqrt{2} - 1/\sqrt{2}\, i)$

$= -4(1+i)$ Ans

$$\boxed{\frac{5\pi}{4} = 2(\pi/2) + \pi/4}$$

MATRICES

Different types of matrices -

* matrices is tabular arrangement of data.
* is in the form of rows & columns
* It is given by (a_{ij}) $i \leq r$, $j \leq c$

→ $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $R = 2$, $C = 3$

It is a matrix of order 2×3 $(2/3)$

Ex Write the matrix (a_{ij}) such that

$i, j \leq 2$ and $a_{ij} = i \times j$

→ Since $i, j \leq 2$, matrix order 2×2

ie $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_{2 \times 2}$ & $a_{ij} = i \times j$

$\therefore a_{11} = 1 \times 1 = 1$

$a_{12} = 1 \times 2 = 2$

$a_{21} = 2 \times 1 = 2$

$a_{22} = 2 \times 2 = 4$

ie $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

Types of matrices :-

① Zero / Null matrix : all $a_{ij} = 0$

ie All the elements of the matrix are zero

② Identity matrix.

A square matrix whose main diagonal elements are 1 and rest are zero. It is denoted by I.

$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(3) Square matrix = (no. of rows = no. of columns)

same rows and columns Number that is square matrix.

(4) Singular matrix :-

A square matrix with determinant value as zero (0) is called singular matrix.

Ex $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}_{2\times 2}$

$\det(A) = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 1\times 4 - 2\times 2$

$= 4 - 4 = 0$

A is a singular matrix.

(5) Non-Singular matrix :-

A square matrix whos determinat value is non-zero is called non-singular matrix.

Ex $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}_{2\times 2}$

$|A| = \det(A) = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 1\times 3 - 2\times 2$

$= 3 - 4 = -1 \neq 0$

$\therefore$ A is a Non-singular matrix.

(6) uppertriangular matrix :-

uppertriangular matrix is a square matrix, where $a_{ij} \neq 0$ if $i \leq j$

Ex $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{pmatrix}$

A square matrix whose element below the main diagonal are zero.

ie $a_{ij} \neq 0$ if $i \leq j$

7) Lower triangular matrix :-

Lower triangular matrix is a square matrix when elements above the diagonals are zero.

Ex $\begin{pmatrix} 1 & 0 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & 9 \end{pmatrix}$

Rank of the Matrix :-

Rank of the matrix is difined as the number of independent rows / columns present is the matrix (in the ecolone form of the matrix has non-zero rows/columns) in cas of square matrices of order $n \times n$ rank $\leq n$.

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* In case of square matrix of order $n \times n$ rank $\leq n$
* In case of rectangular matrix of order $m \times n$ rank $\leq$ minimum $\{n, n\}$

ex Rank of $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ $R_2 \leftarrow 2R_1 - R_2$

$\begin{pmatrix} 1 & 2 \\ 2\times1-2 & 2\times2-4 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$

$\therefore$ Rank no. of nonzero rows $= 1$

Ex

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1\times4-4 & 4\times2-5 & 4\times3-6 \\ 1\times7-7 & 7\times2-8 & 7\times3-9 \end{pmatrix} \begin{matrix} \leftarrow R_2 \to 4R_1 - R_2 \\ \leftarrow R_3 \to 7R_1 - R_3 \end{matrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 0 & 6 & 12 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 2\times0-0 & 2\times3-6 & 2\times6-12 \end{pmatrix} \leftarrow R_3 \to 2\times R_2 - R_3$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ Rank $= 2$

Ex

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 3 & 8 & 2 \end{pmatrix}_{4\times3}$$

rank $\leq$ mini $(4,3)$

$\Rightarrow$ rank ≤ 3

$$= \begin{pmatrix} 1 & 2 & 3 \\ 1\times4-4 & 2\times4-5 & 3\times4-6 \\ 1\times7-7 & 2\times7-8 & 3\times7-9 \\ 1\times3-3 & 2\times3-8 & 3\times2-2 \end{pmatrix} \begin{matrix} R_2 \to 4R_1 - R_2 \\ R_3 \to 7R_1 - R_3 \\ R_4 \to 3R_1 - R_4 \end{matrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 0 & 6 & 12 \\ 0 & -2 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 0 & 2\times3-6 & 2\times6-12 \\ 0 & 2\times3+(-2)\times3 & 2\times6+3\times7 \end{pmatrix} \quad \begin{array}{l} R_3 \to 2R_3 - R_3 \\ R_4 \to 2R_2 + 3R_4 \end{array}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 36 \end{pmatrix}$$

$\Rightarrow$ Rank = no. of non zero rows = 3

Minor Method :-

The highest order nonvanishing / nonzero minor of the matrix.

Ex $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \left.\begin{matrix} a \neq 0 \\ b \neq 0 \\ c \neq 0 \\ d \neq 0 \end{matrix}\right\} \rho(A) \geq 1$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

If ad-bc is zero and the rank of the matrix will be 1

If $ad - bc = 0$, $\rho(A) = 1$

If $ad - bc \neq 0$, $\rho(A) = 2$

Ex $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ by using minor method find $\rho(A)$.

$\rightarrow$ $|A|$ $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = +1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$

$= 1(45-48) - 2(36-42) + 3(32-35)$

$= -3 - 2(-6) + 3(-3)$

$= -3 + 12 - 9$

$= 12 + 12$

$= 0$

$\det(A_{3\times 3}) = 0$

$\Rightarrow \rho(A) < 3$ —— ①

$$\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 5 - 8 = -3 \neq 0$$

$\therefore \rho(A) \geq 2$ —— (ii)

$\therefore$ from (i) & (ii)

$2 \leq \rho(A) < 3$

$\Rightarrow \rho(A) = 2$

Q. Find the rank of the matrix.

$$\begin{pmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{pmatrix}$$

$\rightarrow \begin{pmatrix} 0 & 1 & -3 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix}$ by using the minor method.

$$|A_1| = \begin{pmatrix} 0 & 1 & -3 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix} = +0 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} - 3 \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix}$$

$$= 0 - 1(0-3) - 3(1-0)$$

$$= 3 - 3$$

$$= 0$$

$\det(A_{3\times3}) = 0$

$\Rightarrow \rho(A) < 3$ —— (i)

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1 \neq 0$$

$\rho(A) \geq 2$ —— (ii)

$\therefore$ from (i) & (ii)

$\rho(A) = 2$

$$
* \quad \begin{array}{cccc} + & - & + & - \end{array}
$$

$$
\begin{pmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{pmatrix}
$$

rank of

$\therefore \rho(A) < 4$

$$
|A| = -1 \begin{vmatrix} 1 & 1 & 1 \\ 3 & 0 & 2 \\ 1 & -2 & 0 \end{vmatrix} + (-3) \begin{vmatrix} 1 & 0 & 1 \\ 3 & 1 & 2 \\ 1 & 1 & 0 \end{vmatrix} - (-1) \begin{vmatrix} 1 & 0 & 1 \\ 3 & 1 & 0 \\ 1 & 1 & -2 \end{vmatrix}
$$

$$
= -\left[1 \begin{vmatrix} 0 & 2 \\ -2 & 0 \end{vmatrix} - 1 \begin{vmatrix} 3 & 2 \\ 1 & 0 \end{vmatrix} + 1 \begin{vmatrix} 3 & 0 \\ 1 & -2 \end{vmatrix}\right]
$$

$$
= -3\left[1 \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} - 0 \begin{vmatrix} 3 & 2 \\ 1 & 0 \end{vmatrix} + 1 \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}\right]
$$

$$
= +1\left[1 \begin{vmatrix} 1 & 0 \\ 1 & -2 \end{vmatrix} - 0 \begin{vmatrix} 3 & 0 \\ 1 & -2 \end{vmatrix} + 1 \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}\right]
$$

$$
= -\left[(0+4) - (0-2) + 1(-6-0)\right]
$$

$$
-3\left[(0-2) + (3-1)\right]
$$

$$
+1\left[(-2-0) + 1(3-1)\right]
$$

$$
= (4+2-6) - 3(2-2) + 1(-2+2)
$$

$$
= 6-6-0-0
$$

$$
= 0
$$

$$
A_1 = \begin{pmatrix} 0 & 1 & -3 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix}
$$

$$
|A_1| = 0 - 1 \begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} + (-3) \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix}
$$

$$
= -1(0-3) - 3(1-0)
$$

$$
= 3-3
$$

$$
= 0
$$

$$A_2 = \begin{pmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$$

$$|A_2| = 1\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + 3\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= 1(2-0) + 3(0-1) - 1(0-1)$$

$$= 2 - 3 + 1$$

$$= 3 - 3$$

$$= 0$$

$$|A_3| = 1\begin{vmatrix} 1 & 0 \\ 1 & -2 \end{vmatrix} + 0 - 1\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= 1(-2-0) - 1(3-1)$$

$$= -2 - 2$$

$$= 0$$

$A_3, A_4 \cdots A_{16} = 0$

Since all the 3×3 minors are zero

$\rho(A) \leq 3$ —— (ii)

$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1 \neq 0$

$\therefore \rho(A) \geq 2$ —— (iii)

$\Rightarrow \rho(A) = 2$

* Ecolon form :-

$$\begin{pmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -2 & 0 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 0 & 1 & -3 & -1 \end{pmatrix} \quad R_1 \leftrightarrow R_4$$

$$= \begin{pmatrix} 1 & 1 & -2 & 0 \\ 1-1 & 1-0 & -2-1 & 0-1 \\ 3-3 & 3-1 & -6-0 & 0-2 \\ 0 & 1 & -3 & -1 \end{pmatrix} \quad \begin{matrix} R_2 \to R_1 - R_2 \\ R_3 \to 3R_1 - R_3 \end{matrix}$$

$$= \begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 2 & -6 & -2 \\ 0 & 1 & -3 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 2-2 & -6-(-6) & -2-(-2) \\ 0 & 1-1 & -3-(-3) & -1-(-1) \end{pmatrix} \quad \begin{matrix} R_3 \to 2R_2 - R_3 \\ R_4 \to R_2 - R_4 \end{matrix}$$

$$= \begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore \rho(A) = 2$

Consistency of a System of linear Equation :-

A system of equition / setup equation system is said to be Consistent if its solution exists. otherwise it is called inconsistent.

→ Let the System be

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}.$$

'A' equal to the coefficient matrix and it is the coefficient matrix 'k' is is the augumented matrix.

$$k = \text{augumented matrix} = \left(\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right)$$

→ To check the Consistency of the System we used the Rouche's Theorem.

<u>Case-1</u>:- If Rank of matrix 'A' is not equal to Rank of matrix 'k' then system is not consistent and it has no Solution

$$\boxed{\text{Rank } (A) \neq \text{Rank } (K)}$$

<u>Case-1</u> :- If Rank of matrix 'A' equal to Rank of matrix 'k' and its the same number of variable then system is Consistent and it has unique solution.

$$\boxed{\text{Rank } (A) = \text{rank } (K)}$$

case-3 :- If Rank of matrix 'A' equal to Rank of matrix 'K' but it is less then the number of variable, then system is consistent but it has infinite solution.

$$\boxed{\text{Rank}(A) = \text{Rank}(K) <}$$

Ex. check the Consistency of the System

$2x - 3y + z = 1$

$x + 2y - 3z = 4$

$4x - y - 2z = 5$

$$\rightarrow K = \left(\begin{array}{ccc|c} 2 & -3 & 1 & 1 \\ 1 & 2 & -3 & 4 \\ 4 & -1 & -2 & 5 \end{array}\right) \quad (R_1 \text{ is fixed})$$

$$= \left(\begin{array}{ccc|c} 2 & -3 & 1 & 1 \\ 2-2 & -3-4 & 1-(-6) & 1-8 \\ 4-4 & -6-(-1) & 2-(-2) & 2-5 \end{array}\right) \quad \begin{array}{l} R_2 \rightarrow R_1 - 2R_2 \\ R_3 \rightarrow 2R_1 - R_3 \end{array}$$

$$= \left(\begin{array}{ccc|c} 2 & -3 & 1 & 1 \\ 0 & -7 & 7 & -7 \\ 0 & -5 & 4 & -6 \end{array}\right) \quad (R_2 \text{ is fixed})$$

$$= \left(\begin{array}{ccc|c} 2 & -3 & 1 & 1 \\ 0 & -7 & 7 & -7 \\ 0 & -35-(-35) & 35-28 & -35(-(-42)) \end{array}\right) \quad R_3 \rightarrow 5R_2 - 7R_3$$

$$= \left(\begin{array}{ccc|c} 2 & -3 & 1 & 1 \\ 0 & -7 & 7 & -7 \\ 0 & 0 & 7 & 7 \end{array}\right)$$

$\rho(K) = 3, \ \rho(A) = 3$

$\Rightarrow \rho(A) = \rho(K) = 3 \Rightarrow$ number of variable System is Consistent and it has unique solution.

$$\left.\begin{aligned} 7z &= 7 \\ -7y - 7z &= -7 \\ 2x - 3y - z &= 1 \end{aligned}\right\}$$

solving $z = 1$

$-7y + 7z = -7$

$\Rightarrow -7y + 7 = -7$
$\qquad\qquad\quad -14$

$\Rightarrow -7y = -14$

$\Rightarrow y = -14/7$

$y = 2$

$\Rightarrow 2x - 3y + z = 1$

$\Rightarrow 2x - 6 + \cancel{1} = \cancel{1}$

$\Rightarrow 2x = 6 \Rightarrow x = 2/6 = 3$

$\therefore$ Solution is $x = 3,\ y = 2,\ z = 1$

Ex. Check the consistency of equition.

$2x - y + 3z = 3$

$x + 2y - z - 5w = 4$

$x + 3y - 2z - 7w = 5$

$\longrightarrow$

$$K = \left(\begin{array}{cccc|c} 2 & -1 & 3 & 0 & 3 \\ 1 & 2 & -1 & -5 & 4 \\ 1 & 3 & -2 & -7 & 5 \end{array}\right) \text{(R_1 is fixed)}$$

$$= \left(\begin{array}{cccc|c} 2 & -1 & 3 & 0 & 3 \\ 2-2 & -1-4 & 3+2 & 0+10 & 3-8 \\ 2-2 & -1-6 & 3+4 & 0+14 & 3-10 \end{array}\right) \begin{array}{l} R_2 \to R_1 - 2R_2 \\ R_3 \to R_1 - 2R_2 \end{array}$$

$$= \left(\begin{array}{cccc|c} 2 & -1 & 3 & 0 & 3 \\ 0 & -5 & 5 & 10 & -5 \\ 0 & -7 & 7 & 14 & -7 \end{array}\right) \text{R_2 is fixed}$$

$$= \left(\begin{array}{cccc|c} 2 & -1 & 3 & 0 & 3 \\ 0 & -5 & 5 & 10 & -5 \\ 0 & -35+35 & 35-35 & 70-70 & -35+35 \end{array}\right) \quad R_3 \to 7R_2 - 5R_3$$

$$= \left(\begin{array}{cccc|c} 2 & -1 & 3 & 0 & 3 \\ 0 & -5 & 5 & 10 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow$ Rank of 'k' $\rho(k) = 2$

Rank of 'A' $\rho(A) = 2$

$\Rightarrow$ But Rank is less then number of variables

$\Rightarrow$ The system has infinite number of solutions

Let $z = k_1$, $w = k_2$

$2x - y + 3z = 3$

$-5y + 5z + 10w = -5$

$-5y = -5 - 5z - 10w$

$\Rightarrow y = 1 + 2 + 2w$

$\boxed{y = 1 + k_1 + 2k_2}$

$2x - y - 3z = 3$

$\Rightarrow 2x = 3 + y - 3z$

$= 3 + 1 + k_1 + 2k_2 - 3k_1$

$\Rightarrow 2x = 4 - 2k_1 + 2k_2$

$\Rightarrow \boxed{x = 2 - k_1 + k_2}$

Q: Check the consistency of the system.

$$\left.\begin{array}{l} 2y + 4z + 5 = 0 \\ 8x - y + 4z = 12 \\ 16x - y + 10z = 1 \end{array}\right\} \quad \begin{array}{l} 2y + 4z = -5 \\ 8x - y + 4z = 12 \\ 16x - y + 10z = 1 \end{array}$$

$$k = \left(\begin{array}{ccc|c} 0 & 2 & 4 & -5 \\ 8 & -1 & 4 & 12 \\ 16 & -1 & 10 & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 16 & -1 & 10 & 1 \\ 8 & -1 & 4 & 12 \\ 0 & 2 & 4 & -5 \end{array}\right) \quad \begin{array}{l} R_1 \leftrightarrow R_3 \\ R_1 \text{ is fixed} \end{array}$$

$$= \left(\begin{array}{ccc|c} 16 & -1 & 10 & 1 \\ 16-16 & -1+2 & 10-8 & 1-24 \\ 0 & 2 & 4 & -5 \end{array}\right) \quad R_1 \to R_1 - 2R_2$$

$$= \left(\begin{array}{ccc|c} 16 & -1 & 10 & 1 \\ 0 & 1 & 2 & -23 \\ 0 & 2 & 4 & -5 \end{array}\right) \quad R_2 \text{ is fixed}$$

$$= \left(\begin{array}{ccc|c} 16 & -1 & 10 & 1 \\ 0 & 1 & 2 & -23 \\ 0 & 2-2 & 4-4 & -46+5 \end{array}\right) \quad R_3 \to 2R_2 - R_3$$

$$= \left(\begin{array}{ccc|c} 16 & -1 & 10 & 1 \\ 0 & 1 & 2 & -23 \\ 0 & 0 & 0 & -41 \end{array}\right)$$

$$\left.\begin{array}{l} \rho(K) = 3 \\ \rho(A) = 2 \end{array}\right\} \Rightarrow \rho(K) \neq \rho(A)$$

$\Rightarrow$ It has no solution or it is in-consistent.

→ No solution:- $\lambda - a = 0$ but $\mu - b \neq 0$

$\Rightarrow \lambda = a$ & $\mu \neq b$

→ Infinite solution:- $\lambda - a = 0$ & $\mu - b = 0$

$\Rightarrow \lambda = a$ & $\mu = b$

→ Unique solution:- $\lambda - a \neq 0$ & $\mu - b =$ can be any value.

$\Rightarrow \lambda \neq a$ & μ is any value.

Q. find the λ and μ value such that the system

$2x+3y+5z=9$

$7x+3y-2z=8$

$2x+3y+\lambda z=\mu$ has

(i) No solution

(ii) unique solution

(iii) infinite solution

$\longrightarrow$

$$K = \left(\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{array}\right)$$ R_1 is fixed

$$= \left(\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 14-14 & 21-6 & 35+4 & 63-16 \\ 2-2 & 3-3 & 5-\lambda & 9-\mu \end{array}\right)$$ $R_2 \to 7R_1 - 2R_2$, $R_3 \to R_1 - R_3$

$$= \left(\begin{array}{ccc|c} 2 & 2 & 5 & 9 \\ 0 & 15 & 39 & 47 \\ 0 & 0 & 5-\lambda & 9-\mu \end{array}\right)$$

Case-I (No. Solution)

If $\rho(A) \neq \rho(K)$, then it will have no solution.

then, $5-\lambda=0$ & $9-\mu \neq 0$

$\lambda = 0$, $\mu \neq 9$

Case-II (unique solution)

If $\rho(A) = \rho(K)$ = number of variable $=3$

then it has unique solution.

ie $5-\lambda \neq 0$ & $9-\mu$ any value

$\Rightarrow \lambda \neq 5$ & μ is any value.

Case - III (Infinite Solution)

If $\rho(K) = \rho(A) < $ no. of variable $= 3$

then it has infinite number of solution.

ie $5 - \lambda = 0$ and $9 - \mu = 0$

$\Rightarrow \lambda = 5$ & $\mu = 9$

Q Evaluate the Rank of the matrix.

$$\begin{pmatrix} 1 & 3 & 2 & 5 & 1 \\ 2 & 2 & -1 & 6 & 3 \\ 1 & 1 & 2 & 3 & -1 \\ 0 & 2 & 5 & 2 & -3 \end{pmatrix} \quad R_1 \text{ is fixed}$$

$$= \begin{pmatrix} 1 & 3 & 2 & 5 & 1 \\ 2-2 & 6-2 & 4+1 & 10-6 & 2-3 \\ 1-1 & 3-1 & 2-2 & 5-3 & 1+1 \\ 0 & 2 & 5 & 2 & -3 \end{pmatrix} \quad \begin{matrix} R_2 \to 2R_1 - R_2 \\ R_3 \to R_1 - R_3 \end{matrix}$$

$$= \begin{pmatrix} 1 & 3 & 2 & 5 & 1 \\ 0 & 4 & 5 & 4 & -1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 2 & 5 & 2 & -3 \end{pmatrix} \quad R_2 \text{ is fixed}$$

$$= \begin{pmatrix} 1 & 3 & 2 & 5 & 1 \\ 0 & 4 & 5 & 4 & -1 \\ 0 & 4-4 & 5-0 & 4-4 & -1-4 \\ 0 & 4-4 & 5-10 & 4-4 & -1+6 \end{pmatrix} \quad \begin{matrix} R_3 \to R_2 - 2R_3 \\ R_4 \to R_2 - 2R_4 \end{matrix}$$

$$= \begin{pmatrix} 1 & 3 & 2 & 5 & 1 \\ 0 & 4 & 5 & 4 & -1 \\ 0 & 0 & 5 & 0 & -5 \\ 0 & 0 & -5 & 0 & 5 \end{pmatrix}$$

R_3 is fixed

$R_4 \to R_3 + R_4$

$$= \begin{pmatrix} 1 & 3 & 2 & 5 & 1 \\ 0 & 4 & 5 & 4 & -1 \\ 0 & 0 & 5 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ Rank of the matrix.

$\rho(A) = 3$